COINCIDENT SPECTRAL LINES

for the particle-in-a-box & hydrogen

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Introduction. If an eigenvalue E_n of $\mathbf{H}$ is degenerate in the familiar sense that

$$\mathbf{H}|\psi) = E_n|\psi) \quad \text{possesses multiple solutions}$$

then one (*i*) is motivated to look for one or more simultantaneous observables that collectively serve to "resolve the degeneracy," and (*ii*) expects to encounter some awkwardness in the associated perturbation theory. That, however, is *not* the kind of degeneracy that will concern me here.

Suppose the spectrum of $\mathbf{H}$ to be (let us assume) non-degenerate. And that the physics of the system is such that it is capable of radiative emission/stimulation: the transition $|m) \mapsto |n)$ entails emission/absorption of a photon with angular frequency

$$\nu_{m,n} = (E_m - E_n)/\hbar$$

With our spectroscopes we observe only *energy differences*, not the spectral E_n themselves. What I will call "coincident spectral lines" arise when

$$\nu_{m_1,n_1} = \nu_{m_2,n_2}$$

and reflect *degeneracies within the set of spectral differences*. It is those that interest me (slightly).

For an oscillator the spectrum runs

$$E_n = \mathcal{E}(n + \tfrac{1}{2}) \quad : \quad n = 0, 1, 2, \ldots$$

so

$$E_m - E_n = \mathcal{E}(m - n)$$

and it is immediate that

$$\nu_{m+a,n+a} = \nu_{m,n} \quad : \quad \begin{cases} a \text{ any integer such that } m + a \\ \text{and } n + a \text{ are both non-negative} \end{cases}$$

So our problem is in this case trivial (except for the spectroscopist, who has, in this instance, *no* idea what state was created by the transition he has observed).

Even simpler, in many respects, than the oscillator is the particle-in-a-box, which has a spectrum of the form

$$E_n = \mathcal{E}n^2 \quad : \quad n = 1, 2, 3, \ldots$$

Spectral coincidence requires that

$$m_1^2 - n_1^2 = m_2^2 - n_2^2 \tag{1}$$

and it becomes natural to inquire also after coincidences of higher order

$$m_1^2 - n_1^2 = m_2^2 - n_2^2 = m_3^2 - n_3^2 = \cdots$$

(of which the oscillator provides infinitely many examples).

It was my on-going work on aspects of the particle-in-a-box problem that inspired my casual interest[1] in the solutions of the Diophantine equation (1), and that called again to mind the fact that in the Asim O. Barut Festschrift issues of *Foundations of Physics* I had once encountered a little paper dealing with a related—as will emerge, an intimately related—question. The authors of that paper[2] recall that "...Barut once, *en passant*, asked the question 'For what transitions of the hydrogen atom do the spectral lines coincide?" ...which they (a physicist and an engineer) proceed very elegantly to solve. My own remarks owe much to theirs.

The hydrogen spectrum runs

$$E_n = \mathcal{E}\frac{1}{n^2} \quad : \quad n = 1, 2, 3, \ldots$$

so the Diophantine equation of interest in that case reads

$$\frac{1}{n_1^2} - \frac{1}{m_1^2} = \frac{1}{n_2^2} - \frac{1}{m_2^2} \tag{2.1}$$

Equivalently

$$(m_1n_2m_2)^2 - (n_1n_2m_2)^2 = (n_1m_1m_2)^2 - (n_1m_1n_2)^2 \tag{2.2}$$

which is seen to present an instance of (1). Which is why Wyss & Wyss, in their §2, discuss my problem as preparation for an attack upon theirs.

Simple coincidences for the box problem. Because I have interest also in higher order coincidences, I have been forced to adjust the notation, but in other respects my argument proceeds along lines sketched by Wyss & Wyss.

[1] See footnote 29 in "Phase space formulation of the quantum mechanical particle-in-a-box problem" (December 2000).

[2] Daniel W. Wyss & Walter Wyss, "Coincidental spectral lines for the hydrogen atom," Foundations of Physics **23**, 465 (1993).

Observe that (1) can be written

$$(m_1 - n_1)(m_1 + n_1) = (m_2 - n_2)(m_2 + n_2)$$

the left and right sides of which must represent distinct organizations of the factors of what must necessarily be a composite number, which we might in general write $a \cdot b \cdot c \cdot d$ with the understanding that some of the factors might be absent (unity) and that any/all might be composite. We then have

$$\begin{aligned} m_1 - n_1 &= cd \\ m_1 + n_1 &= ab \\ m_2 - n_2 &= cb \\ m_2 + n_2 &= ad \end{aligned}$$

giving

$$\begin{aligned} m_1 &= \tfrac{1}{2}(ab + cd) \\ n_1 &= \tfrac{1}{2}(ab - cd) \\ m_2 &= \tfrac{1}{2}(ad + cb) \\ n_2 &= \tfrac{1}{2}(ad - cb) \end{aligned}$$

As $\{a, b, c, d\}$ range on the natural numbers the m's and n's will in some cases be fractional. We must exclude those cases. Each of the numbers a, b, c and d is either even (e) or odd (o). Surveying the $2^4 = 16$ possible cases, we find that we must exclude cases of these four types

$$\begin{aligned} \{a, b, c, d\} \text{ is } \{e, e, o, o\} \text{ or } \{o, o, e, e\} &\ :\ \text{give fractional } m_1 \text{ and } n_1 \\ \{a, b, c, d\} \text{ is } \{e, o, o, e\} \text{ or } \{o, e, e, o\} &\ :\ \text{give fractional } m_2 \text{ and } n_2 \end{aligned}$$

Future work will be simplified if we express the result now in hand as follows:

$$\left.\begin{aligned} m_1 &= |\tfrac{1}{2}(u_1 v_1 + u_2 v_2)| \\ n_1 &= |\tfrac{1}{2}(u_1 v_1 - u_2 v_2)| \\ m_2 &= |\tfrac{1}{2}(u_1 v_2 + u_2 v_1)| \\ n_2 &= |\tfrac{1}{2}(u_1 v_2 - u_2 v_1)| \end{aligned}\right\} \tag{3}$$

Then

$$m_1^2 - n_1^2 = m_2^2 - n_2^2 = u_1 u_2 v_1 v_2 \tag{4}$$

in *all* cases, but to avoid fractional m's and n's we must

$$\text{exclude cases of types } \{u_1, v_1, u_2, v_2\} = \begin{cases} \{e, e, o, o\} \\ \{o, o, e, e\} \\ \{e, o, o, e\} \\ \{o, e, e, o\} \end{cases} \tag{5}$$

Suppose, for example, that $\{u_1, v_1, u_2, v_2\} = \{1, 1, 3, 5\}$; we are led then to $4^2 - 1^2 = 8^2 - 7^2 = 15$, which is in fact the smallest instance of a "simple" coincidence (and the calculation demonstrates the purpose of the absolute value bars). The next smallest cases are

$$\begin{aligned}\{1,1,3,7\} &: \text{ gives } 5^2 - 2^2 = 11^2 - 10^2 = 21\\ \{1,2,2,6\} &: \text{ gives } 5^2 - 1^2 = 7^2 - 5^2 = 24\end{aligned}$$

The $4! = 24$ permutations of $\{u_1, v_1, u_2, v_2\}$ can be assembled from six transpositions. Looking serially to those, we find that

$$\left.\begin{matrix}\{u_1, v_1, u_2, v_2\}\\ \{u_2, v_1, u_1, v_2\}\\ \{u_1, v_2, u_2, v_1\}\end{matrix}\right\}\text{ generate the same pair}$$

$$\left.\begin{matrix}\{v_1, u_1, u_2, v_2\}\\ \{u_1, v_1, v_2, u_2\}\end{matrix}\right\}\text{ generate the same other pair}$$

$$\left.\begin{matrix}\{v_2, v_1, u_2, u_1\}\\ \{u_1, u_2, v_1, v_2\}\end{matrix}\right\}\text{ generate the same yet other pair}$$

except that generally distinct pairs will merge when equalities amongst the u's and v's serve to neutralize the effect of a transposition. Examples tell the story most simply: look first to

$$\left.\begin{matrix}\{2,3,4,6\}\\ \{4,3,2,6\}\\ \{2,6,4,3\}\end{matrix}\right\} : \text{ give } \ 15^2 - 9^2 = 12^2 - 0^2 = 144$$

$$\left.\begin{matrix}\{3,2,4,6\}\\ \{2,3,6,4\}\end{matrix}\right\} : \text{ give } \ 15^2 - 9^2 = 13^2 - 5^2 = 144$$

$$\left.\begin{matrix}\{6,3,4,2\}\\ \{2,4,3,6\}\end{matrix}\right\} : \text{ give } \ 13^2 - 5^2 = 12^2 - 0^2 = 144$$

and then (again) to

$$\left.\begin{matrix}\{1,1,3,5\}\\ \{3,1,1,5\}\\ \{1,5,3,1\}\end{matrix}\right\} : \text{ give } \ 4^2 - 1^2 = 8^2 - 7^2 = 15$$

$$\left.\begin{matrix}\{1,1,3,5\}\\ \{1,1,5,3\}\end{matrix}\right\} : \text{ give } \ 4^2 - 1^2 = 8^2 - 7^2 = 15$$

$$\left.\begin{matrix}\{5,1,3,1\}\\ \{1,3,1,5\}\end{matrix}\right\} : \text{ give } \ 4^2 - 1^2 = 4^2 - 1^2 = 15$$

It is clear that if

$$m_1^2 - n_1^2 = m_2^2 - n_2^2 = N \tag{6}$$

and if

$$\left.\begin{aligned} \mathcal{m}_1 &\equiv \lambda m_1 \\ \mathcal{n}_1 &\equiv \lambda n_1 \\ \mathcal{m}_2 &\equiv \lambda m_2 \\ \mathcal{n}_2 &\equiv \lambda n_2 \end{aligned}\right\} \quad : \quad \lambda = 1, 2, 3, \ldots$$

then

$$\begin{aligned} \mathcal{m}_1^2 - \mathcal{n}_1^2 = \mathcal{m}_2^2 - \mathcal{n}_2^2 = & \mathcal{N} \\ & \mathcal{N} \equiv \lambda^2 N \end{aligned}$$

The transformation just described can be accomplished

$$\{u_1, v_1, u_2, v_2\} \longmapsto \{\mathcal{u}_1, v_1, \mathcal{u}_2, v_2\} \equiv \{\lambda u_1, v_1, \lambda u_2, v_2\}$$

An examination of cases shows that such an adjustment can never cast an admissible case into the excluded category, but may (will if λ is even) lift an excluded case into the admissible category. Every such infinite "tower" can be considered to be supported by its smallest member (least N).[3]

Coincidences of higher order. We have already seen by explicit example that permutational tinkering with the solution (3) of the "simple coincidence" problem can lead to the identification of triplet solutions. But the smallest such triplet

$$11^2 - 1^2 = 13^2 - 7^2 = 17^2 - 13^2 = 120$$

escapes detection by such means, or at least presents a problem: noting that

$$120 = 2^3 \cdot 3 \cdot 5$$

we confront four possibilities, namely

$$\begin{aligned} \{2, 2, 2, 15\} &\ : \ \text{leads permutationally to} \ 17^2 - 13^2 = 120 \\ \{2, 2, 3, 10\} &\ : \ \text{leads permutationally to} \ 13^2 - 7^2 = 17^2 - 13^2 = 120 \\ \{2, 2, 5, 6\} &\ : \ \text{leads permutationally to} \ 11^2 - 1^2 = 17^2 - 13^2 = 120 \\ \{2, 3, 4, 5\} &\ : \ \text{leads permutationally to} \ 11^2 - 1^2 = 13^2 - 7^2 = \tfrac{23^2 - 7^2}{4} = 120 \end{aligned}$$

In no individual case (no particular partition of the factors of 120) were we in this instance led to three solutions . . . though collectively we were led to three solutions plus a fractional sport.

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[3] Wyss & Wyss, on the basis of a faulty odd/even analysis, mistakenly conclude that the "excluded case" problem entails simply that the $\frac{1}{2}$'s should be dropped from (3). In place of (4) they therefore obtain

$$m_1^2 - n_1^2 = m_2^2 - n_2^2 = 4u_1u_2v_1v_2$$

which implies that the composite numbers N that support solutions of (6) are necessarily multiples of 4, which $15 = 4^2 - 1^2 = 8^2 - 7^2$ manifestly is not.

Look now to this straightforward 6-parameter generalization of (3):

$$\left.\begin{aligned} m_1 &= |\tfrac{1}{2}(u_1v_1w_1 + u_2v_2w_2)| \\ n_1 &= |\tfrac{1}{2}(u_1v_1w_1 - u_2v_2w_2)| \\ m_2 &= |\tfrac{1}{2}(u_1v_2w_1 + u_2v_1w_2)| \\ n_2 &= |\tfrac{1}{2}(u_1v_2w_1 - u_2v_1w_2)| \\ m_3 &= |\tfrac{1}{2}(u_1v_2w_2 + u_2v_1w_1)| \\ n_3 &= |\tfrac{1}{2}(u_1v_2w_2 - u_2v_1w_1)| \end{aligned}\right\} \tag{7}$$

Automatically

$$m_1^2 - n_1^2 = m_2^2 - n_2^2 = m_3^2 - n_3^2 = u_1u_2v_1v_2w_1w_2 \tag{8}$$

in *all* cases, but the odd/even analysis required to identify and exclude fractional cases has become tedious (one must examine six objects in each of $2^6 = 64$ cases) the permutational possibilities have become abruptly more numerous ($6! = 720$, resolvable into 15 basic transpositions). Rather than pursuing this topic I will simply report this result of quick experimentation:

$$\big\{u_1, v_1, w_1, u_2, v_2, w_2\big\} = \big\{1, 1, 2, 2, 3, 10\big\}$$

generates

$$13^2 - 7^2 = 17^2 - 13^2 = 31^2 - 29^2 = 120 \tag{9}$$

We missed $11^2 - 1^2$, but picked up the unexpected $31^2 - 29^2$: the "smallest triple" is actually quadruple!

From coincidences to spectral degeneracy in higher dimension. The physical condition from which we have proceeded

$$\mathcal{E}(m_1^2 - n_1^2) = \mathcal{E}(m_2^2 - n_2^2)$$

can be written

$$\mathcal{E}(m_1^2 + n_2^2) = \mathcal{E}(m_2^2 + n_1^2)$$

But this can be read as the statement that two of the eigenvalues of a particle in a square 2-dimensional box are equal:

$$E_{m_1,n_2} = E_{m_2,n_1}$$

The solution of one physical problem has transmuted spontaneously into the solution of quite another. Note, however, that the problems of *higher-order coincidence* and *higher-order degeneracy* are distinct problems. For example: from (9) we conclude that

$$\begin{aligned} 13^2 + 13^2 &= 17^2 + \ \ 7^2 = 338 \\ 13^2 + 29^2 &= 31^2 + \ \ 7^2 = 1010 \\ 17^2 + 29^2 &= 31^2 + 13^2 = 1130 \end{aligned}$$

but are told nothing concerning the possible higher-order degeneracy of the numbers on the right.

I am sure that somebody (Gauss?) has long ago answered this question: *In how many ways can N be written as the sum of two squares, and what are they*? That said, I am reminded that, though I possess a technique for *generating* high order coincidences of the form

$$m_1^2 - n_1^2 = m_2^2 - n_2^2 = m_3^2 - n_3^2 = \cdots$$

the technique provides no protection from collapse into redundancy, and it certainly does not provide a sharp answer to the analogous question: *In how many ways can N be written as the difference between two squares, and what are they*?

Lorentz equivalence of coincident spectral lines for the box problem. Whatever a mathematician may see when he looks at

$$m_1^2 - n_1^2 = m_2^2 - n_2^2$$

a physicist is certain to see *Lorentz invariance*, and to observe that there must exist a Lorentz matrix

$$\mathbb{L} = \begin{pmatrix} p & q \\ q & p \end{pmatrix} \quad \text{with} \quad p^2 - q^2 = 1$$

such that

$$\mathbb{L}\begin{pmatrix} m_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} m_2 \\ n_2 \end{pmatrix}$$

Look to the illustrative case $4^2 - 1^2 = 8^2 - 7^2$: we solve

$$\begin{pmatrix} p & q \\ q & p \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}$$

for p and q and obtain

$$\begin{pmatrix} p \\ q \end{pmatrix} = \tfrac{1}{15}\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 8 \\ 7 \end{pmatrix}$$

giving

$$\mathbb{L} = \begin{pmatrix} \frac{5}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{5}{3} \end{pmatrix}$$

which is seen to be in fact a Lorentz matrix, and to do the work intended.

Hydrogen. Since the letters m and n are now preempted, let equations (2) be written

$$\frac{1}{p_1^2}-\frac{1}{q_1^2}=\frac{1}{p_2^2}-\frac{1}{q_2^2}$$

or again

$$(q_1p_2q_2)^2-(p_1p_2q_2)^2=(p_1q_1q_2)^2-(p_1q_1p_2)^2$$

where it will be understood that $q_1\geqslant p_1+1$ and $q_2\geqslant p_2+1$. To gain access to the results now in hand we set

$$\begin{aligned} m_1&=q_1p_2q_2\\ n_1&=p_1p_2q_2\\ m_2&=p_1q_1q_2\\ n_2&=p_1q_1p_2 \end{aligned}$$

to obtain $m_1p_1=n_1q_1=m_2p_2=n_2q_2=p_1q_1p_2q_2$ whence

$$\begin{aligned} q_1&=\frac{m_1}{n_1}\,p_1\\ p_2&=\frac{m_1}{m_2}\,p_1\\ q_2&=\frac{m_1}{n_2}\,p_1 \end{aligned}$$

To insure that the objects on the left are in fact integers we set $p_1=n_1m_2n_2$ and obtain

$$\left.\begin{aligned} p_1&=\color{red}{m_1}n_1m_2n_2\\ q_1&=m_1\color{red}{n_1}m_2n_2\\ p_2&=m_1n_1\color{red}{m_2}n_2\\ q_2&=m_1n_1m_2\color{red}{n_2} \end{aligned}\right\}\quad\text{drop the red factors}\tag{10}$$

where the m's and n's are taken to be the result of introducing any admissible $\{u_1,v_1,u_2,v_2\}$ into (3).

Look to an example. In the case $\{1,1,3,5\}$ we compute

$$p_1=-28,\quad q_1=-32,\quad p_2=-56,\quad q_2=224$$

(the signs are artifacts) and indeed

$$\tfrac{1}{28^2}-\tfrac{1}{32^2}=\tfrac{1}{56^2}-\tfrac{1}{224^2}=\tfrac{15}{50176}$$

From the quadruple coincidence

$$11^2-1^2=13^2-7^2=17^2-13^2=31^2-29^2=120$$

we can extract variously

$$\{m_1,n_1,m_2,n_2\}=\begin{cases}\{11,\ \ 1,13,\ \ 7\}\\ \{11,\ \ 1,17,13\}\\ \{11,\ \ 1,31,29\}\\ \{13,\ \ 7,17,13\}\\ \{13,\ \ 7,31,29\}\\ \{17,13,31,29\}\end{cases}$$

from which by (10) we obtain the hydrogenic coincidences

$$
\begin{aligned}
\tfrac{1}{91^2}-\tfrac{1}{1001^2}&=\tfrac{1}{77^2}-\tfrac{1}{143^2}=\tfrac{120}{1002001}\\
\tfrac{1}{221^2}-\tfrac{1}{2431^2}&=\tfrac{1}{143^2}-\tfrac{1}{187^2}=\tfrac{120}{5909761}\\
\tfrac{1}{899^2}-\tfrac{1}{9889^2}&=\tfrac{1}{319^2}-\tfrac{1}{341^2}=\tfrac{120}{97792321}\\
\tfrac{1}{1547^2}-\tfrac{1}{2873^2}&=\tfrac{1}{1183^2}-\tfrac{1}{1547^2}=\tfrac{120}{404452321}\\
\tfrac{1}{6293^2}-\tfrac{1}{11687^2}&=\tfrac{1}{2639^2}-\tfrac{1}{2821^2}=\tfrac{120}{6692712481}\\
\tfrac{1}{11687^2}-\tfrac{1}{15283^2}&=\tfrac{1}{6409^2}-\tfrac{1}{6851^2}=\tfrac{120}{39473345041}
\end{aligned}
$$

And, of course, each such example can be infinitely replicated by scaling:

$$
\left.
\begin{aligned}
p_1 &\mapsto p_1 \equiv \lambda p_1\\
q_1 &\mapsto q_1 \equiv \lambda q_1\\
p_2 &\mapsto p_2 \equiv \lambda p_2\\
q_2 &\mapsto q_2 \equiv \lambda q_2
\end{aligned}
\right\} \quad : \quad \lambda = 1,2,3,\dots
$$

To proceed in the opposite direction—to the "base of the tower"—one factors out the greatest common divisor of $\{p_1, q_1, p_2, q_2\}$. In the preceding example the GCD's are all unity, except for `GCD[1547,2873,1183,1547]=13`, so the example—thus processed—becomes

$$
\begin{gathered}
\text{unchanged}\\
\text{unchanged}\\
\text{unchanged}\\
\tfrac{1}{119^2}-\tfrac{1}{221^2}=\tfrac{1}{91^2}-\tfrac{1}{119^2}=\tfrac{120}{2393209}\\
\text{unchanged}\\
\text{unchanged}
\end{gathered}
\tag{11}
$$

And (11) might be rewritten

$$
\tfrac{1}{17^2}\Big\{\tfrac{1}{7^2}-\tfrac{1}{13^2}\Big\}=\tfrac{1}{7^2}\Big\{\tfrac{1}{13^2}-\tfrac{1}{17^2}\Big\}=\tfrac{120}{7^2 13^2 17^2}
$$

to emphasize that—quite typically—the Rydberg differences on left and right stand on different bases. The last of the examples at the top of the page, if similarly processed, assumes this much less mystifying appearance:

$$
\tfrac{1}{29^2 31^2}\Big\{\tfrac{1}{13^2}-\tfrac{1}{17^2}\Big\}=\tfrac{1}{13^2 17^2}\Big\{\tfrac{1}{29^2}-\tfrac{1}{31^2}\Big\}=\tfrac{120}{13^2 17^2 29^2 31^2}
$$

One can easily think of further questions that might be explored: Do higher-order degeneracies exist within the hydrogen spectrum? What is the smallest example? But I have squandered already two afternoons on this topic; it's time to get back to serious work.

ADDENDUM. It occurred to me only belatedly that one can use (for example) the quadruple coincidence

$$11^2 - 1^2 = 13^2 - 7^2 = 17^2 - 13^2 = 31^2 - 29^2 = 120$$

in combination with the idea embodied in the last of the preceding equations to construct a *quadruple hydrogenic coincidence*:

$$\begin{aligned}\frac{1}{7^2 13^2 17^2 29^2 31^2}\Big\{\frac{1}{1^2}-\frac{1}{11^2}\Big\} &= \frac{1}{1^2 11^2 17^2 29^2 31^2}\Big\{\frac{1}{7^2}-\frac{1}{13^2}\Big\}\\ &= \frac{1}{1^2 7^2 11^2 29^2 31^2}\Big\{\frac{1}{13^2}-\frac{1}{17^2}\Big\}\\ &= \frac{1}{1^2 7^2 11^2 13^2 17^2}\Big\{\frac{1}{29^2}-\frac{1}{31^2}\Big\}\\ &= \frac{120}{1^2 7^2 11^2 13^2 17^2 29^2 31^2}\end{aligned}$$

I cannot resist the temptation to spell those out in terrifying detail:

$$\begin{aligned}\frac{1}{1390753^2}-\frac{1}{15298283^2} &= \frac{1}{1176791^2}-\frac{1}{2185469^2}\\ &= \frac{1}{899899^2}-\frac{1}{1176791^2}\\ &= \frac{1}{493493^2}-\frac{1}{527527^2}\\ &= \frac{120}{234037462748089}\end{aligned}$$

More generally:

$$n^{\text{th}}\text{-order box coincidences} \iff n^{\text{th}}\text{-order hydrogenic coincidences}$$

The first of the questions posed at the bottom of the preceding page is thus resolved , and I think I have in fact presented the smallest such example.

I get the impression, from an hour spent perusing the (intimidating) number theory shelves, that mathematicians have traditionally been more interested sums of squares than differences. But Richard Mollin, in §6.2 of his *Fundamentals of Number Theory with Applications* (1998), does treat the subject in some depth. And I encountered somewhere the claim that "every n not of the form $4k + 2$ can be written as a difference of squares," to which was appended this corollary: "every odd prime can be written as a difference of consecutive squares." Thus $3 = 2^2 - 1^2$, $5 = 3^2 - 2^2$, $7 = 4^2 - 3^2$, . . . but that is trivial: $(n + 1)^2 - n^2 = 2n + 1$ so *every* odd n (whether prime or not) can be so expressed. Only Mollin, among the sources I consulted, seemed interested in the *number* of such representations, in the general case.

Miscellaneous number-theoretic doodles. I resent the intrusion of number theory—an area in which my knowledge is surpassed by many a high school student—into my thought about "more serious" matters. But I can't seem to let go of the subject discussed recreationally in the preceding pages. All of the material presented there is elementary, but I feel that the simplicity of it all was in some respects not made as plain as it could/should be, and that in other respects it would have been helpful to set the points at issue in broader context. I look upon the following paragraphs as a cemetary for my thoughts on this subject.

☐ Let a, b, c and d be natural numbers such that

$$a - b = c - d \tag{12.1}$$

Divide by $abcd$ and obtain

$$\frac{1}{A} - \frac{1}{B} = \frac{1}{C} - \frac{1}{D} \tag{12.2}$$

with

$$\begin{aligned} A &= \cdot bcd \\ B &= a \cdot cd \\ C &= ab \cdot d \\ D &= abc\,\cdot \end{aligned}$$

Multiplication by $ABCD$ gives back an equation of the original design

$$a' - b' = c' - d'$$

with

$$\begin{aligned} A &= \cdot bcd \\ B &= a \cdot cd \\ C &= ab \cdot d \\ D &= abc\,\cdot \end{aligned}$$

$$\begin{aligned} a' &= \cdot BCD = (abcd)^2 \cdot a \\ b' &= A \cdot CD = (abcd)^2 \cdot b \\ c' &= AB \cdot D = (abcd)^2 \cdot c \\ d' &= ABC\,\cdot = (abcd)^2 \cdot d \end{aligned}$$

Each of the equations (12) supports a tower of similar equations, got by

$$\big\{a, b, c, d\big\} \mapsto \big\{\lambda a, \lambda b, \lambda c, \lambda d\big\} \quad \text{else} \quad \big\{A, B, C, D\big\} \mapsto \big\{\lambda A, \lambda B, \lambda C, \lambda D\big\}$$

and each such tower rests on a base, got by setting

$$\lambda = g.c.d\big\{a, b, c, d\big\} \quad \text{else} \quad \lambda = g.c.d\big\{A, B, C, D\big\}$$

And the upshot of the preceding line of argument is that (12.1) and (12.2) are interconvertible: an instance of either supplies an instance of the other.

☐ Preceding remarks continue to hold if one assigns specialized interpretations to a, b, c and d. Suppose, for example, that we set

$$\begin{aligned} a &= a^n \\ b &= b^n \\ c &= c^n \\ d &= d^n \end{aligned}$$

From instances of

$$a^n - b^n = c^n - d^n \tag{13.1}$$

we then obtain instances of

$$\frac{1}{A^n} - \frac{1}{B^n} = \frac{1}{C^n} - \frac{1}{D^n} \tag{13.2}$$

with

$$\begin{aligned} A &= \ \cdot bcd \\ B &= a \cdot cd \\ C &= ab \cdot d \\ D &= abc \cdot \end{aligned}$$

In the text my remarks were (for physical reasons) restricted to the case $n = 2$.

☐ Number theorists have much to say about the conditions under which N can be represented

$$N = \text{sum of two squares}$$

and also about a population of related problems:

$$\begin{aligned} N &= \text{sum of } n > 2 \text{ squares} \\ N &= \text{sum of two cubes} \\ N &= \text{sum of two } n^{\text{th}} \text{ powers} \\ N &= \text{sum of } n > 2 \text{ cubes } n^{\text{th}} \text{ powers} \end{aligned}$$

Fermat's Last Theorem (sum of two n^{th} powers is never an n^{th} power if $n > 2$) falls into this broad class. The collateral problem of greater interest to me: Under what conditions can one write

$$N = \text{sum of two } n^{\text{th}} \text{ powers } \underline{\text{in more than one way}}$$

Suppose we had an instance of

$$r^n + s^n = t^n + u^n \tag{14}$$

We are then supplied with (unless $t = u$) *two* instances of (13.1)

$$N_1 = r^n - t^n = u^n - s^n \quad \text{and} \quad N_2 = r^n - u^n = t^n - s^n$$

and (therefore) with two instances also of (13.2). Conversely, from any instance of (13.2)—or of (13.2)—one can extract one (!) instance of (14).

☐ We possess a technique for *mindlessly generating* instances of (13.1) in the case $n = 2$, therefore for mindlessly generating instances of

$$r^2 + s^2 = t^2 + u^2 \tag{15}$$

The technique permits one to construct instances also of higher order difference coincidences

$$a^2 - b^2 = c^2 - d^2 = e^2 - f^2 = \cdots$$

but each such equality gives rise generally to a *different* instance of (15). Conversely,

$$r^2 + s^2 = t^2 + u^2 = v^2 + w^2$$

would give rise generally to six different instances of $a^2 - b^2 = c^2 - d^2$ but to no difference coincidences of higher order.

☐ I describe in modified (more easily extendable) notation the construction that led to (7):

$$\begin{aligned} m(p_1, q_1) &= \tfrac{1}{2}(p_1 + q_1) \\ n(p_1, q_1) &= \tfrac{1}{2}(p_1 - q_1) \end{aligned}$$

$$\begin{aligned} m_{12}(p_1, q_1, p_2, q_2) &= m(p_1 p_2, q_1 q_2) \\ n_{12}(p_1, q_1, p_2, q_2) &= \ n(p_1 p_2, q_1 q_2) \\ m_{22}(p_1, q_1, p_2, q_2) &= m(p_1 q_2, q_1 p_2) \\ n_{22}(p_1, q_1, p_2, q_2) &= \ n(p_1 q_2, q_1 p_2) \end{aligned}$$

$$\begin{aligned} m_{13}(p_1, q_1, p_2, q_2, p_3, q_3) &= m(p_1 p_2 p_3, q_1 q_2 q_3) \\ n_{13}(p_1, q_1, p_2, q_2, p_3, q_3) &= \ n(p_1 p_2 p_3, q_1 q_2 q_3) \\ m_{23}(p_1, q_1, p_2, q_2, p_3, q_3) &= m(p_1 q_2 p_3, q_1 p_2 q_3) \\ n_{23}(p_1, q_1, p_2, q_2, p_3, q_3) &= \ n(p_1 q_2 p_3, q_1 p_2 q_3) \\ m_{33}(p_1, q_1, p_2, q_2, p_3, q_3) &= m(p_1 q_2 q_3, q_1 p_2 p_3) \\ n_{33}(p_1, q_1, p_2, q_2, p_3, q_3) &= \ n(p_1 q_2 q_3, q_1 p_2 p_3) \\ &\ \vdots \end{aligned}$$

One can ascend from third order to second by setting $p_3 = q_3 = 1$. *Et cetera.* Automatically

$$\begin{aligned} m_{11}^2 - n_{11}^2 &= p_1 q_1 \\ m_{12}^2 - n_{12}^2 = m_{22}^2 - n_{22}^2 &= p_1 q_1 p_2 q_2 \\ m_{13}^2 - n_{13}^2 = m_{23}^2 - n_{23}^2 = m_{33}^2 - n_{33}^2 &= p_1 q_1 p_2 q_2 p_3 q_3 \\ &\ \vdots \end{aligned}$$

One can avoid the winnowing required to eliminate fractional and/or redundant m's and n's by taking the p's and q's to be distinct odd primes, which would certainly be acceptable if one's objective were (as mine was) to establish the existence of coincidences of arbitrarily high order. Example:

$$\begin{aligned} m_{13}(1,3,5,7,11,13) &= 164 \\ n_{13}(1,3,5,7,11,13) &= -109 \\ m_{23}(1,3,5,7,11,13) &= 136 \\ n_{23}(1,3,5,7,11,13) &= -59 \\ m_{33}(1,3,5,7,11,13) &= 128 \\ n_{33}(1,3,5,7,11,13) &= -37 \end{aligned}$$

The signs would have been eliminated if we had installed | bars, and in all cases we have

$$m^2 - n^2 = 15015$$

Corollaries of this result are

$$\begin{aligned} 164^2 + 59^2 &= 136^2 + 109^2 = 30377 \\ 164^2 + 37^2 &= 128^2 + 109^2 = 28265 \\ 136^2 + 37^2 &= 128^2 + \ \ 59^2 = 19865 \end{aligned}$$

and

$$\begin{aligned} \tfrac{1}{136^2 59^2 128^2 37^2}\Big\{\tfrac{1}{109^2} - \tfrac{1}{164^2}\Big\} &= \tfrac{1}{164^2 109^2 128^2 37^2}\Big\{\tfrac{1}{59^2} - \tfrac{1}{136^2}\Big\} \\ &= \tfrac{1}{164^2 109^2 136^2 59^2}\Big\{\tfrac{1}{37^2} - \tfrac{1}{128^2}\Big\} \\ &= \tfrac{15015}{4614725995740189908008960} \end{aligned}$$

If we kill the last p_3 and q_3 we obtain

$$\left.\begin{aligned} m_{13}(1,3,5,7,1,1) &= 13 \\ n_{13}(1,3,5,7,1,1) &= -8 \\ m_{23}(1,3,5,7,1,1) &= 11 \\ n_{23}(1,3,5,7,1,1) &= -4 \\ m_{33}(1,3,5,7,1,1) &= 11 \\ n_{33}(1,3,5,7,1,1) &= -4 \end{aligned}\right\} \quad : \quad 13^2 - 8^2 = 11^2 - 4^2 = 105$$

and if we kill also p_2 and q_2 we obtain the even more redundant output

$$\left.\begin{aligned} m_{13}(1,3,1,1,1,1) &= 2 \\ n_{13}(1,3,1,1,1,1) &= -1 \\ m_{23}(1,3,1,1,1,1) &= 2 \\ n_{23}(1,3,1,1,1,1) &= -1 \\ m_{33}(1,3,1,1,1,1) &= 2 \\ n_{33}(1,3,1,1,1,1) &= -1 \end{aligned}\right\} \quad : \quad 2^2 - 1^2 = 3$$

One need not kill the *last* arrivals, as the following example makes clear:

$$\left.\begin{aligned} m_{13}(1,1,5,7,11,13) &= 73\\ n_{13}(1,1,5,7,11,13) &= -18\\ m_{23}(1,1,5,7,11,13) &= 71\\ n_{23}(1,1,5,7,11,13) &= 6\\ m_{33}(1,1,5,7,11,13) &= 73\\ n_{33}(1,1,5,7,11,13) &= +18 \end{aligned}\right\} \;:\quad 73^2-18^2=71^2-6^2=5005$$

Such examples become joined when one brings into play *permutations* of the arguments.

□ G. H. Hardy told a story, repeated on page 312 of Robert Kanigel's *The Man Who Knew Infinity: A Life of the Genious Ramanujan* (1991), to this effect: Hardy noticed that the number of the taxi that took him to visit the dying Srinivasa Ramanujan, one day in 1919, was 1729. "Rather a dull number" he remarked, to which Ramanujan responded "No, Hardy. It is the smallest number expressible as the sum of two cubes in two different ways." What he had in mind, and had recorded in his notebooks years previous, is that

$$1729 = 12^3 + 1^3 = 10^3 + 9^3$$

Implications are that

$$\begin{aligned} 728 &= 9^3 - 1^3 = 12^3 - 10^3\\ 999 &= 10^3 - 1^3 = 12^3 - 9^3 \end{aligned}$$

I know, however, of no way to *generate* coincident cubic differences, and certainly of no way to generate them *ad infinitum*. It does not seem to help much that

$$a^2 - b^2 = (a-b)(a+b)$$

generalizes to become[4]

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2+ab+b^2)\\ a^4 - b^4 &= (a-b)(a^3+a^2b+ab^2+b^3)\\ &\;\;\vdots \end{aligned}$$

□ Nor does $a^4 - b^4 = (a^2-b^2)(a^2+b^2)$ provide access to the methods that served us so well in the case $n = 2$. Richard Crandall reports that

$$635318657 = 133^4 + 134^4 = 158^4 + 59^4$$

is an example that was known already to Euler, and that there are known to be infinitely many such double coincidences, but that the question of whether triple coincidences exist remains open.

[4] See the first equations in G. S. Carr's *A Synopsis of Results in Pure & Applied Mathematics* (1880), which had exerted such a profound effect on Ramanujan's early mathematical development.