# Classical/Quantum Motion in a Uniform Gravitational Field

*Reed College Physics Seminar*
*19 February 2003*

Based on work done in loose collaboration with Richard Crandall (March 1994) and work done during May/June 2003 in anticipation of Tomoko Ishihara's thesis project.

- CLASSICAL FREE FALL
- QUANTUM FREE FALL
- CLASSICAL BOUNCER
- QUANTUM BOUNCER
- BOUNCING GAUSSIAN WAVEPACKET
- LESSONS & QUESTIONS

- ADDENDUM

## Part One: Classical Free Fall

Elementary textbook systems:

| | |
|---|---|
| FREE PARTICLE | $\ddot{x} = 0$ |
| FREE FALL | $\ddot{x} = -g$ |
| HARMONIC OSCILLATOR | $\ddot{x} = -\omega^2 x$ |

$$L(x, \dot{x}) = \begin{cases} \frac{1}{2}m\dot{x}^2 \\ \frac{1}{2}m\dot{x}^2 - mgx \\ \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2 \end{cases}$$

$$H(p, x) = \begin{cases} \frac{1}{2m}p^2 \\ \frac{1}{2m}p^2 + mgx \\ \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2 \end{cases}$$

Never stop studying the inexhaustible physics of free particles and oscillators, but tend to neglect free fall after 3$^{\text{rd}}$ week of Physics 100.

Go to *Mathematica*

## Part Two: Quantum Mechanical Free Fall

Schrödinger equation:

$$\left\{\frac{\hbar^2}{2m}\partial_x^2 + mgx\right\}\psi(x) = E\psi(x)$$

Availability of $\hbar$ leads by dimensional analysis to

$$\text{NATURAL LENGTH} \quad \ell_g \equiv \left(\frac{\hbar^2}{2m^2 g}\right)^{\frac{1}{3}} \equiv k^{-1}$$

$$\text{NATURAL ENERGY} \quad \mathcal{E}_g \equiv \left(\frac{mg^2\hbar^2}{2}\right)^{\frac{1}{3}}$$

$$\text{NATURAL TIME} \quad \tau_g \equiv \left(\frac{2\hbar}{mg^2}\right)^{\frac{1}{3}}$$

$$\text{NATURAL FREQUENCY} \quad \omega_g \equiv \mathcal{E}_g/\hbar = 1/\tau_g$$

Set $g = 9.80665\,\text{m/s}^2$, find

$$\ell_g = \begin{cases} 0.0880795\,\text{cm} & : \quad \text{electron} \\ 0.0005874\,\text{cm} & : \quad \text{proton} \\ \approx 10^{-21}\,\text{cm} & : \quad \text{one gram} \end{cases}$$

Pass to dimensionless variables

$$
\begin{aligned}
y &\equiv \left(\tfrac{2m^2g}{\hbar^2}\right)^{\frac{1}{3}}\left(x - \tfrac{E}{mg}\right)\\
&= k(x-a)\\
&\qquad\qquad a \equiv \tfrac{E}{mg} = \begin{cases}\text{maximal height achieved by a}\\ \text{particle lofted with energy } E\end{cases}\\
&\equiv z - \alpha
\end{aligned}
$$

and **NOTE** that value of $E$ has been absorbed into the definition of $y$. Schrödinger equation becomes

$$\left(\tfrac{d}{dy}\right)^2\psi(y) = y\,\psi(y)$$

which is *Airy's differential equation*. Arises from many physical problems, leads to **Airy functions** that have many wonderful properties—all nicely described (in French) in a recent monograph by O. Vallée.

Go to *Mathematica*

$$\psi_{\mathcal{E}}(z) = f(z,\alpha) = \mathrm{Ai}(z-\alpha)$$

Use integral representation to show that

$$\int_{-\infty}^{+\infty} \mathrm{Ai}(z-\alpha)\mathrm{Ai}(z-\beta)\,dz = \delta(\alpha-\beta)$$

Free fall eigenfunctions are orthonormal & complete and all have the same shape! Quantum manifestation of classical translational equivalence, and curiously consonant with the essential wavelet transform idea.

• Instructive to show how free particle exponentials become free fall Airy functions when viewed from an accelerated frame.

## CONSTRUCTION OF THE PROPAGATOR

$$\Psi(x,t_0) \longmapsto \Psi(x,t) = \int K(x,t;x_0,t_0)\Psi(x_0,t_0)\,dx_0$$

$$K(x_1,t_1;x_0,t_0) \equiv \sum_n \Psi_n(x_1)\Psi_n^*(x_0)e^{-\frac{i}{\hbar}E_n(t_1-t_0)}$$

Working in dimensionless variables

$$\mathcal{K}(z,t;z_0,0) = \int_{-\infty}^{+\infty} \psi_{\mathcal{E}}(z)\psi^*_{\mathcal{E}}(z_0)e^{-i\mathcal{E}\theta}\,d\mathcal{E}$$

Use integral representation to obtain finally

$$\begin{aligned} K &= \sqrt{\tfrac{m}{2\pi i\hbar t}}\,\exp\Big\{\tfrac{i}{\hbar}\Big[\tfrac{m}{2t}(x-x_0)^2 - \tfrac{1}{2}mg(x+x_0)t \\ &\qquad\qquad\qquad\qquad - \tfrac{1}{24}mg^2t^3\Big]\Big\} \\ &= \sqrt{\tfrac{m}{2\pi i\hbar t}}\,\exp\Big\{\tfrac{i}{\hbar}\Big[\text{classical action!}\Big]\Big\} \end{aligned}$$

## DROPPED GAUSSIAN WAVEPACKET

Use propagator to study motion of

$$\psi(z,0) = \frac{1}{\sqrt{\sigma\sqrt{2\pi}}}\,e^{-\frac{1}{4}[z/\sigma]^2}$$

Integrals are manageable, get

$$|\Psi(x,t)|^2 = \frac{1}{s(t)\sqrt{2\pi}}\,\exp\left\{-\frac{1}{2}\left[\frac{x+\frac{1}{2}gt^2}{s(t)}\right]^2\right\}$$

with $s \equiv \ell_g\sigma$ and $s(t) \equiv s\sqrt{1+\left(\frac{\hbar t}{2ms^2}\right)^2}$.

Looks just like a "diffusing free particle Gaussian," as viewed from an accelerating frame.

## Part Three: Classical Bouncer

Ball lofted with energy $E$ will rise to height

$$a = \frac{E}{mg}$$

and bounce with period

$$\text{bounce period } \tau = \sqrt{8a/g} = \sqrt{8E/mg^2}$$

$$x(t) = \tfrac{1}{2}gt(\tau - t) \quad : \quad 0 < t < \tau$$

To describe bounce-bounce-bounce... idea, write

$$x(t) = \tfrac{1}{2}g\sum_n [t - n\tau][(n+1)\tau - t] \cdot \texttt{UnitStep}[[t - n\tau][(n+1)\tau - t]]$$

Go to *Mathematica*

## Part Four: Quantum Bouncer Eigenstates

**Bouncer theory according to PLANCK:**

$$\oint p\,dx = nh \quad : \quad n = 1, 2, 3, \ldots$$

$$\therefore \quad \tau_n = \left[12nh/mg^2\right]^{\frac{1}{3}}$$

$$a_n = \ell \cdot \left[\tfrac{3\pi}{2}n\right]^{\frac{2}{3}}$$
$$E_n = \mathcal{E} \cdot \left[\tfrac{3\pi}{2}n\right]^{\frac{2}{3}}$$

**Bouncer theory according to SCHRÖDINGER:**

Again have

$$\left(\tfrac{d}{dy}\right)^2\psi(y) = y\,\psi(y)$$

with $y \equiv k(x-a) \equiv z - \alpha$, but now require

$$\psi(y) = 0 \quad \text{at} \quad x = 0$$

$$\therefore \quad \Psi_n(z) = N_n \cdot \text{Ai}(z - z_n)$$

Acquire interest in zeros of Airy function, which are given asymptotically

$$z_n \approx \left[\tfrac{3\pi}{2}\left(n - \tfrac{1}{4}\right)\right]^{\frac{2}{3}} + \cdots$$

$$\text{SCHRÖDINGER} \quad : \quad E_n \approx mg\ell \cdot \left[\tfrac{3\pi}{2}\left(n - \tfrac{1}{4}\right)\right]^{\frac{2}{3}}$$

$$\text{PLANCK} \quad : \quad E_n = mg\ell \cdot \left[\tfrac{3\pi}{2} n\right]^{\frac{2}{3}}$$

Go to *Mathematica*

## Part Five: Dropped Gaussian Wavepacket

Take $\psi_{\text{initial}}$ to be Gaussian:

$$\psi(z,0) \equiv \frac{1}{\sqrt{\sigma\sqrt{2\pi}}}\, e^{-\frac{1}{4}\left[\frac{z-\alpha}{\sigma}\right]^2}$$

Objective is to compute $\psi(z,t)$ and to examine the motion of $\langle z\rangle$, $\langle z^2\rangle$ and $\Delta z$. Have first to develop

$$\psi(z,0) = \sum_n c_n f_n(z)$$

Go to *Mathematica*

In dimensionless time $\theta$

$$f_n(z) \longrightarrow f_n(z,\theta) \equiv f_n(z) \cdot e^{-i\,z_n\theta}$$

—see what has become of the classical fact that

energy ∼ height of the flight

—so

$$\psi(z,\theta) = \sum_n c_n f_n(z) \cdot e^{-i\,z_n\theta}$$

and therefore

$$|\psi(z,\theta)|^2 = \sum_n \big[c_n f_n\big]^2 + 2\sum_{m>n}\sum_n c_m c_n f_m f_n \cos(z_m - z_n)\theta$$

Go to *Mathematica*

Periodicity a deceptive artifact of my film loop. A film tracing Gaussian $\psi^*\psi$ through first 20 bounces has been posted by Julio Gea-Banacloche, a condensed matter physicist at the University of Arkansas

The motion of $P(z,\theta) \equiv |\psi(z,\theta)|^2$ is conveys more information than we can grasp , so look to motion of the expected position:

$$\begin{aligned}\langle z\rangle_\theta &\equiv \int_0^\infty zP(z,\theta)\,dz\\ &= \sum_n c_n c_n Z^{(1)}_{nn}\\ &\quad + 2\sum_{m>n}\sum_n c_m c_n Z^{(1)}_{mn}\cos(z_m - z_n)\theta\end{aligned}$$

where it can be shown that the matrix elements

$$\begin{aligned}Z^{(1)}_{mn} &\equiv \int_0^\infty f_m(z) z f_n(z\,dz)\\ &= \begin{cases}\frac{2}{3}z_n & \text{if } m=n\\ -2(-)^{m-n}/(z_m - z_n)^2 & \text{otherwise}\end{cases}\end{aligned}$$

Similarly, we might watch $\langle z^2\rangle_\theta$ and use

$$\begin{aligned}Z^{(2)}_{mn} &\equiv \int_0^\infty f_m(z) z^2 f_n(z\,dz)\\ &= \begin{cases}\frac{8}{15}z_n^2 & \text{if } m=n\\ -24(-)^{m-n}/(z_m - z_n)^4 & \text{otherwise}\end{cases}\end{aligned}$$

Elegant proof of these exact results provided by David Goodmanson in 2000.

Go to *Mathematica*

## Lessons & Questions

★ Classical/quantum serves usefully as a "theoretical laboratory:" physically non-trivial, yet analytically accessible.

★ EHRENFEST'S THEOREM is popularly/wrongly claimed to assert that *"quantum motion of the mean is classical."* Have shown that claim to be untenable. But when the classical physics permits construction of a time-independent classical distribution function it appears to be the case that (in all orders?)

time-averaged quantum moment

= classical moment

★ Bouncer exhibits "extinction & recurrence" phenomena, which both free particle & oscillator are (for separate reasons) too simple to capture. Recent research—by Carlos R. Stroud and many others—suggests these are universal features of quantum systems in the semi-classical regime.

★ Crandall has managed to write down the exact propagator for the bouncer. Remains to extact that result from Feynman's sum-over-paths formalism

. . . which was original source of my interest in this problem area.

★ Work would have been impossible without the assistance of a resource like *Mathematica*, and underscores the fact that Airy functions—called by some physicists "rainbow functions"—are wonderful things.

★ Recent interest/activity in the area mainly by the BEC people, for whom gravity has become a fact of their laboratory life. Many web sites relate to this work: John Essick has directed me to a site prepared by physicists at the University of Hanover.

## Basic References

1. J. J. Sakurai, *Modern Quantum Mechanics* (1994), pages 107–109.

2. M. Wadati, "The free fall of quantum particles," J. Phys. Soc. of Japan **68**, 2543 (1999).

3. J. Gea-Banacloche, "A quantum bouncing ball," AJP **68**, 672 (2000).

4. D. Goodmanson, "A recursion relation for matrix elements of the quantum bouncer,"AJP **68**,866 (2000).

5. N. Wheeler, "Classical/quantum motion in a uniform gravitational field," a long essay in three parts that can be found (together with the pdf file and *Mathematica* notebook I used today) in the courses server at PHYSICS > WHEELER STUFF > BOUNCER.

## Acknowledgements

Am indebted—as always—to Richard Crandall for conversation and sharing with me some of his own provocative results. I am also indebted...

- to Oz Bonfim for conversation, and for taking the trouble to discover valuable references on the web;

- to David Griffiths for sitting patiently when I know he had other/better things to do;

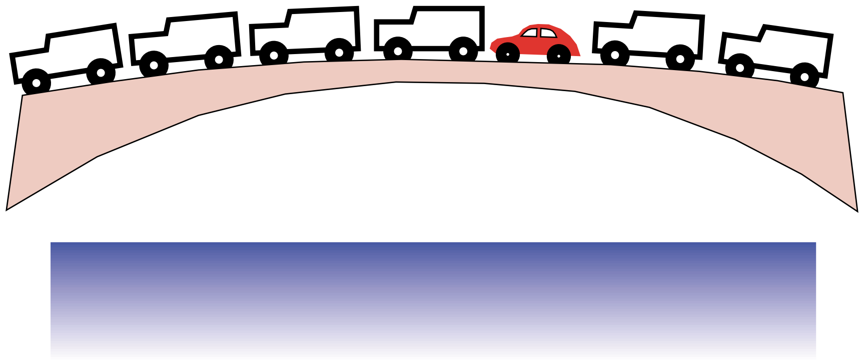
- to John Essick for directions to a web site;

- to David Goodmanson and to Olivier Valleé for correspondence and for supplying indispensable materials; (anybody interested in preparing an English translation of a French masterpiece?);

Finally, I owe much to Tomoko Ishihara, my coworker, for supplying some critical references... and (unwittingly) for motivating my return to this pretty problem area.


Go to *Mathematica*

But at 7:40 a.m. Friday 7 February 2003, as I crossed the Sellwood bridge on my way to Reed...



...I was led to ask:

**Is it, perhaps, misguided to compare the motion of the quantum mean with the motion of a single classical particle?**

Go to *Mathematica*