Mathematical Methods for Analysis of

Composite Quantum Systems with Infinite-dimensional State Spaces

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Introduction. Yoshihisa Yamamoto & Ataç İmamağlu, in §**1.3.4** of their *Mesoscopic Quantum Optics* (1999), discuss aspects of the quantum theory of system/probe interaction in language that considers system and probe (or measurement device/meter) to be component parts of a composite system, and that assumes both system and probe are rich enough to support definitions of "conjugate observables" that satisfy $[\mathbf{q}, \mathbf{p}] = i\hbar\mathbf{I}$. An implication of the latter assumption is that the state spaces $\mathcal{H}_s$ and $\mathcal{H}_m$ of system and probe are, of necessity, infinite-dimensional. We must therefore sacrifice a simplifying assumption standard to the quantum theory of composite systems; namely, that all relevant state spaces—all vectors and matrices—are finite-dimensional. We therefore lose the Kronecker product. My objective here is to develop the mathematical resources that permit us to live with that loss.

Tensor products in the infinite-dimensional case. Familiarly,

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_1 b_3 \\ a_2 b_1 \\ a_2 b_2 \\ a_2 b_3 \end{pmatrix}$$

where on the right we see components of $\boldsymbol{a}$ joined with components of $\boldsymbol{b}$ in all possible ways, and the population of such products presented in a specific order. It is the latter convention that becomes unworkable—must be sacrificed—if either $\boldsymbol{a}$ or $\boldsymbol{b}$ is ∞-dimensional.

Let vectors $\{|s)\}$ comprise an orthonormal basis in $\mathcal{H}_s$, and $\{|m)\}$ comprise an orthonormal basis in $\mathcal{H}_m$. Then every $|a)$ in $\mathcal{H}_s$ can be developed

$$|a) = \sum a_s|s) \quad \text{with} \quad a_s = (s|a)$$

and every $|b)$ in $\mathcal{H}_m$ can be developed

$$|b) = \sum b_m|m) \quad \text{with} \quad b_m = (m|b)$$

We stipulate that $\mathcal{H}_s \otimes \mathcal{H}_m$ is an inner product space, with induced inner product structure

$$\big((a\,|\otimes(b\,|\big)\big(|c)\otimes|d)\big) = (a|c)\cdot(b|d)$$

Then

$$\big((r\,|\otimes(m\,|\big)\big(|s)\otimes|n)\big) = (r|s)\cdot(m\,|n) = \begin{cases}\delta_{rs}\cdot\delta_{mn}\\ \delta(r-s)\cdot\delta(m-n)\end{cases}$$

establishes the orthonormality of the basis vectors

$$|s,m)\equiv|s)\otimes|m)\quad : \quad \text{elements of } \mathcal{H}\equiv\mathcal{H}_s\otimes\mathcal{H}_m$$

and

$$\sum_{s,m}|s,m)(s,m| = \sum_{s,m}\big(|s)\otimes|m)\big)\big((s|\otimes(m|\big) = \mathbf{I}\equiv\mathbf{I}_s\otimes\mathbf{I}_m$$

establishes their completeness.

If $|\psi)$ and $|\phi)$ describe the quantum state of system/meter respectively, then

$$|\Psi) = |\psi)\otimes|\phi) = \sum_{s,m}\big(|s)\otimes|m)\big)\psi_s\phi_m$$

where $\psi_s = (s|\psi)$ and $\phi_m = (m|\phi)$. But the state of the composite system has more generally to be described

$$|\Psi) = \sum_{s,m}\big(|s)\otimes|m)\big)\Psi_{s,m}$$

where $\Psi_{s,m} = (s,m|\Psi)$. The state of the composite system is "entangled" unless—exceptionally—the numbers $\Psi_{s,m}$ can be factored: $\Psi_{s,m} = \psi_s\phi_m$.

Passing to density matrix language, we write

$$\boldsymbol{\rho}_s = |\psi)(\psi| = \sum_{r,s}\psi_r|r)(s|\psi_s^*$$

to describe the disentangled pure state of the system, and a similar expression to describe the disentangled pure state $\boldsymbol{\rho}_m = |\phi)(\phi|$ of the probe. Observe that

$$\begin{aligned}\boldsymbol{\rho}_s\cdot\boldsymbol{\rho}_s &= \sum_{r,s}\sum_{r',s'}\psi_r|r)(s|\psi_s^*\psi_{r'}|r')(s'|\psi_{s'}^*\\ &= \sum_{r,s}\sum_{s'}\psi_r|r)\psi_s^*\psi_s(s'|\psi_{s'}^*\\ &= \sum_{r}\sum_{s'}\psi_r|r)(s'|\psi_{s'}^* \quad \text{by} \quad \textstyle\sum\psi_s^*\psi_s = 1\\ &= \boldsymbol{\rho}_s\end{aligned}$$

and

$$\mathrm{tr}\boldsymbol{\rho}_s = \sum_q \sum_{r,s} \psi_r (q|r)(s|q)\psi_s^* = \sum_q \psi_q \psi_q^* = 1$$

and that both statements are immnediate if one works from $\boldsymbol{\rho}_s = |\psi)(\psi|$.

If the system and probe are only "mentally conjoined" (their respective quantum states disentangled) the density operator of the conjoint systems is

$$\begin{aligned}\boldsymbol{\rho} &= \Big(\sum \psi_r \phi_m \,|r) \otimes |m)\Big) \cdot \Big(\sum (s\,| \otimes (n\,|\, \psi_s^* \phi_n^*\Big)\\ &= \Big(\sum \psi_r |r)(s\,|\psi_s^*\Big) \otimes \Big(\sum \phi_m |m)(n\,|\phi_n^*\Big)\\ &= \boldsymbol{\rho}_s \otimes \boldsymbol{\rho}_m\end{aligned}$$

We can recover either factor by using the *partial trace* to "reduce" $\boldsymbol{\rho}$ by "tracing out" the unwanted factor:

$$\begin{aligned}\mathrm{tr}_1\,\boldsymbol{\rho} &\equiv \sum_q \big((q\,| \otimes \mathbf{I}_m\big)\,\boldsymbol{\rho}\,\big(|q\,) \otimes \mathbf{I}_m\big)\\ &= \underbrace{\Big(\sum_q \sum_{rs} \psi_r (q|r)(s|q)\,|\psi_s^*\Big)}_{1} \otimes \underbrace{\Big(\sum_{mn} \phi_m |m)(n\,|\phi_n^*\Big)}_{\boldsymbol{\rho}_m}\\ &= \boldsymbol{\rho}_m\end{aligned}$$

$$\begin{aligned}\mathrm{tr}_2\,\boldsymbol{\rho} &\equiv \sum_p \big(\mathbf{I}_s \otimes (p\,|\big)\,\boldsymbol{\rho}\,\big(\mathbf{I}_s \otimes |\,p)\big)\\ &= \boldsymbol{\rho}_s\end{aligned}$$

The partial trace concept remains in force (and acquires special importance) even when the state of the $\boldsymbol{\rho}$ of the composite system is mixed or entangled. One then has

$$\boldsymbol{\rho} = \sum \rho_{rm;sn}\,\big(|r) \otimes |m)\big) \cdot \big((s\,| \otimes (n\,|\big)$$

with $\rho^*_{rm,sn} = \rho_{sn;rm}$ and $\sum_{rm} \rho_{rm;rm} = 1$ and defines

$$\begin{aligned}\mathrm{tr}_1\,\boldsymbol{\rho} &\equiv \sum_p \big((p\,| \otimes \mathbf{I}_m\big)\,\boldsymbol{\rho}\,\big(|p\,) \otimes \mathbf{I}_m\big)\\ &= \sum_p \sum_{mn} \rho_{pm;pn}\,|m)(n\,|\end{aligned}$$

$$\begin{aligned}\mathrm{tr}_2\,\boldsymbol{\rho} &\equiv \sum_q \big(\mathbf{I}_s \otimes (q\,|\big)\,\boldsymbol{\rho}\,\big(\mathbf{I}_s \otimes |\,q)\big)\\ &= \sum_q \sum_{rs} \rho_{rq;sq}\,|r)(s\,|\end{aligned}$$

Clearly

$$\mathrm{tr}\,\boldsymbol{\rho} = \mathrm{tr}\big(\mathrm{tr}_1\,\boldsymbol{\rho}\big) = \mathrm{tr}\big(\mathrm{tr}_2\,\boldsymbol{\rho}\big) = \sum_{pq} \rho_{pq;pq} = 1$$

The operators $\boldsymbol{\rho}$,

$$\begin{aligned} \mathsf{S} &\equiv \mathrm{tr}_2\,\boldsymbol{\rho} = \sum_q \sum_{rs} \rho_{rq;sq}\,|r)(s\,| \\ \mathsf{M} &\equiv \mathrm{tr}_1\,\boldsymbol{\rho} = \sum_p \sum_{mn} \rho_{pm;pn}\,|m)(n\,| \end{aligned}$$

are self-adjoint, so can be brought to diagonal (spectral representative) form

$$\begin{aligned} \boldsymbol{\rho} &= \sum_u |R_u)R_u(R_u| &&: \quad |R_u) \text{ live in } \mathcal{H}_s \otimes \mathcal{H}_m \\ \mathsf{S} &= \sum_i |S_i)S_i(S_i| &&: \quad |S_i) \text{ live in } \mathcal{H}_s \\ \mathsf{M} &= \sum_j |M_j)M_j(M_j| &&: \quad |M_j) \text{ live in } \mathcal{H}_m \end{aligned}$$

by unitary transformation. One has

$$\mathrm{tr}\,\mathsf{S} = \sum_i S_i = 1$$

$$\mathrm{tr}\,\mathsf{S}^2 = \sum_i S_i^2 = \sum_{pq} \sum_{rs} \rho_{rp;sp}\,\rho_{sq;rq} \leqslant 1$$

and can say similar things about $\mathrm{tr}\,\mathsf{M}$ and $\mathrm{tr}\,\mathsf{M}^2$.

If we had had the foresight to work in the eigenbases of the reduced density matrices S and M we would have had

$$\boldsymbol{\rho} = \sum_{ijkl} R_{ik\,;jl}\,\big(|S_i) \otimes |M_k)\big) \cdot \big((S_j\,| \otimes (M_l\,|\big)$$

which if $\boldsymbol{\rho}$ referred to a disentangled pure state of the composite system would have assumed the form

$$\begin{aligned} &= \Big(\sum_i S_i|S_i)(S_i|\Big) \otimes \Big(\sum_k M_k|M_k)(M_k|\Big) \\ &= \sum_{ik} S_i M_k\big(|S_i) \otimes |M_k)\big) \cdot \big((S_i\,| \otimes (M_k\,|\big) \end{aligned}$$

which would entail $R_{ik\,;jl} = S_i M_k \delta_{ij}\delta_{kl}$.