Classical/Quantum Dynamics in a Uniform Gravitational Field:

C. Populations of Bouncing Balls

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February 2003

Introduction. In Part B, beginning at Figure 23, I was led to make much of the fact that the quantum mechanical moving mean

$$\langle\text{position}\rangle_{\text{bouncing wavepacket}}$$

does not much resemble—except at brief, irregularly-spaced intervals—the *classical* motion of a bouncing ball. It has occurred to me—but only after "completing" preparations for the seminar (19 February 2003) that was to have been based upon Parts A & B—that the comparison may be unfair: perhaps it would be more appropriate to compare the quantum statistics with statistical properties of a *population* of classical bouncing balls. It is my intention here to explore aspects of that idea.

1. Preliminaries. As was remarked on page 3 of Part B, the maximal height a of the bounce and the period τ stand in the relation

$$a = \tfrac{1}{2}g\big(\tfrac{1}{2}\tau\big)^2$$

so cannot be specified independently. If we find it most convenient to take a to be the independent parameter, then we have

$$\tau = \sqrt{8a/g}$$

If (by convenient convention) we <u>start the clock at a moment of maximality</u> then

$$x(t) = -\tfrac{1}{2}g(t+\tfrac{1}{2}\tau)(t-\tfrac{1}{2}\tau) \quad : \quad -\tfrac{1}{2}\tau < t < +\tfrac{1}{2}\tau$$

which in a-parameterization becomes

$$x(t) = -\tfrac{1}{2}g\big(t^2 - 2a/g\big) = a - \tfrac{1}{2}g\,t^2 \quad : \quad -\sqrt{2a/g} < t < +\sqrt{2a/g}$$

On the indefinitely extended time line we therefore have

$$\begin{aligned} x(t) = \sum_n \big[a - \tfrac{1}{2}g\,\big(t - n\sqrt{8a/g}\,\big)^2\big] \\ \cdot\,\texttt{UnitStep[}\big[a - \tfrac{1}{2}g\,\big(t - n\sqrt{8a/g}\,\big)^2\big]\texttt{]} \end{aligned} \tag{1.1}$$

Fourier analytic methods are shown in §2 of Part B to lead to this alternative description of that same bouncing ballistic motion:

$$x(t) = \tfrac{1}{8}g\,\tau^2\Big[\tfrac{2}{3} + \tfrac{4}{\pi^2}\Big\{\tfrac{1}{1^2}\cos\big[2\pi\tfrac{t}{\tau}\big] - \tfrac{1}{2^2}\cos\big[4\pi\tfrac{t}{\tau}\big] + \tfrac{1}{3^2}\cos\big[6\pi\tfrac{t}{\tau}\big] - \cdots\Big\}\Big]$$

In a-parameterization we therefore have

$$\begin{aligned} x(t) &= a\Big[\tfrac{2}{3} + \tfrac{4}{\pi^2}\Big\{\tfrac{1}{1^2}\cos\big[\pi\sqrt{\tfrac{g}{2a}}t\big] - \tfrac{1}{2^2}\cos\big[2\pi\sqrt{\tfrac{g}{2a}}t\big] \\ &\qquad\qquad + \tfrac{1}{3^2}\cos\big[3\pi\sqrt{\tfrac{g}{2a}}t\big] - \cdots\Big\}\Big] \\ &= a\Big[\tfrac{2}{3} - \tfrac{4}{\pi^2}\sum_{k=1}^{\infty}(-)^k\,\tfrac{1}{k^2}\cos\big[k\pi\sqrt{g/2a}\,t\big]\Big] \end{aligned} \tag{1.2}$$

Equation (1.1) is exact, while truncated instances of (1.2) are only approximate, but it has been my numerical experience that *Mathematica* finds it marginally easier to work with the Fourier series, and that if the series is carried to order ten or greater the imprecision is insignificant.

2. Dropped population of bouncers. Let balls be released simultaneously from initial heights a_1, a_2, . . . , a_N. Their respective positions at time t can, by (1.2), be described

$$x_i(t) = a_i\Big[\tfrac{2}{3} - \tfrac{4}{\pi^2}\sum_{k=1}^{\infty}(-)^k\,\tfrac{1}{k^2}\cos\big[k\pi\sqrt{g/2a_i}\,t\big]\Big] \;\;:\;\; i = 1, 2, \ldots, N \tag{2}$$

which has been used to construct Figure 1.[1]

3. Motion of the mean. Let non-negative weights w_1, w_2, . . . , w_N be assigned to the members of such a population, and let it be required that they sum to

[1] Equation (2) requires—and *Mathematica* confirms—that

$$-\tfrac{4}{\pi^2}\sum_{k=1}^{\infty}(-)^k\,\tfrac{1}{k^2} = \tfrac{1}{3}$$

unity: $\sum w_i = 1$. The instantantaneous location of the *mean* (or 1st moment, or "center of mass") of the population is defined

$$\mathcal{X}(t) \equiv \langle x(t) \rangle \equiv \sum_i w_i x_i(t) \tag{3}$$

and higher moments are defined similarly:

$$\langle [x(t)]^p \rangle \equiv \sum_i w_i [x_i(t)]^p \quad : \quad p = 2, 3, \ldots$$

To explore the implications of the simple construction (3) I assign to the heights a_i and weights[2] w_i illustrative values taken from the discussion in Part B of a particular "bouncing Gaussian wavepacket," and rely upon the graphical resources of *Mathematica*. More particularly: at page 32 in Part B we encounter

$$\Psi(x, 0; A, s) = \frac{1}{\sqrt{s\sqrt{2\pi}}} e^{-\frac{1}{4}\left[\frac{x-A}{s}\right]^2} \quad : \quad A \gg s$$

and elect to set $A = 15$ and $s = 1.75 = \frac{7}{4}$. So we write

$$P(x) \equiv \left|\Psi\big(x, 0; 15, \tfrac{7}{4}\big)\right|^2 = \frac{4}{7\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{4}{7}(x-15)\right]^2} \tag{4}$$

which is plotted below:

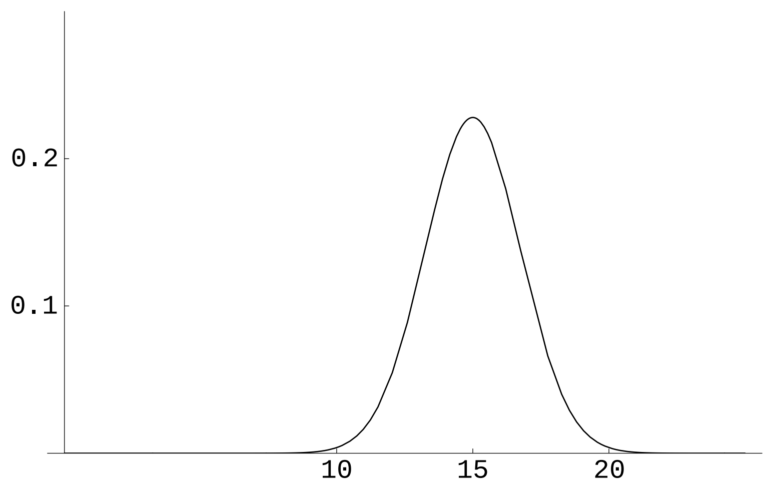

FIGURE 1: *Graph of the normalized Gaussian* (4) *that we have borrowed from* Part B.

Looking to the figure, it becomes natural to

$$\text{let the } a_i \text{ range on } \big\{10, 11, \ldots, \mathbf{15}, \ldots, 19, 20\big\}$$

[2] It is a curious fact that the "assignment of weights" is, in the present physical context, literally that: an assignment of *weights*!

and to set

$$w_{10} = \int_{9.5}^{10.5} P(x)\,dx, \quad w_{11} = \int_{10.5}^{11.5} P(x)\,dx, \quad \ldots \quad , \quad w_{20} = \int_{19.5}^{20.5} P(x)\,dx$$

Thus do we obtain

$$\left.\begin{aligned} w_{10} &= 0.0042\\ w_{11} &= 0.0177\\ w_{12} &= 0.0538\\ w_{13} &= 0.1193\\ w_{14} &= 0.1923\\ w_{15} &= 0.2254\\ w_{16} &= 0.1923\\ w_{17} &= 0.1193\\ w_{18} &= 0.0538\\ w_{19} &= 0.0177\\ w_{20} &= 0.0042 \end{aligned}\right\} \tag{5}$$

for which (because I have, near the center of the list, done some fudging in the last decimal places) it is exactly the case that

$$\sum_{i=10}^{20} w_i = 1.0000$$

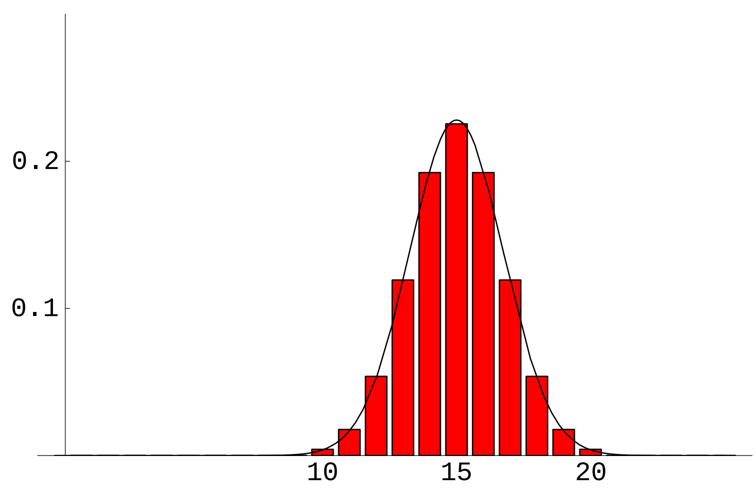

FIGURE 2: *Bar chart display of the data* (5), *on which has been superimposed the Gaussian* (4) *from which the data were obtained.*

Look now to the following figures and their captions:

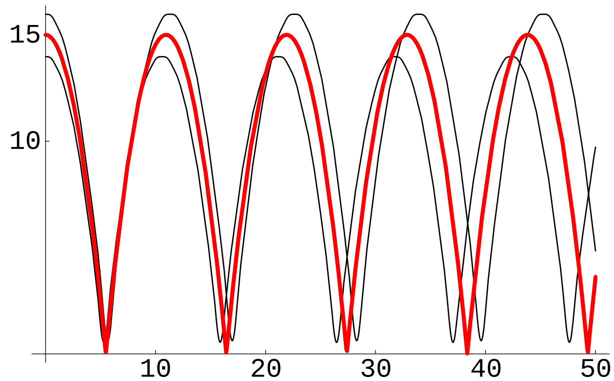

FIGURE 3: *Superimposed orbits of balls dropped from $a = 14, 15, 16$. The first and last were computed in 10-term Fourier approximation* (1.2)*, while the case $a = 15$—shown in* **red**—*was obtained from the exact equation* (1.1).

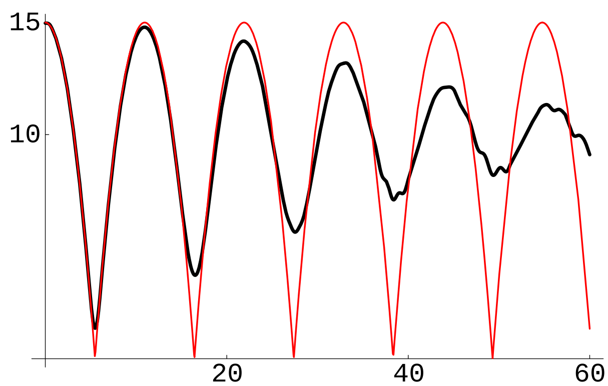

FIGURE 4: **Initial motion of the center of mass** *of a Gaussian population of classical bouncing balls. The red orbit—obtained as explained in the preceding caption—serves as a "clock."*

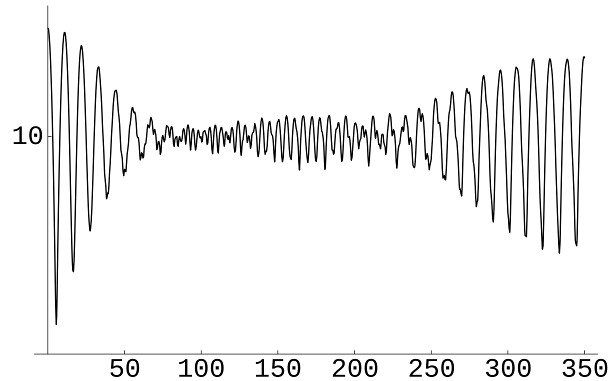

FIGURE 5: *Longer-term motion of the mean position or center of mass, showing* **extinction & revival**.

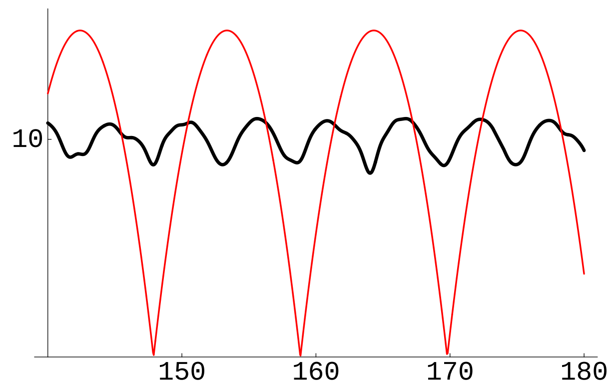

FIGURE 6: *Magnified view of motion of the mean during a typical extinction era. Note the* **frequency-doubling**.

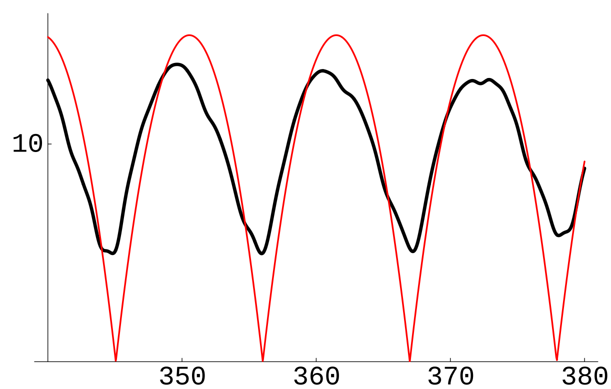

FIGURE 7: *Magnified view of motion of the mean during a typical revival era. The central frequency* $(a = 15)$ *is clearly predominant; neighboring frequencies have distorted the shape of the oscillation, and have conspired to produce a noticeable* **phase shift**.

4. Discussion of the experimental results in hand. Observation & experiment can suggest theories, can contradict theories, can support theories but can never "confirm" theories: that is true of physical experiment, and no less true of "mathematical experiments" of the sort reported above. That said, it remains nevertheless the case that Figures 4–7 do bear a striking resemblance to figures encountered in §13 of Part B. See in this same connection the following figures:

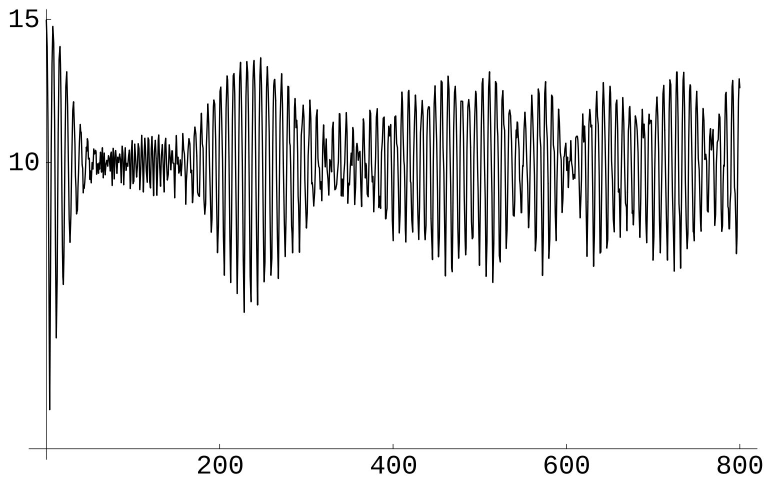

FIGURE 8: *Motion of the center of mass of a* **classical population** *of bouncing balls, the masses of which were at the outset normally distributed.*

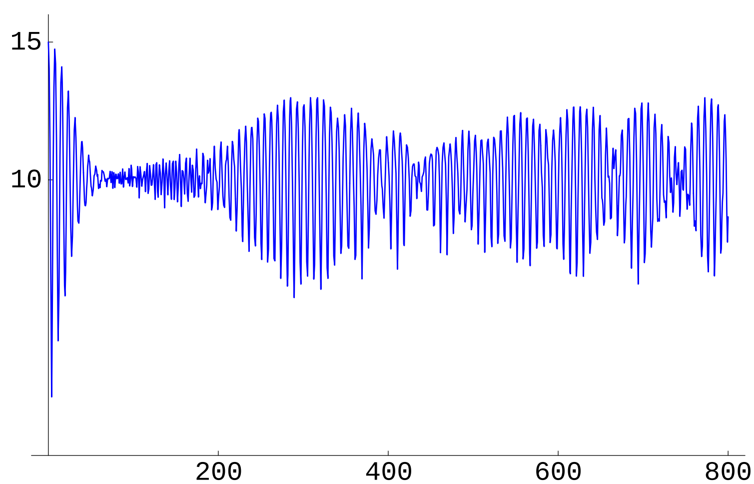

FIGURE 9: **Quantum** *motion of the expected position of a* **single particle** *that was initially in a Gaussian state.*

It is impossible not to infer that

- the classical motion of the center of mass of a *population* of bouncers, and
- the quantum mechanical motion of the expected position $\langle \mathsf{x} \rangle$

are in qualitatively agreement, impossible to suppose that that agreement is an "accidental/exceptional" consequence of the specific numbers that we fed into our calculations—that it is not, in short, *typical of the general case*. More refined experiments (not reported here) suggest, moreover, that as we

- carry the classical Fourier series (1.2) to higher order, and
- " refine the population" in the sense described below

the classical/quantum agreement becomes *precise*.

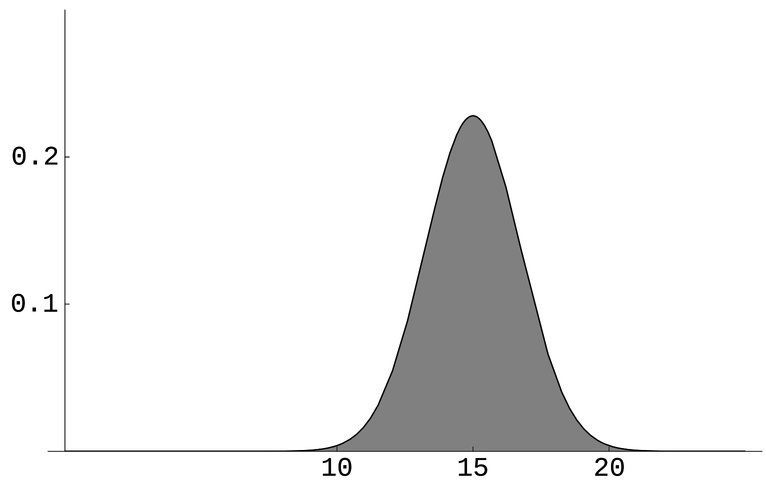

FIGURE 10: *The "course grained" population of eleven particles to which Figure 2 refers has here been replaced by a population of unlimitedly many particles, each of which has infinitesimal weight, but the distribution and collective weight of which is the same as it was before. "Refinement of the population" is identical in essence to the "lattice refinement" that is basic to one common approach to the classical field theory: see Figure 3 in Chapter 1 of* CLASSICAL THEORY OF FIELDS (1999).

How are we to account for the seeming fact that two radically different theories —one intensely quantum mechanical (relying critically upon detailed spectral properties of a Schrödinger equation, upon the principle of superposition), the other entirely classical (no $\hbar$'s, no matrix elements or quantum interference terms)—give identical results?

In the quantum theory of a bouncing ball (Part B) we worked in the **Schrödinger picture** and x-representation, writing

$$\langle \mathsf{x} \rangle_t = \int \psi^*(x,t)\, x\, \psi(x,t)\, dx \tag{6}$$

In our classical theory of a bouncing population we used $|\psi^*(x,0)\,\psi(x,0)|$ to configure the initial design of the population, and let the laws of *classical* mechanics take over. To symbolize the latter procedure, let

$$x(t;a) \equiv \text{expression on the right side of (1)}$$

describe the classical motion of the bouncer that was dropped at time $t = 0$ from height a.[3] To the ball dropped from height a_i we assigned weight

$$w_i = \int_{\frac{1}{2}(a_{i-1}+a_i)}^{\frac{1}{2}(a_i+a_{i+1})} |\psi(a,0)|^2\,da$$

and were led thus to write an equation that can be notated

$$x_{\text{center of mass}}(t) = \sum_i \int_{\frac{1}{2}(a_{i-1}+a_i)}^{\frac{1}{2}(a_i+a_{i+1})} \psi(a,0)^* x(t;a_i)\,\psi(a,0)\,da$$

and in the refined limit becomes

$$= \int_0^\infty \psi(a,0)^* x(t;a)\,\psi(a,0)\,da \tag{7}$$

Notice now that in the **Heisenberg picture** (6) becomes

$$\langle \mathsf{x}\rangle_t = \int_0^\infty\!\!\int_0^\infty \psi^*(x,0)\,dx\,(x|\mathsf{x}(t)|y)\,dy\,\psi(y,0) \tag{8}$$

For *unobstructed* free fall (Part A) the Hamiltonian is $\mathsf{H} = \frac{1}{2m}\mathsf{p}^2 + mg\mathsf{x}$ and the Heisenberg equations of motion read

$$\begin{aligned}\dot{\mathsf{x}} &= -\tfrac{1}{i\hbar}\big[\mathsf{H},\mathsf{x}\big] = \tfrac{1}{m}\mathsf{p}\\ \dot{\mathsf{p}} &= -\tfrac{1}{i\hbar}\big[\mathsf{H},\mathsf{p}\big] = -mg\,\mathsf{I}\end{aligned}$$

which give $\mathsf{p}(t) = \mathsf{p}_0 - mg\,t\,\mathsf{I}$ whence $\dot{\mathsf{x}} = \frac{1}{m}\mathsf{p}_0 - g\,t\,\mathsf{I}$ whence

$$\mathsf{x}(t) = \mathsf{x}_0 + \tfrac{1}{m}t\,\mathsf{p}_0 - \tfrac{1}{2}g\,t^2\,\mathsf{I}$$

Since we plan to "drop" (not to "launch") our particles we set $\mathsf{p}_0 = \mathsf{0}$ and obtain

$$(x|\mathsf{x}(t)|y) = (x|\big\{\mathsf{x}_0 - \tfrac{1}{2}g\,t^2\big\}|y) = \big\{y - \tfrac{1}{2}g\,t^2\big\}\cdot\delta(x-y)$$

The free-fall analog of (7) could on this basis be written

$$\langle \mathsf{x}\rangle_t^{\text{free fall}} = \int \psi^*(y,0)\big\{y - \tfrac{1}{2}g\,t^2\big\}\psi(y,0)\,dy \tag{9}$$

[3] Note that the subsequent **motion of the bouncer is mass-independent!**

where $x(t;y)=\{y-\frac{1}{2}g\,t^2\}$ describes the classical motion of a ball that at time $t=0$ was dropped from height y. For a quantum bouncer we expect therefore to have

$$\langle\mathbf{x}\rangle_t^{\text{bouncer}}=\int\psi^*(y,0)\big\{x(t;y)\big\}\,\psi(y,0)\,dy \tag{10.1}$$

where (see again equations (1))

$$\begin{aligned} x(t;y) &= \sum_n\big[y-\tfrac{1}{2}g\big(t-n\sqrt{8y/g}\,\big)^2\big] \\ &\qquad\cdot\texttt{UnitStep[}\big[y-\tfrac{1}{2}g\big(t-n\sqrt{8y/g}\,\big)^2\big]\texttt{]} \end{aligned}\tag{10.21}$$

$$= y\Big[\tfrac{2}{3}-\tfrac{4}{\pi^2}\sum_{k=1}^\infty(-)^k\tfrac{1}{k^2}\cos\big[k\pi\sqrt{g/2y}\,t\big]\Big] \tag{10.22}$$

describes the classical motion of a *bouncer* that at time $t=0$ was dropped from height y. But in the quantum mechanical equation (10.1) we have recovered precisely the equation that at (7) served to describe the center of mass motion of a *classical population*: quite unexpectedly, **quantum mechanics has acquired a classical face**.[4]

Our prior experience was with an illustrative instance of the Gaussian distribution

$$\begin{aligned}\psi^*(y,0)\,\psi(y,0) &= \tfrac{1}{s\sqrt{2\pi}}\,e^{-\frac{1}{2}\left[\frac{y-A}{s}\right]^2} \\ &\downarrow \\ &= \tfrac{4}{7\sqrt{2\pi}}\,e^{-\frac{1}{2}\left[\frac{4}{7}(x-15)\right]^2}\end{aligned}$$

and when we feed this information into (10) we find that *Mathematica* prefers to work with (10.22) and to integrate numerically; I omit the results of those experiments, but can report that they do tend to confirm the correctness of (10).

5. Steady distributions. I begin with some general remarks. Let $|\psi)_0$ be, in particular, an eigenstate of the Hamiltonian $\mathbf{H}$: $\mathbf{H}|n)=E_n|n)$. Then in the Schrödinger picture

$$|n)_0\longrightarrow|n)_t=e^{-\frac{i}{\hbar}E_nt}\cdot|n)_0$$

and it becomes obvious that

$$(n|\mathbf{A}|n)\text{ is a constant of the motion}$$

[4] The argument has, however, one formal defect: For a quantum bouncer one has

$$\mathbf{H}=\tfrac{1}{2m}\mathbf{p}^2+V_{\text{bouncer}}(\mathbf{x})$$

where the "bouncer potential" $V_{\text{bouncer}}(x)$ is $|\!\!\diagup$-shaped. We have not attempted to extract (10.2) <u>from the operator algebra</u> of the problem, as a solution of the Heisenberg equations of motion. This was easy for unrestricted free fall, but for the bouncer seems at the moment to be almost unapproachable.

for *every* time-independent observable $\mathbf{A}$—this whether or not $\mathbf{A}$ happens to refer to a "constant of the motion:" $\big[\mathbf{H},\mathbf{A}\big]=\mathbf{0}$. To make the same point another way: $\underline{\text{the quantum motion of } (\psi|\mathbf{A}|\psi) \text{ is an interference effect}}$, a consequence of two or more eigencomponents buzzing at different frequencies.[5] In x-representation we write $\psi_n(x,t)\equiv(x|n)_t$ and, in the cases $\mathbf{A}=\mathbf{x}^\nu$, have

$$\int\psi_n^*(x,t)\,x^\nu\psi_n(x,t)\,dx=\int\psi_n^*(x,0)\,x^\nu\psi_n(x,0)\,dx\quad:\quad\nu=0,1,2,\ldots$$

Not only the mean but the *position moments of all orders are seen to be constant* when the quantum system is in an energy eigenstate.

Return now from generalities to the bouncer specifics. We expect—for the unsurprising reason just explained—to have

$$\langle\mathbf{x}\rangle_n=\int_0^\infty\psi_n^*(y,t)\,y\,\psi_n(y,t)\,dx=\text{unchanging mean}\tag{11}$$

if $\psi_n(x,t)$ refers to the buzzing n^{th} bouncer eigenfunction. Which brings me to my question: Can we—as an instance of (10.1)—expect to have

$$=\int_0^\infty\psi_n^*(y,0)\,x(t,y)\,\psi_n(y,0)\,dy\tag{12}$$

and to interpret that construction as having to do with the "invariable center of mass of a suitably-constructed dropped population"? **No!** For when

$$\mathbf{x}(t)=\mathbf{U}^+(t)\,\mathbf{x}\,\mathbf{U}(t)=\sum_{a,b}|a)e^{+\frac{i}{\hbar}E_at}(a|\mathbf{x}|b)e^{+\frac{i}{\hbar}E_bt}(b|$$

is hit with $(n|$ on the left and $|n)$ on the right the t-dependence drops away: one is left with $(n|\mathbf{x}(t)|n)=(n|\mathbf{x}|n)$ which in x-representation is not (12) but (11).

I have, however, performed—and will now describe—some numerical experiments based upon the faulty construction (12) which lead to a curious conjecture. If n is large then $|\psi_n(y,0)|^2$ is so wildly oscillatory that *Mathematica* has difficulty with the $\int$, so I confine my remarks to the low-order case

$$\psi_4(y,0)=1.09787\,\mathrm{Ai}(y-6.78671)$$

The distribution $|\psi_4(y,0)|^2$ is shown in Figure 11. *Mathematica* complained a lot when asked to list values of the numerical integrals

$$\int_0^{12}|\psi_4(y,0)|^2x(t_k,y)\,dy\quad\text{with}\quad t_k\equiv\tfrac{1}{5}k\;:\;k=0,1,2,\ldots,100\tag{13}$$

[5] The point at issue become equally obvious in the Heisenberg picture if one writes

$$\mathbf{U}(t)=\sum_m|m)\,e^{-\frac{i}{\hbar}E_nt}(m|$$

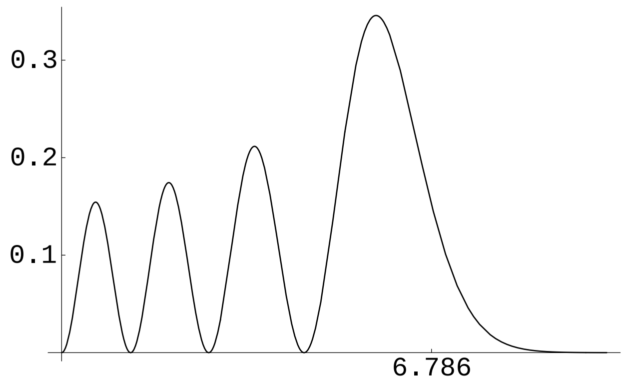

FIGURE 11: *Graph of the distribution* $|\psi_4(y,0)|^2$ *on which were based the experiments described in the text.*

but seemed indifferent to whether I used (10.21) or (10.22) to define $x(t;y)$: in practice I used an 11-term Fourier series, and convinced myself that no significant error would be introduced if the upper limit on the integral were taken to be 12 instead of ∞. The data thus generated is displayed in Figure 12. That the classical center of mass initially descends is very easy to understand, and no great mystery attaches to the fact that it subsequently exhibits damped oscillations. What I do find a little surprising is that after awhile (see Figure 13) the oscillations appear to have damped to extinction: center of mass has come to rest at 3.01685. It was anticipated that

$$(4|\mathsf{x}|4) = \int_0^\infty y\,|\psi_4(y,0)|^2\,dy = 4.42447 = \tfrac{2}{3}\cdot 6.78671$$

but I did not anticipate—and can provide no explanation for the fact—that seemingly

$$\lim_{t\to\infty}\int_0^\infty |\psi_4(y,0)|^2 x(t_k,y)\,dy = 3.01685 \approx 3.01632 = \tfrac{2}{3}\cdot 4.42447$$

Because other examples gave similar results I am led to **conjecture** that for classical populations that have been structured by quantum mechanical energy eigenfunctions one—in all cases—has

$$\lim_{t\to\infty}\int_0^\infty |\psi_n(y,0)|^2 x(t_k,y)\,dy = \tfrac{2}{3}\cdot(n|\mathsf{x}|n) \tag{14.1}$$

The striking stability of the asymptotic value seems to suggest that *bouncer anharmonicity has brought the population to some kind of statistical equilibrium.* Looking to the second moment, we anticipate/confirm[6] that

$$(4|\mathsf{x}^2|4) = \int_0^\infty y^2\,|\psi_4(y,0)|^2\,dy = 24.56500 = \tfrac{8}{15}\cdot(6.78671)^2$$

[6] See Part B, page 9.

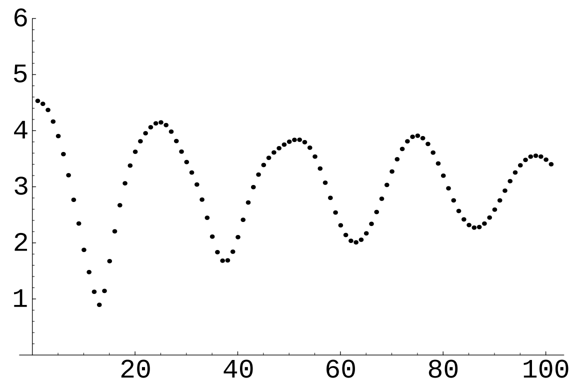

FIGURE 12: *Initial center of mass motion, computed from* (13). *The center of mass of the quantum mechanically structured classical population initially drops, then rebounds, but the oscillations become progressively more indistinct as—owing to the anharmonicity of classical bouncing—the population grows increasingly incoherent.*

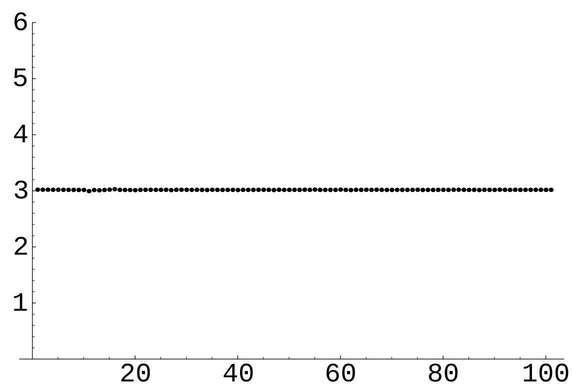

FIGURE 13: *Subsequent center of mass motion, computed from* (13) *with* $k = 200, 201, \ldots, 300$. *By this time the phase incoherence appears to have become complete, and to have achieved a steady state. Add 200 to each of the numbers that decorate the horizontal axis:* Mathematica *was indicating* placement in a list *rather than* value of k. *The averaged point value is* 3.01685.

and on the basis of some relatively rough calculation obtain

$$\lim_{t\to\infty}\int_0^\infty |\psi_4(y,0)|^2\,[x(t_k,y)]^2\,dy = 13.0146 \approx 13.1013 = \tfrac{8}{15}\cdot(4|\mathsf{x}^2|4)$$

—the suggestion being that in the general 2[nd]-order case

$$\lim_{t\to\infty}\int_0^\infty |\psi_n(y,0)|^2\,[x(t_k,y)]^2\,dy = \tfrac{8}{15}\cdot(n|\mathsf{x}^2|n) \tag{14.2}$$

It seems reasonable to hope that we might extend (14) to *all* orders and, on the basis of that information, construct a detailed description of the equilibrated classical population that is latent in the quantum distribution $|\psi_n(y,0)|^2$.

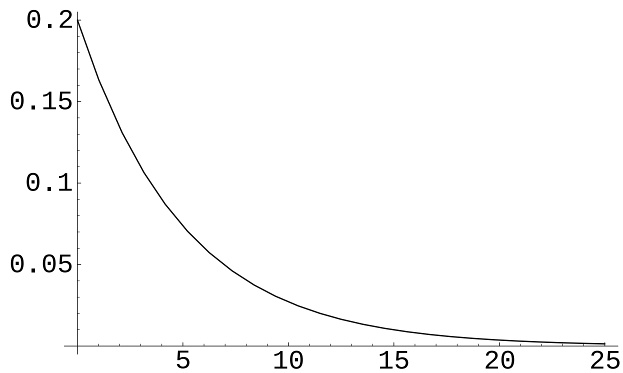

FIGURE 14: *Normalized exponential distribution*

$$B(x;\beta) \equiv \beta\, e^{-\beta x} \quad : \quad \beta > 0$$

in the case $\beta = 5$. *In that case* $\langle x\rangle = 5$ *and* $\langle x^2\rangle = 50$.

With thermal physics in the back of our minds it becomes natural to ask: What can be said concerning asymptotic moment motion if the initial population is *exponentially* weighted (Figure 14)? Numerical experimentation (Figures 15 & 16) suggest that here again the moments of all orders rapidly stabilize

$$\begin{aligned}\text{asymptotic first moment} &= \tfrac{2}{3}\cdot\text{initial first moment}\\ \text{asymptotic second moment} &= \tfrac{8}{15}\cdot\text{initial second moment}\\ &\ \vdots\end{aligned}$$

...where the factors are those latent in the classical distribution[7]

$$Q(x;a) \equiv \frac{1}{2\sqrt{a}\sqrt{a-x}} \quad : \quad 0 \leqslant x \leqslant a \tag{15}$$

which is readily found to supply

$$\begin{aligned}\langle x^1\rangle &= \tfrac{2}{3}\cdot a\\ \langle x^2\rangle &= \tfrac{8}{15}\cdot a^2\\ \langle x^3\rangle &= \tfrac{16}{35}\cdot a^3\\ &\ \vdots\\ \langle x^n\rangle &= \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots(2n+1)}\cdot a^n\end{aligned}$$

[7] See Part B, §3, equation (8). It is of physical interest that this distribution is independent of both m and g.

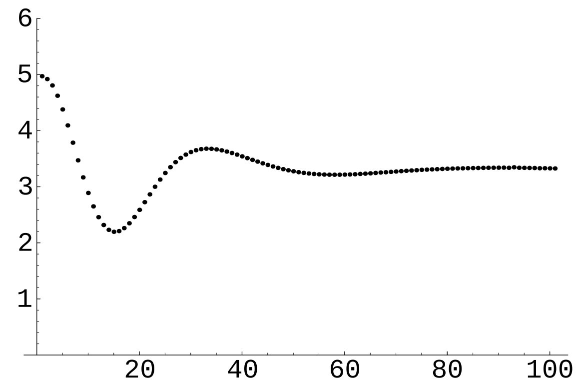

FIGURE 15: *Data obtained by numerical evaluation of*

$$\int_0^{35} B(y,5)\,x(y,t_k)\,dy \quad \text{with} \quad t_k = \tfrac{1}{5}k \;\;:\;\; k = 0,1,2,\ldots 100$$

and with $x(y,t)$ approximated by the first eleven terms of (10.22) *with g set equal to* 2. *Evidently*

$$\text{asymptotic mean} \approx 3.32631 \approx 3.33333 = \tfrac{2}{3}\cdot 5$$

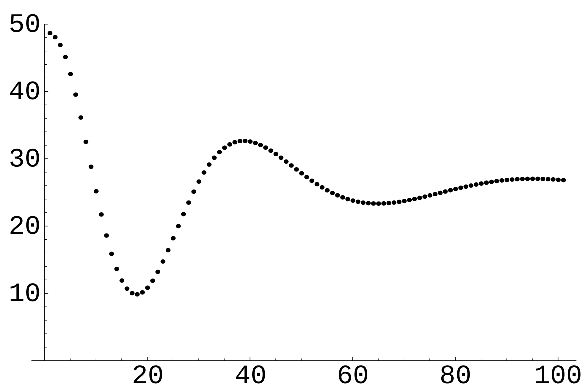

FIGURE 16: *Data obtained by numerical evaluation of*

$$\int_0^{35} B(y,5)[x(y,t_k)]^2\,dy$$

Evidently

$$\text{asymptotic } 2^{\text{nd}} \text{ moment} \approx 26.8188 \approx 26.6667 = \tfrac{8}{15}\cdot 50$$

What we presently lack is a theory on the basis of which we might have *predicted* this experimental result, and that would enable us to understand what has acquired now the status of an urgent problem: *Why does the Gaussian population not stabilize? Why does it exhibit extinction and revival?*

6. Gravitational applications of some operator algebra. Hans Zassenhaus, in unpublished work done sometime prior to 1954, has established[8] that for all A and all B

$$e^{\mathsf{A}+\mathsf{B}} = e^{\mathsf{A}}\cdot e^{\mathsf{B}}\cdot e^{\mathsf{C}_2}\cdot e^{\mathsf{C}_3}\cdots$$

with

$$\begin{aligned}\mathsf{C}_2 &= -\tfrac{1}{2}[\mathsf{A},\mathsf{B}]\\ \mathsf{C}_3 &= \tfrac{1}{6}[\mathsf{A},[\mathsf{A},\mathsf{B}]]+\tfrac{1}{3}[\mathsf{B},[\mathsf{A},\mathsf{B}]]\\ &\;\;\vdots\end{aligned}$$

With $\mathsf{H}=\frac{1}{2m}\mathsf{p}^2+mg\,\mathsf{x}$ in mind we set $\mathsf{A}=\alpha\mathsf{p}^2$, $\mathsf{B}=\beta\mathsf{x}$ and, using $[\mathsf{x},\mathsf{p}]=i\hbar\mathsf{I}$, compute $[\mathsf{A},\mathsf{B}]=-2i\hbar\alpha\beta\mathsf{p}$, $\mathsf{C}_2=i\hbar\alpha\beta\mathsf{p}$, $\mathsf{C}_3=\frac{2}{3}\hbar^2\alpha\beta^2\mathsf{I}$, $\mathsf{C}_{n>3}=\mathsf{0}$ giving

$$e^{\alpha\mathsf{p}^2+\beta\mathsf{x}} = e^{\alpha\mathsf{p}^2}\cdot e^{\beta\mathsf{x}}\cdot e^{i\hbar\alpha\beta\mathsf{p}}\cdot e^{\frac{2}{3}\hbar^2\alpha\beta^2}$$

It is, however, an implication of Zassenhaus' identity that

$$e^{\beta\mathsf{x}}\cdot e^{i\hbar\alpha\beta\mathsf{p}} = e^{i\hbar\alpha\beta\mathsf{p}}\cdot e^{\beta\mathsf{x}}\cdot e^{-\hbar^2\alpha\beta^2}$$

so we have the $\mathsf{p}\,\mathsf{x}$-ordered expression

$$= e^{\alpha\mathsf{p}^2}\cdot e^{i\hbar\alpha\beta\mathsf{p}}\cdot e^{\beta\mathsf{x}}\cdot e^{-\frac{1}{3}\hbar^2\alpha\beta^2} \tag{16}$$

Setting $\alpha=-\frac{i}{\hbar}\frac{1}{2m}t$ and $\beta=-\frac{i}{\hbar}mgt$ we use (16) to obtain

$$\begin{aligned}\mathsf{U}_{\text{free fall}}(t) &\equiv e^{-\frac{i}{\hbar}\{\frac{1}{2m}\mathsf{p}^2+mg\mathsf{x}\}t}\\ &= e^{-\frac{i}{\hbar}\{\frac{1}{2m}\mathsf{p}^2+\frac{1}{2}gt\mathsf{p}\}t}\cdot e^{-\frac{i}{\hbar}\{mg\mathsf{x}\}t}\cdot e^{-\frac{i}{\hbar}\{\frac{1}{6}mg^2t^3\}}\end{aligned} \tag{17}$$

which presents the free-fall propagation operator in $\mathsf{p}\,\mathsf{x}$-ordered form.

To illustrate the utility of (17) I look now with its aid to the construction of the propagator. We use the "mixed representation trick,"[9] writing

$$\begin{aligned}(x|\mathsf{U}(t)|y) &= \int (x|p)\,dp(p|\mathsf{U}(t)|y)\\ &= \int (x|p)\,dp\,e^{-\frac{i}{\hbar}\{\frac{1}{2m}p^2+\frac{1}{2}gtp\}t}\cdot e^{-\frac{i}{\hbar}\{mgy\}t}\cdot e^{-\frac{i}{\hbar}\{\frac{1}{6}mg^2t^3\}}(p|y)\end{aligned}$$

But[10]

$$(x|p)(p|y) = \tfrac{1}{h}e^{\frac{i}{\hbar}p(x-y)}$$

[8] See ADVANCED QUANTUM TOPICS (2001), Chapter 0, page 33.
[9] See pages 38–43 in the notes just cited.
[10] See equation (80) on page 36 of those same notes.

So if we allow ourselves to pretend for the moment that $\frac{it}{\hbar 2m}$ has a positive real part (which it is our intention to "turn off" at the end of the calculation) then we confront a Gaussian integral, formal execution of which yields

$$(x|\,\mathsf{U}(t)|y) = \sqrt{\tfrac{m}{iht}}\exp\Big\{\tfrac{i}{\hbar}\big[\tfrac{m}{2t}(x-y)^2 - \tfrac{1}{2}mgt(x+y) - \tfrac{1}{24}mg^2t^3\big]\Big\} \tag{18}$$

We recognize $[etc.]$ to be the *classical* free-fall action $S(x,t;y,0)$ encountered at (9) in Part A. And we recognize (18) to be in precise agreement with the description of the free-fall propagator $K(x,t;y,0)$ to which at (61) in Part A we were led by quite a different line of argument—an argument that made heavy use of properties of the Airy function $\mathrm{Ai}(z)$.

I press (17) now into service toward a different objective: In the Heisenberg picture (see again page 9) we have

$$\left.\begin{aligned} \mathsf{x} &\longrightarrow \mathsf{x}(t) = \mathsf{U}^{-1}(t)\,\mathsf{x}\,\mathsf{U}(t)\\ \mathsf{p} &\longrightarrow \mathsf{p}(t) = \mathsf{U}^{-1}(t)\,\mathsf{p}\,\mathsf{U}(t)\end{aligned}\right\} \tag{19}$$

and are placed by (17) in position to develop explicit descriptions of $\mathsf{x}(t)$ and $\mathsf{p}(t)$. Writing

$$\mathsf{x}\,e^{-\frac{i}{\hbar}\{\frac{1}{2m}\mathsf{p}^2+\frac{1}{2}gt\mathsf{p}\}t} = {}_{\mathsf{x}}\Big[x\,e^{-\frac{i}{\hbar}\{\frac{1}{2m}p^2+\frac{1}{2}gtp\}t}\Big]_{\mathsf{p}}$$

I use a technique described at (15) on page 7 of Chapter 2 in the notes cited previously[8] to *reverse the ordering* of the expression on the right:

$$\begin{aligned} &= {}_{\mathsf{p}}\Big[\exp\Big\{-\tfrac{\hbar}{i}\tfrac{\partial^2}{\partial x\partial p}\Big\}x\,e^{-\frac{i}{\hbar}\{\frac{1}{2m}p^2+\frac{1}{2}gtp\}t}\Big]_{\mathsf{x}}\\ &= {}_{\mathsf{p}}\Big[e^{-\frac{i}{\hbar}\{\frac{1}{2m}p^2+\frac{1}{2}gtp\}t}\Big\{x+\tfrac{1}{m}tp+\tfrac{1}{2}gt^2\Big\}\Big]_{\mathsf{x}}\\ &= e^{-\frac{i}{\hbar}\{\frac{1}{2m}\mathsf{p}^2+\frac{1}{2}gt\mathsf{p}\}t}\Big\{\mathsf{x}+\tfrac{1}{m}t\mathsf{p}+\tfrac{1}{2}gt^2\,\mathsf{I}\Big\}\end{aligned}$$

Therefore

$$\mathsf{x}(t) = \mathsf{x} + \tfrac{1}{2}gt^2\,\mathsf{I} + e^{+\frac{i}{\hbar}\{mg\mathsf{x}\}t}\cdot\tfrac{1}{m}t\mathsf{p}\cdot e^{-\frac{i}{\hbar}\{mg\mathsf{x}\}t}$$

But by the same argument

$$\begin{aligned}\mathsf{p}\,e^{-\frac{i}{\hbar}\{mg\mathsf{x}\}t} &= {}_{\mathsf{p}}\Big[p\,e^{-\frac{i}{\hbar}\{mgx\}t}\Big]_{\mathsf{x}}\\ &= {}_{\mathsf{x}}\Big[\exp\Big\{+\tfrac{\hbar}{i}\tfrac{\partial^2}{\partial x\partial p}\Big\}p\,e^{-\frac{i}{\hbar}\{mgx\}t}\Big]_{\mathsf{p}}\\ &= {}_{\mathsf{x}}\Big[e^{-\frac{i}{\hbar}\{mgx\}t}\Big\{p-mgt\Big\}\Big]_{\mathsf{p}}\\ &= e^{-\frac{i}{\hbar}\{mg\mathsf{x}\}t}\Big\{\mathsf{p}-mgt\,\mathsf{I}\Big\}\quad:\quad\left\{\begin{array}{l}\text{elaborate derivation of}\\ \text{a simple "shift rule"}\end{array}\right.\end{aligned}$$

so

$$\begin{aligned}\mathsf{x}(t) &= \mathsf{x} + \tfrac{1}{2}gt^2\,\mathsf{I} + \tfrac{1}{m}t\,\mathsf{p} - gt^2\,\mathsf{I}\\ &= \mathsf{x} - \tfrac{1}{2}gt^2\,\mathsf{I} + \tfrac{1}{m}t\,\mathsf{p} \end{aligned}\tag{20.1}$$

Similarly

$$\begin{aligned}\mathsf{p}(t) &= e^{+\frac{i}{\hbar}\{mg\,\mathsf{x}\}t}\,\mathsf{p}\,e^{-\frac{i}{\hbar}\{mg\,\mathsf{x}\}t}\\ &= \mathsf{p} - mgt\,\mathsf{I}\end{aligned}\tag{20.2}$$

Equations (20) spell out the detailed substance of "t-parameterized unitary similarity transformation rule" (19).

We are reassured but not surprised by the observation that

$$\big[\mathsf{x}(t),\mathsf{p}(t)\big] = \big[\mathsf{x},\mathsf{p}\big] + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} = i\hbar\,\mathsf{I} \quad : \quad \text{all } t$$

I say "not surprised" because it is (in the Heisenberg picture) *universally* the case that

$$\big[\mathsf{U}^{-1}(t)\,\mathsf{x}\,\mathsf{U}(t),\mathsf{U}^{-1}(t)\,\mathsf{p}\,\mathsf{U}(t)\big] = \mathsf{U}^{-1}(t)\big[\mathsf{x},\mathsf{p}\big]\,\mathsf{U}(t) = i\hbar\,\mathsf{I}$$

From (20) it follows that

$$\begin{aligned}\tfrac{d}{dt}\mathsf{x}(t) &= \tfrac{1}{m}\big\{\mathsf{p} - mgt\,\mathsf{I}\big\}\\ &= \tfrac{1}{m}\,\mathsf{p}(t) &\quad(21.1)\\ \tfrac{d}{dt}\mathsf{p}(t) &= -mg\,\mathsf{I} &\quad(21.2)\end{aligned}$$

which are precisely *Heisenberg's equations of free fall motion*, as encountered already on page 9. They are structurally identical to their classical counterparts: Hamilton's canonical equations of unobstructed free fall. Differentiating once again we obtain

$$\tfrac{d^2}{dt^2}\mathsf{x}(t) = -g\,\mathsf{I} \quad : \quad \text{all } m$$

which might be said to describe "quantum mechanical free fall according to Newton"!

All of which, though gratifying, is surprising in no respect . . . but does inspire confidence in the accuracy of (17), and in the effectiveness of our operator management techniques.

Pursuant to the discussion in §10, §11 and (especially) §23 of Part A—all of which radiated from the notion that "free fall" is *free motion referred to a uniformly accelerated frame*—we observe that while (17), written

$$\mathsf{U}_g(t) = e^{-\frac{i}{\hbar}\{\frac{1}{6}mg^2t^3\}}\cdot e^{-\frac{i}{\hbar}\{\frac{1}{2m}\mathsf{p}^2+\frac{1}{2}gt\,\mathsf{p}\}t}\cdot e^{-\frac{i}{\hbar}\{mg\,\mathsf{x}\}t}$$

describes quantum dynamical motion in the *presence* of a uniform gravitational field, we have only to set $g = 0$ to obtain the unitary operator

$$\mathsf{U}_0(t) = e^{-\frac{i}{\hbar}\{\frac{1}{2m}\mathsf{p}^2\}t}$$

that describes free motion, motion in the *absence* of such a field. And that we have

$$\mathsf{U}_g(t) = \mathcal{G}(t;g)\,\mathsf{U}_0(t) \tag{22}$$

provided we set

$$\begin{aligned}\mathcal{G}(t;g) &= \mathsf{U}_g(t)\,\mathsf{U}_0^{-1}(t)\\ &= e^{-\frac{i}{\hbar}\{\frac{1}{6}mg^2t^3\}}\cdot e^{-\frac{i}{\hbar}\{\frac{1}{2m}\mathsf{p}^2+\frac{1}{2}gt\,\mathsf{p}\}t}\cdot e^{-\frac{i}{\hbar}\{mg\,\mathsf{x}\}t}\cdot e^{+\frac{i}{\hbar}\{\frac{1}{2m}\mathsf{p}^2\}t}\end{aligned}$$

Drawing once again on the operator reordering procedure employed already twice before, we have

$$\begin{aligned}e^{-\frac{i}{\hbar}\{mg\,\mathsf{x}\}t}\cdot e^{+\frac{i}{\hbar}\{\frac{1}{2m}\mathsf{p}^2\}t} &= {}_{\mathsf{x}}\Big[e^{-\frac{i}{\hbar}\{mgx\}t}\cdot e^{+\frac{i}{\hbar}\{\frac{1}{2m}p^2\}t}\Big]_{\mathsf{p}}\\ &= {}_{\mathsf{p}}\Big[\exp\Big\{-\frac{\hbar}{i}\frac{\partial^2}{\partial x\partial p}\Big\}e^{-\frac{i}{\hbar}\{mgx\}t}\cdot e^{+\frac{i}{\hbar}\{\frac{1}{2m}p^2\}t}\Big]_{\mathsf{x}}\\ &= {}_{\mathsf{p}}\Big[e^{-\frac{i}{\hbar}\{mgx\}t}\cdot \exp\Big\{mgt\frac{\partial}{\partial p}\Big\}e^{+\frac{i}{\hbar}\{\frac{1}{2m}p^2\}t}\Big]_{\mathsf{x}}\\ &= {}_{\mathsf{p}}\Big[e^{-\frac{i}{\hbar}\{mgx\}t}\cdot e^{+\frac{i}{\hbar}\{\frac{1}{2m}(p+mgt)^2\}t}\Big]_{\mathsf{x}}\\ &= e^{+\frac{i}{\hbar}\{\frac{1}{2m}(\mathsf{p}+mgt\,\mathsf{I})^2\}t}\cdot e^{-\frac{i}{\hbar}\{mg\,\mathsf{x}\}t}\\ &= e^{+\frac{i}{\hbar}\{\frac{1}{2m}\mathsf{p}^2\}t}\cdot e^{+\frac{i}{\hbar}\{gt^2\,\mathsf{p}+\frac{1}{2}mg^2t^3\,\mathsf{I}\}t}\cdot e^{-\frac{i}{\hbar}\{mg\,\mathsf{x}\}t}\end{aligned}$$

giving

$$\begin{aligned}\mathcal{G}(t;g) &= e^{\frac{i}{\hbar}\{(\frac{1}{2}-\frac{1}{6})mg^2t^3\}}\cdot e^{\frac{i}{\hbar}\{(1-\frac{1}{2})gt^2\,\mathsf{p}\}}\cdot e^{-\frac{i}{\hbar}\{mg\,\mathsf{x}\}t}\\ &= e^{\frac{i}{\hbar}\{\frac{1}{2}gt^2\,\mathsf{p}\}}\cdot e^{-\frac{i}{\hbar}\{mgt\,\mathsf{x}\}}\cdot e^{+\frac{i}{\hbar}\{\frac{1}{3}mg^2t^3\}} \quad : \quad \mathsf{p}\,\mathsf{x}\text{-ordered} \qquad (23.1)\\ &= e^{-\frac{i}{\hbar}\{mgt\,\mathsf{x}\}}\cdot e^{\frac{i}{\hbar}\{\frac{1}{2}gt^2\,\mathsf{p}\}}\cdot e^{-\frac{i}{\hbar}\{(\frac{1}{2}-\frac{1}{3})mg^2t^3\}}\\ &= e^{-\frac{i}{\hbar}\{\frac{1}{6}mg^2t^3\}}\cdot e^{-\frac{i}{\hbar}\{mgt\,\mathsf{x}\}}\cdot e^{\frac{i}{\hbar}\{\frac{1}{2}gt^2\,\mathsf{p}\}} \quad : \quad \mathsf{x}\,\mathsf{p}\text{-ordered} \qquad (23.2)\end{aligned}$$

in connection with which we notice that

$$\left.\begin{aligned}\mathcal{G}(0;g) &= \mathsf{I} \quad : \quad \text{all } g\\ \mathcal{G}(t;0) &= \mathsf{I} \quad : \quad \text{all } t\end{aligned}\right\} \tag{24}$$

What, by these exertions, have we gained? Suppose that $|\Psi)$ <u>evolves freely</u>:

$$|\Psi)_0 \longrightarrow |\Psi)_t = \mathsf{U}_0(t)|\Psi)_0$$

Hit that statement with the unitary operator $\mathcal{G}(t;g)$, introduce the notation

$$|\Psi)_t \equiv \mathcal{G}(t;g)|\Psi)_t$$

and—by (22)—obtain

$$\begin{aligned}|\Psi)_0 \longrightarrow |\Psi)_t &= \mathsf{U}_g(t)\,|\Psi)_0\\ |\Psi)_0 &= |\Psi)_0\end{aligned}$$

according to which $|\Psi)$ evolves by free fall. In the $\mathbf{x}$-representation we have

$$\Psi(x,t) = e^{-\frac{i}{\hbar}\{mgtx+\frac{1}{6}mg^2t^3\}}\cdot\Psi(x+\tfrac{1}{2}gt^2,t) \tag{25}$$

as an immediate consequence of (23.2). Evidently $\mathbf{G}(t;g)$ provides a (picture-independent, also representation-independent) description of the quantum mechanical effect

- of "turning on g," or equivalently
- of adopting a uniformly accelerated reference frame

and, of course, $\mathbf{G}^{-1}(t;g)$ describes the reverse of those operations. We have been brought here by operator manipulations to conclusions identical to those reached in §23 of Part A. The present line of argument is, I would argue, to be preferred: it is swifter, and avoids explicit reference to "gauge factors."

7. The "dropped eigenfunction" problem—revisited. I return now to the problem addressed in §25 of Part A. Consider the p-indexed xt-separated functions

$$\Psi_p(x,t) \equiv \tfrac{1}{\sqrt{h}}e^{\frac{i}{\hbar}px}\cdot e^{-\frac{i}{\hbar}\frac{1}{2m}p^2t}$$

—each of which satisfies the free particle Schrödinger equation

$$-\tfrac{\hbar^2}{2m}\partial_x^2\psi = i\hbar\partial_t\psi$$

and describes what might be called a "buzzing eigenfunction" of the system $\mathbf{H} = \frac{1}{2m}\mathbf{p}^2$. We note that Ψ_{+p} and Ψ_{-p} associate with the *same* spectral value $E = p^2/2m$. The free particle energy spectrum is (except at $p=0$) *doubly degenerate*: the system supports both left-running and right-running monochromatic waves of every frequency.

Now use (25) to "drop" such a function $\Psi_p(x,t)$. It becomes

$$\begin{aligned}\Psi_p(x,t) &= \tfrac{1}{\sqrt{h}}\,e^{-\frac{i}{\hbar}\{mgtx+\frac{1}{6}mg^2t^3\}}\cdot e^{\frac{i}{\hbar}p(x+\frac{1}{2}gt^2)}\cdot e^{-\frac{i}{\hbar}\frac{1}{2m}p^2t}\\ &= \tfrac{1}{\sqrt{h}}\,\exp\Big\{\tfrac{i}{\hbar}\Big[x(p-mgt)+\tfrac{1}{6m^2g}(p-mgt)^3-\tfrac{1}{6m^2g}p^3\Big]\Big\}\end{aligned} \tag{26}$$

which (ask *Mathematica*) demonstrably satisfies

$$\Big\{-\tfrac{\hbar^2}{2m}\partial_x^2+mgx\Big\}\psi = i\hbar\partial_t\psi \tag{27}$$

but does *not* possess the xt-separated form of a "buzzing eigenfunction." Nor, for that matter, does t enter *linearly* into the exponent, though linear entry is the *sine qua non* of buzzery. To circumvent those interrelated difficulties—to construct buzzing *eigenfunctions* of the free-fall Hamiltonian—we must take the dropped functions in suitably contrived linear combinations[11]

$$\Psi^{\text{free fall}}(x,t) = \sqrt{k}\,\tfrac{1}{2\pi}\int c(p)\cdot\sqrt{h}\Psi_p(x,t)\cdot\big(\tfrac{1}{2m^2g\hbar}\big)^{\frac{1}{3}}dp \tag{28}$$

[11] Here $\wp \equiv (2m^2g\hbar)^{\frac{1}{3}} = \hbar/\ell_g = \hbar k$ is (see pages 22 and 51 in Part A) a "natural momentum." It and the other factors have been introduced for the dimensional reasons discussed on the next page.

And it is pretty clear how to proceed: set

$$c(p) = \exp\left\{\tfrac{i}{\hbar}\tfrac{1}{6m^2g}p^3\right\}$$

and obtain

$$\begin{aligned}\Psi_0^{\text{free fall}}(x,t) &= \left(\tfrac{1}{2m^2g\hbar}\right)^{\frac{1}{3}}\sqrt{k}\,\tfrac{1}{2\pi}\int \exp\left\{\tfrac{i}{\hbar}\Big[x(p-mgt)+\tfrac{1}{6m^2g}(p-mgt)^3\Big]\right\}dp\\ &= \sqrt{k}\,\tfrac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left\{i\Big[kxu+\tfrac{1}{3}u^3\Big]\right\}du\\ &= \sqrt{k}\cdot \text{Ai}(kx)\quad \text{by definition: see Part A, page 25}\end{aligned}$$

The maneuver

$$p\longmapsto u \quad \text{with} \quad u^3 \equiv \tfrac{1}{2m^2g}(p-mgt)^3$$

that was intended to kill the extraneous t-dependence has in fact killed *all* the t's: we have been led to a *static* solution of (27), a solution that just stands there (doesn't fall, doesn't buzz), a <u>zero energy solution</u> (whence the $_0$).

More generally, we set

$$c(p) = \exp\left\{\tfrac{i}{\hbar}\Big[\tfrac{1}{6m^2g}p^3 - ap\Big]\right\} \tag{29}$$

and (by $x(p-mgt)-ap = (x-a)(p-mgt)-mgat$) obtain

$$\Psi_a^{\text{free fall}}(x,t) = \sqrt{k}\cdot \text{Ai}(k[x-a])\cdot e^{-\frac{i}{\hbar}mgat} \tag{30}$$

which buzzes all right, but—remarkably—*does not fall*. Equations (29) and (30) are identical to results achieved in §25 of Part A, but have been obtained here by what I consider to be a more transparent line of argument. The relative efficiency of the present argument can be attributed to the availability of the *operator identities* that culminate in the construction (23) of the "drop operator" $\mathfrak{G}(t;g)$. Notice particularly that **the free particle spectral degeneracy has been lifted** by the integration process (28).

One delicate point merits comment: the free-particle eigenfunctions $\Psi_p(x,t)$ are not normalizable, do not describe quantum states, are intended to be assembled into normalized *wavepackets*

$$\psi(x,t) = \int \Psi_p(x,t)\,dp\,(p|\psi)$$

We have $[(p|\psi)] = \frac{1}{\sqrt{\text{momentm}}}$ and have contrived to have $[\Psi_p] = \frac{1}{\sqrt{\text{momentm}\cdot\text{length}}}$ so as to insure that $[\psi(x,t)] = \frac{1}{\sqrt{\text{length}}}$. The dropped free-particle eigenfunctions $\Psi_p(x,t)$, even after assembly into free-fall eigenfunctions $\Psi_a^{\text{free fall}}(x,t)$, suffer from similar defects: they await assembly into normalized wavepackets, but by a procedure of the dimensionally distictive design

$$\psi^{\text{free fall}}(x,t) = \int \Psi_a^{\text{free fall}}(x,t)\,da\,f(a)$$

If we adopt the convention that $da\,f(a)$ is dimensionless then we must have

$$[\Psi_a^{\text{free fall}}(x,t)] = \frac{1}{\sqrt{\text{length}}}$$

This is accomplished by the $\sqrt{k}$-factor in (30).[12]

We have several times had occasion to remark[13] the striking fact that the free-fall eigenfunctions are—for reasons foreshadowed already in the classical physics—*rigid translates* of one another. In §18 of Part A it was remarked that, for the reason just stated, the x-translation operator serves also/simultaneously as the energy-translation or "ladder" operator. The objects of most recent interest have been *buzzing* eigenfunctions, the translation properties of which are a bit more intricate: it follows from (30) that

$$\begin{aligned}
\Psi_a^{\text{free fall}}(x-\xi,t) &= \sqrt{k}\cdot \mathrm{Ai}(k[x-\xi-a])\cdot e^{-\frac{i}{\hbar}mgat}\\
&= \sqrt{k}\cdot \mathrm{Ai}(k[x-(a+\xi)])\cdot e^{-\frac{i}{\hbar}mg(a+\xi)t}\cdot e^{+\frac{i}{\hbar}mg\xi t}\\
&= \Psi_{a+\xi}^{\text{free fall}}(x,t)\cdot e^{+\frac{i}{\hbar}mg\xi t}
\end{aligned}$$

which is to say:

$$\Psi_{a+\xi}^{\text{free fall}}(x,t) = \exp\Big\{-\xi\big[\tfrac{\partial}{\partial x}+\tfrac{i}{\hbar}mgt\big]\Big\}\cdot\Psi_a^{\text{free fall}}(x,t) \tag{31}$$

8. Can "Schwinger's trick" be adapted to the free fall problem? Julian Schwinger was a man of famously *many* tricks: the one I have now in mind is the clever way in which he once exploited operator ordering techniques to extract the eigenvalues, eigenfunctions and other properties of quantum oscillators from the construction

$$\mathsf{U}_{\text{osc}}(t) = e^{-\frac{i}{\hbar}\{\frac{1}{2m}\mathsf{p}^2+\frac{1}{2}m\omega^2\mathsf{x}^2\}t}$$

Schrödinger himself had noticed that $\mathsf{H}_{\text{osc}} \equiv \frac{1}{2m}\mathsf{p}^2+\frac{1}{2}m\omega^2\mathsf{x}^2$ can be written

$$\mathsf{H}_{\text{osc}} = \hbar\omega(\mathsf{a}^+\mathsf{a}+\tfrac{1}{2}\mathsf{I})$$

with

$$\begin{aligned}
\mathsf{a} &\equiv \sqrt{m\omega/2\hbar}\,(\mathsf{x}+i\tfrac{1}{m\omega}\mathsf{p})\\
\mathsf{a}^+ &\equiv \sqrt{m\omega/2\hbar}\,(\mathsf{x}+i\tfrac{1}{m\omega}\mathsf{p})
\end{aligned}$$

Schwinger was clever enough to establish that

$$\begin{aligned}
\mathsf{U}_{\text{osc}}(t) &= e^{-i\omega(\mathsf{a}^+\mathsf{a}+\frac{1}{2}\mathsf{I})t}\\
&= \sum_n e^{-\frac{i}{\hbar}(n+\frac{1}{2})\hbar\omega t}\tfrac{1}{\sqrt{n!}}(\mathsf{a}^+)^n|0)(0|(\mathsf{a})^n\tfrac{1}{\sqrt{n!}}\ :\ \text{ordered}
\end{aligned}\tag{32}$$

[12] See in this regard also (50) in Part A.

[13] See Part A, page 26.

where $|0)(0| \equiv \sum_k \frac{1}{k!}(-)^k(\mathsf{a}^+)^k(\mathsf{a})^k$ projects onto a state $|0)$ that is annihilated by a: $\mathsf{a}|0) = 0$. It became then a simple matter to establish that

$$E_n = (n + \tfrac{1}{2})\hbar\omega \quad : \quad n = 0, 1, 2, \ldots$$

and that the functions

$$\psi_n(x) \equiv (x|n) \quad \text{with} \quad |n) \equiv \tfrac{1}{\sqrt{n!}}(\mathsf{a}^+)^n|0)$$

are precisely the normalized oscillator eigenfunctions (Hermite functions) described in every quantum text.[14] The question before us: What might Schwinger have to say about the system

$$\mathsf{U}_g(t) = e^{-\frac{i}{\hbar}\{\frac{1}{2m}\mathsf{p}^2 + mg\mathsf{x}\}t}$$

(where I have reverted to an abbreviated notation introduced on page 18)?

The propagator $(x|\mathsf{U}_g(t)|x_0)$ was subject to detailed development in §15 of Part A, and more recently in the discussion that led to (18). But our interest here lies not in the possibility of writing

$$(x|\mathsf{U}_g(t)|x_0) = \sqrt{\tfrac{m}{iht}}\, e^{\frac{i}{\hbar}\cdot\text{classical action}}$$

but in the “spectral” representation of $\mathsf{U}_g(t)$. The translation operator $\mathsf{T}(a)$ was defined

$$\mathsf{T}(a) \equiv e^{-\frac{i}{\hbar}a\,\mathsf{p}}$$

at page 33 in Part A, and has obviously the properties

$$\mathsf{T}^+(a) = \mathsf{T}^{-1}(a) = \mathsf{T}(-a)$$

From (17)

$$\mathsf{U}_g(t) = e^{-\frac{i}{\hbar}\{\frac{1}{6}mg^2t^3\}} \cdot e^{-\frac{i}{\hbar}\{\frac{1}{2m}\mathsf{p}^2 + \frac{1}{2}gt\,\mathsf{p}\}t} \cdot e^{-\frac{i}{\hbar}\{mg\mathsf{x}\}t}$$

it follows by

$$\begin{aligned} e^{-\frac{i}{\hbar}\{mg\mathsf{x}\}t} \cdot e^{-\frac{i}{\hbar}a\,\mathsf{p}} &= e^{-\frac{i}{\hbar}a\,\mathsf{p}} \cdot e^{-\frac{i}{\hbar}\{mg\mathsf{x}\}t} \cdot e^{-(\frac{i}{\hbar})^2 mgat[\mathsf{x},\mathsf{p}]} \\ &= e^{-\frac{i}{\hbar}a\,\mathsf{p}} \cdot e^{-\frac{i}{\hbar}\{mg\mathsf{x}\}t} \cdot e^{-\frac{i}{\hbar}mgat} \end{aligned}$$

that

$$\mathsf{T}^+(a)\,\mathsf{U}_g(t)\,\mathsf{T}(a) = e^{-\frac{i}{\hbar}mgat} \cdot \mathsf{U}_g(t) \tag{33}$$

Observe that if, in rough imitation of (32), we allowed ourselves to write

$$\mathsf{U}_g(t) = \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar}mgbt}\,\mathsf{T}(b)|0)(0|\,\mathsf{T}^+(b)\,db \tag{34}$$

[14] For details and references relating to “Schwinger’s oscillator trick” see ADVANCED QUANTUM TOPICS (2001), Chapter 0, pages 40–42.

then (33) would become automatic:

$$
\begin{aligned}
\mathsf{T}^{+}(a)\,\mathsf{U}_g(t)\,\mathsf{T}(a) &= \int e^{-\frac{i}{\hbar}mgb\,t}\,\mathsf{T}(b-a)|0)(0|\,\mathsf{T}^{+}(b-a)\,db\\
&= e^{-\frac{i}{\hbar}mgat}\cdot\int e^{-\frac{i}{\hbar}mg(b-a)\,t}\,\mathsf{T}(b-a)|0)(0|\,\mathsf{T}^{+}(b-a)\,db\\
&= e^{-\frac{i}{\hbar}mgat}\cdot\mathsf{U}_g(t)
\end{aligned}
$$

The initial condition $\mathsf{U}_g(0)=\mathsf{I}$ requires

$$
\int |b)db(b| = \mathsf{I} \quad \text{with} \quad |b) \equiv \mathsf{T}(b)\,|0) \tag{35.1}
$$

while we would have $\mathsf{U}_g(t_1)\cdot\mathsf{U}_g(t_2)=\mathsf{U}_g(t_1+t_2)$ if

$$
(0|\,\mathsf{T}^{+}(a)\,\mathsf{T}(b)|0) = (a|b) = \delta(a-b) \tag{35.2}
$$

Moreover[15]

$$
\begin{aligned}
\mathsf{H}\,\mathsf{U}_g(t) &= \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar}mga\,t}\,\mathsf{H}\,\mathsf{T}(a)|0)(0|\,\mathsf{T}^{+}(a)\,da\\
&= \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar}mga\,t}\,\mathsf{T}(a)\big\{\mathsf{H}+mga\,\mathsf{I}\big\}|0)(0|\,\mathsf{T}^{+}(a)\,da
\end{aligned}
$$

while

$$
i\hbar\partial_t\,\mathsf{U}_g(t) = \int_{-\infty}^{+\infty} \big\{mga\big\}e^{-\frac{i}{\hbar}mga\,t}\,\mathsf{T}(a)|0)(0|\,\mathsf{T}^{+}(a)\,da
$$

We therefore have $\big\{\mathsf{H}-i\hbar\partial_t\big\}\mathsf{U}_g(t)=\mathbf{0}$ provided

$$
\big\{\tfrac{1}{2m}\mathsf{p}^2+mg\mathsf{x}\big\}|0) = 0|0) \tag{36}
$$

From (36) we are led back directly to Airy's function, while in (35) we read an allusion to the remarkable orthogonality/completeness properties of the translates of that function. I will not belabor the details.

Concluding remarks. This work began as an attempt to account theoretically for the surprising numerical discovery that the center of mass of a classical *population* of bouncing balls mimics the quantum motion of the mean of a "bouncing wavepacket"—this even though the wavepacket refers to the state of a *single* particle. The point at issue was found to be not at all mysterious if one works in the Heisenberg picture. . . where it is not $|\psi)$ but the operators that move. The latter circumstance led me to look in fair detail into some of the operator-algebraic aspects of the free fall system. I was encouraged in the latter effort by the fact that Reece Heineke is (under my direction) writing a thesis on the quantum applications of some operator methods and had—quite coincidentally—elected to take the free fall propagator as a primary test object. I must emphasize that operator methods surveyed in these pages all pertain to *unobstructed* free fall, and that I have at present nothing to say about how they might be adapted to the bouncer problem.

[15] I draw here on equations (69) in Part A.