# 2-DIMENSIONAL "PARTICLE-IN-A-BOX" PROBLEMS IN QUANTUM MECHANICS

*Part II: Eigenfunctions & the method of sections*

**Introduction**. In a previous essay[1] I identified a small population of cases in which the quantum mechanical "particle-in-a-box problem" yields to exact analysis by a variant of the familiar "method of images." Those cases involve

- rectangular boxes (of any proportion);
- right isoceles triangular boxes (45-45-90);
- equilateral boxes (60-60-60);
- bisected equilateral boxes (30-60-90)

and the method proceeds from a simplified Feynman formalism; one writes

$$K(\boldsymbol{x},t;\boldsymbol{y};0) = \tfrac{m}{iht} \sum_{\text{direct \& reflected paths}} (-)^{\text{number of reflection points}} e^{\frac{i}{\hbar}S[\text{path}]}$$

and, taking advantage of the fact that the classical paths $(\boldsymbol{x},t) \leftarrow (\boldsymbol{y},0)$ can in these cases be neatly ennumerated, obtains

$$= \text{sum of a few generalized } \textbf{theta functions}$$

Drawing then upon the associated generalization of a famous identity due to Jacobi

$$\begin{aligned}\vartheta(z,\tau) &\equiv 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz \quad \text{with} \quad q \equiv e^{i\pi\tau} \\ &= A \cdot \vartheta\Big(\frac{z}{\tau}, -\frac{1}{\tau}\Big) \quad \text{with} \quad A \equiv \sqrt{i/\tau}\, e^{x^2/i\pi\tau}\end{aligned}$$

one passes from the preceding "particle representation" of the propagator to the "wave representation"

$$K(\boldsymbol{x},t;\boldsymbol{y};0) = \sum_{\text{points } \boldsymbol{n} \text{ of a certain lattice}} e^{-\frac{i}{\hbar}E(\boldsymbol{n})t} \psi_{\boldsymbol{n}}(\boldsymbol{x}) \psi^*_{\boldsymbol{n}}(\boldsymbol{y})$$

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[1] "2-dimensional particle-in-a-box problems in quantum mechanics, Part I: Propagator & eigenfunctions by the method of images" (1997).

from which the eigenvalues and eigenfunctions can simply be read off. Spectral analysis can be accomplished by straightforward appeal to methods borrowed from algebraic number theory, while study of interrelationships among the eigenfunctions puts one in touch with group representation theory.

The train of argument just summarized possesses a kind of crystaline elegance that is unique in my experience. Mathematical topics of remarkable variety (higher analysis, number theory, group theory, geometry) come here harmoniously together in the service of some fairly fundamental physics. I find it difficult to escape the feeling that we stand in the presence not of a small population of wonderful accidents, but of something deep. One would like to penetrate the diamond surface of our subject, the better to explore the geology that supports it, and of which the surface is only a pretty symptom. But I have, thus far, found that a daunting task.[2] The ideas explored in these pages relate to that overarching effort, but are much more particular in their primary focus.

It is a notable fact that the method sketched above leads to eigenfunctions which, though assembled from elementary functions, are (except in the almost trivial rectangular case) *non-separable*. My objective here will be to clarify how it comes to pass that those functions manage to satisfy both the Schrödinger equation and the imposed boundary conditions, and to explore the feasibility of an idea relating to their direct construction. In Part I the object at center stage was the propagator (Green's function); here the propagator steps into the shadows, yielding the stage to a chorus line of eigenfunctions. . . about which the method of images has nothing individually to say.

**1. What makes the special cases special?** In Figure 1 I show a point in a rectangular box, together with their“reflective images;” that the box and its images tesselate the plane—producing a design that can be rendered as a two-color map—is a fact central to the argument developed in Part I. Figure 2 draws attention to the fact that the design thus generated can be reproduced by inscribing two (orthogonal) families of parallel lines on the plane. A similar remark pertains to the right isoceles, equilateral and bisected equilateral plane tesselations, as illustrated in Figures 3–5. These elementary remarks acquire interest from the circumstance that

- while other reflective tesselations of the plane do exist (see Figure 6), they cannot be generated by linear inscription;
- while linear inscriptions exist in unlimited abundance (see Figure 7), they do not generally yield reflective tesselations.

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[2] Evidence that variants of my dream are shared also by others—and of how daunting is the task—can be found by perusal of C. Grosche, *Path Integrals, Hyperbolic Spaces and Selberg Trace Formulae* (1996); M. Waldschmidt *et al* (eds.), *From Number Theory to Physics* (1992); Audrey Terras, *Harmonic Analysis on Symmetric Spaces & Applications I & II* (1985); M. Brack & R. K. Bhaduri, *Semiclassical Physics* (1997). . . and of the many publications cited therein.

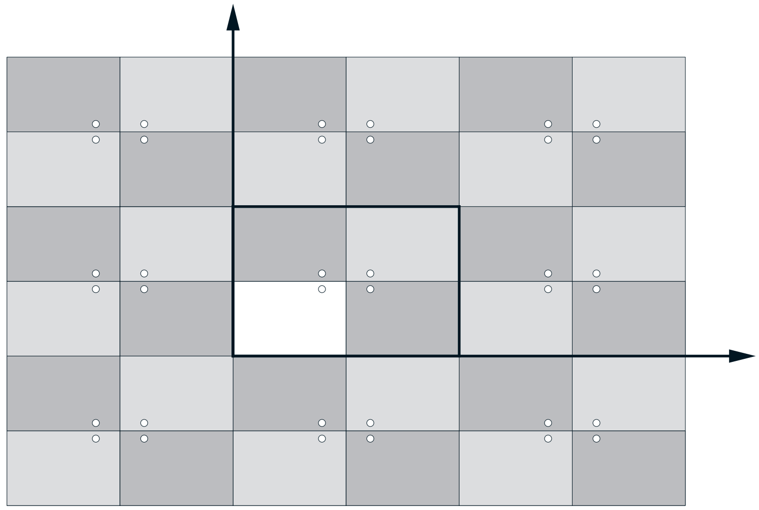

FIGURE 1: *Point in a rectangular box, together with its reflective images. Special importance attached in Part I—but not in the present discussion—to the "fundamental unit," which here has four elements (outlined), and gives rise translationally to the remainder of the figure.*

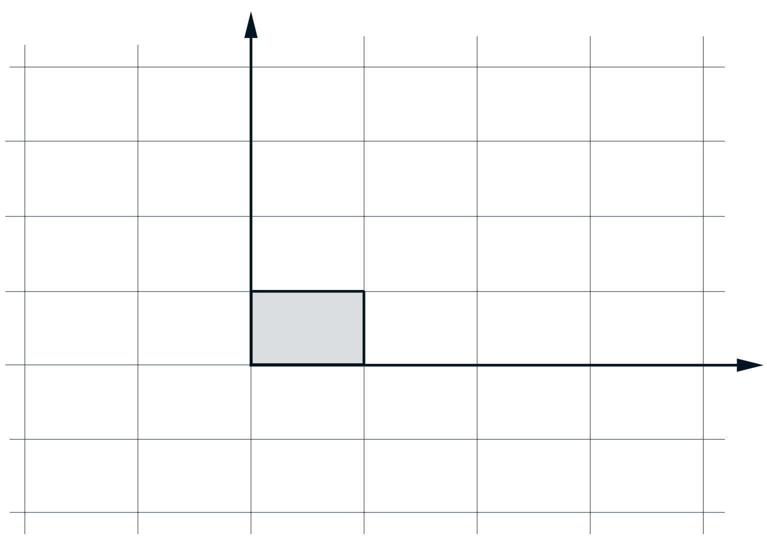

FIGURE 2: *Reproduction of the preceding figure by superimposed linear inscripton on the plane.*

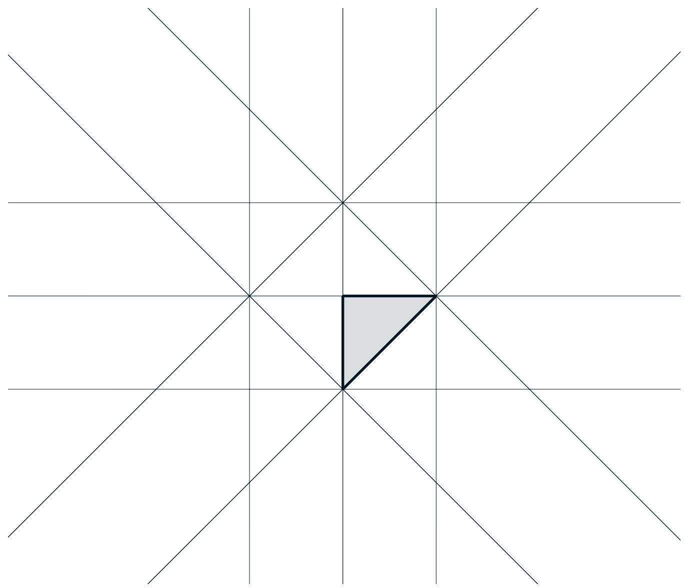

Figure 3: *Production of the 45-45-90 tesselation by superposition of four families of inscribed lines.*

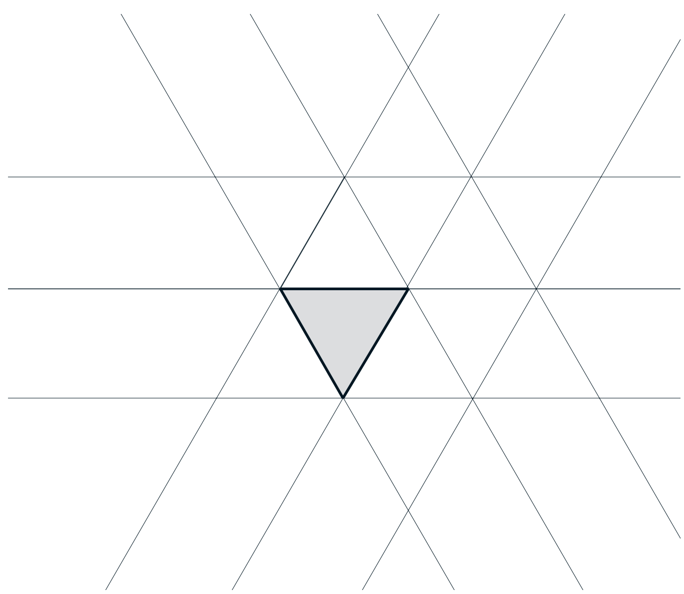

Figure 4: *Production of the 60-60-60 tesselation by superposition of three families of inscribed lines.*

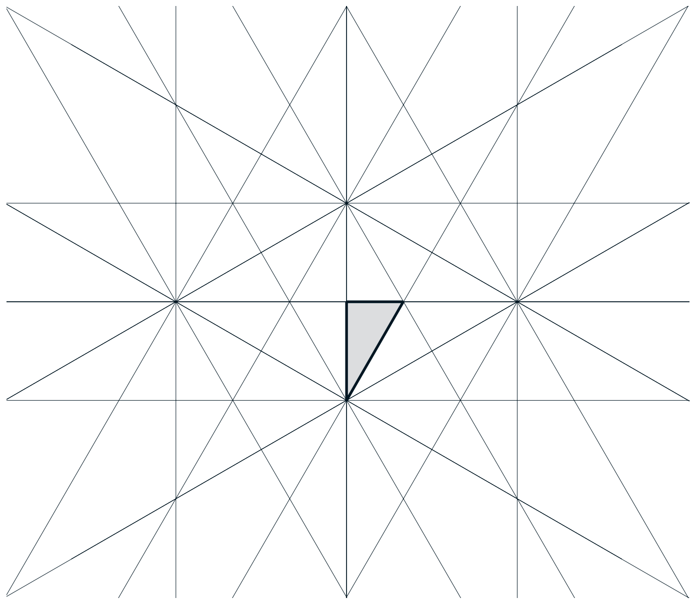

FIGURE 5: *Production of the 30-60-90 tesselation by superposition of six families of inscribed lines.*

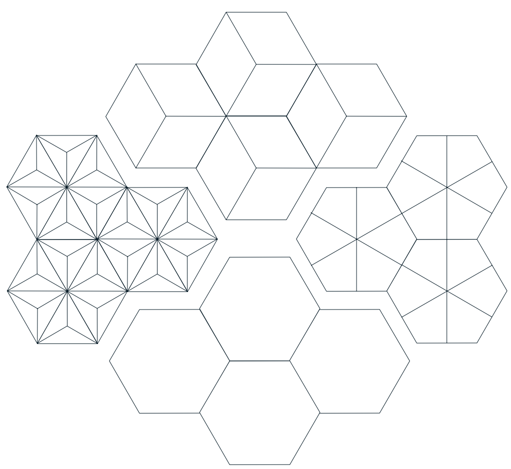

FIGURE 6: *Reflective tesselations that* cannot *be constructed by superposition of families of inscribed lines. Each contains vertices of order three, and therefore cannot be displayed as a two-color map.*

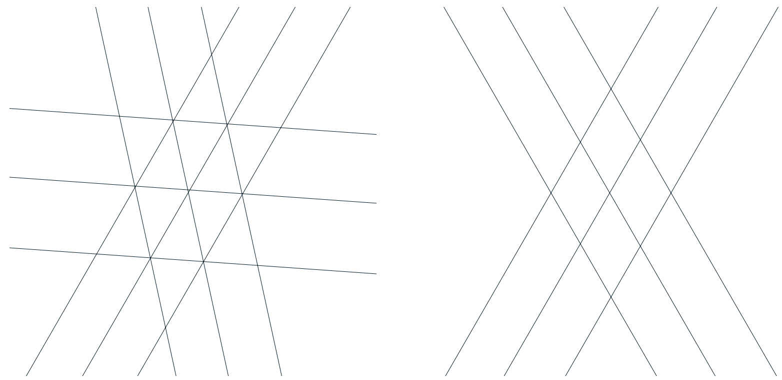

FIGURE 7: *Ruled tesselations that are* not *reflective. An analog of the figure on the left can be constructed for* any *triangle. The rhombus on the right supports also a reflective tesselation (see again the preceding figure).*

The special cases of interest to us are special therefore in (amongst others) this sense: for them—and for them alone—is the reflective tesselation ruled, the ruled tesselation reflective. Reflective tesselation is essential to the method of images, but ruled tesselation relates most directly to the line of argument developed below. But the rectangle and its three triangular friends are special also in several other respects:

It has been known for a long time[3] that "diffraction in the corner of a polygon (exceptionally) does not occur whenever the corner angle is $\pi$ divided by an integer." The following cases

$$
\begin{aligned}
45+45+90 &= 180 &&: \text{case of the right isoceles triangle}\\
60+60+60 &= 180 &&: \text{case of the equilateral triangle}\\
30+60+90 &= 180 &&: \text{case of the bisected equilateral triangle}\\
90+90+90+90 &= 360 &&: \text{case of the rectangle}
\end{aligned}
$$

therefore exhaust the list of "diffractionless boxes." The argument here (which involves nothing more sophisticated than direct inspection of the arithmetic possibilities) is similar to the argument which in 1980 led Thomas Wieting to a list of all the (ruled/unruled) reflective tesselations of the plane. Here we see again the confluence of physics and geometry which distinguishes the method of images in all of its manifestations.

[3] A. Sommerfeld, *Math. Ann.* **47**, 317 (1896); F. Oberhettinger, *J. Res. Natl. Bur. Stand.* **61**,343 (1958). More immediately germane (because rooted in the Feynman formalism) is R. E. Crandall, "Exact propagator for motion confined to a sector," *J. Phys. A: Math. Gen.* **16**, 513 (1983), which provides several additional references.

Our special cases are (though I will not, on the present occasion, attempt to formalize the remark) distinguished from other reflective tesselations also in the geometrical respect illustrated in Figure 8. This fact brings to mind the

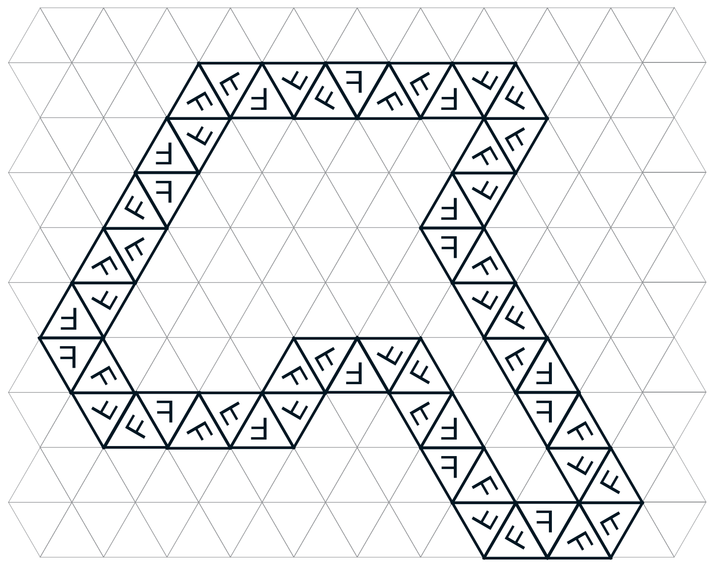

FIGURE 8: *When an equilateral box moves reflectively from one location to another its orientation upon arrival is independent of the path taken; when it returns to its original location it is returned also—in all cases—to its original orientation. Each image of the original box can be assigned therefore an unambiguous "name." The other "special boxes" possess also that property, which is basic to the successful application of the method of images. For a demonstration that this is a property not shared by (for example) hexagonal boxes, see Figure 8 in Part I.*

condition $\oint \boldsymbol{F}\cdot d\boldsymbol{x} = 0$ characteristic of "conservative" forces, and is reminiscent also of the parallel transport property that serves to characterize the "flatness" of a Riemannian manifold.

**2. Non-separable eigenfunctions in the equilateral case**. Near the end of §8 in Part I, we obtain eigenfunctions which, relative to the coordinates of Figure 9, can be described

$$\Psi_{\hat{\boldsymbol{n}}}(\boldsymbol{x}) \equiv \sqrt{\tfrac{4}{3\cdot\text{area}}}\Big\{G_{\hat{\boldsymbol{n}}}(\boldsymbol{x}) + iF_{\hat{\boldsymbol{n}}}(\boldsymbol{x})\Big\}$$

where the quantum numbers $\{\hat{n}_1, \hat{n}_2\}$ range on the interior of the wedge shown in Figure 10, and where the real/imaginary parts of $\Psi_{\hat{\boldsymbol{n}}}(\boldsymbol{x})$ can be described[4]

[4] See Part I, pp. 45 & 46.

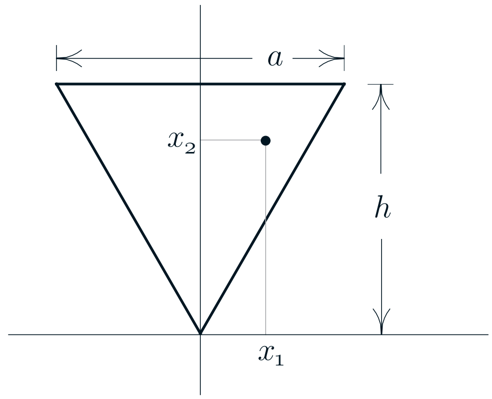


FIGURE 9: *Coordinates employed in connection with the equilateral box problem. The box has*

$$\text{height } h = \tfrac{1}{2}\sqrt{3}a$$
$$\text{area} = \tfrac{1}{4}\sqrt{3}a^2$$

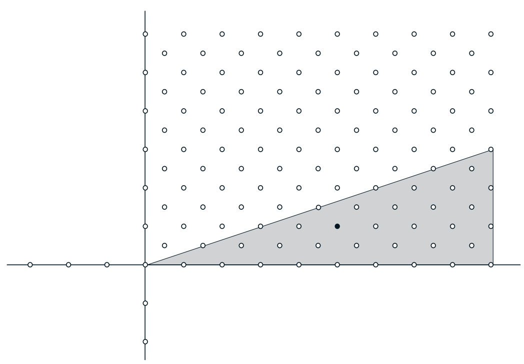

FIGURE 10: *The quantum numbers* $\hat{\boldsymbol{n}}$ *that arise in connection with the equilateral box problem range (as do those of the closely related 30-60-90 box problem) on the interior of the shaded wedge, and always have the same parity. See Figure 28 in Part I for more detailed information.*

$$G_{\hat{\boldsymbol{n}}}(x_1, x_2) \equiv \cos[2\hat{n}_1\xi_1]\sin[2\hat{n}_2\xi_2] + \cos[\hat{n}_1(\xi_1+\xi_2)]\sin[\hat{n}_2(3\xi_1-\xi_2)] - \cos[\hat{n}_1(\xi_1-\xi_2)]\sin[\hat{n}_2(3\xi_1+\xi_2)] \tag{1.1}$$

$$F_{\hat{\boldsymbol{n}}}(x_1, x_2) \equiv \sin[2\hat{n}_1\xi_1]\sin[2\hat{n}_2\xi_2] - \sin[\hat{n}_1(\xi_1+\xi_2)]\sin[\hat{n}_2(3\xi_1-\xi_2)] + \sin[\hat{n}_1(\xi_1-\xi_2)]\sin[\hat{n}_2(3\xi_1+\xi_2)] \tag{1.2}$$

or again (by appeal to some wonderful—if little known—identities)

$$
\begin{aligned}
G_{\hat{\boldsymbol{n}}}(x_1,x_2) &= \cos[2\hat{n}_1\xi_1]\sin[2\hat{n}_2\xi_2] + \cos[2\tfrac{-\hat{n}_1+3\hat{n}_2}{2}\xi_1]\sin[2\tfrac{-\hat{n}_1-\hat{n}_2}{2}\xi_2] \\
&\qquad + \cos[2\tfrac{-\hat{n}_1-3\hat{n}_2}{2}\xi_1]\sin[2\tfrac{+\hat{n}_1-\hat{n}_2}{2}\xi_2] \qquad (2.1)\\
F_{\hat{\boldsymbol{n}}}(x_1,x_2) &= \sin[2\hat{n}_1\xi_1]\sin[2\hat{n}_2\xi_2] + \sin[2\tfrac{-\hat{n}_1+3\hat{n}_2}{2}\xi_1]\sin[2\tfrac{-\hat{n}_1-\hat{n}_2}{2}\xi_2] \\
&\qquad + \sin[2\tfrac{-\hat{n}_1-3\hat{n}_2}{2}\xi_1]\sin[2\tfrac{+\hat{n}_1-\hat{n}_2}{2}\xi_2] \qquad (2.2)
\end{aligned}
$$

where the dimensionless variables $\xi_1$ and $\xi_2$ are defined

$$
\xi_1 \equiv \tfrac{\pi}{3a}x_1 \quad \text{and} \quad \xi_2 \equiv \tfrac{\pi}{3a}\sqrt{3}x_2 \tag{3}
$$

The adjustment (1) $\rightarrow$ (2) was basic to the progress of the argument developed in Part I, but in the present context it is useful to have both variants at our disposal.

How do the functions $G_{\hat{\boldsymbol{n}}}(x_1,x_2)$ and $F_{\hat{\boldsymbol{n}}}(x_1,x_2)$ manage to vanish on the boundary of the equilateral box? The box is bounded by (segments of) lines that can be described

$$
\begin{aligned}
x_2 &= h &&: \text{ top}\\
x_2 &= +(2h/a)x_1 &&: \text{ right side}\\
x_2 &= -(2h/a)x_1 &&: \text{ left side}
\end{aligned}
$$

which in dimensionless variables read

$$
\left.
\begin{aligned}
\xi_2 &= \tfrac{\pi}{2}\\
\xi_2 &= +3\xi_1\\
\xi_2 &= -3\xi_1
\end{aligned}
\right\} \tag{4}
$$

respectively. Working from (2), we have

$$
\begin{aligned}
G_{\hat{\boldsymbol{n}}}(\boldsymbol{x}_{\text{top}}) &= \cos[2\hat{n}_1\xi_1]\sin[\hat{n}_2\pi] + \cos[2\tfrac{-\hat{n}_1+3\hat{n}_2}{2}\xi_1]\sin[\tfrac{-\hat{n}_1-\hat{n}_2}{2}\pi]\\
&\qquad + \cos[2\tfrac{-\hat{n}_1-3\hat{n}_2}{2}\xi_1]\sin[\tfrac{+\hat{n}_1-\hat{n}_2}{2}\pi]\\
&= 0+0+0\\
F_{\hat{\boldsymbol{n}}}(\boldsymbol{x}_{\text{top}}) &= \sin[2\hat{n}_1\xi_1]\sin[\hat{n}_2\pi] + \sin[2\tfrac{-\hat{n}_1+3\hat{n}_2}{2}\xi_1]\sin[\tfrac{-\hat{n}_1-\hat{n}_2}{2}\pi]\\
&\qquad + \sin[2\tfrac{-\hat{n}_1-3\hat{n}_2}{2}\xi_1]\sin[\tfrac{+\hat{n}_1-\hat{n}_2}{2}\pi]\\
&= 0+0+0
\end{aligned}
$$

because $\hat{n}_1$ and $\hat{n}_2$ both even or both odd $\Longrightarrow \frac{\mp\hat{n}_1-\hat{n}_2}{2}$ is (to within a sign) an integer. Working now from (1), we have

$$
\begin{aligned}
G_{\hat{\boldsymbol{n}}}(\boldsymbol{x}_{\text{right side}}) &= \cos[2\hat{n}_1\xi_1]\sin[6\hat{n}_2\xi_1] + \cos[4\hat{n}_1\xi_1]\sin[\hat{n}_2(3\xi_1-3\xi_1)]\\
&\qquad - \cos[2\hat{n}_1\xi_1]\sin[\hat{n}_2(3\xi_1+3\xi_1)]\\
&= \text{term} + 0 - \text{ same term} = 0\\
F_{\hat{\boldsymbol{n}}}(\boldsymbol{x}_{\text{right side}}) &= \sin[2\hat{n}_1\xi_1]\sin[6\hat{n}_2\xi_1] - \sin[4\hat{n}_1\xi_1]\sin[\hat{n}_2(3\xi_1-3\xi_1)]\\
&\qquad - \sin[2\hat{n}_1\xi_1]\sin[\hat{n}_2(3\xi_1+3\xi_1)]\\
&= \text{term} + 0 - \text{ same term} = 0
\end{aligned}
$$

$G_{\hat{\boldsymbol{n}}}(\boldsymbol{x}_{\text{left side}})$ and $F_{\hat{\boldsymbol{n}}}(\boldsymbol{x}_{\text{left side}})$ vanish by an identical mechanism. Evidently the eigenfunctions vanish "straightforwardly" at the top of the equilateral box, but "by conspiracy" on its sides.

How do the functions $G_{\hat{\boldsymbol{n}}}(x_1, x_2)$ and $F_{\hat{\boldsymbol{n}}}(x_1, x_2)$ manage to satisfy the Schrödinger equation

$$\nabla^2 \Psi_{\hat{\boldsymbol{n}}} = -\tfrac{2m}{\hbar^2} E(\hat{\boldsymbol{n}}) \Psi_{\hat{\boldsymbol{n}}} \tag{5}$$

in the interior of the equilateral box? It is an implication of (3) that

$$\nabla^2 \equiv \Big(\frac{\partial}{\partial x_1}\Big)^2 + \Big(\frac{\partial}{\partial x_2}\Big)^2 = \Big(\frac{\pi}{3a}\Big)^2 \Big\{ \Big(\frac{\partial}{\partial \xi_1}\Big)^2 + 3\Big(\frac{\partial}{\partial \xi_2}\Big)^2 \Big\}$$

And it follows by elementary calculation (which is to say: by inspection) from (2) that

$$\Big\{ \Big(\frac{\partial}{\partial \xi_1}\Big)^2 + 3\Big(\frac{\partial}{\partial \xi_2}\Big)^2 \Big\} G_{\hat{\boldsymbol{n}}} = -4\big[\hat{n}_1^2 + 3\hat{n}_2^2\big] G_{\hat{\boldsymbol{n}}} \tag{6}$$

and that $F_{\hat{\boldsymbol{n}}}$ satisfies an identical equation. We conclude that the functions $G_{\hat{\boldsymbol{n}}}(\boldsymbol{x})$ and $F_{\hat{\boldsymbol{n}}}(\boldsymbol{x})$ individually satisfy the Schrödinger equation (5) with

$$\begin{aligned} E(\hat{\mathbf{n}}) &= \tfrac{\hbar^2}{2m}\big(\tfrac{\pi}{3a}\big)^2 4\big[\hat{n}_1^2 + 3\hat{n}_2^2\big] \\ &= \tfrac{h^2}{18ma^2}\big[\hat{n}_1^2 + 3\hat{n}_2^2\big] \end{aligned} \tag{7}$$

and that they do so because each is assembled additatively from functions which[5] individually satisfy (6). At (7) we have reproduced a result which in Part I (see especially equations (61) and (69)) was obtained by a variety of alternative means.

**3. Properties of a class of elementary non-separable functions**. Let $\boldsymbol{x}$ refer to an $N$-dimensional Cartesian coordinate system. It is an elementary fact—familiar from the theory of plane waves—that functions of the form

$$f(\boldsymbol{x}; \boldsymbol{k}) \equiv \sin \boldsymbol{k}\cdot\boldsymbol{x}$$

satisfy the Helmholtz equation

$$\nabla^2 f = -k^2 f \quad \text{with} \quad k^2 \equiv \boldsymbol{k}\cdot\boldsymbol{k}$$

The equation $f(\boldsymbol{x}; \boldsymbol{k}) = 0$ serves to inscribe on $N$-space a family of parallel planes, as illustrated in Figure 11. Products

$$F(\boldsymbol{x}; \boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_\mu) \equiv f(\boldsymbol{x}; \boldsymbol{k}_1) \cdot f(\boldsymbol{x}; \boldsymbol{k}_2) \cdots f(\boldsymbol{x}; \boldsymbol{k}_\mu)$$

of such functions inscribe superimposed families of such planes, as illustrated in Figure 12, but do, in general, *not* satisfy the Helmnoltz equation; instead,

[5] One draws here upon the elementary identities

$$\big[(2\hat{n}_1)^2 + 3(2\hat{n}_2)^2\big] = \big[(\hat{n}_1 - 3\hat{n}_2)^2 + 3(\hat{n}_1 + \hat{n}_2)^2\big] = \big[(\hat{n}_1 + 3\hat{n}_2)^2 + 3(\hat{n}_1 - \hat{n}_2)^2\big]$$

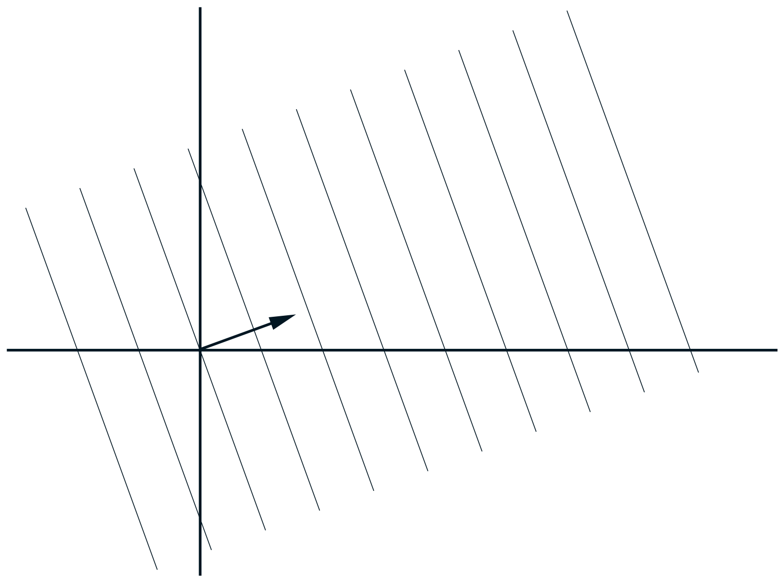

FIGURE 11: *Two-dimensional representation of the null planes of* $f(\boldsymbol{x};\boldsymbol{k})$*. The planes (lines) are normal to the "wave-vector"* $\boldsymbol{k}$ *and are separated by a distance* $s = \pi/k$*.*

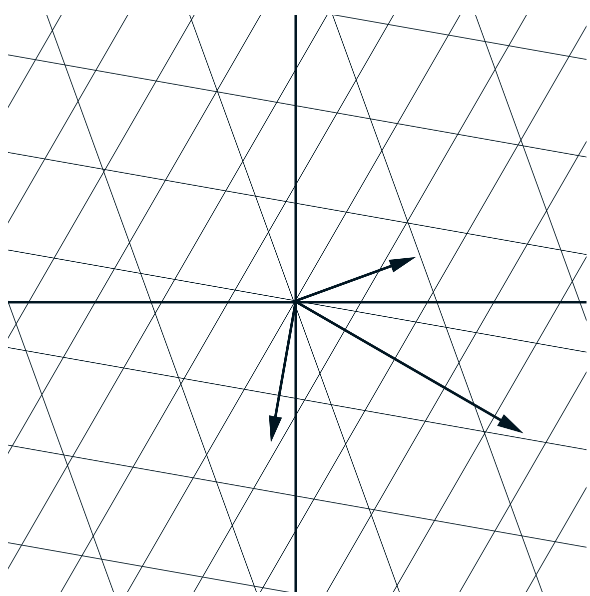

FIGURE 12: *Superimposed null lines typical of* $F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$*. The absence of periodicity is conspicuous.*

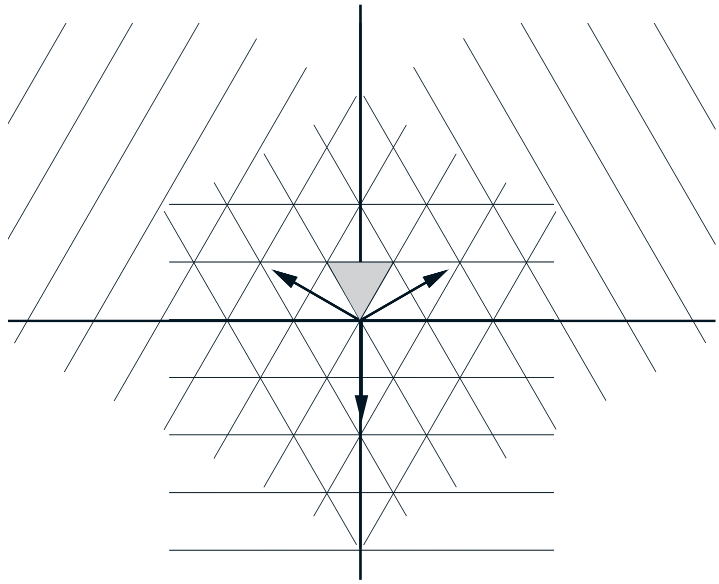

FIGURE 13: *Variant of the preceding figure, in which I have set*

$$\boldsymbol{k}_1 = \begin{pmatrix} 0 \\ -k \end{pmatrix}, \quad \boldsymbol{k}_2 = \mathbb{R}\,\boldsymbol{k}_1, \quad \boldsymbol{k}_3 = \mathbb{R}^2\,\boldsymbol{k}_1$$

*with*

$$\mathbb{R} \equiv -\tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} = 120^\circ \circlearrowleft \textit{ rotation matrix}$$

*and* $k = 2\pi/\sqrt{3}a$, *thus to achieve (see again Figure 9)*

$$\textit{separation distance} \;=\; \textit{height } \tfrac{1}{2}\sqrt{3}a \textit{ of equilateral triangle}$$

*Thus do we recover Figure 4 as a diagram of the set of points at which a certain* $F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ *function vanishes.*

one has

$$\begin{aligned}
\nabla^2 F(\boldsymbol{x};\boldsymbol{k}_1) &= -(k_1^2)F(\boldsymbol{x};\boldsymbol{k}_1) + \text{no dangling term}\\
\nabla^2 F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2) &= -(k_1^2+k_2^2)F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2) + \text{dangling term}_2\\
\nabla^2 F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) &= -(k_1^2+k_2^2+k_3^2)F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) + \text{dangling term}_3\\
&\;\;\vdots\\
\nabla^2 F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots,\boldsymbol{k}_\mu) &= -\Big(\sum_{i=1}^{\mu} k_i^2\Big)F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots,\boldsymbol{k}_\mu) + \text{dangling term}_\mu
\end{aligned}$$

where

$$\begin{aligned}
\text{dangling term}_2 &= \quad 2\boldsymbol{k}_1\!\cdot\!\boldsymbol{k}_2 \cos(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\cos(\boldsymbol{k}_2\!\cdot\!\boldsymbol{x})\\
\text{dangling term}_3 &= \quad 2\boldsymbol{k}_1\!\cdot\!\boldsymbol{k}_2 \cos(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\cos(\boldsymbol{k}_2\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_3\!\cdot\!\boldsymbol{x})\\
&\quad + 2\boldsymbol{k}_1\!\cdot\!\boldsymbol{k}_3 \cos(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_2\!\cdot\!\boldsymbol{x})\cos(\boldsymbol{k}_3\!\cdot\!\boldsymbol{x})\\
&\quad + 2\boldsymbol{k}_2\!\cdot\!\boldsymbol{k}_3 \sin(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\cos(\boldsymbol{k}_2\!\cdot\!\boldsymbol{x})\cos(\boldsymbol{k}_3\!\cdot\!\boldsymbol{x})\\
&\ \vdots\\
\text{dangling term}_\mu &= \text{etc.}
\end{aligned}$$

The question now arises: Under what conditions do the "dangling terms" vanish? Clearly

$$\begin{aligned}
&\text{dangling term}_2 = 0 \text{ if } \boldsymbol{k}_1 \perp \boldsymbol{k}_2, \text{ which requires } N \geq 2\\
&\text{dangling term}_3 = 0 \text{ if } \boldsymbol{k}_1, \boldsymbol{k}_2 \text{ and } \boldsymbol{k}_3 \text{ are mutually } \perp, \text{ which requires } N \geq 3\\
&\qquad\qquad \vdots
\end{aligned}$$

These are conditions that come naturally into play when the "particle in a (hyperdimensional) rectangular box" problem is approached by separaton of variables. I will not, on this occasion, linger to discuss the evidently more difficult question "Are the conditions just stated also necessary?"

It is important to notice that for $F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ to be a solution of the Helmholtz equation[6] it is not strictly necessary for the associated dangling term actually to vanish; it is sufficient to achieve

$$\text{dangling term}_3 \sim F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) \text{ itself}$$

To illustrate the point, I assign to the wave vectors $\boldsymbol{k}_1, \boldsymbol{k}_2$ and $\boldsymbol{k}_3$ the values indicated in Figure 13:

$$\boldsymbol{k}_1 = \begin{pmatrix} 0 \\ -k \end{pmatrix}, \quad \boldsymbol{k}_2 = \tfrac{1}{2}\begin{pmatrix} +\sqrt{3}k \\ k \end{pmatrix}, \quad \boldsymbol{k}_2 = \tfrac{1}{2}\begin{pmatrix} -\sqrt{3}k \\ k \end{pmatrix} \tag{8.1}$$

Then

$$k_1^2 = k_2^2 = k_3^2 = k^2 \quad \text{and} \quad \boldsymbol{k}_1\cdot\boldsymbol{k}_2 = \boldsymbol{k}_1\cdot\boldsymbol{k}_3 = \boldsymbol{k}_2\cdot\boldsymbol{k}_3 = -\tfrac{1}{2}k^2 \tag{8.2}$$

Moreover

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = \boldsymbol{0} \tag{8.3}$$

which is, by the way, clear from the figure, and consistent with the dot product data just presented. Drawing upon (8), we have (by some commonplace trickery: introduce a term only to subtract it again)

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[6] It is, of course, only in the service of expository concreteness that I have here assigned $\mu$ the value 3; the point at issue holds quite generally. But it is, I admit, (and as will soon emerge) a particular instance of the case $\mu = 3$ that serves primarily to motivate this discussion.

$$
\begin{aligned}
\text{dangling term}_3 = -k^2\big\{ &\ \ \cos(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\cos(\boldsymbol{k}_2\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_3\!\cdot\!\boldsymbol{x})\\
&+\cos(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_2\!\cdot\!\boldsymbol{x})\cos(\boldsymbol{k}_3\!\cdot\!\boldsymbol{x})\\
&+\sin(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\cos(\boldsymbol{k}_2\!\cdot\!\boldsymbol{x})\cos(\boldsymbol{k}_3\!\cdot\!\boldsymbol{x})\\
&-\sin(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_2\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_3\!\cdot\!\boldsymbol{x})\\
&+\sin(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_2\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_3\!\cdot\!\boldsymbol{x})\big\}\\
= -k^2\big\{ &\ \ \cos(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\sin([\boldsymbol{k}_2+\boldsymbol{k}_3]\!\cdot\!\boldsymbol{x})\\
&+\sin(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\cos([\boldsymbol{k}_2+\boldsymbol{k}_3]\!\cdot\!\boldsymbol{x})\\
&+\sin(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_2\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_3\!\cdot\!\boldsymbol{x})\big\}\\
= -k^2\big\{ &\sin(\underbrace{[\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3]}_{\mathbf{0}}\!\cdot\!\boldsymbol{x}) + \underbrace{\sin(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_2\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_3\!\cdot\!\boldsymbol{x})}_{F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)}\big\}
\end{aligned}
$$

Here the dangling term has in fact not vanished, but has returned a weighted replica of $F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ itself; we conclude that *in the equilateral triangular case* (Figure 13)

$$\nabla^2 F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = -(3+1)k^2 F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$$

The function $F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ vanishes on the boundary of the trangle, and satisfies the time-independent Schrödinger equation

$$-\tfrac{\hbar^2}{2m}\nabla^2 F = E\cdot F$$

with

$$E = \tfrac{\hbar^2}{2m}4k^2 = \tfrac{h^2}{18ma^2}\cdot 12$$

This is in some respects a curious result. We note, for example, that

$$12 = 3^2 + 3\cdot 1^2$$

is associated with the lowest-lying point on the upper *edge* of the wedge shown in Figure 10, and edge points are excluded according to arguments developed in Part I, where we found the ground state of a particle in an equilateral box to energy given by

$$E_{\text{ground}} = \tfrac{h^2}{18ma^2}\cdot(5^2+3\cdot 1^2)$$

We acquire, therefore, an obligation to make sense of the result in now in hand.

As a first step in that direction, I observe that

$$F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) \equiv \sin(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_2\!\cdot\!\boldsymbol{x})\sin(\boldsymbol{k}_3\!\cdot\!\boldsymbol{x})$$

can, with the aid of standard trigonometric identities, be recast

$$
\begin{aligned}
F(\boldsymbol{x};\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) &= \tfrac{1}{2}\sin(\boldsymbol{k}_1\!\cdot\!\boldsymbol{x})\big\{\cos([\boldsymbol{k}_2-\boldsymbol{k}_3]\!\cdot\!\boldsymbol{x}) - \cos([\boldsymbol{k}_2+\boldsymbol{k}_3]\!\cdot\!\boldsymbol{x})\big\}\\
&= \tfrac{1}{4}\big\{\ \ \sin([\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3]\!\cdot\!\boldsymbol{x}) + \sin([\boldsymbol{k}_1-\boldsymbol{k}_2+\boldsymbol{k}_3]\!\cdot\!\boldsymbol{x})\\
&\qquad -\sin(\underbrace{[\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3]}_{\mathbf{0}\ \text{ in the equilateral case}}\!\cdot\!\boldsymbol{x}) - \sin([\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3]\!\cdot\!\boldsymbol{x})\big\}\\
&= -\tfrac{1}{4}\big\{\sin(2\boldsymbol{k}_1\!\cdot\!\boldsymbol{x}) + \sin(2\boldsymbol{k}_2\!\cdot\!\boldsymbol{x}) + \sin(2\boldsymbol{k}_3\!\cdot\!\boldsymbol{x})\big\} \qquad (9)
\end{aligned}
$$

From this quite a striking result[7] follows by inspection a result which previously we were able to obtain only by some fairly heavy calculation:

$$\begin{aligned}\nabla^2 F &= -\tfrac{1}{4}\big\{4k_1^2\sin(2\boldsymbol{k}_1\cdot\boldsymbol{x}) + 4k_2^2\sin(2\boldsymbol{k}_2\cdot\boldsymbol{x}) + 4k_3^2\sin(2\boldsymbol{k}_3\cdot\boldsymbol{x})\big\}\\ &\sim F \quad \text{only if } k_1^2 = k_2^2 = k_3^2 \equiv k^2, \text{ when we recover}\\ &= -4k^2 F\end{aligned}$$

To require of vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$ that

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = \boldsymbol{0} \quad \text{and} \quad k_1^2 = k_2^2 = k_3^2 \equiv k^2$$

forces them to lie in a plane, and on that plane to stand in the equilateral relationship shown in the following figure. It is by now evident that the functions

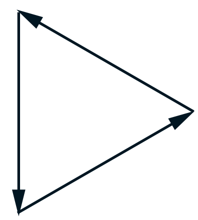

FIGURE 14: *Equilateral triangle in $\boldsymbol{k}$-space, an induced companion of the equilateral box in physical space. Also associated with that same physical box are the scaled copies obtained from*

$$\boldsymbol{k} \longrightarrow \boldsymbol{k}' \equiv n\boldsymbol{k} \quad : \quad n = 1, 2, 3, \ldots$$

*which induces a scaled refinement of the wave function. Rotation of the physical box induces an identical rotation in $\boldsymbol{k}$-space. But the tesselated companions of the physical box are not evident in $\boldsymbol{k}$-space.*

$F(\boldsymbol{x}; n_1\boldsymbol{k}_1, n_2\boldsymbol{k}_2, n_3\boldsymbol{k}_3)$ satisfy the equilateral box boundary conditions for all assignments of positive integral values to $\{n_1, n_2, n_3\}$, but will be eigenfunctions if and only if $n_1 = n_2 = n_3 \equiv n$. The associated eigenvalues assume then the form

$$E_n = \tfrac{h^2}{18ma^2} \cdot 12n^2$$

in which connection we observe that

$$12n^2 = \hat{n}_1^2 + 3\hat{n}_2^2 \quad \text{with} \quad \hat{n}_1 = 3n \text{ and } \hat{n}_2 = n$$

engages precisely the points $\{\hat{n}_1, \hat{n}_2\}$ that in Figure 10 fall on the upper "edge of the wedge." Our obligation now is to understand how these "edge states" (see the following figures) relate to the "interior states" (1) which in Part I were

---

[7] Some of the mystery is removed if one expands

$$\Big(\frac{e^{iK_1} - e^{-iK_1}}{2i}\Big)\Big(\frac{e^{iK_2} - e^{-iK_2}}{2i}\Big)\Big(\frac{e^{iK_3} - e^{-iK_3}}{2i}\Big)$$

and imposed the side condition $K_1 + K_2 + K_3 = 0$.

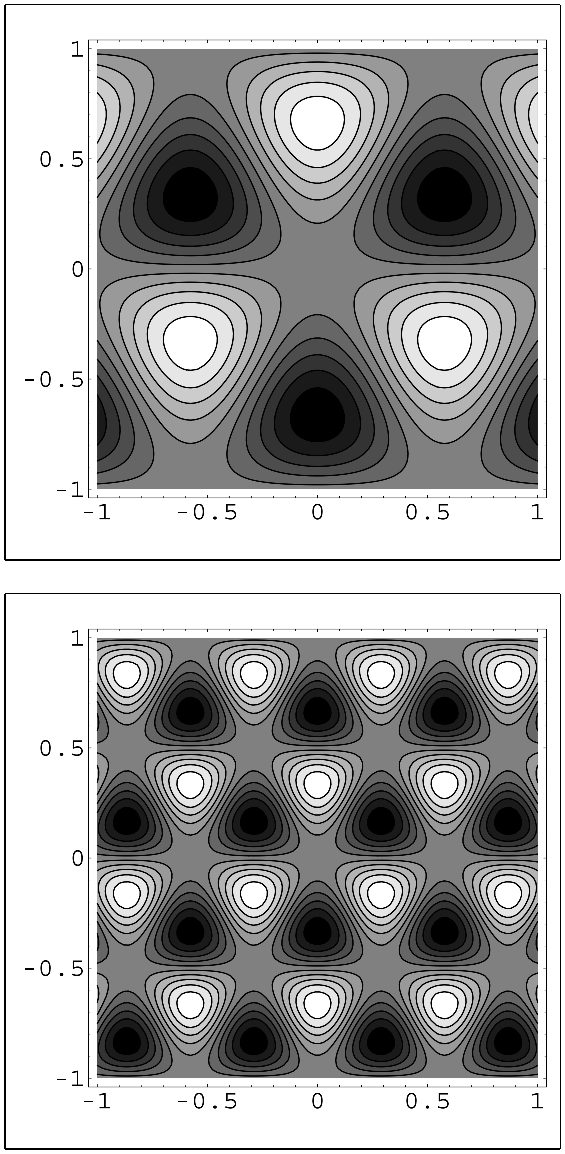


FIGURE 15: *Contour plots (see p. 34 of Part I for a description of the Mathematica routine) of the equilateral box functions*

$$\sin(kx_2)\sin(\tfrac{1}{2}k[\sqrt{3}x_1 + x_2])\sin(\tfrac{1}{2}k[\sqrt{3}x_1 - x_2])$$

*with $k = \pi$ (above) and $k = 2\pi$ (below). The top figure looks much more like a ground state than the real/imaginary parts of the purported ground state shown in Figure 30 of Part I.*

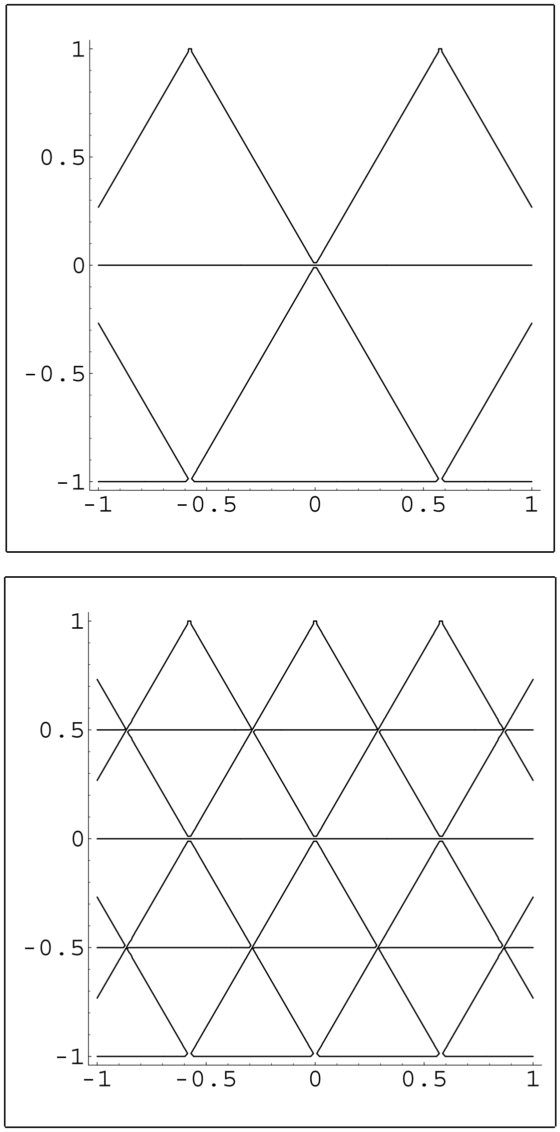


FIGURE 16: *Nodal curves (lines) of the equilateral box states shown in the preceding figure. The figures are self-similar in the sense discussed on p. 83 of Part I, and display the design anticipated already in Figures 4 & 13.*

were obtained by the method of images. To facilitate discussion of that topic, we observe that introduction of (8.1) into (9) gives

$$F = \tfrac{1}{4}\big\{\sin(2kx_2) + \sin(k[\sqrt{3}x_1 - x_2]) - \sin(k[\sqrt{3}x_1 + x_2])\big\}$$

which in the dimensionless $\xi$-variables introduced at (3) becomes

$$F = \tfrac{1}{4}\big\{\sin(2k\tfrac{3a}{\pi\sqrt{3}}\xi_2) + \sin(k[\sqrt{3}\tfrac{3a}{\pi}\xi_1 - \tfrac{3a}{\pi\sqrt{3}}\xi_2]) - \sin(k[\sqrt{3}\tfrac{3a}{\pi}\xi_1 + \tfrac{3a}{\pi\sqrt{3}}\xi_2])\big\}$$

Finally (see again the captions to Figures 14 & 13) we make the replacements

$$k \longrightarrow nk \longrightarrow n\tfrac{2\pi}{\sqrt{3}a}$$

and are led to the functions

$$\left.\begin{aligned} H_n(x_1,x_2) &\equiv \sin[2n(2\xi_2)] + \sin[2n(3\xi_1 - \xi_2)] - \sin[2n(3\xi_1+\xi_2)] \\ &= \sin[4n\xi_2] - 2\cos[6n\xi_1]\cdot\sin[2n\xi_2] \end{aligned}\right\} \qquad (10)$$

**4. Relationship to eigenfunctions obtained by method of images**. Looking back again to (1) it becomes immediately evident that we have only to set

$$\hat{\boldsymbol{n}} = \begin{pmatrix} 0 \\ 2n \end{pmatrix}$$

to obtain

$$\begin{aligned} G_{\hat{\boldsymbol{n}}}(x_1,x_2) &= H_n(x_1,x_2) \\ F_{\hat{\boldsymbol{n}}}(x_1,x_2) &= 0 \end{aligned}$$

But in §8 of Part I we were at pains to establish that the “axial” lattice point $\binom{0}{2n}$ and the “upper edge of the wedge” point $\binom{3n}{n}$ are equivalent in the duplex

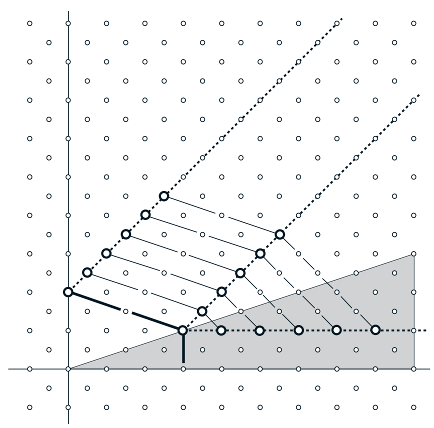

FIGURE 17: *The axial point $\binom{0}{2n}$ is equivalent to the “wedge-edge” point $\binom{3n}{n}$, and is the “seed” from which sprout a population of associated points $\binom{3n}{n+2k}$. For finer details, see Figure 27 in Part I and associated text.*

sense that the eigenvalues

$$\begin{aligned}E(\hat{\boldsymbol{n}}) &= \tfrac{h^2}{18ma^2}[\hat{n}_1^2 + 3\hat{n}_2^2]\\ &\downarrow\\ &= \tfrac{h^2}{18ma^2}12n^2 \quad \text{when } \hat{\boldsymbol{n}} = \left(\begin{smallmatrix}0\\2n\end{smallmatrix}\right)\end{aligned}$$

are invariant under $\left(\begin{smallmatrix}0\\2n\end{smallmatrix}\right) \longrightarrow \left(\begin{smallmatrix}3n\\n\end{smallmatrix}\right)$, and so also (to within an overall sign) are the associated eigenfunctions[8]

$$\begin{aligned}G_{\hat{\boldsymbol{n}}}(x_1, x_2) = \cos[2\hat{n}_1\xi_1]\sin[2\hat{n}_2\xi_2] &+ \cos[2\tfrac{-\hat{n}_1+3\hat{n}_2}{2}\xi_1]\sin[2\tfrac{-\hat{n}_1-\hat{n}_2}{2}\xi_2]\\ &+ \cos[2\tfrac{-\hat{n}_1-3\hat{n}_2}{2}\xi_1]\sin[2\tfrac{+\hat{n}_1-\hat{n}_2}{2}\xi_2]\\ \downarrow\qquad&\\ = H_n(x_1, x_2) \quad \text{when } \hat{\boldsymbol{n}} = \left(\begin{smallmatrix}0\\2n\end{smallmatrix}\right)&\end{aligned}$$

To summarize: drawing motivation from the observation that reflective tesselations of the plane yield to analysis by the method of images if and only if they are at the same time *ruled* tesselations (and *vice versa*: ruled tesselations are tractable if and only if they are reflective), we looked in §3 to some elementary functions whose zeros rule the plane, and were led at length to a population $H_n(x_1, x_2)$ of equilateral box eigenfunctions which are none other than the $G_{\hat{\boldsymbol{n}}}(x_1, x_2)$ associated with lattice points $\hat{\boldsymbol{n}}$ that are positioned on the upper edge of the wedge. The argument did serve to expose a major error in Part I, which stands therefore in need of revision,[9] but was found to suffer from its own intrinsic limitations: it provides no direct insight into the orthonormality properties of the $H_n(x_1, x_2)$ functions, and it fails to account for the eigenfunctions identified by lattice points *interior* to the wedge. It is clear that the functions $G_{\hat{\boldsymbol{n}}}(x_1, x_2)$ and $F_{\hat{\boldsymbol{n}}}(x_1, x_2)$, since known on other grounds to be (when $\hat{\boldsymbol{n}}$ lies interior to the wedge) linearly independent of the functions $H_n(x_1, x_2)$, cannot be developed as linear combinations of the latter. Nor can they contain $H_n(x_1, x_2)$ functions as factors—else they would exhibit equilaterally arranged patterns of nodal lines, which manifestly they do not do.

I turn now to discussion of a train of thought that first occurred to me while constructing figures[10] in quite another connection, but which has a acquired new interest as a possible means of escape from the limitations just ennumerated.

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[8] I quote here from (2). Similarly invariant—in the trivial sense $0 = 0$—are the companion eigenfunctions $F_{\hat{\boldsymbol{n}}}(x_1, x_2)$.

[9] It follows by inspection from (1) that $G_{\hat{\boldsymbol{n}}}(x_1, x_2)$ and $F_{\hat{\boldsymbol{n}}}(x_1, x_2)$ both vanish when $\hat{\boldsymbol{n}}$ lies on the lower edge of the wedge ($\hat{n}_2 = 0$), and that $F_{\hat{\boldsymbol{n}}}(x_1, x_2)$ vanishes also on the upper edge (equivalent to $\hat{n}_1 = 0$), but I was in error when (on p. 51 of Part I) I claimed that $G_{\hat{\boldsymbol{n}}}(x_1, x_2)$ vanishes on the upper edge; it manifestly does not. I was, for this reason, wrong when (on that same page) I asserted that every equilateral box eigenvalue "is, in the absence of accidental degeneracy, doubly degenerate;" wrong when, in Figure 30, I claimed to providing represention the equilateral ground state(s); wrong so far as concerns some of the spectral density details presented in §9.

[10] See especially Figure 48 in Part I.

**5. Method of sections**. Let our mass point $m$ be constrained now to move freely within the rectangular *3-dimensional* box shown in the following figure. Holding $\boldsymbol{x}$-variables in reserve, we write $\boldsymbol{y}$ to describe points in the 3-space

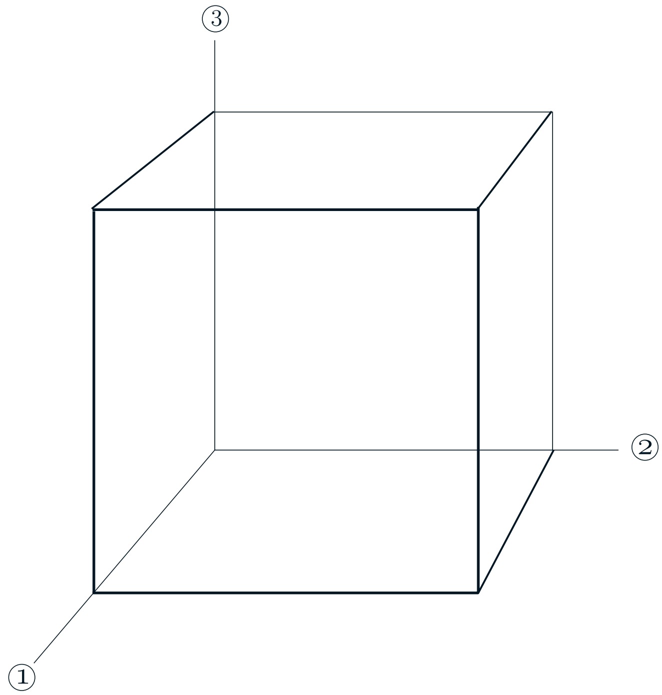


FIGURE 18: *Generic rectangular box, of volume $b_1 b_2 b_3$, within which a mass point $m$ moves freely. It is by refinement of this elementary construction that we are led to the systems to which the "method of sections" pertains. We will have interest mainly in the cubic case $b_1 = b_2 = b_3 \equiv b$.*

within which the box resides. Looking to the elementary time-independent quantum mechanics of such a system, one is led—whether one proceeds by separation of variables, by the method of images, or (perhaps most efficiently) by the methods developed in §3—to eigenfunctions of the form

$$\begin{aligned}\Psi(\boldsymbol{y}) &\sim \sin(n_1 \tfrac{\pi}{b_1} y_1) \sin(n_2 \tfrac{\pi}{b_2} y_2) \sin(n_3 \tfrac{\pi}{b_3} y_3) \\ &\downarrow \\ &= \sin(n_1 \tfrac{\pi}{b} y_1) \sin(n_2 \tfrac{\pi}{b} y_2) \sin(n_3 \tfrac{\pi}{b} y_3) \quad \text{in the cubic case} \end{aligned} \tag{11}$$

The associated eigenvalues are given by

$$E(\boldsymbol{n}) = \frac{h^2}{8mb^2}[n_1^2 + n_2^2 + n_3^2]$$

Such product functions serve to partition $\boldsymbol{y}$-space into stacked rectangles, which in the cases of immediate interest become stacked cubes (see Figure 19).

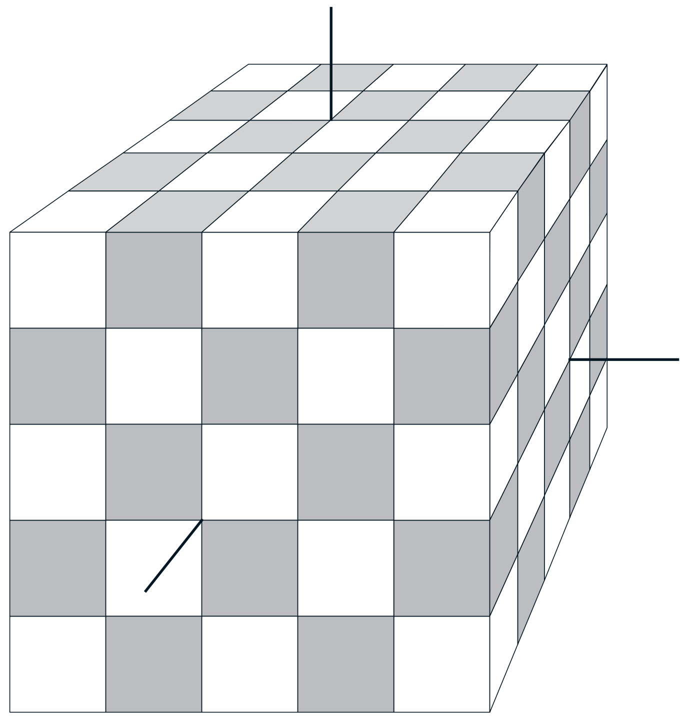

FIGURE 19: *Reflective tesselation of 3-space by a cube. Such a construction comes implicitly into play whichever of the methods mentioned in the text are used to obtain the eigenfunctions (11). Each of those functions* vanishes on each of the faces of each of the cubes.

The essence of what I call the "method of sections" is (in the particular instance of immediate interest) illustrated in Figure 20. Our assignment now is to lend analytical substance to the simple idea in question. Drawing upon the notational conventions set forth in Figure 21, we note first that $\boldsymbol{y}$ will be co-planar with $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ if and only if $\{\boldsymbol{y}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ define a tetrahedron of zero volume

$$\frac{1}{3!}\begin{vmatrix} 1 & y_1 & y_2 & y_3 \\ 1 & b & 0 & 0 \\ 1 & 0 & b & 0 \\ 1 & 0 & 0 & b \end{vmatrix} = 0$$

and are led thus to this "equation of the plane:"

$$y_1 + y_2 + y_3 - b = 0 \tag{12}$$

We have now to describe the relationship between the $(y_1, y_2, y_3)$-coordinates and the $(x_1, x_2)$-coordinates of points on the plane. It is clear on general grounds that the equations in question have the inhomogeneous linear form

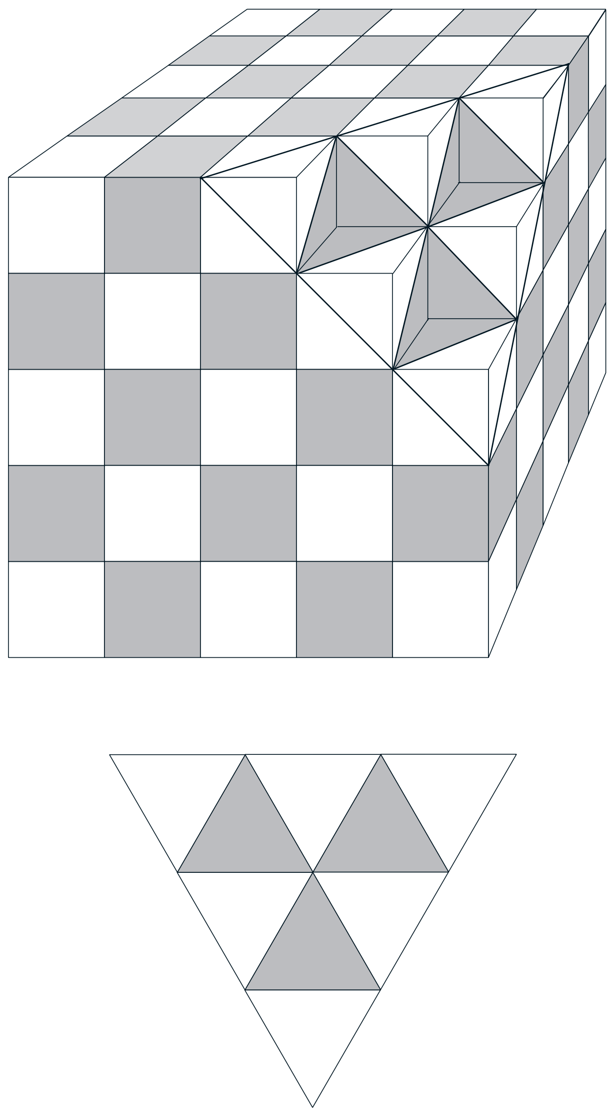

FIGURE 20: *The equilaterally tesselated plane (below) has been obtained by "sanding off the corner" of (i.e., as a plane section of) the cubically tesselated 3-space shown above (see also Figure 19). The figure exposes the central idea of what I call the "method of sections."*

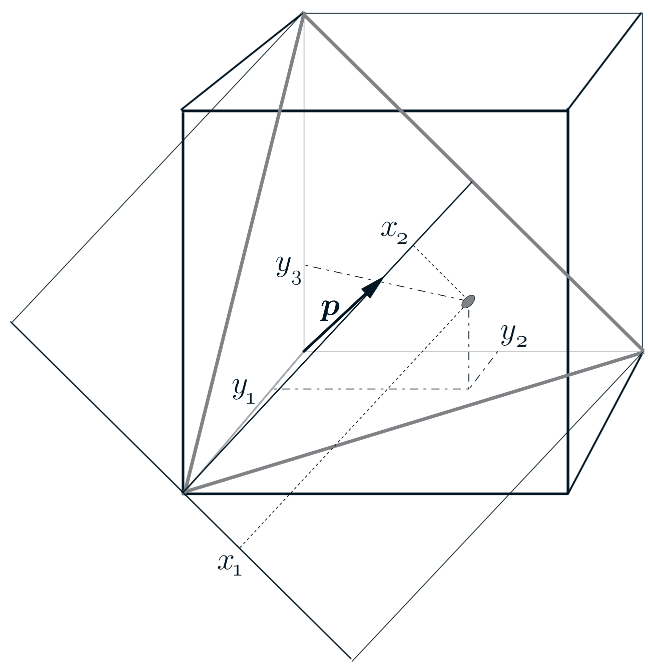


FIGURE 21: *Notations used in discussion of the method of sections, as it relates to the equilateral box problem. The cube has sides of length $b$ and vertices at the points*

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \underbrace{\begin{pmatrix}b\\0\\0\end{pmatrix}, \begin{pmatrix}0\\b\\0\end{pmatrix}, \begin{pmatrix}0\\0\\b\end{pmatrix}}_{\text{vertices } \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \text{ of the } \triangle}, \begin{pmatrix}0\\b\\b\end{pmatrix}, \begin{pmatrix}b\\0\\b\end{pmatrix}, \begin{pmatrix}b\\b\\0\end{pmatrix} \text{ and } \begin{pmatrix}b\\b\\b\end{pmatrix}$$

*The plane is perpendicular to the semi-diagonal vector*

$$\boldsymbol{p} \equiv \tfrac{1}{2}\begin{pmatrix}b\\b\\b\end{pmatrix}$$

*The triangle has sides of length $a = \sqrt{2}b$. The $\boldsymbol{x}$-coordinate system (inscribed on the plane, oriented as shown, with origin at $\boldsymbol{v}_1$) stands to the triangle in a relation imitative of Figure 9.*

$$\begin{aligned} y_1 &= A_{11}x_1 + A_{12}x_2 + B_1 \\ y_2 &= A_{21}x_1 + A_{22}x_2 + B_2 \\ y_3 &= A_{31}x_1 + A_{32}x_2 + B_3 \end{aligned}$$

of which, however, we need consider only two, since the third will follow from (12); we look to the last two, since $y_2$ and $y_3$ stand in a symmetrical relationship (and $y_1$ in an eccentric relationship) to the $x$-coordinate system. Looking specifically to the vertices of the triangle, it becomes clear that the coefficients $\{A_{21}, A_{22}, A_{31}, A_{32}, B_2, B_3\}$ must satisfy

$$\left.\begin{aligned} 0 &= \quad A_{21}0 + A_{22}0 + B_2 \\ 0 &= \quad A_{31}0 + A_{32}0 + B_3 \end{aligned}\right\} \quad \text{at } \boldsymbol{v}_1$$

$$\left.\begin{aligned} b &= +A_{21}\tfrac{a}{2} + A_{22}h + B_2 \\ 0 &= +A_{31}\tfrac{a}{2} + A_{32}h + B_3 \end{aligned}\right\} \quad \text{at } \boldsymbol{v}_2$$

$$\left.\begin{aligned} 0 &= -A_{21}\tfrac{a}{2} + A_{22}h + B_2 \\ b &= -A_{31}\tfrac{a}{2} + A_{32}h + B_3 \end{aligned}\right\} \quad \text{at } \boldsymbol{v}_3$$

where (see again the captions to Figures 9 & 21) $\frac{a}{2} = \frac{1}{\sqrt{2}}b$ and $h = \frac{1}{2}\sqrt{6}b$. It follows from the top pair of equations that $B_2 = B_3 = 0$. The remaining equations can therefore be written

$$\begin{pmatrix} +\frac{1}{2}a & h & 0 & 0 \\ 0 & 0 & +\frac{1}{2}a & h \\ -\frac{1}{2}a & h & 0 & 0 \\ 0 & 0 & -\frac{1}{2}a & h \end{pmatrix} \begin{pmatrix} A_{21} \\ A_{22} \\ A_{31} \\ A_{32} \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ 0 \\ b \end{pmatrix}$$

which by marix inversion give

$$\begin{aligned} A_{21} &= +\tfrac{b}{a} &&= +\tfrac{1}{\sqrt{2}} \\ A_{22} &= \tfrac{b}{2h} &&= \tfrac{1}{\sqrt{6}} \\ A_{31} &= -\tfrac{b}{a} &&= -\tfrac{1}{\sqrt{2}} \\ A_{32} &= \tfrac{b}{2h} &&= \tfrac{1}{\sqrt{6}} \end{aligned}$$

Thus do we obtain

$$\left.\begin{aligned} y_1 &= \qquad\qquad - \tfrac{2}{\sqrt{6}}x_2 + b \\ y_2 &= +\tfrac{1}{\sqrt{2}}x_1 + \tfrac{1}{\sqrt{6}}x_2 \\ y_3 &= -\tfrac{1}{\sqrt{2}}x_1 + \tfrac{1}{\sqrt{6}}x_2 \end{aligned}\right\} \tag{13}$$

where the top equation was obtained from the latter pair by appeal to (12). Drawing upon this result, we learn that the functions

$$\mathcal{Z}_{\boldsymbol{n}}(y_1, y_2, y_3) \equiv \sin(n_1\tfrac{\pi}{b}y_1)\sin(n_2\tfrac{\pi}{b}y_2)\sin(n_3\tfrac{\pi}{b}y_3) \tag{14}$$

assume values on the plane which can, in $x$-coordinates, be described

$$\begin{aligned} Z_{\boldsymbol{n}}(x_1,x_2) \equiv \sin\big[n_1\tfrac{\pi}{b}\big(-\tfrac{2}{\sqrt{6}}x_2+b\big)\big]& \\ \cdot\sin\big[n_2\tfrac{\pi}{b}\big(+\tfrac{1}{\sqrt{2}}x_1+\tfrac{1}{\sqrt{6}}x_2\big)\big]& \\ \cdot\sin\big[n_3\tfrac{\pi}{b}\big(-\tfrac{1}{\sqrt{2}}x_1+\tfrac{1}{\sqrt{6}}x_2\big)\big]& \end{aligned} \tag{15}$$

where $\{n_1, n_2, n_3\}$ range independently/unrestrictedly on the positive integers. It becomes appropriate at this point to replace the cube scale parameter $b$ with the triangle scale parameter $a = \sqrt{2}b$; we obtain

$$\begin{aligned} Z_{\boldsymbol{n}}(x_1,x_2) \equiv \sin\big[n_1\tfrac{\pi}{a}\tfrac{2}{\sqrt{3}}x_2 - n_1\pi\big]& \\ \cdot\sin\big[n_2\tfrac{\pi}{a}\big(x_1+\tfrac{1}{\sqrt{3}}x_2\big)\big]\cdot\sin\big[n_3\tfrac{\pi}{a}\big(x_1-\tfrac{1}{\sqrt{3}}x_2\big)\big]& \end{aligned}$$

in which connection we notice that the leading factor

$$\sin\big[n_1\tfrac{\pi}{a}\tfrac{2}{\sqrt{3}}x_2 - n_1\pi\big] = (-)^{n_1}\sin\big[n_1\tfrac{\pi}{a}\tfrac{2}{\sqrt{3}}x_2\big]$$

In terms of the dimensionless $\xi$-variables introduced at (3) we have

$$Z_{\boldsymbol{n}}(x_1,x_2) = (-)^{n_1}W_{\boldsymbol{n}}(x_1,x_2)$$

where

$$W_{\boldsymbol{n}}(x_1,x_2) \equiv \sin\big[2n_1\xi_2\big]\cdot\sin\big[n_2\big(3\xi_1+\xi_2\big)\big]\cdot\sin\big[n_3\big(3\xi_1-\xi_2\big)\big] \tag{16.1}$$

can, in the notation of §3, be written

$$= \sin(\boldsymbol{k}_1\cdot\boldsymbol{x})\cdot\sin(\boldsymbol{k}_2\cdot\boldsymbol{x})\cdot\sin(\boldsymbol{k}_3\cdot\boldsymbol{x}) \tag{16.2}$$

where

$$\boldsymbol{k}_1 = n_1k\begin{pmatrix}0\\-2\end{pmatrix}\ ,\ \boldsymbol{k}_2 = n_2k\begin{pmatrix}+\sqrt{3}\\1\end{pmatrix}\quad\text{and}\quad\boldsymbol{k}_3 = n_3k\begin{pmatrix}-\sqrt{3}\\1\end{pmatrix}$$

stand (or at least they do when $n_1 = n_2 = n_3 = n$ )in the relationship familiar from Figure 14; here as before, $k \equiv \frac{\pi}{\sqrt{3}a}$.

It is by now abundantly clear that the "method of sections," as thus far developed, has led us back again to precisely the population of functions, and to the set of ideas, familiar from §3. Back again, but no farther. Both lines of argument supply infinite populations of functions *each of which satisfies the equilateral box boundary conditions* and some of which (the functions $H_n(x_1,x_2)$ discussed in §4) are in fact eigenfunctions as they stand. The question still open: Is it possible, with the material thus provided, to *assemble* the remainder of the "interior" eigenfunctions (1)?

**6. Assembly of eigenfunctions in the equilateral case**. We undertake now to explore the possibility of representing functions of type (1)—eigenfunctions of the equilateral box problem—as linear combinations of functions to type (16.1). To simplify the discussion, I will for the moment restrict my remarks to eigenfunctions of "type $G$," as defined at (1.1); eigenfunctions of "type $F$" will be discussed separately. As a preparatory step, intended to facilitate comparisons, we render the functions of interest to us into a common (which is to say, a shared) language. Drawing upon *Mathematica*'s `TrigReduce[`*expr*`]` resource, we obtain

$$G_{\hat{\boldsymbol{n}}}(x_1,x_2) \equiv \cos[2\hat{n}_1\xi_1]\sin[2\hat{n}_2\xi_2] + \cos[\hat{n}_1(\xi_1+\xi_2)]\sin[\hat{n}_2(3\xi_1-\xi_2)] - \cos[\hat{n}_1(\xi_1-\xi_2)]\sin[\hat{n}_2(3\xi_1+\xi_2)]$$

$$\begin{aligned} = \tfrac{1}{2}\big\{ &-\sin[2\hat{n}_1\xi_1 - 2\hat{n}_2\xi_2] + \sin[2\hat{n}_1\xi_1 + 2\hat{n}_2\xi_2] \\ &-\sin[\hat{n}_1(\xi_1+\xi_2) - \hat{n}_2(3\xi_1-\xi_2)] + \sin[\hat{n}_1(\xi_1+\xi_2)+\hat{n}_2(3\xi_1-\xi_2)] \\ &+\sin[\hat{n}_1(\xi_1-\xi_2)-\hat{n}_2(3\xi_1+\xi_2)] - \sin[\hat{n}_1(\xi_1-\xi_2)+\hat{n}_2(3\xi_1+\xi_2)]\big\} \end{aligned} \tag{19.1}$$

$$\begin{aligned} = \tfrac{1}{2}\big\{ -\sin[2\hat{n}_1\xi_1 - 2\hat{n}_2\xi_2] + \sin[2\hat{n}_1\xi_1 + 2\hat{n}_2\xi_2] \\ -\sin[(\hat{n}_1-3\hat{n}_2)\xi_1 + (\hat{n}_1+\hat{n}_2)\xi_2] \\ +\sin[(\hat{n}_1+3\hat{n}_2)\xi_1 + (\hat{n}_1-\hat{n}_2)\xi_2] \\ +\sin[(\hat{n}_1-3\hat{n}_2)\xi_1 - (\hat{n}_1+\hat{n}_2)\xi_2] \\ -\sin[(\hat{n}_1+3\hat{n}_2)\xi_1 - (\hat{n}_1-\hat{n}_2)\xi_2]\big\} \end{aligned} \tag{19.2}$$

and

$$\begin{aligned} W_{\boldsymbol{n}}(x_1,x_2) &\equiv \sin\big[-2n_1\xi_2\big]\cdot\sin\big[n_2\big(+3\xi_1+\xi_2\big)\big]\cdot\sin\big[n_3\big(-3\xi_1+\xi_2\big)\big] \\ &= \tfrac{1}{4}\big\{W_1+W_2+W_3+W_4\big\} \end{aligned} \tag{20.1}$$

where

$$\begin{aligned} W_1 &\equiv -\sin[2n_1\xi_2 + n_2(3\xi_1+\xi_2) + n_3(3\xi_1-\xi_2)] \\ &= -\sin[(3n_2+3n_3)\xi_1 + (2n_1+n_2-n_3)\xi_2] \\ W_2 &\equiv -\sin[2n_1\xi_2 - n_2(3\xi_1+\xi_2) - n_3(3\xi_1-\xi_2)] \\ &= +\sin[(3n_2+3n_3)\xi_1 - (2n_1-n_2+n_3)\xi_2] \\ W_3 &\equiv +\sin[2n_1\xi_2 + n_2(3\xi_1+\xi_2) - n_3(3\xi_1-\xi_2)] \\ &= +\sin[(3n_2-3n_3)\xi_1 + (2n_1+n_2+n_3)\xi_2] \\ W_4 &\equiv +\sin[2n_1\xi_2 - n_2(3\xi_1+\xi_2) + n_3(3\xi_1-\xi_2)] \\ &= -\sin[(3n_2-3n_3)\xi_1 - (2n_1-n_2-n_3)\xi_2] \end{aligned} \tag{20.2}$$

When editing and transcribing results reported by *Mathematica* it is easy to make typographic errors; evidence that my work has been accurate is, however, supplied by the following observations:

Whether one proceeds from (19.1) or from (19.2), one obtains easily the familiar "bottom of the wedge" statement $G(2n,0)=0$,[11] which is comforting, but too simple to be very informative. Similarly direct are the statements

$$\begin{aligned} G(0,2n) &= \sin[4n\xi_2]+\sin[6n\xi_1-2n\xi_2]-\sin[6n\xi_1+2n\xi_2] \\ &= 4W(n,n,n) \end{aligned}$$

which are familiar from the discussion that culminated in (10). Recalling

$$\nabla^2\equiv\Big(\frac{\partial}{\partial x_1}\Big)^2+\Big(\frac{\partial}{\partial x_2}\Big)^2=\Big(\frac{\pi}{3a}\Big)^2\Big\{\Big(\frac{\partial}{\partial \xi_1}\Big)^2+3\Big(\frac{\partial}{\partial \xi_2}\Big)^2\Big\}$$

from near the end of §2, we find it to be—for the interesting reason that

$$(2\hat n_1)^2+3(2\hat n_2)^2=(\hat n_1-3\hat n_2)^2+3(\hat n_1+\hat n_2)=(\hat n_1+3\hat n_2)^2+3(\hat n_1-\hat n_2)$$

—an almost immediate implication of (19.2) that

$$\Big\{\Big(\frac{\partial}{\partial \xi_1}\Big)^2+3\Big(\frac{\partial}{\partial \xi_2}\Big)^2\Big\}G=-4\big[\hat n_1^2+3\hat n_2^2\big]G \tag{21}$$

which reproduces (6). Proceeding similarly from (20) we obtain

$$\begin{aligned} \Big\{\Big(\frac{\partial}{\partial \xi_1}\Big)^2+3\Big(\frac{\partial}{\partial \xi_2}\Big)^2\Big\}W =&-12[n_1^2+n_2^2+n_3^2+(+n_1n_2+n_2n_3-n_3n_1)]\tfrac14 W_1 \\ &-12[n_1^2+n_2^2+n_3^2+(-n_1n_2+n_2n_3+n_3n_1)]\tfrac14 W_2 \\ &-12[n_1^2+n_2^2+n_3^2+(+n_1n_2-n_2n_3+n_3n_1)]\tfrac14 W_3 \\ &-12[n_1^2+n_2^2+n_3^2+(-n_1n_2-n_2n_3-n_3n_1)]\tfrac14 W_4 \\ =-12[n_1^2+n_2^2+n_3^2]W &-3(+n_1n_2+n_2n_3-n_3n_1)W_1 \\ &-3(-n_1n_2+n_2n_3+n_3n_1)W_2 \\ &-3(+n_1n_2-n_2n_3+n_3n_1)W_3 \\ &-3(-n_1n_2-n_2n_3-n_3n_1)W_4 \end{aligned} \tag{22}$$

The expression on the right is so intricate and—on its face—strange looking that we can expect only with unaccustomed effort to establish its equivalence to results already in hand. We begin by observing that on the upper edge of the wedge (21) reads

$$\Big\{\Big(\frac{\partial}{\partial \xi_1}\Big)^2+3\Big(\frac{\partial}{\partial \xi_2}\Big)^2\Big\}G(0,2n)=-48n^2G(0,2n)$$

[11] I will write $G(\hat n_1,\hat n_2)$ when I want to emphasize the $\boldsymbol{\hat n}$-dependence of $G_{\boldsymbol{\hat n}}(x_1,x_2)$, and $G(\xi_1,\xi_2)$ to indicate that I am looking to the $\boldsymbol{x}$-dependence but have switched to $\boldsymbol{\xi}$ variables. The notations $W(n_1,n_2,n_3)$ and $W(\xi_1,\xi_2)$ will be used with similar intent.

while from (22) we obtain

$$\begin{aligned}\Big\{\Big(\frac{\partial}{\partial\xi_1}\Big)^2+3\Big(\frac{\partial}{\partial\xi_2}\Big)^2\Big\}W(n,n,n)=&-12[3+1]n^2\tfrac{1}{4}W_1(n,n,n)\\&-12[3+1]n^2\tfrac{1}{4}W_2(n,n,n)\\&-12[3+1]n^2\tfrac{1}{4}W_3(n,n,n)\\&-12[3-3]n^2\tfrac{1}{4}W_4(n,n,n)\end{aligned}$$

$W_4$ is killed on the right, but it is an implication of (20.2) that $W_4(n,n,n)=0$, so $W_4$ is in fact absent also on the left; we have

$$\Big\{\Big(\frac{\partial}{\partial\xi_1}\Big)^2+3\Big(\frac{\partial}{\partial\xi_2}\Big)^2\Big\}W(n,n,n)=-48n^2W(n,n,n)$$

as anticipated. The relevant *general* "result already in hand" was developed in §3, and reads

$$\begin{aligned}&\nabla^2\sin(\boldsymbol{k}_1\cdot\boldsymbol{x})\cdot\sin(\boldsymbol{k}_2\cdot\boldsymbol{x})\cdot\sin(\boldsymbol{k}_3\cdot\boldsymbol{x})\\&\qquad=-[k_1^2+k_2^2+k_3^2]\sin(\boldsymbol{k}_1\cdot\boldsymbol{x})\cdot\sin(\boldsymbol{k}_2\cdot\boldsymbol{x})\cdot\sin(\boldsymbol{k}_3\cdot\boldsymbol{x})+\text{dangling term}\end{aligned}$$

where

$$\begin{aligned}\text{dangling term}=&\quad 2\boldsymbol{k}_1\cdot\boldsymbol{k}_2\cos(\boldsymbol{k}_1\cdot\boldsymbol{x})\cos(\boldsymbol{k}_2\cdot\boldsymbol{x})\sin(\boldsymbol{k}_3\cdot\boldsymbol{x})\\&+2\boldsymbol{k}_1\cdot\boldsymbol{k}_3\cos(\boldsymbol{k}_1\cdot\boldsymbol{x})\sin(\boldsymbol{k}_2\cdot\boldsymbol{x})\cos(\boldsymbol{k}_3\cdot\boldsymbol{x})\\&+2\boldsymbol{k}_2\cdot\boldsymbol{k}_3\sin(\boldsymbol{k}_1\cdot\boldsymbol{x})\cos(\boldsymbol{k}_2\cdot\boldsymbol{x})\cos(\boldsymbol{k}_3\cdot\boldsymbol{x})\end{aligned}$$

Taking our $\boldsymbol{k}$-vectors from p. 25, we have

$$\begin{aligned}[k_1^2+k_2^2+k_3^2]&=4k^2[n_1^2+n_2^2+n_3^2]\\2\boldsymbol{k}_1\cdot\boldsymbol{k}_2&=-4k^2n_1n_2\\2\boldsymbol{k}_1\cdot\boldsymbol{k}_3&=-4k^2n_1n_3\\2\boldsymbol{k}_2\cdot\boldsymbol{k}_3&=-4k^2n_2n_3\end{aligned}$$

where again $k\equiv\frac{\pi}{\sqrt{3}a}$. Changing variables $\boldsymbol{x}\longrightarrow\boldsymbol{\xi}$, we on the basis of these remarks have

$$\begin{aligned}&\tfrac{1}{3}\Big\{\Big(\frac{\partial}{\partial\xi_1}\Big)^2+3\Big(\frac{\partial}{\partial\xi_2}\Big)^2\Big\}\sin\big[-2n_1\xi_2\big]\cdot\sin\big[n_2\big(3\xi_1+\xi_2\big)\big]\cdot\sin\big[n_3\big(-3\xi_1+\xi_2\big)\big]\\&\quad=-4[n_1^2+n_2^2+n_3^2]\sin\big[-2n_1\xi_2\big]\cdot\sin\big[n_2\big(3\xi_1+\xi_2\big)\big]\cdot\sin\big[n_3\big(-3\xi_1+\xi_2\big)\big]\\&\qquad\quad-4\Big\{n_1n_2P_3+n_1n_3P_2+n_2n_3P_1\Big\}\end{aligned}$$

where the $\frac{1}{3}$ arose from $(\frac{\pi}{3a})^2=\frac{1}{3}k^2$ (the $k^2$ was then abandoned both left and right) and where the product functions

$$\begin{aligned}P_3&\equiv\cos\big[-2n_1\xi_2\big]\cdot\cos\big[n_2\big(+3\xi_1+\xi_2\big)\big]\cdot\sin\big[n_3\big(-3\xi_1+\xi_2\big)\big]\\P_2&\equiv\cos\big[-2n_1\xi_2\big]\cdot\sin\big[n_2\big(+3\xi_1+\xi_2\big)\big]\cdot\cos\big[n_3\big(-3\xi_1+\xi_2\big)\big]\\P_1&\equiv\sin\big[-2n_1\xi_2\big]\cdot\cos\big[n_2\big(+3\xi_1+\xi_2\big)\big]\cdot\cos\big[n_3\big(-3\xi_1+\xi_2\big)\big]\end{aligned}$$

Drawing again upon *Mathematica*'s `TrigReduce[`*expr*`]` resource, we find

$$\begin{aligned} P_3 &= \tfrac{1}{4}\big\{ + W_1 - W_2 + W_3 - W_4 \big\} \\ P_2 &= \tfrac{1}{4}\big\{ - W_1 + W_2 + W_3 - W_4 \big\} \\ P_1 &= \tfrac{1}{4}\big\{ + W_1 + W_2 - W_3 - W_4 \big\} \end{aligned}$$

Returning with this information to the equation in which $P_1$, $P_2$ and $P_3$ made their first appearance, we obtain

$$\begin{aligned} \left\{\Big(\frac{\partial}{\partial\xi_1}\Big)^2 + 3\Big(\frac{\partial}{\partial\xi_2}\Big)^2\right\}W &= -12[n_1^2+n_2^2+n_3^2]W \\ &\qquad - 3n_1n_2\big\{ + W_1 - W_2 + W_3 - W_4 \big\} \\ &\qquad - 3n_1n_3\big\{ - W_1 + W_2 + W_3 - W_4 \big\} \\ &\qquad - 3n_2n_3\big\{ + W_1 + W_2 - W_3 - W_4 \big\} \\ \\ &= -12[n_1^2+n_2^2+n_3^2]W \\ &\qquad - 3(+n_1n_2 + n_2n_3 - n_3n_1)W_1 \\ &\qquad - 3(-n_1n_2 + n_2n_3 + n_3n_1)W_2 \\ &\qquad - 3(+n_1n_2 - n_2n_3 + n_3n_1)W_3 \\ &\qquad - 3(-n_1n_2 - n_2n_3 - n_3n_1)W_4 \end{aligned}$$

which does in fact precisely reproduce (22). I proceed in confidence that equations (19) and (20) are indeed correct; though they were obtained by what *Mathematica* calls "trigonometric reduction," they provide what are in fact simply Fourier sine expansions of the functions $G$ and $W$.

My objective is to construct a formula of the type

$$G(\hat{n}_1,\hat{n}_2) = \sum_{n_1n_2n_3} \text{weighted } W\text{-functions} \tag{23}$$

but I lack any straightforwardly computional means for getting from here to there; I have seemingly no option but to proceed by "incremental insight," and it is in that spirit that I assemble the following miscellaneous remarks.

It was established in §8 of Part I that—independently of any assumption that $\hat{n}_1$ and $\hat{n}_2$ be integers—the function

$$N(\hat{n}_1,\hat{n}_2) \equiv \hat{n}_1^2 + 3\hat{n}_2^2$$

is invariant under the linear transformations $\hat{\boldsymbol{n}} \rightarrow \mathbb{A}\hat{\boldsymbol{n}} \rightarrow \mathbb{A}^2\hat{\boldsymbol{n}}$ where

$$\mathbb{A} \equiv \tfrac{1}{2}\begin{pmatrix} -1 & +3 \\ -1 & -1 \end{pmatrix}$$

has the properties

$$\mathbb{A}^3 = \mathbb{I} \quad\text{and}\quad \mathbb{A}^{\mathsf{T}}\mathbb{G}\mathbb{A} = \mathbb{G} \quad\text{with}\quad \mathbb{G} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

Specifically

$$\begin{pmatrix} \hat n_1 \\ \hat n_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{2}[-\hat n_1 + 3\hat n_2] \\ \frac{1}{2}[-\hat n_1 - \ \ \hat n_2] \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{2}[-\hat n_1 - 3\hat n_2] \\ \frac{1}{2}[+\hat n_1 - \ \ \hat n_2] \end{pmatrix} \tag{24.1}$$

$N(\hat n_1, \hat n_2)$ is invariant also under the reflective transformations

$$\begin{pmatrix} \hat n_1 \\ \hat n_2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\hat n_1 \\ +\hat n_2 \end{pmatrix} \quad \text{else} \quad \begin{pmatrix} +\hat n_1 \\ -\hat n_2 \end{pmatrix} \quad \text{else} \quad \begin{pmatrix} -\hat n_1 \\ -\hat n_2 \end{pmatrix} \tag{24.2}$$

The properties that attach to $N(\hat n_1, \hat n_2)$ attach also to the functions $G(\hat n_1, \hat n_2)$ (which were, after all, generated by a process that involved “summing over spectral symmetries”—summing, that is to say, over the symetries of $N(\hat n_1, \hat n_2)$); *Mathematica*, working from (19.2), readily confirms that

$$\begin{aligned} G(\hat n_1, \hat n_2) &= G(\tfrac{1}{2}[-\hat n_1 + 3\hat n_2], \tfrac{1}{2}[-\hat n_1 - \hat n_2]) = G(\tfrac{1}{2}[-\hat n_1 - 3\hat n_2], \tfrac{1}{2}[+\hat n_1 - \hat n_2]) \\ &= +G(-\hat n_1, +\hat n_2) = -G(+\hat n_1, -\hat n_2) = -G(-\hat n_1, -\hat n_2) \end{aligned}$$

and—remarkably—does so independently of any assumption that $\hat n_1$ and $\hat n_2$ be integers. So far as concerns the $\boldsymbol{\xi}$-dependence of $G$, *Mathematica* confirms that

$$G(\xi_1, \xi_2) = G(-\xi_1, \xi_2)$$

The $G$-properties assembled above do not (or at least do not in their totality) attach to the functions $W$, but must perforce attach to the anticipated linear combinations of $W$-functions; I propose to use that fact as a design principle.

Passing over now from the $G$ to the $W$ side of the street, we encounter the circumstance that $\{n_1, n_2, n_3\}$ are too numerous. Tinkering (inspired partly by the symmetry evident in (8.3)) leads me tentatively to require

$$n_1 + n_2 + n_3 = 0 \tag{25}$$

and to automate that condition by writing

$$\left.\begin{aligned} n_1 &= n_1(\hat n_1, \hat n_2) \equiv \qquad\ \ 2\hat n_2 \\ n_2 &= n_2(\hat n_1, \hat n_2) \equiv +\hat n_1 - \hat n_2 \\ n_3 &= n_3(\hat n_1, \hat n_2) \equiv -\hat n_1 - \hat n_2 \end{aligned}\right\} \tag{26}$$

Then

$$n_1^2 + n_2^2 + n_3^2 = 2[\hat n_1^2 + 3\hat n_2^2]$$

The transformations described at the top of the page induce

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \longrightarrow \begin{pmatrix} n_3 \\ n_1 \\ n_2 \end{pmatrix} \longrightarrow \begin{pmatrix} n_2 \\ n_3 \\ n_1 \end{pmatrix} \quad : \quad \text{cyclic} \tag{27.1}$$

$$\longrightarrow \begin{pmatrix} +n_1 \\ +n_3 \\ +n_2 \end{pmatrix} \quad \text{else} \quad \begin{pmatrix} -n_1 \\ -n_3 \\ -n_2 \end{pmatrix} \quad \text{else} \quad \begin{pmatrix} -n_1 \\ -n_2 \\ -n_3 \end{pmatrix} \tag{27.2}$$

Looking to the structure of (22) we are motivated to define

$$\left.\begin{aligned} w_1(n_1,n_2,n_3) &\equiv +n_1n_2+n_2n_3-n_3n_1\\ w_2(n_1,n_2,n_3) &\equiv -n_1n_2+n_2n_3+n_3n_1\\ w_3(n_1,n_2,n_3) &\equiv +n_1n_2-n_2n_3+n_3n_1\\ w_4(n_1,n_2,n_3) &\equiv -n_1n_2-n_2n_3-n_3n_1 \end{aligned}\right\} \tag{28}$$

in which notation (22) reads

$$\begin{aligned}\Big\{\Big(\frac{\partial}{\partial\xi_1}\Big)^2+3\Big(\frac{\partial}{\partial\xi_2}\Big)^2\Big\}W &= -12[n_1^2+n_2^2+n_3^2]W\\ &\qquad -3\big\{w_1W_1+w_2W_2+w_3W_3+w_4W_4\big\}\end{aligned}$$

We observe in this connection that $\hat{\boldsymbol{n}}\to\mathbb{A}\hat{\boldsymbol{n}}\to\mathbb{A}^2\hat{\boldsymbol{n}}$ induces

$$\begin{pmatrix}w_1\\w_2\\w_3\\w_4\end{pmatrix}\longrightarrow\begin{pmatrix}w_3\\w_1\\w_2\\w_4\end{pmatrix}\longrightarrow\begin{pmatrix}w_2\\w_3\\w_1\\w_4\end{pmatrix} \tag{29}$$

which is again cyclic except in this detail:

$$w_4=\hat{n}_1^2+3\hat{n}_2^2 \quad \text{transforms by invariance}$$

Nor is this last equation a surprise; it follows from

$$\begin{aligned}(n_1+n_2+n_3)^2&=0^2\\ &=\underbrace{(n_1^2+n_2^2+n_3^2)}_{2[\hat{n}_1^2+3\hat{n}_2^2]}+2\underbrace{(n_1n_2+n_2n_3+n_3n_1)}_{-\,w_4}\end{aligned}$$

At (19) we encounter the display

$$G=\begin{cases}\text{sum of eigenfunctions, each with the \textit{same eigenvalue},}\\ \text{which \textit{collectively} satisfy the imposed boundary conditions}\end{cases}$$

while at (20) we have

$$W=\begin{cases}\text{sum of eigenfunctions with \textit{distinct} eigenvalues,}\\ \text{which collectively satisfy the imposed boundary conditions}\end{cases}$$

In exploratory work (not reported here) I have been tripping over implications of the latter fact, and am led to look now therefore to properties of $W_4$, which is, as we have several times had occasion to notice, a "distinguished member" of the population $\{W_1,W_2,W_3,W_4\}$. We have

$$\begin{aligned}\Big\{\Big(\frac{\partial}{\partial\xi_1}\Big)^2+3\Big(\frac{\partial}{\partial\xi_2}\Big)^2\Big\}W_4 &= -12[n_1^2+n_2^2+n_3^2-n_1n_2-n_2n_3-n_3-n_1]W_4\\ &= -12[\tfrac{3}{2}(n_1^2+n_2^2+n_3^2)]W_4 \quad \text{when } n_1+n_2+n_3=0\end{aligned}$$

Let us, in structural imitation of (26), write

$$\begin{aligned} n_1 &= \qquad 2m_2 \\ n_2 &= +m_1 - m_2 \\ n_3 &= -m_1 - m_2 \end{aligned}$$

We then have

$$\begin{aligned} \Big\{\Big(\frac{\partial}{\partial\xi_1}\Big)^2 + 3\Big(\frac{\partial}{\partial\xi_2}\Big)^2\Big\}W_4 &= -36[m_1^2 + 3m_2^2]W_4 \\ &= -4[\hat n_1^2 + 3\hat n_2^2]W_4 \quad \text{if} \quad m_1 \equiv \tfrac{1}{3}\hat n_1 \text{ and } m_2 \equiv \tfrac{1}{3}\hat n_2 \end{aligned}$$

with

$$\begin{aligned} W_4 &= W_4\big(\tfrac{2\hat n_2}{3}, \tfrac{+\hat n_1 - \hat n_2}{3}, \tfrac{-\hat n_1 - \hat n_2}{3}\big) \\ &= -\sin[2\hat n_1\xi_1 - 2\hat n_2\xi_2] \end{aligned}$$

and are led by this result—taken in conjunction with (27)—to the observation that[12]

$$\begin{aligned} W_4\big(\tfrac{2\hat n_2}{3}, \tfrac{+\hat n_1 - \hat n_2}{3}, \tfrac{-\hat n_1 - \hat n_2}{3}\big) &= -\sin[2\hat n_1\xi_1 - 2\hat n_2\xi_2] \\ W_4\big(\tfrac{-\hat n_1 - \hat n_2}{3}, \tfrac{2\hat n_2}{3}, \tfrac{+\hat n_1 - \hat n_2}{3}\big) &= +\sin[(\hat n_1 - 3\hat n_2)\xi_1 - (\hat n_1 + \hat n_2)\xi_2] \\ W_4\big(\tfrac{+\hat n_1 - \hat n_2}{3}, \tfrac{-\hat n_1 - \hat n_2}{3}, \tfrac{2\hat n_2}{3}\big) &= +\sin[(\hat n_1 + 3\hat n_2)\xi_1 + (\hat n_1 - \hat n_2)\xi_2] \\[6pt] W_4\big(\tfrac{2\hat n_2}{3}, \tfrac{-\hat n_1 - \hat n_2}{3}, \tfrac{+\hat n_1 - \hat n_2}{3}\big) &= +\sin[2\hat n_1\xi_1 + 2\hat n_2\xi_2] \\ W_4\big(\tfrac{-\hat n_1 - \hat n_2}{3}, \tfrac{+\hat n_1 - \hat n_2}{3}, \tfrac{2\hat n_2}{3}\big) &= -\sin[(\hat n_1 - 3\hat n_2)\xi_1 + (\hat n_1 + \hat n_2)\xi_2] \\ W_4\big(\tfrac{+\hat n_1 - \hat n_2}{3}, \tfrac{2\hat n_2}{3}, \tfrac{-\hat n_1 - \hat n_2}{3}\big) &= -\sin[(\hat n_1 + 3\hat n_2)\xi_1 - (\hat n_1 - \hat n_2)\xi_2] \end{aligned}$$

The functions on the right are precisely the six functions which (see again (19.2)) when added together give $G(\hat n_1, \hat n_2)$.

What, in the same vein, can one say of the functions $W_1$, $W_2$ and $W_3$ that collaborate with $W_4$ in the assembly of $W$? Looking to the definitions (28) and

[12] Here I omit six equations on these grounds made evident by (20.2):

$$W_i(-n_1, -n_2, -n_3) = -W_i(n_1, n_2, n_3) \quad : \quad i = 1, 2, 3, 4$$

This "reflection principle"—elementary though it is—will soon acquire some importance.

(20.2) of the functions $w_i(n_1, n_2, n_3)$ and $W_i(n_1, n_2, n_3)$, we notice that

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \to \begin{pmatrix} -n_1 \\ +n_2 \\ +n_3 \end{pmatrix} \quad \text{induces} \quad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \to \begin{pmatrix} w_2 \\ w_1 \\ w_4 \\ w_3 \end{pmatrix} \quad \& \quad \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix} \to \begin{pmatrix} -W_2 \\ -W_1 \\ -W_4 \\ -W_3 \end{pmatrix}$$

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \to \begin{pmatrix} +n_1 \\ -n_2 \\ +n_3 \end{pmatrix} \quad \text{induces} \quad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \to \begin{pmatrix} w_4 \\ w_3 \\ w_2 \\ w_1 \end{pmatrix} \quad \& \quad \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix} \to \begin{pmatrix} -W_4 \\ -W_3 \\ -W_2 \\ -W_1 \end{pmatrix}$$

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \to \begin{pmatrix} +n_1 \\ +n_2 \\ -n_3 \end{pmatrix} \quad \text{induces} \quad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \to \begin{pmatrix} w_3 \\ w_4 \\ w_1 \\ w_2 \end{pmatrix} \quad \& \quad \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix} \to \begin{pmatrix} -W_3 \\ -W_4 \\ -W_1 \\ -W_2 \end{pmatrix}$$

The preceding equations describe consequences of what are in effect *improper rotations in 3-dimensional* $\boldsymbol{n}$*-space*. A clearer sense of what is going on can be obtained if (by compounding the preceding transformations) one looks to the associated *proper* rotations; the induced transformations are then *permutational* (no intrusive signs):

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \to \begin{pmatrix} +n_1 \\ -n_2 \\ -n_3 \end{pmatrix} \quad \text{induces} \quad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \to \begin{pmatrix} w_2 \\ w_1 \\ w_4 \\ w_3 \end{pmatrix} \quad \& \quad \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix} \to \begin{pmatrix} W_2 \\ W_1 \\ W_4 \\ W_3 \end{pmatrix}$$

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \to \begin{pmatrix} -n_1 \\ +n_2 \\ -n_3 \end{pmatrix} \quad \text{induces} \quad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \to \begin{pmatrix} w_4 \\ w_3 \\ w_2 \\ w_1 \end{pmatrix} \quad \& \quad \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix} \to \begin{pmatrix} W_4 \\ W_3 \\ W_2 \\ W_1 \end{pmatrix}$$

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \to \begin{pmatrix} -n_1 \\ -n_2 \\ +n_3 \end{pmatrix} \quad \text{induces} \quad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \to \begin{pmatrix} w_3 \\ w_4 \\ w_1 \\ w_2 \end{pmatrix} \quad \& \quad \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix} \to \begin{pmatrix} W_3 \\ W_4 \\ W_1 \\ W_2 \end{pmatrix}$$

That these induced transformation can be interpreted as having to do with a *subgroup of the tetrahedral group* is demonstrated in Figure 22.[13] The figure serves at the same time to cast new light on other matters as well; it becomes natural, for example, to interpret (29) as having to do with certain other rotations in $\boldsymbol{n}$-space—a different subgroup of the tetrahedral group. It is to lend substance to that remark that I now digress:

To describe (relative to a right-handed frame in 3-space) a rotation through angle $\varphi$ (in the right-handed sense) about the unit vector $\boldsymbol{\lambda}$ one writes

$$\boldsymbol{x} \to \boldsymbol{x}' = \mathbb{R}(\boldsymbol{\lambda}, \varphi)\boldsymbol{x}$$

[13] The idea embodied in the figure, we note in passing, springs quite naturally from the "method of sections," but might have escaped our notice had we had at our disposal only the methods of §3.

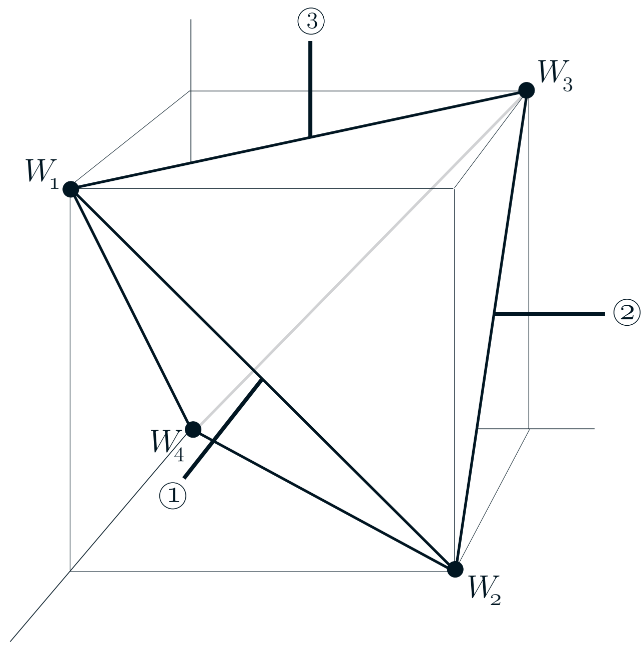


FIGURE 22: *The functions* $W_1, W_2, W_3$ *and* $W_4$ *have been associated with the vertices of a tetrahedron inscribed within the cube familiar from Figure 21 (except that the present figure lives not in* $\boldsymbol{x}$*-space but in* $\boldsymbol{n}$*-space). The boldface coordinate system has its origin at the shared center of the cube and tetrahedron ("center of mass" of the construction).* 180° *rotations about the* ①, ② *and* ③ *axes give rise to the transformations described at the bottom of the preceding page.*

with[14]

$$\mathbb{R}(\boldsymbol{\lambda}, \varphi) = e^{\varphi\mathbb{A}} = \mathbb{P} + (\cos\varphi \cdot \mathbb{I} + \sin\varphi \cdot \mathbb{A})(\mathbb{I} - \mathbb{P})$$

where the antisymmetric matrix $\mathbb{A}$ inherits its structure from $\boldsymbol{\lambda}$

$$\mathbb{A} \equiv \begin{pmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{pmatrix}$$

where

$$\mathbb{P} \equiv \mathbb{A}^2 + \mathbb{I} = \begin{pmatrix} \lambda_1\lambda_1 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_2\lambda_1 & \lambda_2\lambda_2 & \lambda_2\lambda_3 \\ \lambda_3\lambda_1 & \lambda_3\lambda_2 & \lambda_3\lambda_3 \end{pmatrix}$$

[14] See, for example, CLASSICAL DYNAMICS (1964-1965), Chapter I, p. 84.

projects onto the ray defined by $\boldsymbol{\lambda}$, and where $(\mathbb{I}-\mathbb{P})$ projects onto the plane $\perp$ to $\boldsymbol{\lambda}$. Suppose, by way of illustration (and to test the accuracy of our signs), we had

$$\begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Then

$$\mathbb{A}=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}\quad\text{and}\quad\mathbb{P}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

give

$$\mathbb{R}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}+\cos\varphi\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}+\sin\varphi\cdot\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}$$

which sends

$$\boldsymbol{x}\equiv\begin{pmatrix}x\\y\\z\end{pmatrix}\longrightarrow\boldsymbol{x}'=\begin{pmatrix}x\cos\varphi-y\sin\varphi\\y\cos\varphi+x\sin\varphi\\z\end{pmatrix}$$
$$=\text{familiar result of rotation around } z\text{-axis}$$

Now—with an eye to the implications of Figure 22—let

$$\boldsymbol{\lambda}=\tfrac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\quad\text{and}\quad\varphi=180^\circ,\ \text{so}\ \begin{cases}\cos\varphi=-\frac{1}{2}\\[2ex]\sin\varphi=+\frac{\sqrt{3}}{2}\end{cases}$$

Then

$$\mathbb{A}=\tfrac{1}{\sqrt{3}}\begin{pmatrix}0&-1&1\\1&0&-1\\-1&1&0\end{pmatrix}\quad\text{and}\quad\mathbb{P}=\begin{pmatrix}\frac13&\frac13&\frac13\\\frac13&\frac13&\frac13\\\frac13&\frac13&\frac13\end{pmatrix}$$

give

$$\mathbb{R}=\begin{pmatrix}\frac13&\frac13&\frac13\\\frac13&\frac13&\frac13\\\frac13&\frac13&\frac13\end{pmatrix}-\tfrac12\cdot\begin{pmatrix}\frac23&-\frac13&-\frac13\\-\frac13&\frac23&-\frac13\\-\frac13&-\frac13&\frac23\end{pmatrix}+\tfrac{\sqrt{3}}{2}\cdot\tfrac{1}{\sqrt{3}}\begin{pmatrix}0&-1&1\\1&0&-1\\-1&1&0\end{pmatrix}$$

$$=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\quad\text{which entails}\quad\mathbb{R}^2=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\ \text{and}\ \mathbb{R}^3=\mathbb{I}$$

This result is so strikingly simple that it could well have been written down directly, without calculation; its action (considered to take place in $\boldsymbol{n}$-space) is permutational

$$\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}\xrightarrow[\mathbb{R}]{}\begin{pmatrix}n_3\\n_1\\n_2\end{pmatrix}\xrightarrow[\mathbb{R}]{}\begin{pmatrix}n_2\\n_3\\n_1\end{pmatrix}\xrightarrow[\mathbb{R}]{}\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}\ \text{back again}$$

and in fact precisely reproduces (27.1), which we found to be induced by (24.1), i.e., by $\hat{\boldsymbol{n}}\to\mathbb{A}\hat{\boldsymbol{n}}\to\mathbb{A}^2\hat{\boldsymbol{n}}$.

**Memo to myself**: I must take temporary leave of this project to write on couple of other topics. I have yet to extract $G$ from $W$. Maybe I should look more closely to $W_1, W_2, W_3$ to see whether they, after summation, also happen to satisfy the boundary conditions (as $W_4$ turned out to do). Am in position to exploit representation theory of the tetrahedral group (treated by Lomaont in his *Applications of Finite Groups*), should that turn out to be useful. Still looks like I will—owing to the $\frac{1}{3}$ factors—have limited success at best. And I have no idea yet as to how I will get the $F$ functions. Must do all with such generalizable clarity that I will know how to treat the hexagonal case. Have also to address orthonormality. Can that be imported from the known orthonormal completeness of the cube functions?