TOY QUANTUM FIELD THEORY[‡]

Populations of indistinguishable finite-state systems

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Introduction. Quantum mechanics, as it sprang from the brows of Heisenberg and Schrödinger, was a theory of mechanical systems endowed—like, most notably, the hydrogen atom—with finitely many degrees of freedom. But it had been obvious to Bohr already in 1913 that it is by the absorption/emission of "photons" (Einstein's invention: 1905) that atoms reveal their mechanical designs, and was obvious to the founding fathers of the "new quantum mechanics" that their creation called urgently for another—the creation of a "dynamical theory of photons," a "quantum electrodynamics"... and, more generally, of a "quantum field theory."

Such an effort began almost immediately. Photons move relativistically (i.e., emerge from a Maxwellian electrodynamics known to be Lorentz covariant), had been known since 1925 to obey Einstein-Bose statistics, are radiated and absorbed (created and annihilated). By broad implication one might on those grounds expect quantum field theory to be *relativistic*, and to embrace the notions of *indistinguishability* and *ephemerality*. But it was not immediately appreciated that those seemingly distinct features are, in fact, intimately intertwined, largely inseparable. It took roughly twenty years for relativistic quantum field theory—the work of many first-rate minds—to achieve a kind of tentative completion.[1]

The resulting ediface lies at the foundation of particle physics and plays an important role also in modern statistical mechanics. But quantum field theory is generally considered to be a "difficult" subject, and is accorded no place in the standard undergraduate curriculum ... which is why I have given scarcely

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1 See Chapter I in S. S. Schweber, *QED and the Men Who Made It: Dyson, Feynman, Schwinger, and Tomonaga* (1994) for a good account of the intricate historical details.

any thought to the subject for nearly forty years, and why I am a little surprised to find myself speaking about it today before such an audience.[2] But recent work in quite another connection (to which I will allude near the end of the hour) has brought to my attention an approach to the subject which, it seems to me, *is* readily accessible to undergraduates. It is that approach, and the "toy quantum field theory" to which it leads, that I take this opportunity to describe.

Setting the stage. Think of some mechanical system $\mathfrak{S}$—it might, for example, be an oscillator, or an atom—upon which we propose to do quantum mechanics. The states $|\psi)$ of $\mathfrak{S}$ are elements of a complex vector space $\mathcal{V}$ which is usually understood to be ∞-dimensional... and (as can be shown) *must* be so if the theory is to accommodate the fundamental statement

$$\big[\mathbf{x}, \mathbf{p}\big] = i\hbar\mathbf{I} \tag{1}$$

But there exist contexts (some merely expository, some deeply physical) in which infinite-dimensionality is replaced by finite-dimensionality,[3] and it is the pullback $N \longleftarrow \infty$ that lies at the heart of what I will have to say.

If $\mathfrak{S}$ is an N-state system;[4] i.e., if $\mathcal{V}$ is N-dimensional... then the state of the system can be described by an ordinary complex N-vector $\boldsymbol{\psi}$. Erect—arbitrarily—within $\mathcal{V}$ an orthonormal basis $\big\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N\big\}$ and obtain

$$\boldsymbol{\psi} = \psi_1\boldsymbol{e}_1 + \psi_2\boldsymbol{e}_2 + \cdots + \psi_N\boldsymbol{e}_N = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \tag{2}$$

where $\boldsymbol{\psi}^{\mathsf{t}}\boldsymbol{\psi} = \psi_1^*\psi_1 + \psi_2^*\psi_2 + \cdots + \psi_N^*\psi_N = 1$ is understood.

Consider now the composite system $\mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_n$ assembled from n (individually identifiable, non-interactive) copies of $\mathfrak{S}$. It is natural to write $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_n$ to describe the respective states of the individual members of the composite, but what should we write to describe the state of the composite itself? The natural answer, I claim, is provided by...

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[2] David Griffiths, in the very helpful introduction to his *Introduction to Elementary Particles* (1987), writes (with a dismissive "by the way") "quantum field theory in all its glory is difficult and deep, but don't be alarmed: Feynman invented a beautiful and intuitively satisfying formulation that is not hard to learn... (The *derivation* of Feynman's rules from the underlying quantum field theory is a different matter, which can easily consume the better part of an advanced graduate course, but this need not concern us here.)"

[3] See, for example, §9.1 in Griffiths' *Introduction to Quantum Mechanics* (1994), or Chapter 1 in my ADVANCED QUANTUM TOPICS (2000).

[4] Such systems are, in the case $N = 2$, often called "spin systems." See §4.4 in Griffiths' text.[3]

The Kronecker product. Let $\mathbb{A}$ be an $m \times n$ rectangular matrix, and let $\mathbb{B}$ be $p \times q$. The "Kronecker product" of $\mathbb{A}$ on $\mathbb{B}$—otherwise known as the "direct," or "outer," or "tensor" product—is denoted/defined

$$\mathbb{A} \otimes \mathbb{B} \equiv \begin{pmatrix} a_{11}\,\mathbb{B} & a_{12}\,\mathbb{B} & \ldots & a_{1n}\,\mathbb{B} \\ a_{21}\,\mathbb{B} & a_{22}\,\mathbb{B} & \ldots & a_{2n}\,\mathbb{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbb{B} & a_{m2}\mathbb{B} & \ldots & a_{mn}\mathbb{B} \end{pmatrix}$$

and is $mp \times nq$. The following properties of the Kronecker product follow almost immediately from the definition[5]

$$k(\mathbb{A} \otimes \mathbb{B}) = (k\mathbb{A}) \otimes \mathbb{B} = \mathbb{A} \otimes (k\mathbb{B}) \tag{3.1}$$

$$\left.\begin{aligned} (\mathbb{A} + \mathbb{B}) \otimes \mathbb{C} &= \mathbb{A} \otimes \mathbb{C} + \mathbb{B} \otimes \mathbb{C} \\ \mathbb{A} \otimes (\mathbb{B} + \mathbb{C}) &= \mathbb{A} \otimes \mathbb{B} + \mathbb{A} \otimes \mathbb{C} \end{aligned}\right\} \tag{3.2}$$

$$\mathbb{A} \otimes (\mathbb{B} \otimes \mathbb{C}) = (\mathbb{A} \otimes \mathbb{B}) \otimes \mathbb{C} \equiv \mathbb{A} \otimes \mathbb{B} \otimes \mathbb{C} \tag{3.3}$$

$$(\mathbb{A} \otimes \mathbb{B})^{\mathsf{T}} = \mathbb{A}^{\mathsf{T}} \otimes \mathbb{B}^{\mathsf{T}} \tag{3.4}$$

$$\mathrm{tr}(\mathbb{A} \otimes \mathbb{B}) = \mathrm{tr}\mathbb{A} \cdot \mathrm{tr}\,\mathbb{B} \tag{3.5}$$

and are valid except when meaningless. Less obvious—but of special importance in what follows—is the identity

$$(\mathbb{A} \otimes \mathbb{B})(\mathbb{C} \otimes \mathbb{D}) = \mathbb{A}\mathbb{C} \otimes \mathbb{B}\mathbb{D} \quad \text{if} \quad \begin{cases} \mathbb{A} \text{ is } m \times p \\ \mathbb{B} \text{ is } n \times q \\ \mathbb{C} \text{ is } p \times u \\ \mathbb{D} \text{ is } q \times v \end{cases} \tag{3.6}$$

from which one can extract

$$\mathbb{A} \otimes \mathbb{B} = (\mathbb{A} \otimes \mathbb{I}_n)(\mathbb{I}_m \otimes \mathbb{B}) \tag{3.7}$$

$$\det(\mathbb{A} \otimes \mathbb{B}) = (\det \mathbb{A})^n (\det \mathbb{B})^m \tag{3.8}$$

$$(\mathbb{A} \otimes \mathbb{B})^{-1} = \mathbb{A}^{-1} \otimes \mathbb{B}^{-1} \tag{3.9}$$

Here I have used $\mathbb{I}_m$ to designate the $m \times m$ identity matrix, and at (3.6) I have spelled out the "multiplicative conformability" conditions in the absence of which the identity would become meaningless. Notice that the Kronecker product is *associative*

$$(\mathbb{A} \otimes \mathbb{B}) \otimes \mathbb{C} = \mathbb{A} \otimes (\mathbb{B} \otimes \mathbb{C}) \quad : \quad \text{both written } \mathbb{A} \otimes \mathbb{B} \otimes \mathbb{C}$$

but *not commutative*: $\mathbb{A} \otimes \mathbb{B} \neq \mathbb{B} \otimes \mathbb{A}$ except under special conditions. It is useful to be aware also that *Mathematica* constructs the Kronecker product in response to the command **`Outer[Times,A,B]`**.

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[5] A version of this list appears as (63) in Chapter 1 of ADVANCED QUANTUM TOPICS, where references to relevant literature can be found.

The proposal now is that the state of $\mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_n$ be described

$$\boldsymbol{\psi}_{\text{composite}} = \boldsymbol{\psi}_1 \otimes \boldsymbol{\psi}_2 \otimes \cdots \otimes \boldsymbol{\psi}_n$$

which is an $N^n \times 1$ matrix: in short, an N^n-vector. Look, for example, to the case $n = 2$, $N = 3$ where (relative to a prescribed basis) we have

$$\boldsymbol{\psi}_{\text{composite}} = \boldsymbol{\psi} \otimes \boldsymbol{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \otimes \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \psi_1\psi_1 \\ \psi_1\psi_2 \\ \psi_1\psi_3 \\ \psi_2\psi_1 \\ \psi_2\psi_2 \\ \psi_2\psi_3 \\ \psi_3\psi_1 \\ \psi_3\psi_2 \\ \psi_3\psi_3 \end{pmatrix} = \sum_{i,j} \psi_i\psi_j \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

which is a $(3^2 = 9)$-vector.

The following little argument serves (among other things) to illustrate the power of the identities (3): $\boldsymbol{\psi}_1 \otimes \boldsymbol{\psi}_2$ is $N^2 \times 1$, so $(\boldsymbol{\psi}_1 \otimes \boldsymbol{\psi}_2)^{\mathsf{t}} = \boldsymbol{\psi}_1^{\mathsf{t}} \otimes \boldsymbol{\psi}_2^{\mathsf{t}}$ is $1 \times N^2$ and $(\boldsymbol{\psi}_1 \otimes \boldsymbol{\psi}_2)^{\mathsf{t}}(\boldsymbol{\psi}_1 \otimes \boldsymbol{\psi}_2)$ is 1×1; i.e., a (real) number. Appealing to (3.6) we have $(\boldsymbol{\psi}_1 \otimes \boldsymbol{\psi}_2)^{\mathsf{t}}(\boldsymbol{\psi}_1 \otimes \boldsymbol{\psi}_2) = (\boldsymbol{\psi}_1^{\mathsf{t}}\boldsymbol{\psi}_1) \cdot (\boldsymbol{\psi}_2^{\mathsf{t}}\boldsymbol{\psi}_2)$. This result generalizes straightforwardly, and informes us that

$$\text{if } \boldsymbol{\psi}_i^{\mathsf{t}}\boldsymbol{\psi}_i = 1 \text{ for } i = 1, 2, \ldots, n \text{ then so also does } \boldsymbol{\psi}^{\mathsf{t}}\boldsymbol{\psi} = 1$$

i.e., that the normalization of the component state vectors implies that of the composite state $\boldsymbol{\psi}_{\text{composite}}$.

Accommodating indistinguishability. Return for a moment to ordinary quantum mechanics, where one would (in the $\boldsymbol{x}$-representation) write $\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$ to describe the state of a composite sytem, and

$$P(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) = |\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)|^2$$

to describe the associated probability density. Suppose now that the constituent systems are “indistinguishable” because *too simple to support identifying marks* (as pool balls do). $P(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$ must then be a *symmetric* function of its arguments; i.e., it must be invariant under all permutations $\wp$ of the $\boldsymbol{x}_i$. By a bit of a leap—well supported by the physical evidence—one standardly takes that to mean that the ensuing theory comes in two flavors

$$\begin{aligned} \text{BOSONIC CASE} &: \ \Psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) \text{ is totally symmetric} \\ \text{FERMIONIC CASE} &: \ \Psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) \text{ is totally antisymmetric} \end{aligned}$$

and discards all intermediate (mixed-symmetry) possibilities. How are such assumptions to be incorporated into the N-state formalism?

In the *absence* of indistinguishability we wrote

$$\boldsymbol{\psi}_{\text{composite}} = \sum_{i_1 i_2 \cdots i_n} \psi_{i_1 i_2 \cdots i_n} \boldsymbol{e}_{i_1} \otimes \boldsymbol{e}_{i_2} \otimes \cdots \otimes \boldsymbol{e}_{i_n}$$

where the N^n-tuple of N^n-vectors comprise an "induced basis" in

$$\mathcal{V}^n = \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_n$$

Drawing again upon (3.4) and (3.6), we have

$$(\boldsymbol{e}_{i_1} \otimes \boldsymbol{e}_{i_2} \otimes \cdots \otimes \boldsymbol{e}_{i_n})^{\mathsf{t}} (\boldsymbol{e}_{j_1} \otimes \boldsymbol{e}_{j_2} \otimes \cdots \otimes \boldsymbol{e}_{j_n}) = \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_n j_n} \tag{4}$$

which shows how the induced basis inherits orthonormality from the assumed orthonormality $\boldsymbol{e}_i^{\mathsf{t}} \boldsymbol{e}_j = \delta_{ij}$ of the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N\}$ we deposited in $\mathcal{V}$.

In the presence of *bosonic indistinguishability* we will write

$$\boldsymbol{\psi}_{\text{bosonic}} \in \mathcal{V}^n_{\text{bosonic}}$$

where $\mathcal{V}^n_{\text{bosonic}} \subset \mathcal{V}^n$ is spanned by the *symmetrizations* of $\{\boldsymbol{e}_{i_1} \otimes \boldsymbol{e}_{i_2} \otimes \cdots \otimes \boldsymbol{e}_{i_n}\}$:

$$\boldsymbol{e}_{i_1} \circ \boldsymbol{e}_{i_2} \circ \cdots \circ \boldsymbol{e}_{i_n} \equiv \frac{1}{\sqrt{n!}} \sum_{\wp} \boldsymbol{e}_{i_1} \otimes \boldsymbol{e}_{i_2} \otimes \cdots \otimes \boldsymbol{e}_{i_n} \tag{5}$$

We can/will now assume without loss of generality that the indices are in dictionary order: $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n$. The only details that really matter are *how many* of the i's are 1's, how many are 2's, etc. We will adopt this notation

$$\left.\begin{aligned} n_1 &\equiv \text{number of 1's} \\ n_2 &\equiv \text{number of 2's} \\ &\vdots \\ n_N &\equiv \text{number of } N\text{'s} \end{aligned}\right\} \quad : \quad n_1 + n_2 + \cdots + n_N = n$$

and call the n_k's "occupation numbers." To see how this works in a concrete case, take $N = 3$, $n = 2$. Then

$$\boldsymbol{e}_i \circ \boldsymbol{e}_j = \tfrac{1}{\sqrt{2}} \{\boldsymbol{e}_i \otimes \boldsymbol{e}_j + \boldsymbol{e}_j \otimes \boldsymbol{e}_i\}$$

supplies this 6-tuple of 9-vectors: $\{\boldsymbol{e}_1 \circ \boldsymbol{e}_1,\, \boldsymbol{e}_1 \circ \boldsymbol{e}_2,\, \boldsymbol{e}_1 \circ \boldsymbol{e}_3,\, \boldsymbol{e}_2 \circ \boldsymbol{e}_2,\, \boldsymbol{e}_2 \circ \boldsymbol{e}_3,\, \boldsymbol{e}_3 \circ \boldsymbol{e}_3\}$. If we take

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

then we have these explicit descriptions of those six vectors

$$\begin{aligned}
(\boldsymbol{e}_1\circ\boldsymbol{e}_1)^{\rm t} &= \tfrac{1}{\sqrt{2}}\,(\,2\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\,)\\
(\boldsymbol{e}_1\circ\boldsymbol{e}_2)^{\rm t} &= \tfrac{1}{\sqrt{2}}\,(\,0\quad 1\quad 0\quad 1\quad 0\quad 0\quad 0\quad 0\quad 0\,)\\
(\boldsymbol{e}_1\circ\boldsymbol{e}_3)^{\rm t} &= \tfrac{1}{\sqrt{2}}\,(\,0\quad 0\quad 1\quad 0\quad 0\quad 0\quad 1\quad 0\quad 0\,)\\
(\boldsymbol{e}_2\circ\boldsymbol{e}_2)^{\rm t} &= \tfrac{1}{\sqrt{2}}\,(\,0\quad 0\quad 0\quad 0\quad 2\quad 0\quad 0\quad 0\quad 0\,)\\
(\boldsymbol{e}_2\circ\boldsymbol{e}_3)^{\rm t} &= \tfrac{1}{\sqrt{2}}\,(\,0\quad 0\quad 0\quad 0\quad 0\quad 1\quad 0\quad 1\quad 0\,)\\
(\boldsymbol{e}_3\circ\boldsymbol{e}_3)^{\rm t} &= \tfrac{1}{\sqrt{2}}\,(\,0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 2\,)
\end{aligned}$$

where it is for typographic reasons that I display the adjoints (transposes) of the column vectors in question, rather than the column vectors themselves. These vectors are orthogonal by inspection, but also the argument that gave (4); they are, however, normalized only in the cases $i \neq j$. A combinatorial argument leads easily to the general conclusions that

$$(\boldsymbol{e}_{i_1}\circ\boldsymbol{e}_{i_2}\circ\cdots\circ\boldsymbol{e}_{i_n}) \perp (\boldsymbol{e}_{j_1}\circ\boldsymbol{e}_{j_2}\circ\cdots\circ\boldsymbol{e}_{j_n}) \tag{6.1}$$

unless their respective occupation numbers are identical, in which case one has

$$(\boldsymbol{e}_{i_1}\circ\boldsymbol{e}_{i_2}\circ\cdots\circ\boldsymbol{e}_{i_n})^{\rm t}(\boldsymbol{e}_{i_1}\circ\boldsymbol{e}_{i_2}\circ\cdots\circ\boldsymbol{e}_{i_n}) = n_1!n_2!\cdots n_N! \tag{6.2}$$

The vectors

$$|n_1 n_2 \dots n_N) \equiv \frac{1}{\sqrt{n_1!n_2!\cdots n_N!}}\;\boldsymbol{e}_{i_1}\circ\boldsymbol{e}_{i_2}\circ\cdots\circ\boldsymbol{e}_{i_n} \tag{7}$$

are therefore orthonormal:

$$(m_1 m_2 \dots m_N|n_1 n_2 \dots n_N) = \delta_{m_1 n_1}\delta_{m_2 n_2}\cdots\delta_{m_N n_N} \tag{8}$$

On the other hand...in the presence of *fermionic indistinguishability* we write

$$\boldsymbol{\psi}_{\text{fermionic}} \in \mathcal{V}^n_{\text{fermionic}}$$

where $\mathcal{V}^n_{\text{fermionic}}$ is spanned by the *antisymmetrizations* of $\big\{\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\cdots\otimes\boldsymbol{e}_{i_n}\big\}$:

$$\boldsymbol{e}_{i_1}\wedge\boldsymbol{e}_{i_2}\wedge\cdots\wedge\boldsymbol{e}_{i_n} \equiv \frac{1}{\sqrt{n!}}\sum_{\wp}(-)^{\wp}\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\cdots\otimes\boldsymbol{e}_{i_n} \tag{9}$$

The $\wedge$ notation has been borrowed from exterior algebra, which we are now, in effect, reproducing. Our former assumption that the indices are in dictionary order $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n$ now—for the first time—acquires some significant "standardizing" consequences, but since

antisymmetry forces all occupation numbers to be either 0 or 1 it acquires this sharper form: $i_1 < i_2 < \cdots < i_n$. To see how antisymmetrization works in a concrete case, let us again take $N = 3$, $n = 2$. Then

$$\boldsymbol{e}_i \wedge \boldsymbol{e}_j = \tfrac{1}{\sqrt{2}}\big\{\boldsymbol{e}_i \otimes \boldsymbol{e}_j - \boldsymbol{e}_j \otimes \boldsymbol{e}_i\big\}$$

supplies now only a 3-tuple of 9-vectors: $\big\{\boldsymbol{e}_1 \wedge \boldsymbol{e}_2,\, \boldsymbol{e}_1 \wedge \boldsymbol{e}_3,\, \boldsymbol{e}_2 \wedge \boldsymbol{e}_3\big\}$. Explicitly

$$\begin{aligned}
(\boldsymbol{e}_1 \circ \boldsymbol{e}_2)^{\text{t}} &= \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\\
(\boldsymbol{e}_1 \circ \boldsymbol{e}_3)^{\text{t}} &= \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}\\
(\boldsymbol{e}_2 \circ \boldsymbol{e}_3)^{\text{t}} &= \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}
\end{aligned}$$

These vectors are seen by inspection to be not only orthogonal but also already normalized. An argument based again upon (3.4) and (3.6) leads to the more general conclusions that

$$(\boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \wedge \cdots \wedge \boldsymbol{e}_{i_n}) \perp (\boldsymbol{e}_{j_1} \wedge \boldsymbol{e}_{j_2} \wedge \cdots \wedge \boldsymbol{e}_{j_n})$$

unless their respective occupation numbers are identical, in which case one has

$$\begin{aligned}
(\boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \wedge \cdots \wedge \boldsymbol{e}_{i_n})^{\text{t}}(\boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \wedge \cdots \wedge \boldsymbol{e}_{i_n}) &= 1\\
&= n_1!\, n_2! \cdots n_N! \quad \text{by} \quad 0! = 1! = 1
\end{aligned}$$

The vectors

$$|n_1 n_2 \ldots n_N) \equiv \frac{1}{\sqrt{n_1! n_2! \cdots n_N!}}\, \boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \wedge \cdots \wedge \boldsymbol{e}_{i_n} \tag{10}$$

(in which the prefactor is merely cosmetic, and can be abandoned) are therefore orthonormal. One seldom has occasion to speak of bosons and fermions in the same breath, so there is little chance that confusion will arise when one drops the pedantic labels from (for example)

$$\begin{aligned}
|1,1)_{\text{bosonic}} &\equiv \tfrac{1}{\sqrt{2}}\, \boldsymbol{e}_1 \circ \boldsymbol{e}_2\\
|1,1)_{\text{fermionic}} &\equiv \tfrac{1}{\sqrt{2}}\, \boldsymbol{e}_1 \wedge \boldsymbol{e}_2
\end{aligned}$$

We are in position now to observe that

$$\begin{aligned}
[N,n]_{\text{bosonic}} &\equiv \text{dimension of } \mathcal{V}^n_{\text{bosonic } N\text{-state}}\\
&= \text{number of terms in } (x_1 + x_2 + \cdots + x_N)^n\\
&= \begin{cases} \text{number of “words” } {*}{*}\,|\,{*}\,||\,{*} \cdots {*}\,|{*} \\ \text{constructable from } n\ {*}\text{'s and } (N-1)\ |\text{'s} \end{cases}\\
&= \frac{(N+n-1)!}{(N-1)!\; n!}
\end{aligned} \tag{11.1}$$

On the other hand, state spaces $\mathcal{V}^{n>N}_{\text{fermionic}}$ are empty: it is (by antisymmetry) *impossible to construct a fermionic population containing more subsystems than each system has states*. Generally

$$
\begin{aligned}
[N,n]_{\text{fermionic}} &\equiv \text{dimension of } \mathcal{V}^n_{\text{fermionic } N\text{-state}} \\
&= \begin{cases} \text{number of ``words'' } **|*||*\cdots*|* \\ \text{constructable from } n\ *\text{'s and } (N-1)\ |\text{'s} \\ \text{if adjacent } *\text{'s are disallowed} \end{cases} \\
&= \begin{cases} \binom{N}{n} \text{ if } n = 1,2,\ldots,N \\ \\ \ \ 0 \quad \text{if } n > N \end{cases}
\end{aligned} \tag{11.2}
$$

Since "it takes two to (anti)symmetrize" there is, in the present theory, no distinction between $\mathcal{V}^1_{\text{bosonic}}$ and $\mathcal{V}^1_{\text{fermionic}}$. Nor, for other reasons, is there a distinction between

$$\mathcal{V}^0_{\text{bosonic}} = \mathcal{V}^0_{\text{fermionic}} = \mathcal{V}_{\text{vacuum}}$$

which is a formal device, the residence of

$$|\text{vac}) = |0,0,\ldots,0) \quad : \quad (\text{vac}|\text{vac}) = 1 \tag{12}$$

which we must be careful to distinguish from the null vector that lives in *every* vector space.

Creation & annihilation operators: accommodating ephemerality. We turn now to the description of some operators the intended effect of which is very easy to describe:

$$\mathbf{b}_k\,|n_1,n_2,\ldots,n_k,\ldots,n_N) \sim |n_1,n_2,\ldots,n_k+1,\ldots,n_N) \tag{13.1}$$

From the definition of the adjoint

$$(m_1,m_2,\ldots|\overset{\text{acts to the right}\downarrow}{\mathbf{b}_k}|n_1,n_2,\ldots) \equiv (m_1,m_2,\ldots|\overset{\downarrow\text{acts to the left}}{\mathbf{b}_k^{\mathsf{t}}}|n_1,n_2,\ldots)$$

we conclude that $\mathbf{b}_k^{\mathsf{t}}$ must have the opposite effect:

$$\mathbf{b}_k^{\mathsf{t}}\,|n_1,n_2,\ldots,n_k,\ldots,n_N) \sim |n_1,n_2,\ldots,n_k-1,\ldots,n_N) \tag{13.2}$$

Look first to the bosonic case. By design, each $\mathbf{b}$ is intended to achieve the *symmetrized admixture of an* $\boldsymbol{e}$, as illustrated below:

$$
\begin{aligned}
&\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k \\
&\downarrow \\
&\underbrace{\boldsymbol{e}\otimes\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k+\boldsymbol{e}_i\otimes\boldsymbol{e}\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k+\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}\otimes\boldsymbol{e}_k+\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}}_{\text{``symmetrized admixture,'' denoted } \boldsymbol{e}\,Ⓢ\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k}
\end{aligned}
$$

Each of the 3! terms that entered into the definition $\boldsymbol{e}_i \circ \boldsymbol{e}_j \circ \boldsymbol{e}_k$ has expanded by this mechanism into 4 terms. It becomes natural in the light of the example to define

$$\begin{aligned}
\mathsf{b}_k|n_1 \dots n_k \dots n_N) &\equiv \frac{1}{\sqrt{n+1}} \cdot \boldsymbol{e}_k \,\text{Ⓢ}\, |n_1 \dots n_k \dots n_N) \\
&= \frac{1}{\sqrt{n_1! \cdots n_k! \cdots n_N}} \sum_{\wp} \frac{\boldsymbol{e}_k \,\text{Ⓢ}\, \boldsymbol{e}_{i_1} \otimes \boldsymbol{e}_{i_2} \otimes \cdots \otimes \boldsymbol{e}_{i_n}}{\sqrt{n+1}\sqrt{n!}} \\
&= \sqrt{n_k+1} \cdot \frac{1}{\sqrt{n_1! \cdots (n_k+1)! \cdots n_N!}} \sum_{\wp} \frac{\boldsymbol{e}_k \,\text{Ⓢ}\, \boldsymbol{e}_{i_1} \otimes \boldsymbol{e}_{i_2} \otimes \cdots \otimes \boldsymbol{e}_{i_n}}{\sqrt{n+1}\sqrt{n!}} \\
&= \sqrt{n_k+1} \cdot |n_1 \dots n_k + 1 \dots n_N)
\end{aligned}$$

which lends these sharpened meanings to (13):

$$\mathsf{b}_k\, |n_1, n_2, \dots, n_k, \dots, n_N) = \sqrt{n_k+1} \cdot |n_1, n_2, \dots, n_k + 1, \dots, n_N) \qquad (14.1)$$
$$\mathsf{b}_k^{\mathsf{t}}\, |n_1, n_2, \dots, n_k, \dots, n_N) = \sqrt{n_k} \cdot |n_1, n_2, \dots, n_k - 1, \dots, n_N) \qquad (14.2)$$

While b_k accomplishes "symmetrized admixture," $\mathsf{b}_k^{\mathsf{t}}$ achieves "symmetrized extraction" of an $\boldsymbol{e}_k$, which is a rather more intricate process; an argument based again upon (3.6) shows the process to be mechanized by the $N^{n-1} \times N^n$ rectangular matrix

$$\mathbb{B}_k = \sqrt{n}\,\big(\mathbb{I}_{N^{n-1}} \otimes \boldsymbol{e}_k\big)^{\mathsf{t}}$$

but I will not linger to review the elementary details.[6]

Turning now to the fermionic case... we have adopted the convention that

"dictionary order" is the "canonical order"

and it is here that that convention does its work. We have interest now in the *antisymmetrized admixture of an* $\boldsymbol{e}$, as illustrated below:

$$\begin{aligned}
&\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \\
&\downarrow \\
&\underbrace{\boldsymbol{e} \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k - \boldsymbol{e}_i \otimes \boldsymbol{e} \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k + \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e} \otimes \boldsymbol{e}_k - \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}}_{\text{“antisymmetrized admixture,” denoted } \boldsymbol{e}\,\text{Ⓐ}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k}
\end{aligned}$$

[6] Those can be found at (41) in "Comments concerning Julian Schwinger's 'On Angular Momentum' " (October 2000), from which my remarks today have been excerpted.

It becomes in this light natural to define

$$
\begin{aligned}
\mathsf{b}_j|n_1,n_2,\ldots,n_N) &\equiv \frac{1}{\sqrt{n+1}}\cdot \boldsymbol{e}_j \text{Ⓐ} |n_1,n_2,n_3)\\
&= \begin{cases} \mathbf{0} & \text{if } j \text{ already present in the ordered } i\text{-list}\\ \displaystyle\sum_\wp (-)^\wp \frac{\boldsymbol{e}_j \text{Ⓐ}\, \boldsymbol{e}_{i_1}\otimes \boldsymbol{e}_{i_2}\otimes\cdots\otimes \boldsymbol{e}_{i_n}}{\sqrt{n+1}\sqrt{n!}} & \text{if } j \text{ absent}\end{cases}\\
&= \begin{cases} \mathbf{0} \\ \displaystyle(-)^S\sum_\wp (-)^\wp \frac{\boldsymbol{e}_{i_1}\otimes\cdots\otimes \boldsymbol{e}_{i_s}\otimes \boldsymbol{e}_j\otimes \boldsymbol{e}_{k_1}\otimes\cdots\otimes \boldsymbol{e}_{k_{n-s}}}{\sqrt{(n+1)!}}\end{cases}
\end{aligned}
$$

according as $n_j = 1$ or $n_j = 0$. Here we have moved $\boldsymbol{e}_j$ to its canonical place

$$\text{all } i\text{'s} < j < \text{all } k\text{'s}$$

and

$$
\begin{aligned}
s \equiv s_j &\equiv \text{number of } \boldsymbol{e}_i \text{ that conventionally stand left of } \boldsymbol{e}_j\\
&= \sum_{i<j} n_i
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\mathsf{b}_j|n_1,n_2,\ldots,n_j,\ldots,n_N)\\
&\qquad = (-)^{S_j}(1-n_j)\cdot|n_1,n_2,\ldots,n_j+1,\ldots,n_N)
\end{aligned}
\tag{15.1}
$$

where we have installed

$$1-n_j = \begin{cases} 0 & \text{if} \quad n_j = 1\\ 1 & \text{if} \quad n_j = 0\end{cases}$$

as a nifty "switch" to distinguish one case from the other.[7] It follows already from (15.1) by appeal to the *meaning* of the adjoint that

$$
\begin{aligned}
&\mathsf{b}_j^{\mathsf{t}}\,|n_1,n_2,\ldots,n_j,\ldots,n_N)\\
&\qquad = (-)^{S_j}\, n_j\cdot|n_1,n_2,\ldots,n_j-1,\ldots,n_N)
\end{aligned}
\tag{15.2}
$$

We are in possession now of apparatus sufficient to produce (along lines already sketched) a "mechanized" account of $\mathsf{b}_j^{\mathsf{t}}$, but I won't.

---

[7] The complementary switch—needed in a moment—is even simpler:

$$n_j = \begin{cases} 1 & \text{if} \quad n_j = 1\\ 0 & \text{if} \quad n_j = 0\end{cases}$$

I have found it easier, from a constructive standpoint, to discuss the $\mathsf{b}$irth of an $\boldsymbol{e}$-factor than its demise, and have elected to assign the simpler name to the simpler object. But now that all the work is behind us, we can revert

$$\begin{aligned} \mathsf{b}_j &\longmapsto \mathsf{a}_j^{\mathsf{t}} &:& \quad \text{creation operators} \\ \mathsf{b}_j^{\mathsf{t}} &\longmapsto \mathsf{a}_j &:& \quad \text{annihilation operators} \end{aligned}$$

to the notation that has long been standard. We have

$$\begin{aligned} &\mathsf{a}_j^{\mathsf{t}}|n_1, n_2, \ldots, n_j, \ldots, n_N) \\ &\qquad\qquad = \sqrt{n_j + 1} \cdot |n_1, n_2, \ldots, n_j + 1, \ldots, n_N) \end{aligned} \tag{16.11}$$

$$\begin{aligned} &\mathsf{a}_j\,|n_1, n_2, \ldots, n_j, \ldots, n_N) \\ &\qquad\qquad = \sqrt{\quad n_j \quad} \cdot |n_1, n_2, \ldots, n_j - 1, \ldots, n_N) \end{aligned} \tag{16.12}$$

$$\begin{aligned} &\mathsf{a}_j^{\mathsf{t}}|n_1, n_2, \ldots, n_j, \ldots, n_N) \\ &\qquad\qquad = (-)^{s_j}(1 - n_j) \cdot |n_1, n_2, \ldots, n_j + 1, \ldots, n_N) \end{aligned} \tag{16.21}$$

$$\begin{aligned} &\mathsf{a}_j\,|n_1, n_2, \ldots, n_j, \ldots, n_N) \\ &\qquad\qquad = (-)^{s_j}(\quad n_j \quad) \cdot |n_1, n_2, \ldots, n_j - 1, \ldots, n_N) \end{aligned} \tag{16.22}$$

in the bosonic/fermionic cases, respectively.

Bosonic commutators, fermionic anticommutators. In the former (bosonic) case it is evident that

$$\big[\mathsf{a}_i\,, \mathsf{a}_j\big] = \big[\mathsf{a}_i^{\mathsf{t}}\,, \mathsf{a}_j^{\mathsf{t}}\big] = \mathbf{0} \quad : \quad \text{all } i,\, j \tag{17.1}$$

and

$$\big[\mathsf{a}_i\,, \mathsf{a}_j^{\mathsf{t}}\big] = \mathbf{0} \quad : \quad i \neq j \tag{17.2}$$

while from

$$\begin{aligned} \mathsf{a}_i \mathsf{a}_i^{\mathsf{t}}\,|n_1, n_2, \ldots, n_i\,, \ldots, n_N) &= (n_i + 1)|n_1, n_2, \ldots, n_i\,, \ldots, n_N) \\ \mathsf{a}_i^{\mathsf{t}} \mathsf{a}_i\,|n_1, n_2, \ldots, n_i\,, \ldots, n_N) &= (\quad n_i \quad)|n_1, n_2, \ldots, n_i\,, \ldots, n_N) \end{aligned}$$

we obtain

$$\big[\mathsf{a}_i\,, \mathsf{a}_i^{\mathsf{t}}\big] = \mathbf{I} \quad : \quad \text{all } i \tag{17.3}$$

To develop fermionic analogs of those statements we must take into account the sign factors, which have some curious consequences. Look first to $\mathsf{a}_i \mathsf{a}_j|$etc.) on the assumption that $i < j$. In $\mathsf{a}_i \mathsf{a}_j|$etc.) we encounter $(-)^{s_i + s_j}$ while the a_j in $\mathsf{a}_j \mathsf{a}_i|$etc.) *sees an extra term*, so presents $(-)^{s_i + s_j + 1} = -(-)^{s_i + s_j}$. This upshot of this line of argument is that

$$\big[\mathsf{a}_i\,,\mathsf{a}_j\big]_+ = \big[\mathsf{a}_i^{\mathsf{t}}\,,\mathsf{a}_j^{\mathsf{t}}\big]_+ = \mathbf{0} \quad : \quad \text{all } i,\, j \tag{18.1}$$

$$\big[\mathsf{a}_i\,,\mathsf{a}_j^{\mathsf{t}}\big]_+ = \mathbf{0} \quad : \quad i \neq j \tag{18.2}$$

where the $_+$ signifies *anti*commutation:

$$\big[\mathsf{A},\mathsf{B}\big]_+ \equiv \mathsf{A}\,\mathsf{B} + \mathsf{B}\,\mathsf{A}$$

It follows more simply from

$$\mathsf{a}_i\mathsf{a}_i^{\mathsf{t}}\,|\ldots,n_i,\ldots) = \begin{cases} (-)^{2S_i}\,|\ldots,n_i,\ldots) & \text{if} \quad n_i = 0 \\ \mathbf{0} & \text{if} \quad n_i = 1 \end{cases}$$
$$\mathsf{a}_i^{\mathsf{t}}\mathsf{a}_i\,|\ldots,n_i,\ldots) = \begin{cases} \mathbf{0} & \text{if} \quad n_i = 0 \\ (-)^{2S_i}\,|\ldots,n_i,\ldots) & \text{if} \quad n_i = 1 \end{cases}$$

and $(-)^{2S_i} = 1$ (all cases) that

$$\big[\mathsf{a}_i\,,\mathsf{a}_i^{\mathsf{t}}\big]_+ = \mathbf{I} \quad : \quad \text{all } i \tag{18.3}$$

Fock space, the field operator, second quantization. It is clear from the preceding discussion that

$$\begin{aligned} &\mathsf{a}_i \text{ acts on elements of } \mathcal{V}_N^n \text{ to yield elements of } \mathcal{V}_N^{n-1} \\ &\mathsf{a}_i^{\mathsf{t}} \text{ acts on elements of } \mathcal{V}_N^n \text{ to yield elements of } \mathcal{V}_N^{n+1} \end{aligned}$$

Which is to say: creation/annihilation operators have as their sphere of activity not the state spaces of individual composite systems, but the formal *union* of such spaces—a place called "Fock space"[8]

$$\mathcal{V}_{\text{Fock}} \equiv \begin{cases} \mathcal{V}_N^0 \oplus \mathcal{V}_N^1 \oplus \cdots \oplus \mathcal{V}_N^N \oplus \mathcal{V}_N^{N+1} \oplus \cdots & : \quad \text{bosonic case} \\ \mathcal{V}_N^0 \oplus \mathcal{V}_N^1 \oplus \cdots \oplus \mathcal{V}_N^N & : \quad \text{fermionic case} \end{cases}$$

which is

∞-dimensional in the bosonic case, but only

2^N-dimensional in the fermionic case.

Combinations of creation and annihilation operators, on the other hand, *may* be meaningful on individual $\mathcal{V}$'s: the so-called "number operators"

$$\mathsf{N}_i \equiv \mathsf{a}_i^{\mathsf{t}}\mathsf{a}_i \tag{19.1}$$

[8] V. Fock, "Konfigurationsraum und zweite Quantelung," Zeit. f. Physik **75**, 622 (1932). The idea injected into physics by Fock was already standard to the exterior calculus, where the $\boldsymbol{d}$ and $\boldsymbol{*}$ are similarly "international" in their action; see the final two pages of §2 in "Electrodynamical applications of the exterior calculus" (1996).

$$\mathsf{N}_i\,|\ldots,n_i,\ldots) = n_i\,|\ldots,n_i,\ldots) \quad : \quad \text{bosonic or fermionic}$$

and the "total number operator"

$$\mathsf{N} \equiv \sum_{i=1}^{N} \mathsf{N}_i \tag{19.2}$$

provide (hermitian) examples. We note in passing that

$$\left.\begin{array}{l} \big[\mathsf{N},\mathsf{a}_i\big] \;=\big[\mathsf{N}_i,\mathsf{a}_i\big] \;=-\mathsf{a}_i \\ \big[\mathsf{N},\mathsf{a}_i^{\text{t}}\big] \;=\big[\mathsf{N}_i,\mathsf{a}_i^{\text{t}}\big] \;=+\mathsf{a}_i^{\text{t}} \end{array}\right\} \quad : \quad \text{bosonic case} \tag{20.1}$$

$$\left.\begin{array}{l} \big[\mathsf{N},\mathsf{a}_i\big]_+ = \big[\mathsf{N}_i,\mathsf{a}_i\big]_+ = +\mathsf{a}_i \\ \big[\mathsf{N},\mathsf{a}_i^{\text{t}}\big]_+ = \big[\mathsf{N}_i,\mathsf{a}_i^{\text{t}}\big]_+ = +\mathsf{a}_i^{\text{t}} \end{array}\right\} \quad : \quad \text{fermionic case} \tag{20.2}$$

So many (essentially all) of the properties of $|n_1 n_2 \ldots n_N)$-states entered into the fabrication of the a's and a^{t}'s that we can now turn the process around, and use the established properties of creation/annihilation operators to *recreate* the states—to fashion them "from nothing:"

$$|n_1 n_2 \ldots n_N) = \frac{1}{\sqrt{n_1!\, n_2! \cdots n_N!}}(\mathsf{a}_1^{\text{t}})^{n_1}(\mathsf{a}_2^{\text{t}})^{n_2}\cdots(\mathsf{a}_N^{\text{t}})^{n_N}|\text{vac}) \tag{21}$$

The expression on the right is central to what in oscillator physics is called the theory of "states of minimal dispersion/uncertainty," and in quantum optics is called the "theory of coherent states."[9] Notice that *operator order* on the right is irrelevant in the bosonic case but—because (18.1) entails $\mathsf{a}_i^{\text{t}}\mathsf{a}_j^{\text{t}} = -\mathsf{a}_j^{\text{t}}\mathsf{a}_i^{\text{t}}$ —is of critical importance in the fermionic case; it is in this detail that the connection between "antisymmetry" and "anticommutivity" is most sharply exposed.

We come now to the profound development that (upon abandonment of our characteristic finite-state assumption) earns for this subject the name "quantum field theory," and that gives rise to the concept of "second quantization." As an expository convenience we set $N = 3$.

Let H be a hermitian operator defined on the state space $\mathcal{V}$ of a the solitary 3-state system $\mathfrak{S}$. To describe H relative to a selected orthonormal basis, Dirac would write

$$\mathsf{H} = \sum_{i,j}|i)(i|\mathsf{H}|j)(j| \equiv \sum_{i,j} H_{ij}|i)(j|$$

[9] See Chapter 11 in L. Mandell & E. Wolf, *Optical coherence and quantum optics* (1995) for detailed discussion and references (which date mainly from the 1960's).

When presented to $|\psi) = \sum_k |k)(k|\psi) \equiv \sum_k |k)\,\psi_k$ the operator $H_{ij}|i)(j|$ plucks out the $|j)$-component and turns it into an $|i)$-component, to which it assigns weight H_{ij}. To express the same idea, we might write

$$|\psi) = |1,0,0)(1,0,0|\psi) + |0,1,0)(0,1,0|\psi) + |0,0,1)(0,0,1|\psi)$$
$$|i)(j| = \mathsf{a}_i^{\mathsf{t}}\mathsf{a}_j$$
$$H_{11} = (1,0,0|\mathbf{H}|1,0,0), \quad H_{12} = (1,0,0|\mathbf{H}|0,1,0), \quad \text{etc.}$$

It becomes in this light natural to introduce

$$\boldsymbol{\Psi} \equiv |1,0,0)\,\mathsf{a}_1 + |0,1,0)\,\mathsf{a}_2 + |0,0,1)\,\mathsf{a}_3 \tag{22}$$

and to write

$$\underbrace{\mathcal{H}}_{\text{has become meaningful on } \mathcal{V}_{\text{Fock}}} = \boldsymbol{\Psi}^{\mathsf{t}}\overbrace{\mathbf{H}}^{\text{was initially defined only on } \mathcal{V}^1}\boldsymbol{\Psi} \tag{23}$$

We notice that

$$\boldsymbol{\Psi}\,|\text{vac}) = 0 \tag{24}$$

and that

$$\boldsymbol{\Psi}^{\mathsf{t}}\boldsymbol{\Psi} = \sum_i \mathsf{a}_i^{\mathsf{t}}\mathsf{a}_i = \sum_i \mathbf{N}_i = \mathbf{N} \tag{25}$$

In the ordinary quantum theory of $\mathfrak{S}$ the expression $(\psi|\mathbf{H}|\psi)$ carries the interpretation of a number-valued "expectation value." But the right side of (47) is *operator*-valued: it acts simultaneously on $\mathcal{V}^1$, $\mathcal{V}^2$, $\mathcal{V}^3, \ldots$; it acts, in short, on the elements

$$|\Psi) \in \mathcal{V}_{\text{Fock}} \quad : \quad (\Psi|\Psi) = 1$$

and in

$$\mathcal{H}\,|\Psi) = i\hbar\,\partial_t|\Psi) \tag{26}$$

(where $\mathcal{H}$ has assumed the status of a Hamiltonian) provides *simultaneous accounts* of

$$\begin{aligned}&\text{quantum physics on } \mathfrak{S}\\ &\text{quantum physics on } \mathfrak{S}\times\mathfrak{S}\\ &\text{quantum physics on } \mathfrak{S}\times\mathfrak{S}\times\mathfrak{S}\\ &\qquad\qquad\vdots\end{aligned}$$

Moreover, if we introduce (23) into (26) and draw upon (24) we find that

$$\partial_t|\text{vac}) = 0 \quad : \quad |\text{vac}) \text{ is dynamically invariant}$$

The finite-state Schrödinger equation $\mathbb{H}\boldsymbol{\psi} = i\hbar\partial_t\boldsymbol{\psi}$ can—together with its conjugate— be obtained from the classical (!) Lagrangian

$$L = \boldsymbol{\psi}^{\mathsf{t}}(i\hbar\boldsymbol{\psi}_t - \mathbb{H}\,\boldsymbol{\psi})$$

on the strength of which we are led to define[10]

$$\boldsymbol{\pi} \equiv \frac{\partial L}{\partial\boldsymbol{\psi}_t} = i\hbar\,\boldsymbol{\psi}^{\mathsf{t}} \quad : \quad \text{“conjugate momentum”}$$

Mimicing that result, we define

$$\boldsymbol{\Pi} \equiv i\hbar\,\boldsymbol{\Psi}^{\mathsf{t}}$$

and obtain

$$\left.\begin{aligned} \text{bosonic case} \;:\; \big[\boldsymbol{\Psi},\boldsymbol{\Pi}\big] \;\; &= i\hbar\sum_k \big[\mathbf{a}_k\,,\,\mathbf{a}_k^{\mathsf{t}}\big] \\ \text{fermionic case} \;:\; \big[\boldsymbol{\Psi},\boldsymbol{\Pi}\big]_+ &= i\hbar\sum_k \big[\mathbf{a}_k\,,\,\mathbf{a}_k^{\mathsf{t}}\big]_+ \end{aligned}\right\} = i\hbar\,(\text{number } N \text{ of states})\cdot\mathbf{I}$$

In field theory the “number of states” (degrees of freedom) becomes infinite, and one is led (in the non-relativistic theory) to statements of the form

$$\big[\boldsymbol{\Psi}(\boldsymbol{x}),\boldsymbol{\Pi}(\boldsymbol{y})\big]_\pm = i\hbar\,\delta(\boldsymbol{x}-\boldsymbol{y})$$

In field theory it is entirely natural to call $\boldsymbol{\Psi}(\boldsymbol{x})$ the “field operator.” I will borrow that terminology, though it is a bit of a stretch to think of $\mathfrak{S}\times\mathfrak{S}\times\cdots$ as a “field.”

“Quantum field theory” is the name given to the quantum theory of indefinitely many indistinguishable bosonic/fermionic subsystems (usually understood to be “particles”)...and can be considered to arise by formal “2$^{\text{nd}}$ quantization” of the ψ-field that serves to describe the state of a solitary subsystem (but that is thought of now as a complex *classical* field as susceptible to quantization as any other classical field). It is, in my view, one of the grand and glorious surprises of 20$^{\text{th}}$ Century physics that the first part of the preceding sentence has anything at all to do with the second part.

Where Feynman diagrams come from—roughly. Particularly interesting is the situation that arises when a second population $\mathfrak{S}\times\mathfrak{S}\times\cdots$ is brought into play, and allowed to interact with $\mathfrak{S}\times\mathfrak{S}\times\cdots$ The state of such a compound system lives in a compounded Fock space spanned by orthonormal vectors

$$|n_1 n_2 \dots n_{\scriptscriptstyle N} n_1 n_2 \dots n_{\scriptscriptstyle M})$$

[10] Note that the gauge-equivalent Lagrangian $L = \frac{1}{2}i\hbar(\boldsymbol{\psi}^{\mathsf{t}}\boldsymbol{\psi}_t - \boldsymbol{\psi}_t^{\mathsf{t}}\boldsymbol{\psi}) - \boldsymbol{\psi}^{\mathsf{t}}\,\mathbb{H}\,\boldsymbol{\psi}$ leads to different results.

and is constructed

$$|\Psi) \equiv |\Psi)_{\text{compound}} = |\Psi)_{\text{system \#1}} \otimes |\Psi)_{\text{system \#2}}$$

We have now additional creation/annihilation operators $\mathbf{a}^{\mathsf{t}}$ and $\mathbf{a}$, an additional "field operator" $\boldsymbol{\Psi}$, and in place of (26) write

$$\big\{\mathcal{H} + \mathcal{H} + \mathcal{H}_{\text{interaction}}\big\}|\Psi) = i\hbar\,\partial_t|\Psi) \tag{27}$$

Suppose the designs of $\mathcal{H}$ and $\mathcal{H}$ have been designed to insure the conservation of $\mathbf{N}$ and $\mathbf{N}$

$$\begin{aligned}\big[\{\mathcal{H} + \mathcal{H}\}, \mathbf{N}\big] &= \big[\mathcal{H}, \mathbf{N}\big] = \mathbf{0}\\ \big[\{\mathcal{H} + \mathcal{H}\}, \mathbf{N}\big] &= \big[\mathcal{H}, \mathbf{N}\big] = \mathbf{0}\end{aligned}$$

but that the interaction term is of (say) the form

$$\mathcal{H}_{\text{interaction}} = \lambda \boldsymbol{\Psi}^{\mathsf{t}}\boldsymbol{\Psi}\cdot(\boldsymbol{\Psi}^{\mathsf{t}} + \boldsymbol{\Psi})$$

encountered in quantum electrodynamics (where $\boldsymbol{J}\cdot\boldsymbol{A}$ serves as the classical inspiration). Then $\mathbf{N}$ is still conserved, but $\mathbf{N}$-conservation is violated: we encounter transitions of the types

$$\begin{aligned}&|n_1 \dots n_i \dots\dots\dots n_j \dots\dots\dots n_N n_1 \dots n_k \dots\dots\dots n_M)\\ &\Big\downarrow \text{action typical of the interaction term } \lambda\boldsymbol{\Psi}^{\mathsf{t}}\boldsymbol{\Psi}\,\boldsymbol{\Psi}^{\mathsf{t}}\\ &|n_1 \dots n_i + 1 \dots n_j - 1 \dots n_N n_1 \dots n_k + 1 \dots n_M)\end{aligned}$$

and

$$\begin{aligned}&|n_1 \dots n_i \dots\dots\dots n_j \dots\dots\dots n_N n_1 \dots n_k \dots\dots\dots n_M)\\ &\Big\downarrow \text{action typical of the interaction term } \lambda\boldsymbol{\Psi}^{\mathsf{t}}\boldsymbol{\Psi}\,\boldsymbol{\Psi}\\ &|n_1 \dots n_i + 1 \dots n_j - 1 \dots n_N n_1 \dots n_k - 1 \dots n_M)\end{aligned}$$

which invite diagrammatic representation as in Figure 1. When we expand the exponential that enters into the *solution* of (26)

$$|\Psi)_t = \exp\Big\{-\tfrac{i}{\hbar}\big[\mathcal{H} + \mathcal{H} + \mathcal{H}_{\text{interaction}}\big]t\Big\}|\Psi)_0$$

we encounter an intricate tangle of such terms. It was in connection with the production/management/interpretation of that tangle that Feynman, Dyson, Schwinger, Tomonaga *et al* made their great contributions.

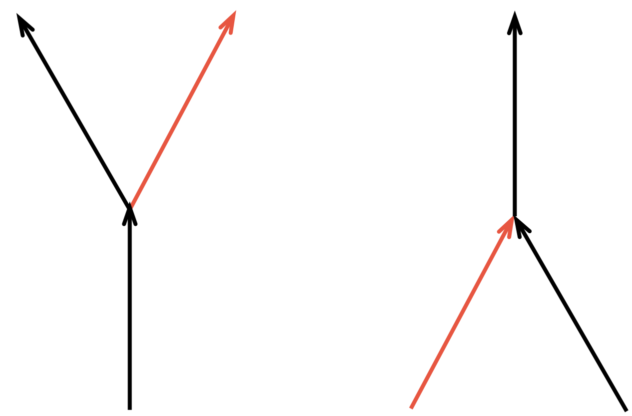

FIGURE 1: *Toy Feynman diagrams illustrating the action of* $\boldsymbol{\Psi}^t\boldsymbol{\Psi}\,\boldsymbol{\Psi}^t$ *on the left, and of* $\boldsymbol{\Psi}^t\boldsymbol{\Psi}\,\boldsymbol{\Psi}$ *on the right.*

I conclude these sketchy remarks with a cautionary remark: There is a tendency in this field—which the use of Feynman diagrams seems to encourage —to confuse mathematical "terms" with physical "events." People point to a diagram and say "this is a picture of an electron emitting a photon." It is not, and such statements illustrate what I call "misplaced concreteness." One would not confuse the vector $\boldsymbol{A} = \sum a^i \boldsymbol{e}_i$ that describes some physics with the coordinates a^i that describe the vector, for the latter have meaning only relative to the selection of a basis, which is arbitrary/conventional. The "physics in the mathematics" can attach only to those mathematical features that are convention-independent, invariant with respect to adjustments of those conventions. So it is here: the detailed meaning of $|n_1 n_2 \ldots n_N)$ and of the associated a and a^t operators hinges on the selection of a basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N\}$ in $\mathcal{V}$; alter that selection and all those detailed meanings change. Feynman diagrams become linear combinations of other Feynman diagrams. It is, in this light, interesting to notice that the field operator $\boldsymbol{\Psi}$, since assembled at (22) from two objects with (as couild easily be shown) complementary transformation properties, *is* stable with respect to basis adjustments, so *can* be held to possess a kind of "direct physical significance."

Application to the quantum theory of angular momentum. I start this discussion in a funny place; namely, with the observation that the Hamiltonian of a classical isotropic oscillator

$$H(p_1, p_2, x_1, x_2) \equiv \tfrac{1}{2m}\Big\{(p_1^2 + p_2^2) + m^2\omega^2(x_1^2 + x_2^2)\Big\}$$

can be written

$$H = \hbar\omega\Big\{a_1^* a_1 + a_2^* a_2\Big\}$$

where

$$a \equiv \tfrac{1}{\sqrt{2}}\big\{y + iq\big\} \quad \text{and} \quad a^* \equiv \tfrac{1}{\sqrt{2}}\big\{y - iq\big\}$$

have been assembled from the dimensionless variables

$$y \equiv \sqrt{\tfrac{m\omega^2}{\hbar\omega}}\,x \quad \text{and} \quad q \equiv \sqrt{\tfrac{1}{m\cdot\hbar\omega}}\,p$$

and subscripts have been surpressed. From $\big[x_1, p_1\big] = \big[x_2, p_2\big] = 1$ (all other Poisson brackets constructable from x_1, p_1, x_2 and p_2 vanish) one obtains

$$\big[a_1^*, a_1\big] = \big[a_2^*, a_2\big] = \tfrac{i}{\hbar} \quad : \quad \text{all other } a\text{-brackets vanish}$$

A line of argument can now be constructed that leads fairly naturally[6] to the introduction of constants of the motion

$$\left.\begin{aligned} J_1 &= \tfrac{\hbar}{2}(a_1^* a_2 + a_2^* a_1) \\ J_2 &= -i\tfrac{\hbar}{2}(a_1^* a_2 - a_2^* a_1) \\ J_3 &= \tfrac{\hbar}{2}(a_1^* a_1 - a_2^* a_2) \end{aligned}\right\} \tag{28}$$

and to the observation that

$$\left.\begin{aligned} \big[J_1, J_2\big] &= J_3 \\ \big[J_2, J_3\big] &= J_1 \\ \big[J_3, J_1\big] &= J_2 \end{aligned}\right\} \tag{29}$$

These are identical to the Poisson brackets supplied by the classical theory of angular momentum; the implication is that the 3-dimensional rotation group $O(3)$ is an "accidental symmetry group"[11] of the 2-dimensional isotropic oscillator.

All of which transfers directly to the associated quantum theory; one has

$$\begin{aligned} \mathsf{H} &\equiv \tfrac{1}{2m}\Big\{(\mathsf{p}_1^2 + \mathsf{p}_2^2) + m^2\omega^2(\mathsf{x}_1^2 + \mathsf{x}_2^2)\Big\} \\ &= \hbar\omega\Big\{\mathsf{a}_1^+\mathsf{a}_1 + \mathsf{a}_2^+\mathsf{a}_2 + \big(\tfrac{1}{2} + \tfrac{1}{2}\big)\mathsf{I}\Big\} \end{aligned}$$

and

$$\big[\mathsf{a}_1, \mathsf{a}_1^{\mathsf{t}}\big] = \big[\mathsf{a}_2, \mathsf{a}_2^{\mathsf{t}}\big] = \mathsf{I} \;:\; \text{all other } \mathsf{a}\text{-commutators vanish} \tag{30}$$

from which it follows that

$$\left.\begin{aligned} \mathsf{J}_1 &= \tfrac{\hbar}{2}(\mathsf{a}_1^{\mathsf{t}}\mathsf{a}_2 + \mathsf{a}_2^{\mathsf{t}}\mathsf{a}_1) \\ \mathsf{J}_2 &= -i\tfrac{\hbar}{2}(\mathsf{a}_1^{\mathsf{t}}\mathsf{a}_2 - \mathsf{a}_2^{\mathsf{t}}\mathsf{a}_1) \\ \mathsf{J}_3 &= \tfrac{\hbar}{2}(\mathsf{a}_1^{\mathsf{t}}\mathsf{a}_1 - \mathsf{a}_2^{\mathsf{t}}\mathsf{a}_2) \end{aligned}\right\} \tag{31}$$

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[11] For a good short introduction to this concept see H. V. McIntosh, "On accidental degeneracy in classical & quantum mechanics," AJP **27**, 620 (1959). See also footnote 20 in "Classical/quantum theory of 2-dimensional hydrogen," (February 1999).

are constants of the quantum motion (commute with $\mathbf{H}$) and satisfy precisely the commutation relations that lie at the foundation of the quantum theory of angular momentum:

$$\left.\begin{aligned}[\mathbf{J}_1, \mathbf{J}_2] &= i\hbar \mathbf{J}_3\\ [\mathbf{J}_2, \mathbf{J}_3] &= i\hbar \mathbf{J}_1\\ [\mathbf{J}_3, \mathbf{J}_1] &= i\hbar \mathbf{J}_2\end{aligned}\right\}\tag{32}$$

At this point in the latter theory one standardly introduces (as computational aids) certain non-hermitian operators

$$\left.\begin{aligned}\mathbf{J}_+ &\equiv \mathbf{J}_1 + i\,\mathbf{J}_2 = \hbar\,\mathbf{a}_1^{\mathsf{t}}\mathbf{a}_2\\ \mathbf{J}_- &\equiv \mathbf{J}_1 - i\,\mathbf{J}_2 = \hbar\,\mathbf{a}_2^{\mathsf{t}}\mathbf{a}_1\end{aligned}\right\}\tag{33}$$

and observes that

$$[\mathbf{J}^2, \mathbf{J}_+] = [\mathbf{J}^2, \mathbf{J}_-] = \mathbf{0}\tag{34}$$

$$\left.\begin{aligned}[\mathbf{J}_3, \mathbf{J}_+] &= +\hbar \mathbf{J}_+\\ [\mathbf{J}_3, \mathbf{J}_-] &= -\hbar \mathbf{J}_-\end{aligned}\right\}\tag{35}$$

It is from (34) that (a few steps down the road) $\mathbf{J}_+$ and $\mathbf{J}_-$ acquire significance as "raising and lowering operators," from which the entire quantum theory of angular momentum radiates with elegant efficiency.[12]

Notice, however, that at (33) the raising/lowering operators of angular momentum theory have been expressed in terms of more primitive objects, the raising/lowering operators of isotropic oscillator theory. That "factorization" permits one to achieve a significant sharpening of the standard algebraic theory, as was first appreciated by Julian Schwinger.[13]

Schwinger (without saying as much in plain words to his few readers) elected, however, to work not in the language of oscillator theory but in the language of our toy quantum field theory—what he called the "language of second quantization." Look in particular to the toy theory of a

bosonic population $\mathfrak{S} \times \mathfrak{S} \times \cdots$ of 2-state systems

Since 2-state systems can be looked upon as spin-$\frac{1}{2}$ systems, and it has been established (within certain broad contexts, and subject to certain assumptions) that[14]

half-integer spin $\Rightarrow$ Fermi-Dirac statistics $\Rightarrow$ anticommutation

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[12] See D. Griffiths[3] §4.3 or A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (1957).

[13] "On angular momentum," unpublished except as Technical Report NYO-3071 of the United States Atomic Energy Commission (1952).

[14] See E. C. G. Sudarshan & I. Duck, *Pauli and the Spin-Statistics Theorem* (1997).

this might seem a strange thing to do. . . but nothing prevents us from proceeding *formally* down that road. We find ourselves confronting a theory that

- presents creation/annihilation operators of only two flavors;
- that those operators satisfy precisely the commutation relations (30)

. . . a theory that, in short, serves as well to support the quantum theory of angular momentum as does the theory of isotropic oscillators (from which it is formally indistinguishable). Looking back again to (11.1), we observe that

$$\begin{aligned}
[2,n]_{\text{bosonic}} &\equiv \text{dimension of } \mathcal{V}^n_{\text{bosonic 2-state}} \\
&= \frac{(2+n-1)!}{(2-1)!\;n!} \\
&= n+1
\end{aligned}$$

is of just the right size to accommodate all the states in the n^{th} level of the following familiar diagram:

$$\begin{array}{cccccccccccccl}
\bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & \mathfrak{S}\times\mathfrak{S}\times\mathfrak{S}\times\mathfrak{S}\times\mathfrak{S}\times\mathfrak{S} \\
 & \circ & & \circ & & \circ & & \circ & & \circ & & \circ & & \mathfrak{S}\times\mathfrak{S}\times\mathfrak{S}\times\mathfrak{S}\times\mathfrak{S} \\
 & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & & \mathfrak{S}\times\mathfrak{S}\times\mathfrak{S}\times\mathfrak{S} \\
 & & & \circ & & \circ & & \circ & & \circ & & & & \mathfrak{S}\times\mathfrak{S}\times\mathfrak{S} \\
 & & & & \bullet & & \bullet & & \bullet & & & & & \mathfrak{S}\times\mathfrak{S} \\
 & & & & & \circ & & \circ & & & & & & \mathfrak{S} \\
 & & & & & & \bullet & & & & & & &
\end{array}$$

One is led thus to the conception that

$$\text{spin } j \text{ states are bosonic assemblies of spin } \tfrac{1}{2} \text{ states}$$

Something very like that notion was casually advanced in a brief paper written by E. Majorana in 1932.[15] In a reminiscence entitled "The Majorana formula" which Schwinger contributed to *A Festschrift for I. I. Rabi*[16] he reports that when that paper (which was fundamental to the atomic beam work in which Rabi was engaged, and for which he was to receive the Nobel Prize) was brought to his attention—he was only seventeen at the time!—he found it "baffling," and that each of his (ultimately four) contributions to angular momentum theory radiated from that experience. Much more recently, Majorana's little paper influenced thought of Roger Penrose[17] having to do with the foundations of quantum mechanics. It was my interest in Penrose's work—brought to my attention by Thomas Wieting—that led me back to Schwinger (whose notes I had purchased for 60 ¢ from the Office of Technical Services, Department of Commerce in 1958), and my attempt to understand Schwinger that led me to the subject of today's talk.

[15] "Atomi orientati in campo magnetico variabile," Nuovo Cimento **9**, 43.
[16] Transactions of the New York Academy of Sciences, Series II Volume 38 (1977).
[17] *Shadows of the Mind* (1994), Chapter 5, Appendix C.