JACOBI'S THETA TRANSFORMATION, & MEHLER'S FORMULA:

Their interrelation, and their role in the quantum theory of angular momentum

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Introduction. Schwinger has used to powerful effect the fact that a formal replica of the quantum theory is angular momentum is embedded within the quantum theory of an isotropic 2-dimensional oscillator. I have recently been motivated (by the use made by Penrose of ideas injected into angular momentum theory by Majorana) to reexamine Schwinger's work,[1] and the present discussion is a spin-off from that activity.

Oscillator theory provides a valuable laboratory in which to examine (amongst much else) the relationship between the "standard" formulation of quantum mechanics Feynman's "sum-over-paths" formulation. That topic assigns central importance to an identity

$$\sum_{n=0}^{\infty} \tfrac{1}{n!}\big(\tfrac{1}{2}\tau\big)^n H_n(x)H_n(y) = \frac{1}{\sqrt{1-\tau^2}}\exp\left\{\frac{2xy\tau-(x^2+y^2)\tau^2}{1-\tau^2}\right\} \tag{1.1}$$

which is encountered only occasionally in handbooks,[2] with attribution to one F. G. Mehler...concerning whom I have been able, with much searching, to discover almost nothing.[3] At that same formal interface the quantum theory

[1] See "Comments concerning Julian Schwinger's 'On angular momentum,' " (October 2000) and "Toy quantum field theory: populations of indistinguishable finite-state systems," (Physics Seminar Notes: 1 November 2000).

[2] I quote above from page 194 in A. Erdélyi, *Higher Transendental Functions, Volume II* (1953).

[3] Ferdinand Gustav Mehler (1835–1895) pursued his obscure academic career in Breslau, Berlin and Danzig. He was active mainly during the 1860's and 1870's, when his papers appeared with fair regularity in Crelle's Journal and Mathematische Annalen. "Mehler's formula" made its first appearance on

of a *free* particle is found to hinge on the innocent-looking identity

$$\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\tau p^2}e^{ipx}dp = \tfrac{1}{\sqrt{\tau}}e^{-\frac{1}{2\tau}x^2} \tag{1.2}$$

while the problem of a *constrained* free particle (particle-in-a-box) leads to an identity which looks less innocent

$$\vartheta_3(z,\tau) = \sqrt{i/\tau}e^{z^2/i\pi\tau}\cdot\vartheta_3(\tfrac{z}{\tau},-\tfrac{1}{\tau}) \tag{1.3}$$

but concerning which Richard Bellman has this to say: "[the preceding] identity has amazing ramifications in [many branches of pure/applied] mathematics; in fact, it is not easy to find another identity of comparable significance."[4] I will, in a moment, review *how it comes about* that the identities (1) acquire importance in connection with the quantum mechanics of some commonly encountered and closely related physical systems. But here I wish only to draw attention to the fact that the identities (1) appear—if your squint—to be *structurally* similar in this respect: each presents τ "upstairs on the left, downstairs on the right." In this regard (1) resembles a population of "transformation formulæ" encountered at various points within the theory of higher functions—

$$F(a,b;c;z) = (1-z)^{-a}F(a,c-b;c;\tfrac{z}{z-1})$$

provides an example, selected almost at random[5]—so the question arises: Does there exist a sense in which all such statements are instances of the *same* over-arching abstract statement? Can such statements be unified/generalized?

One "soft" generalization is much less problematic: each of the identities (1) possesses a natural *multivariate companion*. In particular, Mehler's identity (1.1) possess a bivariate companion which stands central to the quantum theory of the aforementioned isotropic 2-dimensional oscillator. And must, therefore, lurk somewhere within the quantum theory of angular momentum. Where?

We have now before us a little tangle of interrelated issues. My objective will be to sort them out.

(continued from the preceding page) p. 174 of "Reihenentwicklungen nach Laplaceschen Functionen hoher Ordnung," J. reine angew. Math. (Crelle) **66**, 161 (1866), when the Hermite polynomials themselves were only two years old. Today Mehler is best known for his contributions to the theory of Gaussian quadrature. Some of his function-theoretic work seems to have been motivated by an interest in electrodynamics.

[4] *A Brief Introduction to Theta Functions* (1961), page 4.

[5] See page 8 in W. Magnum & F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics* (1954); also §**2.9** in Volume I of Erdélyi[2] and §**60:5** in J. Spanier & K. B. Oldham, *An Atlas of Functions* (1987).

Green's function of an unconstrained free particle. We have

$$\mathsf{H}|\psi) = i\hbar\partial_t|\psi) \quad \text{with} \quad \mathsf{H} = \tfrac{1}{2m}\mathsf{p}^2$$

giving

$$|\psi)_t = \mathsf{U}(t)|\psi)_0 \quad \text{with} \quad \mathsf{U}(t) = \exp\Big\{-\tfrac{1}{2m}\mathsf{p}^2 t\Big\}$$

Therefore

$$(x|\psi)_t = \int (x|\mathsf{U}(t)|y)dy(y|\psi)_0$$

which is usually written

$$\psi(x,t) = \int G(x,t;y,0)\psi(y,0)\,dy$$

The Green's function can be developed

$$G(x,t;y,0) = \int (x|\mathsf{U}(t)|p)dp(p|y) = \int e^{-\frac{i}{\hbar}\frac{1}{2m}p^2t}(x|p)dp(p|y) \tag{2.1}$$

But $\mathsf{p}|p) = p|p)$ entails $\frac{\hbar}{i}\partial_x(x|p) = p(x|p)$ which gives

$$(x|p) = \tfrac{1}{\sqrt{h}}e^{\frac{i}{\hbar}px}$$

These momentum eigenfunctions are normalized in the formal sense that

$$\int (p|x)dx(x|q) = \delta(p-q)$$

and complete in the sense that (to say the same thing another way)

$$\int (x|p)dp(p|y) = \delta(x-y)$$

So we have

$$G(x,t;y,0) = \sqrt{\frac{m}{iht}}\exp\Big\{\frac{i}{\hbar}\frac{m}{2}\frac{(x-y)^2}{t}\Big\} \tag{2.2}$$

as a formal instance of (1.2). Mathematically, (2) is a manifestation simply of the familiar fact that

$$\text{thin/fat Gaussian} \xrightarrow[\text{Fourier transformation}]{} \text{fat/thin Gaussian}$$

but from a physical point of view it is rather more interesting: quick calculation establishes that $G(x,t;y,0)$ is a solution of the Schrödinger equation

$$\Big\{-\tfrac{\hbar^2}{2m}\partial_x^2 - i\hbar\partial_t\Big\}G(x,t;y,0) = 0$$

and that within the population of such solutions it is distinguished by the fact —this follows most quickly from (2.1)—that initially

$$\lim_{t\downarrow 0} G(x,t;y,0) = \delta(x-y)$$

Moreover, (2.2) can be written

$$G(x,t;y,0) = \sqrt{\tfrac{i}{\hbar}\tfrac{\partial^2 S}{\partial x \partial y}}\;\exp\Big\{\tfrac{i}{\hbar}S(x,t;y,0)\Big\} \tag{3}$$

$$S(x,t;y,0) \equiv \frac{m}{2}\frac{(x-y)^2}{t}$$

where $S(x,t;y,0)$ is the dynamical action $S = \int_0^t L d\tau$ associated with the free particle trajectory

$$x(\tau) = y + \frac{x-y}{t}\tau \quad : \quad (x,t) \longleftarrow (y,0)$$

and, as such, constitues the "fundamental solution" of the Hamilton-Jacobi equation

$$\tfrac{1}{2m}S_x^2 + S_t = 0$$

We touch here on ideas that lie close to the roots of the Feynman formalism.

Green's function of a free particle constrained to move on a ring.[6] The mass m is constrained to move now on a circle of radius r. Accordingly: to the theory just sketched we bring the "ring periodicity condition"

$$(x + 2\pi r|p) = (x|p) \sim e^{\frac{i}{\hbar}xp}$$

and find that p has become discrete:

$$p \;\longmapsto\; p_n = n\hbar/r \quad : \quad n = 0, \pm 1, \pm 2, \ldots$$

This fact is more neatly formulated as a quantization of angular momentum[7]

$$\ell = rp \;\longmapsto\; \ell_n = n\hbar$$

and entails quantization of energy:

$$E = \tfrac{1}{2m}p^2 = \tfrac{1}{2mr^2}\ell^2 \;\longmapsto\; E_n = \mathcal{E}n^2$$

$$\mathcal{E} \equiv \tfrac{\hbar^2}{2mr^2}$$

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[6] For a more elaborate discussion of this topic see my FEYNMAN FORMALISM FOR POLYGONAL DOMAINS (1971–1976), pages 227–231.

[7] Notice that free motion on a "twice-around ring" would entail

$$(x + 4\pi r|p) = (x|p)$$

and give

$$\ell_n = n \cdot \tfrac{1}{2}\hbar$$

In place of the $\int$ at (2.1) we now encounter a $\sum$: the normalized (angular) momentum/energy eigenstates are

$$(x|p_n) = \tfrac{1}{\sqrt{2\pi r}} e^{in(x/r)}$$

so we have

$$\begin{aligned} G(x,t;y,0) &= \tfrac{1}{2\pi r}\sum_{-\infty}^{+\infty} e^{-\frac{i}{\hbar}\mathcal{E}n^2 t} e^{in(x-y)/r} \qquad (4.1)\\ &= \tfrac{1}{2\pi r}\Big\{1 + 2\sum_{1}^{\infty} e^{-\frac{i}{\hbar}\mathcal{E}n^2 t}\cos n(\varphi-\varphi_0)\Big\} \end{aligned}$$

with $\varphi \equiv x/r$, $\varphi_0 \equiv y/r$. The theta function $\vartheta_3(z,\tau)$ is defined[8]

$$\begin{aligned} \vartheta_3(z,\tau) &= 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz \quad \text{with} \quad q \equiv e^{i\pi\tau}\\ &= \sum_{-\infty}^{+\infty} e^{i(\pi\tau n^2 - 2nz)} \end{aligned}$$

and by appeal to "Jacobi's theta transformation" (1.3) becomes

$$\begin{aligned} &= \sqrt{i/\tau}\, e^{z^2/i\pi\tau}\sum_{-\infty}^{+\infty}\exp\Big\{-i\Big(\frac{\pi n^2}{\tau} + \frac{2nz}{\tau}\Big)\Big\}\\ &= \sqrt{\tfrac{i}{\tau}}\sum_{-\infty}^{+\infty}\exp\Big\{-\frac{i\pi}{\tau}\Big(\frac{z}{\pi}+n\Big)^2\Big\} \end{aligned}$$

In this notation the Green's function becomes

$$\begin{aligned} G(x,t;y,0) &= \tfrac{1}{2\pi r}\vartheta\big(\tfrac{\varphi-\varphi_0}{2}, -\tfrac{\mathcal{E}t}{\pi\hbar}\big)\\ &= \tfrac{1}{2\pi r}\sqrt{\tfrac{\pi\hbar}{i\mathcal{E}t}}\sum_{-\infty}^{+\infty}\exp\Big\{i\pi\frac{\pi\hbar}{\mathcal{E}t}\Big(\frac{\varphi-\varphi_0+2\pi n}{2\pi}\Big)^2\Big\}\\ &= \sqrt{\tfrac{m}{iht}}\sum_{-\infty}^{+\infty}\exp\Big\{\frac{i}{\hbar}\frac{mr^2}{2t}(\varphi-\varphi_0+2\pi n)^2\Big\}\\ &= \sqrt{\tfrac{m}{iht}}\sum_{-\infty}^{+\infty}\exp\Big\{\frac{i}{\hbar}\frac{m}{2t}(x+n2\pi r-y)^2\Big\} \qquad (4.2) \end{aligned}$$

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[8] As the subscript suggests, $\vartheta_3(z,\tau)$ is the third member of a set (quartet) of closely interrelated functions, introduced in 1829 by the young C. G. Jacobi (1804–1851) as aids to the development of what are now called "Jacobian elliptic functions" (see Chapter 16 in Abramowitz & Stegun, *Handbook of Mathematical Functions* (1964)). One should be aware that the definitions employed by various authors differ in small details, one from another; I adhere here and in "2-dimensional 'particle-in-a-box' problems in quantum mechanics" (1997) to the conventions of Bellman and Abramowitz & Stegun.

Notice now that on a ring (which is to say: on any closed circuit of length $a = 2\pi r$) there are *multiple* paths $x \longleftarrow y$, and that they differ in length by multiples of a. And that

$$\begin{aligned} S_n(x,t;y,0) &\equiv \frac{m}{2t}(x + n2\pi r - y)^2 \\ &= \text{dynamical action of the path with winding number } n \end{aligned}$$

So, in an obvious shorthand, (4.2) becomes—compare (3)—

$$G(x,t;y,0) = \sum_{\text{paths}} \sqrt{\tfrac{i}{\hbar}\tfrac{\partial^2 S[\text{path}]}{\partial x \partial y}}\, \exp\Big\{\tfrac{i}{\hbar}S[\text{path}]\Big\} \tag{5}$$

and brings us yet closer to the essential spirit of the Feynman formalism.

The ring problem provides the simplest instance—and captures all the essential analytical features—of a broad class of "particle-in-a-box" problems, many of which I have discussed elsewhere,[8] but it supplies no direct insight into the origin of the "chaotic" features that lend interest to the much larger class of "stadium" problems[9]. . . though from a physical point of view the two classes of problems are hardly to be distinguished.

Green's function of a simple oscillator. Both classically & quantum mechanically, quadratic Hamiltonians

$$H(x,p) = ap^2 + 2bpx + cx^2$$

are distinguished from other, more general Hamiltonians by the fact that they give rise to *linear equations of motion*, and by the many special features that radiate from that circumstance. The free particle provides the simplest example. Another is provided by the oscillator

$$\begin{aligned} \mathsf{H} &= \tfrac{1}{2m}\mathsf{p}^2 + \tfrac{1}{2}m\omega^2\mathsf{x}^2 \\ &= \tfrac{1}{2}\hbar\omega\big\{\mathsf{q}^2 + \mathsf{y}^2\big\} \\ &= \hbar\omega\big\{\mathsf{a}^+\mathsf{a} + \tfrac{1}{2}\mathsf{I}\big\} \end{aligned}$$

where

$$\mathsf{y} \equiv \sqrt{\tfrac{m\omega^2}{\hbar\omega}}\,\mathsf{x} \quad \text{and} \quad \mathsf{q} \equiv \sqrt{\tfrac{1}{m\cdot\hbar\omega}}\,\mathsf{p}$$

are dimensionless self-adjoint operators and where and the operators

$$\mathsf{a} \equiv \tfrac{1}{\sqrt{2}}\big\{\mathsf{y} + i\mathsf{q}\big\} \quad ; \quad \mathsf{a}^+ \equiv \tfrac{1}{\sqrt{2}}\big\{\mathsf{y} - i\mathsf{q}\big\}$$

are *not* self-adjoint, and therefore do not represent "observables:" they are the familiar step-down/up "ladder operators," which were first introduced into

[9] L. A. Buminovich (1974). See M. C. Gurzwiller, *Chaos in Classical and Quantum Physics* (1990): Figures 35, 44 & 45 and accompanying text. See also O. F. de Alcantara Bonfim, J. Florencio & F. C. Sá Barreto, "Chaotic dynamics in billiards using Bohm's quantum mechanics," Phys. Rev. E **58**, R2693 (1998).

oscillator theory by Dirac[10] and are encountered today in most elementary texts.[11] From $\big[\mathbf{x},\mathbf{p}\big] = i\hbar\mathbf{I}$ it follows that

$$\big[\mathbf{y},\mathbf{q}\big] = i\,\mathbf{I} \quad\text{and}\quad \big[\mathbf{a},\mathbf{a}^+\big] = \mathbf{I} \tag{6}$$

The oscillator Green's function can be described[12]

$$\begin{aligned} G_{\text{osc}}(x,t;y,0) = (x|\,&\mathbf{U}_{\text{osc}}(t)|y) \\ \mathbf{U}_{\text{osc}}(t) &= e^{-i\,\omega(\mathbf{a}^+\mathbf{a}+\frac{1}{2}\mathbf{I})\,t} \\ &= e^{-i\,\frac{1}{2}\omega\,t}\cdot e^{-i\,\omega(\mathbf{a}^+\mathbf{a})\,t} \end{aligned}$$

Schwinger has shown that one can, with sufficient cleverness,[13] use (6) to obtain

$$\begin{aligned} &= e^{-i\,\frac{1}{2}\omega\,t}\cdot {}_{\mathbf{a}^+}\Big[\exp\Big\{\big(e^{-i\omega t}-1\big)a^+a\Big\}\Big]_{\mathbf{a}} \\ &= e^{-i\,\frac{1}{2}\omega\,t}\cdot \sum_n e^{-in\,\omega\,t}\frac{(\mathbf{a}^+)^n}{\sqrt{n!}}|0)(0|\frac{(\mathbf{a})^n}{\sqrt{n!}} \\ &= \sum_0^\infty e^{-i\omega(n+\frac{1}{2})\,t}|n)(n| \end{aligned} \tag{7}$$

where $\mathbf{a}|0) = 0$ and $(0|0) = 1$ and where

$$|n) \equiv \tfrac{1}{\sqrt{n!}}(\mathbf{a}^+)^n|0)$$

Realize (6) by setting $\mathbf{y}\mapsto y\cdot$ and $\mathbf{q}\mapsto -i\frac{d}{dy}$. Then

$$\mathbf{a}\mapsto \tfrac{1}{\sqrt{2}}\big(y+\tfrac{d}{dy}\big) \qquad\text{and}\qquad \mathbf{a}^+\mapsto \tfrac{1}{\sqrt{2}}\big(y-\tfrac{d}{dy}\big)$$

and we have

$$\begin{aligned} (y|n) &= \big(-\tfrac{1}{\sqrt{2}}\big)^n\tfrac{1}{\sqrt{n!}}\big(\tfrac{d}{dy}-y\big)^n(y|0) \\ &= \big(-\tfrac{1}{\sqrt{2}}\big)^n\tfrac{1}{\sqrt{n!}}\,e^{+\frac{1}{2}y^2}\big(\tfrac{d}{dy}\big)^n e^{-\frac{1}{2}y^2}(y|0) \end{aligned}$$

The condition $\mathbf{a}|0) = 0$ has become $\big(y+\frac{d}{dy}\big)(y|0) = 0$ which gives

$$(y|0) = (\text{complex constant})\cdot e^{-\frac{1}{2}y^2}$$

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[10] *Principles of Quantum Mechanics* (2nd edition 1935), §34.

[11] See, for example, David Griffiths' *Introduction to Quantum Mechanics* (1994), §2.3.1.

[12] Be careful: y wears now two distinct hats.

[13] For the full details see "An operator ordering technique with quantum mechanical applications" in COLLECTED SEMINARS 1963–1970; also page 32 of an essay already cited.[1]

The normalization condition (since $dx = \sqrt{\frac{\hbar}{m\omega}}\,dy$) requires that we set

$$\int (0|y)dy(y|0) = |\text{complex constant}|^2\sqrt{\pi} = \sqrt{\tfrac{m\omega}{\hbar}}$$

which entails

$$(\text{complex constant}) = \left(\tfrac{2m\omega}{h}\right)^{\frac{1}{4}} e^{i\,(\text{arbitrary phase})}$$

and (if we abandon the uninteresting phase factor) gives

$$(y|0) = \left(\tfrac{2m\omega}{h}\right)^{\frac{1}{4}}\left(\tfrac{1}{\sqrt{2}}\right)^n \tfrac{1}{\sqrt{n!}}\, e^{-\frac{1}{2}y^2}\cdot e^{+y^2}\left(-\tfrac{d}{dy}\right)^n e^{-y^2}$$

But

$$H_n(y) \equiv e^{+y^2}\left(-\tfrac{d}{dy}\right)^n e^{-y^2}$$

is precisely Rodrigues' construction of the Hermite polynomials.[14] Thus do we recover this familiar description

$$\psi_n(x) = \left(\tfrac{2m\omega}{h}\right)^{\frac{1}{4}} \tfrac{1}{\sqrt{2^n n!}}\, e^{-\frac{1}{2}(m\omega/\hbar)x^2} H_n\left(\sqrt{\tfrac{m\omega}{\hbar}}\,x\right) \tag{8}$$

of the oscillator eigenfunctions, the orthonormality of which

$$\int \psi_m(x)\psi_n(x)\,dx = \delta_{mn}$$

is most readily established by a generating function technique which I have described elsewhere[13] and will not repeat. Returning with this information to (7) we have

$$\begin{aligned} G_{\text{osc}}(x,t;y,0) &= \sum_0^\infty e^{-i\omega(n+\frac{1}{2})\,t}\,\psi_n(x)\psi_n(y) \qquad (9.1)\\ &= \left(\tfrac{2m\omega}{h}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(m\omega/\hbar)(x^2+y^2)} e^{-i\frac{1}{2}\omega t}\\ &\qquad\cdot\sum_0^\infty \tfrac{1}{n!}\left(\tfrac{1}{2}e^{-i\omega t}\right)^n H_n\left(\sqrt{\tfrac{m\omega}{\hbar}}\,x\right) H_n\left(\sqrt{\tfrac{m\omega}{\hbar}}\,y\right) \end{aligned}$$

Mehler's formula (1.1) now supplies

$$\begin{aligned} &= \left(\tfrac{2m\omega}{h}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(m\omega/\hbar)(x^2+y^2)} e^{-i\frac{1}{2}\omega t}\\ &\qquad\cdot\frac{1}{\sqrt{1-\tau^2}}\exp\left\{\tfrac{m\omega}{\hbar}\left[\frac{2xy\tau-(x^2+y^2)\tau^2}{1-\tau^2}\right]\right\}\\ &= \left(\tfrac{2m\omega}{h}\right)^{\frac{1}{2}}\sqrt{\frac{\tau}{1-\tau^2}}\exp\left\{\tfrac{m\omega}{\hbar}\left[2xy\frac{\tau}{1-\tau^2}-(x^2+y^2)\left(\tfrac{1}{2}+\frac{\tau^2}{1-\tau^2}\right)\right]\right\} \end{aligned}$$

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[14] See Spanier & Oldham,[5] page 219.

with $\tau = e^{-i\omega t}$. Straightforward simplifications give

$$G_{\rm osc}(x,t;y,0) = \sqrt{\frac{m\,\omega}{ih\sin\omega\,t}}\;\exp\Big\{\tfrac{i}{\hbar}\tfrac{m\,\omega}{2\sin\omega\,t}\big[(x^2+y^2)\cos\omega\,t - 2xy\big]\Big\} \qquad (9.2)$$

$$= \sqrt{\tfrac{i}{h}\tfrac{\partial^2 S_{\rm osc}}{\partial x\partial y}}\;\exp\Big\{\tfrac{i}{\hbar}S_{\rm osc}(x,t;y,0)\Big\}$$

We note the persistence of (3/5), and verify by computation that $G_{\rm osc}(x,t;y,0)$ satisfies the Schrödinger equation

$$\Big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{\partial}{\partial x}\big)^2 + \tfrac{1}{2}m\omega^2x^2 - i\hbar\tfrac{\partial}{\partial t}\Big\}G = 0$$

while $S_{\rm osc}(x,t;y,0)$ satisfies the associated Hamilton-Jacobi equation

$$\tfrac{1}{2m}\big(\tfrac{\partial S}{\partial x}\big)^2 + \tfrac{1}{2}m\omega^2x^2 S + \tfrac{\partial S}{\partial t} = 0$$

And, moreover, that $S_{\rm osc}(x,t;y,0)$ is the dynamical action function for the harmonic oscillator—the result of inserting the dynamical trajectory

$$x(\tau) = y\cos\omega\tau - \frac{y\cos\omega\,t - x}{\sin\omega\,t}\sin\omega\tau \quad : \quad (x,t) \longleftarrow (y,0)$$

into $S = \int_0^t \big\{\tfrac{1}{2}m\dot{x}^2(\tau) + \tfrac{1}{2}m\omega^2x^2(\tau)\big\}d\tau$.[15]

An argument that proceeded *directly* (9.1) to (9.2) or *vice versa*—without appeal to (1.1)—would amount, in effect, to a *proof* of Mehler's formula. Such an argument can, in fact, be constructed. The idea is to write

$$G_{\rm osc}(x,t;y,0) = \int (x|\,\mathbf{U}_{\rm osc}(t)|p)dp(p|y)$$

Then[16] to use $\big[\,\mathbf{x},\mathbf{p}\big] = i\hbar\mathbf{I}$ to bring $\mathbf{U}_{\rm osc}(t) = e^{-\frac{i}{\hbar}\left\{\frac{1}{2m}\mathbf{p}^2+\frac{1}{2}m\,\omega^2\mathbf{x}^2\right\}t}$ to $\mathbf{xp}$-ordered form

$$e^{-\frac{i}{\hbar}\left\{\frac{1}{2m}\mathbf{p}^2+\frac{1}{2}m\,\omega^2\mathbf{x}^2\right\}t} = {}_{\mathbf{x}}\Big[U(x,p;t)\Big]_{\mathbf{p}}$$

$$U(x,p;t) = \sqrt{\sec\omega t}\cdot e^{-\frac{i}{\hbar}\frac{m\omega}{2}\tan\omega t\cdot x^2}\cdot e^{-\frac{i}{\hbar}(1-\sec\omega t)\cdot xp}\cdot e^{-\frac{i}{\hbar}\frac{1}{2m\omega}\cdot p^2}$$

Notice that

$$\int (x|\,\mathbf{U}_{\rm osc}(t)|p)dp(p|y) = \int U(x,p;t)(x|p)dp(p|y)$$

$$= \frac{1}{h}\int U(x,p;t)e^{\frac{i}{\hbar}(x-y)p}dp$$

Perform the (Gaussian) integral, and after simplifications recover (9.2).

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[15] See QUANTUM MECHANICS (1967/68), Chapter 1, pages 21–23 for the details.

[16] This the tricky part, but yields readily to an elegant technique devised by N. H. McCoy ("Certain expansions in the algebra of quantum mechanics," Proc. Edinburgh Math. Soc. **2**, 205 (1931)), reinvented by Schwinger and described elsewhere by me in old notes[13] that give also all the details absent from the present discussion.

Multivariate generalizations. I was motivated by the 2-dimensional subject matter of some work already cited[6] to develop a theory of "theta functions of several variables." According to Bellman,[4] who devotes his final pages to a sketch of this subject, it was pioneered in the 1930's by E. Hecke and C. L. Siegel, who drew their motivation from algebraic number theory. That application is reflected in the name of the standard add-on package

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<< NumberTheory`SiegelTheta`
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that provides *Mathematica* with capability in this area; the accompanying text informs us that the multivariate function in question "was initially investigated by Riemann and Weierstrass, and [that] further studies were done by Frobenius and Poincaré."[17] In any event, I found the generalization process to be entirely straightforward: one defines

$$\vartheta_3(\boldsymbol{z}, \mathbb{M}) \equiv \sum_{\boldsymbol{n}} e^{i(\pi \boldsymbol{n}\cdot\mathbb{M}\boldsymbol{n} - 2\boldsymbol{n}\cdot\boldsymbol{z})} \tag{10}$$

and obtains this "generalized Jacobi transformation"

$$\vartheta_3(\boldsymbol{z}, \mathbb{M}) = \sqrt{\frac{i^N}{\det\mathbb{M}}}\, e^{-i\frac{1}{\pi}\boldsymbol{n}\cdot\mathbb{W}\boldsymbol{n}} \cdot \vartheta_3(\mathbb{W}\boldsymbol{z}, -\mathbb{W}) \tag{11}$$

Here $\mathbb{M}$ is an $N \times N$ symmetric matrix with a positive-definite imaginary part, $\mathbb{W} \equiv \mathbb{M}^{-1}$, $\boldsymbol{z}$ is a complex N-vector, and $\boldsymbol{n}$ is a N-vector with integer elements of either sign; the $\sum$ ranges over the entire lattice of such $\boldsymbol{n}$-vectors. In the case $N = 1$ we recover Jacobi's (1.3).

The identity (11) was obtained[8] as a corollary of the multivariate Gaussian integral

$$\int \cdots \int_{-\infty}^{+\infty} e^{-(\boldsymbol{y}\cdot\mathbb{A}\boldsymbol{y} + 2\boldsymbol{b}\cdot\boldsymbol{y})} dy_1 dy_2 \cdots dy_N = \sqrt{\frac{\pi^N}{\det\mathbb{A}}}\, e^{\boldsymbol{b}\cdot\mathbb{A}^{-1}\boldsymbol{b}}$$

which upon $\mathbb{A} \mapsto \frac{1}{2}\mathbb{M}$, $\boldsymbol{y} \mapsto \boldsymbol{p}$ and $\boldsymbol{b} \mapsto i\frac{1}{2}\boldsymbol{x}$ becomes the multivariate Fourier transformation formula

$$\int \cdots \int_{-\infty}^{+\infty} e^{i\boldsymbol{x}\cdot\boldsymbol{p}}\, e^{-\frac{1}{2}\boldsymbol{p}\cdot\mathbb{M}\boldsymbol{p}} dp_1 dp_2 \cdots dp_N = \sqrt{\frac{(2\pi)^N}{\det\mathbb{M}}}\, e^{-\frac{1}{2}\boldsymbol{x}\cdot\mathbb{M}^{-1}\boldsymbol{x}} \tag{12}$$

and gives back (1.2) in the case $N = 1$. Equation (12) provides the point of departure also for the following discussion of the multivariate generalization of Mehler's formula, which I have extracted from some old notes.[18]

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[17] *Mathematica 3.0: Standard Add-on Packages* (1996), p. 320.

[18] In 1971 I happened by accident upon a paper by one W. F. Kibble ("An extension of a theorem of Mehler's on Hermite polynomials," Proc. Camb. Phil. Soc. **41**, 12 (1945)), an account of which can be found in TRANSFORMATIONAL PHYSICS & PHYSICAL GEOMETRY (1971–1983), pp. 167–171. Kibble's paper derived from his dissertation ("Analytical properties of certain probability distributions," University of Edinburgh (1938)) and provides a valuable bibliography, but he looks upon Mehler's formula as having to do with properties of multivariate normal distributions. Tom W. B. Kibble has informed me today (14 November 2000) that W. F. Kibble was his father.

We begin with review of the proof of Mehler's formula (which in its original form is already bivariate). I sketched what is in effect one line of proof. Kibble cites a paper by G. N. Watson[19] that provides three alternative proofs, all intended to be simpler than one provided by N. Wiener.[20] Kibble finds it most convenient, for his own clever purposes, to proceed this way:[21] Suppose $\mathbb{M}$ is of the design

$$\mathbb{M} = \begin{pmatrix} 1 & \tau \\ \tau & 1 \end{pmatrix}$$

Then

$$\mathbb{W} \equiv \mathbb{M}^{-1} = \frac{1}{\det \mathbb{M}} \begin{pmatrix} 1 & -\tau \\ -\tau & 1 \end{pmatrix} \quad \text{with} \quad \det \mathbb{M} = 1 - \tau^2$$

and (12) becomes

$$\begin{aligned} &\tfrac{1}{\sqrt{1-\tau^2}}\, e^{-\frac{1}{2(1-\tau^2)}\left\{x_1^2 - 2m\,x_1 x_2 + x_2^2\right\}} \\ &\quad = \tfrac{1}{2\pi}\iint e^{i(x_1p_1+x_2p_2)} e^{-\frac{1}{2}\left\{p_1^2+2\tau\,p_1p_2+p_2^2\right\}}\,dp_1dp_2 \\ &\quad = \tfrac{1}{2\pi}\sum_{n=0}^{\infty}\tfrac{\tau^n}{n!}\Big\{\int e^{ix_1p_1}(-ip_1)^n e^{-\frac{1}{2}p_1^2}dp_1\Big\}\Big\{\int e^{ix_2p_2}(-ip_2)^n e^{-\frac{1}{2}p_2^2}dp_2\Big\} \end{aligned} \tag{13}$$

It proves convenient at this point to adopt the "monic" definition of the Hermite polynomials (which is the definition favored by Magnus & Oberhettinger):

$$He_n(x) \equiv e^{+\frac{1}{2}x^2}\big(-\tfrac{d}{dx}\big)^n e^{-\frac{1}{2}x^2} = \tfrac{1}{\sqrt{2^n}}H_n(x/\sqrt{2})$$

Tricks spelled out on pages 59–68 of Chapter 2 in QUANTUM MECHANICS[15] then supply the integral representations

$$\begin{aligned} He_n(x) = \tfrac{1}{\sqrt{2\pi}}\int (x+ip)^n e^{-\frac{1}{2}p^2}dp &= e^{\frac{1}{2}x^2}\cdot\tfrac{1}{\sqrt{2\pi}}\int e^{-ixp}(ip)^n e^{-\frac{1}{2}p^2}dp \\ &= e^{\frac{1}{2}x^2}\cdot\tfrac{1}{\sqrt{2\pi}}\int e^{ixp}(-ip)^n e^{-\frac{1}{2}p^2}dp \end{aligned}$$

Returning with this information to (13) we have

$$\begin{aligned} &\tfrac{1}{\sqrt{1-\tau^2}}\, e^{-\frac{1}{2(1-\tau^2)}\left\{x_1^2 - 2\tau\,x_1 x_2 + x_2^2\right\}} \\ &\qquad = e^{-\frac{1}{2}\left\{x_1^2+x_2^2\right\}}\cdot\sum_{n=0}^{\infty}\tfrac{\tau^n}{n!}He_n(x_1)He_n(x_2) \end{aligned} \tag{14}$$

---

[19] "Notes on generating functions of polynomials: (2) Hermite polynomials," J. London Math. Soc. **7**, 194 (1932). The companion paper is "Notes on generating functions of polynomials: (1) Laguerre polynomials," J. London Math. Soc. **7**, 187 (1932).

[20] *The Fourier Integral* (1933), pp57–62.

[21] What follows does not much resemble Kibble's own argument, which I have radically recast.

which is Mehler's formula (1.1) in the form preferred by Kibble. Notice that at $m = 0$ the preceding equation reduces to a triviality. Notice also that

$$\begin{aligned}
\sum_{n=0}^{\infty} \tfrac{\tau^n}{n!} He_n(x) &= \tfrac{1}{\sqrt{2\pi}}\int \sum_{n=0}^{\infty} \tfrac{\tau^n}{n!}(x+ip)^n e^{-\frac{1}{2}p^2}dp \\
&= \tfrac{1}{\sqrt{2\pi}}\int e^{\tau(x+ip)}e^{-\frac{1}{2}p^2}dp \\
&= e^{x\tau - \frac{1}{2}\tau^2}
\end{aligned}$$

produces the standard generating function of the (monic) Hermite polynomials. Equivalently

$$e^{-\frac{1}{2}(x-\tau)^2} = e^{-\frac{1}{2}x^2}\cdot \sum_{n=0}^{\infty}\tfrac{\tau^n}{n!}He_n(x)$$

It begins to become evident why Watson calls the expression on the left side of (14) a "generating function."

To illustrate Kibble's method for treating cases $N > 2$ I look to the case $N = 4$, since it is the case of special interest to me. Write

$$\mathbb{M} = \begin{pmatrix} 1 & \tau_{12} & \tau_{13} & \tau_{14} \\ \tau_{21} & 1 & \tau_{23} & \tau_{24} \\ \tau_{31} & \tau_{32} & 1 & \tau_{34} \\ \tau_{41} & \tau_{42} & \tau_{43} & 1 \end{pmatrix} = \mathbb{I} + \mathbb{T} \quad : \quad \tau_{ij} = \tau_{ji}$$

Then (12) gives

$$\begin{aligned}
\sqrt{\frac{1}{\det\mathbb{M}}}\, e^{-\frac{1}{2}\boldsymbol{x}\cdot\mathbb{M}^{-1}\boldsymbol{x}} &= \sqrt{\frac{1}{(2\pi)^4}}\iiiint e^{i\boldsymbol{x}\cdot\boldsymbol{p}}\, e^{-\frac{1}{2}\boldsymbol{p}\cdot(\mathbb{I}+\mathbb{T})\boldsymbol{p}}dp_1dp_2dp_3dp_4 \\
&= \iiiint e^{-\frac{1}{2}\boldsymbol{p}\cdot\mathbb{T}\boldsymbol{p}} \prod_{k=1}^{4} \tfrac{1}{\sqrt{2\pi}} e^{ix_kp_k - \frac{1}{2}p_k^2}dp_k
\end{aligned}$$

But

$$\begin{aligned}
(\tfrac{1}{2}\boldsymbol{p}\cdot\mathbb{T}\boldsymbol{p})^1 &= \tau_{12}p_1p_2 + \tau_{13}p_1p_3 + \tau_{14}p_1p_4 \\
&\qquad\qquad + \tau_{23}p_2p_3 + \tau_{24}p_2p_4 \\
&\qquad\qquad\qquad + \tau_{34}p_3p_4 \\
&\equiv \sum_{\{\nu\}} C^1_{\nu_1\nu_2\nu_3\nu_4} p_1^{\nu_1}p_2^{\nu_2}p_3^{\nu_3}p_4^{\nu_4} \quad : \quad \nu_1+\nu_2+\nu_3+\nu_4 = 2 \\
(\tfrac{1}{2}\boldsymbol{p}\cdot\mathbb{T}\boldsymbol{p})^2 &= \sum_{\{\nu\}} C^2_{\nu_1\nu_2\nu_3\nu_4} p_1^{\nu_1}p_2^{\nu_2}p_3^{\nu_3}p_4^{\nu_4} \quad : \quad \nu_1+\nu_2+\nu_3+\nu_4 = 4 \\
&\vdots
\end{aligned}$$

so we have

$$\sqrt{\frac{1}{\det\mathbb{M}}}\,e^{-\frac{1}{2}\boldsymbol{x}\cdot\mathbb{M}^{-1}\boldsymbol{x}} = e^{-\frac{1}{2}(x_1^2+x_2^2+x_3^2+x_4^2)} \cdot\sum_{n=0}^{\infty}\tfrac{1}{n!}\sum_{\{\nu\}}C^n_{\nu_1\nu_2\nu_3\nu_4}He_{\nu_1}(x_1)He_{\nu_2}(x_2)He_{\nu_3}(x_3)He_{\nu_4}(x_4) \tag{15}$$

which is Kibble's final result. The expression on the left is made somewhat awkward by the occurance of $\mathbb{M}^{-1}$, and the intricacy of the combinatorics that enter into the design of the coefficients $C^n_{\nu_1\nu_2\nu_3\nu_4}$, which are repositories ultimately of information written into the design of $\mathbb{T} = \|\tau_{ij}\|$. I will consider later what (15) has to say about the quantum mechanics of isotropic oscillators.

G. N. Watson[19] has obtained Mehler's formula as a corollary of a nameless but analogous property of the associated Laguerre polynomials

$$L_n^\alpha(x) \equiv \tfrac{1}{n!}e^{-x}x^\alpha\big(\tfrac{d}{dx}\big)^n e^x x^{n+\alpha}$$

He defines

$$\phi_n(x) \equiv \left[\frac{n!e^{-x}x^\alpha}{\Gamma(n+\alpha+1)}\right]^{\frac{1}{2}}L_n^\alpha(x) \quad : \quad \int_0^\infty \phi_m(x)\phi_n(x)\,dx = \delta_{mn}$$

then writes

$$K(x,y,t) \equiv \sum_{n=0}^\infty t^n\phi_n(x)\phi_n(y) \tag{16.1}$$

and—following more or less in the footsteps of Wiegert (1921), Hille (1926) and Hardy (1932)—establishes that

$$K(x,y,t) = \frac{t^{-\frac{1}{2}\alpha}}{1-t}\exp\Big\{-\tfrac{1}{2}(x+y)\tfrac{1+t}{1-t}\Big\}\cdot I_\alpha\Big(2\frac{\sqrt{xyt}}{1-t}\Big) \tag{16.2}$$

Watson's (16) appears as equation (15) in §**19.12** of Chapter 19 "Generating Functions" in A. Erdéyli, *Higher Transcendental Functions: Volume 3* (1955), where it is attributed to Hille/Hardy, with reference also to Myller-Lebedeff, Math. Ann. **64**, 388 (1907). I was led to Erdélyi by Gradshteyn & Ryzhik, who at **8.976.1** present this variant of (16):

$$\sum_{n=0}^{n} t^n n!\frac{L_n^\alpha(x)L_n^\alpha(y)}{\Gamma(n+\alpha+1)} = \frac{(xyt)^{-\frac{1}{2}\alpha}}{1-t}\exp\Big\{-t\frac{x+y}{1-t}\Big\}\cdot I_\alpha\Big(2\frac{\sqrt{xyt}}{1-t}\Big)$$

Watson uses

$$He_{2n}(x) = (-)^n 2^{2n}n!\,L_n^{-\frac{1}{2}}(x^2)$$
$$He_{2n+1}(x) = (-)^n 2^{2n+1}n!\,xL_n^{+\frac{1}{2}}(x^2)$$

and

$$
\begin{aligned}
I_{+\frac{1}{2}}(x) &= \sqrt{\frac{2x}{\pi}}\,\frac{\sinh x}{x}\\
I_{-\frac{1}{2}}(x) &= \sqrt{\frac{2x}{\pi}}\,\frac{\cosh x}{x}
\end{aligned}
$$

to extract (14) from (16). Watson remarks that "this proof of Mehler's formula can be regarded as elementary," but Kibble's argument seems to me to be much more direct and transparent (also more readily generalized). The point to be remarked is that Mehler's formula is not an isolated result, but (on evidence especially of Erdélyi's Chapter 19) one of a sizeable *population* of such formulæ, distinguished by its relative simplicity.

Kibble-Mehler and the isotropic oscillator. The elementary facts are these: the Schrödinger equation, in Cartesian coordinates, reads

$$
\Big\{-\tfrac{\hbar^2}{2m}\nabla^2 + \tfrac{1}{2}m\omega^2(x_1^2+x_2^2)\Big\}\psi(x_1,x_2) = E\,\psi(x_1,x_2)
$$

Separation is complete, and leads to eigenfunctions of the form

$$
\psi(x_1,x_2) = \psi_{n_1}(x_1)\cdot\psi_{n_2}(x_2)
$$

where $\psi_n(x)$ is given by (8). The associated eigenvalue is

$$
E_{n_1,n_2} = \hbar\omega(n_1+n_2+1)
$$

and is (n_1+n_2+1)-fold degenerate: write

$$
\begin{aligned}
n_1 &= \nu\\
n_2 &= n-\nu\\
E_{\nu,n-\nu} &= \hbar\omega(n+1) \equiv E_n
\end{aligned}
$$

with $\nu = 0,1,\ldots,n$ and $n = 0,1,2,\ldots.$ We expect to have

$$
\begin{aligned}
&G(x_1,x_2,t;y_1,y_2,0)\\
&\quad= \sum_{n=0}^\infty e^{-i\omega(n+1)t}\sum_{\nu=0}^n \psi_\nu(x_1)\psi_{n-\nu}(x_2)\psi_\nu(y_1)\psi_{n-\nu}(y_2)\\
&\quad= \big(\tfrac{2m\omega}{h}\big)e^{-\frac{1}{2}(m\omega/\hbar)(x_1^2+x_2^2+y_1^2+y_2^2)}\sum_{n=0}^\infty e^{-i\omega(n+1)t}\sum_{\nu=0}^n \tfrac{1}{\nu!(n-\nu)!}\\
&\qquad\qquad\cdot\tfrac{1}{\sqrt{2^\nu}}H_\nu\big(\sqrt{\tfrac{m\omega}{\hbar}}\,x_1\big)\tfrac{1}{\sqrt{2^\nu}}H_\nu\big(\sqrt{\tfrac{m\omega}{\hbar}}\,y_1\big)\\
&\qquad\qquad\cdot\tfrac{1}{\sqrt{2^{n-\nu}}}H_{n-\nu}\big(\sqrt{\tfrac{m\omega}{\hbar}}\,x_2\big)\tfrac{1}{\sqrt{2^{n-\nu}}}H_{n-\nu}\big(\sqrt{\tfrac{m\omega}{\hbar}}\,y_2\big)
\end{aligned}
$$

which can be written

$$
\begin{aligned}
G = \big(\tfrac{2m\omega}{h}\big)e^{-\frac{1}{4}(x_1^2+y_1^2+x_2^2+y_2^2)}\sum_{n=0}^\infty\big(e^{-i\theta}\big)^{n+1}&\\
\cdot\sum_{\nu=0}^n\tfrac{1}{\nu!(n-\nu)!}He_\nu(x_1)He_\nu(y_1)He_{n-\nu}(x_2)He_{n-\nu}(y_2)&
\end{aligned}
\tag{17.1}
$$

with $x \equiv \sqrt{\frac{2m\omega}{\hbar}}\,x$, $y \equiv \sqrt{\frac{2m\omega}{\hbar}}\,y$ and $\theta \equiv \omega t$. On the other hand, we are led by our experience at (3/5/9) to anticipate

$$G(x_1, x_2, t; y_1, y_2, 0) = (\text{factor}) \cdot e^{\frac{i}{\hbar}S(x_1,x_2,t;y_1,y_2,0)} \tag{17.21}$$

with (as we readily convince ourselves)

$$\begin{aligned}\tfrac{1}{\hbar}S(x_1, x_2, t; y_1, y_2, 0) &= \tfrac{m\omega}{2\hbar\sin\omega t}\Big\{\big[(x_1^2 + y_1^2)\cos\omega t - 2x_1y_1\big]\\ &\qquad\qquad + \big[(x_2^2 + y_2^2)\cos\omega t - 2x_2y_2\big]\Big\}\\ &= \tfrac{1}{4\sin\theta}\Big\{\big[(x_1^2 + y_1^2)\cos\theta - 2x_1y_1\big]\\ &\qquad\qquad + \big[(x_2^2 + y_2^2)\cos\theta - 2x_2y_2\big]\Big\}\end{aligned} \tag{17.22}$$

What does the Kibble-Mehler formula (15) have to say about the relationship between (17.1) and (17.2)? The formula is made a little difficult to apply in specific instances by the occurrance of $\mathbb{M}^{-1}$ on the left, and the complexity of the coefficients $C^n_{\nu_1\nu_2\nu_3\nu_4}$ on the right. So we squint/guess/tinker... and are soon develop interest in the case

$$\mathbb{M} = \mathbb{I} + \mathbb{T} = \begin{pmatrix} 1 & \tau & 0 & 0\\ \tau & 1 & 0 & 0\\ 0 & 0 & 1 & \tau\\ 0 & 0 & \tau & 1\end{pmatrix}$$

Then

$$\det\mathbb{M} = (1-\tau^2)^2$$

$$\mathbb{M}^{-1} = (1-\tau^2)^{-1}\begin{pmatrix} 1 & -\tau & 0 & 0\\ -\tau & 1 & 0 & 0\\ 0 & 0 & 1 & -\tau\\ 0 & 0 & -\tau & 1\end{pmatrix}$$

Write

$$\boldsymbol{x} = \begin{pmatrix} x_1\\ y_1\\ x_2\\ y_2\end{pmatrix} \quad\text{and}\quad \boldsymbol{p} = \begin{pmatrix} p_1\\ q_1\\ p_2\\ q_2\end{pmatrix}$$

and obtain

$$\begin{aligned}(\tfrac{1}{2}\boldsymbol{p}\cdot\mathbb{T}\boldsymbol{p})^n &= \tau^n(x_1y_1 + x_2y_2)^n\\ &= \tau^n\sum_{\nu=0}^{n}\tfrac{n!}{\nu!(n-\nu)!}x_1^\nu y_1^\nu x_2^{n-\nu}y_2^{n-\nu}\end{aligned}$$

These notations place us in position to write

$$\text{LHS of (15)} = \tfrac{1}{1-\tau^2}\exp\Big\{-\tfrac{1}{2(1-\tau^2)}\Big[\big(x_1^2 + y_1^2 - 2\tau x_1y_1\big) + \big(x_2^2 + y_2^2 - 2\tau x_2y_2\big)\Big]\Big\}$$

$$\begin{aligned}\text{RHS of (15)} &= e^{-\frac{1}{2}(x_1^2+y_1^2+x_2^2+y_2^2)}\\ &\quad\cdot\sum_{n=0}^{\infty}\tfrac{\tau^n}{n!}\sum_{\nu=0}^{n}\tfrac{n!}{\nu!(n-\nu)!}He_\nu(x_1)He_\nu(y_1)He_{n-\nu}(x_2)He_{n-\nu}(y_2)\end{aligned}$$

Multiply each of the preceding expressions by $\left(\frac{2m\omega}{h}\right)\tau e^{+\frac{1}{4}(x_1^2+y_1^2+x_2^2+y_2^2)}$ and set $\tau \mapsto e^{-i\theta}$. The latter expression then reproduces (17.1) while the former expression becomes

$$\left(\tfrac{2m\omega}{h}\right)\tfrac{\tau}{1-\tau^2}\exp\Big\{-\tfrac{1}{4}\tfrac{1+\tau^2}{1-\tau^2}\big(x_1^2+y_1^2+x_2^2+y_2^2\big)+\tfrac{\tau}{1-\tau^2}\big(x_1y_1+x_2y_2\big)\Big\}$$

But $\frac{1+\tau^2}{1-\tau^2}=-i\,\frac{\cos\theta}{\sin\theta}$ and $\frac{\tau}{1-\tau^2}=-i\,\frac{1}{2}\frac{1}{\sin\theta}$ so we have

$$\left(\tfrac{m\,\omega}{i\,h\sin\theta}\right)\exp\Big\{i\tfrac{1}{4\sin\theta}\Big[\big(x_1^2+y_1^2+x_2^2+y_2^2\big)\cos\theta-2\big(x_1y_1+x_2y_2\big)\Big]\Big\}$$

This result conforms precisely to (17.2); it places us, moreover, in position to observe that

$$(\text{factor})=\tfrac{m\,\omega}{i\,h\sin\omega t}$$

and that this result, in conformity with a principle fundamental to the Feynman formalism,[22] can be expressed (compare (3/5))

$$\begin{aligned}&=\sqrt{\left(\tfrac{i}{2\pi\hbar}\right)^2\det\begin{pmatrix}\partial^2S/\partial x_1\partial y_1 & \partial^2S/\partial x_1\partial y_2\\ \partial^2S/\partial x_2\partial y_1 & \partial^2S/\partial x_2\partial y_2\end{pmatrix}}\\ &=\sqrt{\left(\tfrac{i}{2\pi\hbar}\right)^2\big[-\big(-\tfrac{m\omega}{\sin\omega t}\big)^2\big]}\end{aligned}$$

The results developed above are in no respect surprising, but that hardly diminishes their charm or importance, and it is gratifying to discover that the details are manageable, and that they work out "just so." I remark once again that the time variable t lives upstairs in (17.1), but downstairs in (17.2).

Symmetries of the action for an isotropic oscillator. At (17.2) we obtained

$$S(x_1,x_2,t;y_1,y_2,0)=\tfrac{m\,\omega}{2\sin\omega t}\Big\{(x_1^2+y_1^2+x_2^2+y_2^2)\cos\omega t-2(x_1y_1+x_2y_2)\Big\}$$

which can be written

$$=\tfrac{m\,\omega}{2\sin\omega t}\Big\{(\boldsymbol{\xi}\cdot\mathbb{I}\,\boldsymbol{\xi})\cos\omega t-2(\boldsymbol{\xi}\cdot\mathbb{J}\,\boldsymbol{\xi})\Big\}\tag{18}$$

with

$$\boldsymbol{\xi}\equiv\begin{pmatrix}x_1\\y_1\\x_2\\y_2\end{pmatrix},\quad \mathbb{I}\equiv\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad \mathbb{J}\equiv\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\tag{19}$$

. . . in which connection we notice that $\mathbb{I}^{-1}=\mathbb{I}$, $\mathbb{J}^{-1}=\mathbb{J}$. It is obvious from (18)

[22] See QUANTUM MECHANICS (1967), Chapter 1, page 50.

that $S(x_1, x_2, t; y_1, y_2, 0)$ is invariant under $\boldsymbol{\xi} \longmapsto \boldsymbol{\xi} : \boldsymbol{\xi} = \mathbb{R}\,\boldsymbol{\xi}$ provided is is simultaneously true of $\mathbb{R}$ that

$$\mathbb{R}^{\mathsf{T}}\mathbb{I}\,\mathbb{R} = \mathbb{I} \qquad \text{and} \qquad \mathbb{R}^{\mathsf{T}}\mathbb{J}\,\mathbb{R} = \mathbb{J} \tag{20}$$

i.e., that

$$\mathbb{I}^{-1}\mathbb{R}^{\mathsf{T}}\mathbb{I} = \mathbb{R}^{-1} \qquad \text{and} \qquad \mathbb{J}^{-1}\mathbb{R}^{\mathsf{T}}\mathbb{J} = \mathbb{R}^{-1}$$

Write $\mathbb{R} = e^{\mathbb{L}}$. The requirement then is that

$$\mathbb{I}^{-1}\mathbb{L}^{\mathsf{T}}\mathbb{I} = -\mathbb{L} \qquad \text{and} \qquad \mathbb{J}^{-1}\mathbb{L}^{\mathsf{T}}\mathbb{J} = -\mathbb{A}$$

which is readily seen to entail that $\mathbb{L}$ be antisymmetric, but of the specialized form

$$\mathbb{L} = \begin{pmatrix} 0 & 0 & \alpha & \beta \\ 0 & 0 & \beta & \alpha \\ -\alpha & -\beta & 0 & 0 \\ -\beta & -\alpha & 0 & 0 \end{pmatrix} \equiv \alpha\mathbb{A} + \beta\mathbb{B} \tag{21}$$

The matrices

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{B} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

commute

$$\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A}$$

and satisfy

$$\mathbb{A}^2 = \mathbb{B}^2 = -\mathbb{I}$$

so we have

$$\begin{aligned} \mathbb{R} = e^{\alpha\mathbb{A}+\beta\mathbb{B}} &= e^{\alpha\mathbb{A}}\,e^{\beta\mathbb{B}} \\ &= \big\{\cos\alpha\cdot\mathbb{I} + \sin\alpha\cdot\mathbb{A}\big\}\big\{\cos\beta\cdot\mathbb{I} + \sin\beta\cdot\mathbb{B}\big\} \end{aligned} \tag{22}$$

Evidently we confront two independent copies of $O(2)$ embedded with $O(4)$. Working from (22) we have

$$\mathbb{R} = \cos\alpha\cos\beta\cdot\mathbb{I} + \sin\alpha\cos\beta\cdot\mathbb{A} + \cos\alpha\sin\beta\cdot\mathbb{B} + \sin\alpha\sin\beta\cdot\mathbb{C}$$

with

$$\mathbb{C} \equiv \mathbb{A}\mathbb{B} = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -\mathbb{J}$$

which is *not* antisymmetric, *not itself the generator* of a symmetry of the action function…though from an algebraic point of view it is a very close relative of $\mathbb{A}$ and $\mathbb{B}$:

$$\left.\begin{aligned} \mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A} &= +\mathbb{C} \\ \mathbb{B}\mathbb{C} = \mathbb{C}\mathbb{B} &= -\mathbb{A} \\ \mathbb{C}\mathbb{A} = \mathbb{A}\mathbb{C} &= -\mathbb{B} \end{aligned}\right\} \tag{23}$$

I find this result to be disappointing, if not perplexing: I had hoped that I would encounter $O(3)$—the "hidden symmetry" of the isotropic oscillator, but that prize is not so easily won.

The $O(3)$-symmetry of the isotropic oscillator lives in phase space. I look now, therefore, for bridges that will take me from (x_1, y_1, x_2, y_2)-space—locus of the preceding discussion—to (x_1, p_1, x_2, p_2)-space.

Translocation to phase space. By way of orientation,look to the fundamental 2-point action function of a free particle:

$$S(x,t;y,0) = \tfrac{m}{2t}(x-y)^2$$

Therefore

$$p = \frac{\partial S}{\partial x} = \tfrac{m}{t}(x-y) \quad\Longrightarrow\quad y = y(x,p;t) \equiv x - \tfrac{1}{m}p\,t$$

and

$$S(x,t;y(x,p;t)) \equiv \mathcal{S}(x,p;t) = \tfrac{1}{2m}p^2\cdot t$$

The result is more interesting than "corrupted Legendre transformation" that gave it.

For an oscillator we have

$$S(x,t;y,0) = \tfrac{m\,\omega}{2\sin\omega t}\big\{(x^2+y^2)\cos\omega t - 2xy\big\}$$

$$p = \frac{\partial S}{\partial x} = m\omega\,\frac{x\cos\omega t - y}{\sin\omega t} \quad\Longrightarrow\quad y = x\cos\omega t - \tfrac{1}{m\omega}p\sin\omega t$$

$$\mathcal{S}(x,p;t) = \tfrac{1}{2m\,\omega}\,p^2\cos\omega t\sin\omega t + px\sin^2\omega t - \tfrac{m\,\omega}{2}\,x^2\cos\omega t\sin\omega t$$

which give back free particle results in the limit $\omega\downarrow 0$, while expansion in powers of t gives

$$\mathcal{S}(x,p;t) = \Big\{\tfrac{1}{2m}p^2 + \tfrac{1}{2}m\omega^2x^2\Big\}\cdot t + \cdots$$

For an isotropic oscillator

$$\mathcal{S}(x_1,p_1,x_2,p_2;t) = \Big\{\tfrac{1}{2m}(p_1^2+p_2^2) + \tfrac{1}{2}m\omega^2(x_1^2+x_2^2)\Big\}\cdot t + \cdots$$

and in general we can expect to have

$$\mathcal{S}(\boldsymbol{p},\boldsymbol{x};t) = H(\boldsymbol{p},\boldsymbol{x})\cdot t + \cdots \tag{24}$$

We are led thus to the curious observation that (for time-independent systems)

$$\lim_{t\downarrow 0}\frac{\partial\mathcal{S}(\boldsymbol{p},\boldsymbol{x};t)}{\partial t} = H(\boldsymbol{p},\boldsymbol{x}) \text{ is a constant of the motion}$$

though the same cannot be said of $\mathcal{S}$ itself (or, generally, of its t-derivatives of ascending order).

To extract the *motion* from $S(x,y;t)$ one proceeds not by the crooked construction $S \longrightarrow \mathcal{S}$ but in the manner standard to the text books; i.e., by treating $S(x,y;t)$ as the Legendre generator of a canonical transformation.[23] Write

$$\left.\begin{aligned} \text{evolved momentum } p &= +\frac{\partial S}{\partial x} = +m\omega\,\frac{x\cos\omega t - y}{\sin\omega t}\\ \text{initial momentum } q &= -\frac{\partial S}{\partial y} = -m\omega\,\frac{y\cos\omega t - x}{\sin\omega t} \end{aligned}\right\} \tag{25}$$

Solve the latter for x

$$x = y\cos\omega t + \tfrac{1}{m\omega}q\sin\omega t \quad : \quad x(0) = y$$

and insert the result into the former:

$$\begin{aligned} p &= q\cos\omega t - m\omega y\sin\omega t \quad : \quad p(0) = q\\ &= m\dot{x} \end{aligned}$$

One then verifies that

$$\tfrac{1}{2m}p^2 + \tfrac{1}{2}m\omega^2x^2 = \tfrac{1}{2m}q^2 + \tfrac{1}{2}m\omega^2y^2$$

I have entrusted all the calculation to *Mathematica*, but the results are entirely elementary.

But close by hover some facts/questions/issues—*general* issues, though I discuss them as they occur in a specific instance (isotropic oscillator)—that, if also "elementary," are less frequently considered, less broadly understood. Consider the following scheme:

$$\begin{array}{ccc} H(p,x) & \xleftrightarrow[\quad t\text{-parameterized “motion”}\quad]{} & S(x,y;t)\\ \Big| {\scriptstyle [H,A]=0} & & \\ A(p,x) & \xleftrightarrow[\quad u\text{-parameterized canonical}\quad]{} & F(x,y;u) \end{array}$$

The Hamiltonian $H(p,x)$ is the Lie generator of the t-parameterized canonical transformation we call "motion," and $S(x,y;t)$ is—as demonstrated above—the Legendre generator of that same transformation: the relationship between the two is provided by the Hamilton-Jacobi equation $H(S_x,x)+S_t=0$. From $\big[H,A\big]=0$ we learn that $A(p,x)$ is a constant of the motion—the Lie generator of a u-parameterized family of canonical transformations that map

$$\text{dynamical orbits} \longmapsto \text{dynamical orbits} \tag{26}$$

and of which $F(x,y;t)$ is the Legendre generator: $A(F_x,x)+F_u=0$. The questions of interest to me: What does $F(x,y;t)$ look like in some representative concrete cases? What statement relating F to S echos the condition $\big[H,A\big]=0$?

[23] See, for example, H. Goldstein, *Classical Mechanics* (2nd edition 1980), equations (9–14).

For an isotropic oscillator we have

$$H = \tfrac{1}{2m}(p_1^2 + p_2^2) + \tfrac{1}{2}m\omega^2(x_1^2 + x_2^2) \tag{27.1}$$

and in connection with that physical system have an acquired[24] interest in

$$\left.\begin{aligned}
A_1 &\equiv \tfrac{1}{m}\,p_1p_2 + m\omega^2 x_1x_2\\
A_2 &\equiv \omega(x_1p_2 - x_2p_1)\\
A_3 &\equiv \tfrac{1}{2m}(p_1^2 - p_2^2) + \tfrac{1}{2}m\omega^2(x_1^2 - x_2^2)
\end{aligned}\right\} \tag{27.2}$$

which are readily shown to possess these properties:

$$A_1^2 + A_2^2 + A_3^2 = H^2 \tag{28.1}$$

$$\big[H, A_1\big] = \big[H, A_2\big] = \big[H, A_3\big] = 0 \ \ : \ \text{each is a constant of the motion} \tag{28.2}$$

$$\left.\begin{aligned}
\big[A_1, A_2\big] &= 2\omega A_3\\
\big[A_2, A_3\big] &= 2\omega A_1\\
\big[A_3, A_1\big] &= 2\omega A_2
\end{aligned}\right\} \therefore \boldsymbol{L} \equiv \tfrac{1}{2\omega}\boldsymbol{A} \ \text{ mimics “angular momentum”} \tag{28.3}$$

Each A-observable is dimensionally an “energy,” so $\boldsymbol{L}$ even mimics the physical dimensionality of angular momentum. This, of course, is the point at which $O(3)$ is standardly considered to sneak into the physics of isotropic oscillators.

Look upon $A_1(x_1, x_2, p_1, p_2)$ as a “Hamiltonian.” The “canonical equations of motion” read

$$\left.\begin{aligned}
\tfrac{d}{du}x_1 &= +\tfrac{1}{m}p_2\\
\tfrac{d}{du}p_1 &= -m\omega^2 x_2\\
\tfrac{d}{du}x_2 &= +\tfrac{1}{m}p_1\\
\tfrac{d}{du}p_2 &= -m\omega^2 x_1
\end{aligned}\right\} \Longrightarrow \left\{\begin{aligned}
\big(\tfrac{d^2}{du^2} + \omega^2\big)x_1 &= 0\\
\big(\tfrac{d^2}{du^2} + \omega^2\big)p_1 &= 0\\
\big(\tfrac{d^2}{du^2} + \omega^2\big)x_2 &= 0\\
\big(\tfrac{d^2}{du^2} + \omega^2\big)p_2 &= 0
\end{aligned}\right. \tag{29.1}$$

Solutions are of the familiar form[15]

$$\begin{aligned}
x(\tau) &= y\cos\omega\tau + \tfrac{1}{m\omega}q\sin\omega\tau\\
&= y\cos\omega\tau - \frac{y\cos\omega u - x}{\sin\omega u}\sin\omega\tau \quad : \quad (x, u) \longleftarrow (y, 0)
\end{aligned}$$

The equivalent “Lagrangian” is

$$\begin{aligned}
B_1(x_1, x_2, \dot x_1, \dot x_2) &= \dot x_1 p_1 + \dot x_2 p_2 - A_1(x_1, x_2, p_1, p_2)\\
&= m\big\{\dot x_1\dot x_2 - \omega^2 x_1x_2\big\}
\end{aligned} \tag{29.2}$$

[24] See (6) in “Comments ...‘On angular momentum,’” (October 2000).

and leads us to

$$\begin{aligned} F_1(x_1,x_2,u;y_1,y_2,0) &= \int_0^u B_1(x_1,x_2,\dot x_1,\dot x_2)\,du \\ &= m\omega\frac{(x_1x_2+y_1y_2)\cos\omega u-(x_1y_2+x_2y_1)}{\sin\omega u} \end{aligned} \tag{29.3}$$

Calculation confirms that F_1 is in fact a solution of

$$\tfrac{1}{m}\tfrac{\partial F}{\partial x_1}\tfrac{\partial F}{\partial x_2}+m\omega^2x_1x_2+\tfrac{\partial F}{\partial u}=0 \tag{29.4}$$

Look next/similarly upon $A_2(x_1,x_2,p_1,p_2)$ as a "Hamiltonian." The "canonical equations of motion" read

$$\left.\begin{aligned} \tfrac{d}{du}x_1 &= -\omega x_2\\ \tfrac{d}{du}p_1 &= -\omega p_2\\ \tfrac{d}{du}x_2 &= +\omega x_1\\ \tfrac{d}{du}p_2 &= +\omega p_1 \end{aligned}\right\}\Longrightarrow\left\{\begin{aligned} \big(\tfrac{d^2}{du^2}+\omega^2\big)x_1&=0\\ \big(\tfrac{d^2}{du^2}+\omega^2\big)p_1&=0\\ \big(\tfrac{d^2}{du^2}+\omega^2\big)x_2&=0\\ \big(\tfrac{d^2}{du^2}+\omega^2\big)p_2&=0 \end{aligned}\right. \tag{30.1}$$

Solutions are obviously (remarkably?) the same as before. But, because of the *bilinear* design of $A_2(x_1,x_2,p_1,p_2)$, equations of the form

$$\left.\begin{aligned} p_1&=p_1(x_1,x_2,\dot x_1,\dot x_2)\\ p_1&=p_2(x_1,x_2,\dot x_1,\dot x_2) \end{aligned}\right\}\quad\text{are impossible in this case}$$

so it is impossible to execute the Legendre transformation that would lead *via*

$$B_2(x_1,x_2,\dot x_1,\dot x_2)=\dot x_1p_1+\dot x_2p_2-A_2(x_1,x_2,p_1,p_2)$$

to an equivalent "Lagrangian," and impossible therefore to lend meaning to the construction

$$F_2(x_1,x_2,u;y_1,y_2,0)=\int_0^u B_2(x_1,x_2,\dot x_1,\dot x_2)\,du$$

Look finally/similarly upon $A_3(x_1,x_2,p_1,p_2)$ as a "Hamiltonian." The "canonical equations of motion" read

$$\left.\begin{aligned} \tfrac{d}{du}x_1 &= +\tfrac{1}{m}p_1\\ \tfrac{d}{du}p_1 &= -m\omega^2 x_1\\ \tfrac{d}{du}x_2 &= -\tfrac{1}{m}p_2\\ \tfrac{d}{du}p_2 &= +m\omega^2 x_2 \end{aligned}\right\}\Longrightarrow\left\{\begin{aligned} \big(\tfrac{d^2}{du^2}+\omega^2\big)x_1&=0\\ \big(\tfrac{d^2}{du^2}+\omega^2\big)p_1&=0\\ \big(\tfrac{d^2}{du^2}+\omega^2\big)x_2&=0\\ \big(\tfrac{d^2}{du^2}+\omega^2\big)p_2&=0 \end{aligned}\right. \tag{31.1}$$

Solutions are once again the same as before. The equivalent "Lagrangian" is

$$\begin{aligned} B_3(x_1,x_2,\dot x_1,\dot x_2) &= \dot x_1p_1+\dot x_2p_2-A_3(x_1,x_2,p_1,p_2)\\ &= \tfrac12 m(\dot x_1^2-\dot x_2^2)+\tfrac12 m\omega^2(x_1^2-x_2^2) \end{aligned} \tag{31.2}$$

so we have (compare (17.2))

$$\begin{aligned} F_3(x_1,x_2,u;y_1,y_2,0) &= \int_0^u B_3(x_1,x_2,\dot{x}_1,\dot{x}_2)\,du \\ &= \tfrac{1}{2}m\omega\frac{(x_1^2+y_1^2-x_2^2-y_2^2)\cos\omega u - 2(x_1y_1-x_2y_2)}{\sin\omega u} \end{aligned} \tag{31.3}$$

Calculation confirms that F_3 is in fact a solution of

$$\tfrac{1}{2m}\Big[\big(\tfrac{\partial F}{\partial x_1}\big)^2 - \big(\tfrac{\partial F}{\partial x_2}\big)^2\Big] + \tfrac{1}{2}m\omega^2(x_1^2-x_2^2) + \tfrac{\partial F}{\partial u} = 0 \tag{31.4}$$

Preceding results are responsive to the first of the questions posed at the bottom of page 19, but to the second question—"What statement relating F to S echos the condition $\big[H,A\big]=0$?"—they serve only to supply evidence that the answer is "not obvious."

Composed Legendre transformations. The issue before us is perhaps most readily grasped when viewed in relation to its quantum mechanical counterpart: write

$$\begin{aligned} \text{orbit} \ :\ |\psi)_0 \longrightarrow |\psi)_t \ &= \mathbf{U}(t)|\psi)_0 \quad\text{with}\quad \mathbf{U}(t) = e^{-\frac{i}{\hbar}\mathbf{H}t} \\ \text{transformed orbit} \ :\ |\psi)_t \longrightarrow |\psi)_{u,t} &= \mathbf{V}(u)|\psi)_t \quad\text{with}\quad \mathbf{V}(u) = e^{-\frac{i}{\hbar}\mathbf{A}u} \end{aligned}$$

The condition that a transformed orbit be itself an orbit (orbit of the transform) can be expressed

$$\mathbf{V}(u)\mathbf{U}(t)|\psi)_0 = \mathbf{U}(t)\mathbf{V}(u)|\psi)_0 \tag{32.1}$$

which if valid for

- all $|\psi)_0$
- all u
- all t

entails $\big[\mathbf{H},\mathbf{A}\big]=\mathbf{0}$. A more "Noetherean" formulation of the same idea is

$$\big[\mathbf{A},\mathbf{U}(t)\big] = \mathbf{0}$$

Pass to the $\mathbf{x}$-representation and obtain statements of the form

$$\begin{aligned} \int (x|\mathbf{V}(u)|z)dz(z|\mathbf{U}(t)|y) &= \int (x|\mathbf{U}(t)|z)dz(z|\mathbf{V}(u)|y) \\ \int (x|\mathbf{A}|z)dz(z|\mathbf{U}(t)|y) &= \int (x|\mathbf{U}(t)|z)dz(z|\mathbf{A}|y) \end{aligned} \tag{32.2}$$

which acquire special interest from the circumstance that

$$(x|\mathbf{U}(t)|y) = \text{Green's function} \sim e^{\frac{i}{\hbar}(\text{action function})}$$

At (32) we composed transformations that are canonical in the unitary sense of quantum mechanics. We look now to the classical analog/precursor of that process, as it presents itself in a specific instance (isotropic oscillator).

We have already examined the Legendre transform mechanism by which

$$S(x_1,x_2,t;y_1,y_2,0)=\tfrac{1}{2}m\omega\frac{(x_1^2+y_1^2+x_2^2+y_2^2)\cos\omega t-2(x_1y_1+x_2y_2)}{\sin\omega t}$$

generates

$$\begin{pmatrix}y_1\\q_1\\y_2\\q_2\end{pmatrix}\xrightarrow[S\text{-generated motion}]{}\begin{pmatrix}x_1=y_1\cos\omega t+\frac{1}{m\omega}q_1\sin\omega t\equiv Sy_1\\p_1=q_1\cos\omega t-m\omega y_1\sin\omega t\equiv Sq_1\\x_2=y_2\cos\omega t+\frac{1}{m\omega}q_2\sin\omega t\equiv Sy_2\\p_2=q_2\cos\omega t-m\omega y_2\sin\omega t\equiv Sq_2\end{pmatrix}$$

and by that same mechanism (I omit the details) we obtain

$$\begin{pmatrix}y_1\\q_1\\y_2\\q_2\end{pmatrix}\xrightarrow[F_1\text{-generated map}]{}\begin{pmatrix}x_1=y_1\cos\omega u+\frac{1}{m\omega}q_2\sin\omega u\equiv F_1y_1\\p_1=q_1\cos\omega u-m\omega y_2\sin\omega u\equiv F_1q_1\\x_2=y_2\cos\omega u+\frac{1}{m\omega}q_1\sin\omega u\equiv F_1y_2\\p_2=q_2\cos\omega u-m\omega y_1\sin\omega u\equiv F_1q_2\end{pmatrix}\tag{33.1}$$

From the latter it follows, by the way, that for *infinitesimal* u we in leading order have

$$\begin{pmatrix}x_1=y_1+\frac{1}{m}q_2\cdot\delta u\\p_1=q_1-m\omega^2y_2\cdot\delta u\\x_2=y_2+\frac{1}{m}q_1\cdot\delta u\\p_2=q_2-m\omega^2y_1\cdot\delta u\end{pmatrix}$$

which by

$$\big[F_1,\bullet\big]=-\tfrac{1}{m}p_2\frac{\partial}{\partial x_1}+m\omega^2x_2\frac{\partial}{\partial p_1}-\tfrac{1}{m}p_1\frac{\partial}{\partial x_2}+m\omega^2x_1\frac{\partial}{\partial p_2}\tag{33.2}$$

can be written

$$\left.\begin{pmatrix}x_1\\p_1\\x_2\\p_2\end{pmatrix}-\Big[F_1,\begin{pmatrix}x_1\\p_1\\x_2\\p_2\end{pmatrix}\Big]\cdot\delta u+\cdots\right|_{x\to y,\,p\to q}$$

Now to the point of this discussion: Entrusting the computational labor to *Mathematica*, we find

$$\left.\begin{aligned}Fy_1(Sy_1,Sq_1,Sy_2,Sq_2)&=Sy_1(Fy_1,Fq_1,Fy_2,Fq_2)\\Fq_1(Sy_1,Sq_1,Sy_2,Sq_2)&=Sq_1(Fy_1,Fq_1,Fy_2,Fq_2)\\Fy_2(Sy_1,Sq_1,Sy_2,Sq_2)&=Sy_2(Fy_1,Fq_1,Fy_2,Fq_2)\\Fq_2(Sy_1,Sq_1,Sy_2,Sq_2)&=Sq_2(Fy_1,Fq_1,Fy_2,Fq_2)\end{aligned}\right\}\tag{33.3}$$

where I have dropped the subscripts $_1$ from all the F's. This result expresses the *commutivity* of the S-map and the F_1-map: the results are the same when they are performed in either order. Specificially we have

$$\begin{aligned}
x_1(t,u) &= \frac{\cos\omega t\big\{m\omega y_1\cos\omega u + q_2\sin\omega u\big\} + \sin\omega t\big\{q_1\cos\omega u - m\omega y_2\sin\omega u\big\}}{m\omega}\\
p_1(t,u) &= \cos\omega t\big\{q_1\cos\omega u - m\omega y_2\sin\omega u\big\} - \sin\omega t\big\{m\omega y_1\cos\omega u + q_2\sin\omega u\big\}\\
x_2(t,u) &= \frac{\cos\omega t\big\{m\omega y_2\cos\omega u + q_1\sin\omega u\big\} + \sin\omega t\big\{q_2\cos\omega u - m\omega y_1\sin\omega u\big\}}{m\omega}\\
p_2(t,u) &= \cos\omega t\big\{q_2\cos\omega u - m\omega y_1\sin\omega u\big\} - \sin\omega t\big\{m\omega y_2\cos\omega u + q_1\sin\omega u\big\}
\end{aligned}$$

from which results appropriate to $u \mapsto \delta u$ are readily extracted.

Working similarly (see again (27.2)) from A_3—which is to say: from the $F_3(x_1, x_2, u; y_1, y_2, 0)$ of (31.3)—we obtain

$$\begin{aligned}
x_1 &= \frac{m\omega y_1\cos\omega u + q_1\sin\omega u}{m\omega}\\
p_1 &= q_1\cos\omega u - m\omega y_1\sin\omega u\\
x_2 &= \frac{m\omega y_2\cos\omega u - q_2\sin\omega u}{m\omega}\\
p_2 &= q_2\cos\omega u + m\omega y_2\sin\omega u
\end{aligned}$$

Composition with the transformation generated by S is again found to be commutative, and produces

$$\begin{aligned}
x_1(t,u) &= y_1\cos\omega(t+u) + \tfrac{1}{m\omega}q_1\sin\omega(t+u)\\
p_1(t,u) &= q_1\cos\omega(t+u) - m\omega\, y_1\sin\omega(t+u)\\
x_2(t,u) &= y_2\cos\omega(t-u) + \tfrac{1}{m\omega}q_2\sin\omega(t-u)\\
p_2(t,u) &= q_2\cos\omega(t-u) - m\omega\, y_2\sin\omega(t-u)
\end{aligned}$$

Look finally to A_2. No $F_2(x_1, x_2, u; y_1, y_2, 0)$ exists, but nothing prevents *direct* solution of the "canonical equations" (30.1); *Mathematica* supplies

$$\begin{aligned}
x_1 &= y_1\cos\omega u - y_2\sin\omega u\\
p_1 &= q_1\cos\omega u - q_2\sin\omega u\\
x_2 &= y_2\cos\omega u + y_1\sin\omega u\\
p_2 &= q_2\cos\omega u + q_1\sin\omega u
\end{aligned}$$

Composition with the transformation generated by S is once again found to be commutative, and produces

$$\begin{aligned}
x_1(t,u) &= \cos\omega t\big\{y_1\cos\omega u - y_2\sin\omega u\big\} + \sin\omega t\,\frac{q_1\cos\omega u - q_2\sin\omega u}{m\omega}\\
p_1(t,u) &= \cos\omega t\big\{q_1\cos\omega u - q_2\sin\omega u\big\} - m\omega\sin\omega t\big\{y_1\cos\omega u - y_2\sin\omega u\big\}\\
x_2(t,u) &= \cos\omega t\big\{y_2\cos\omega u + y_1\sin\omega u\big\} + \sin\omega t\,\frac{q_2\cos\omega u + q_1\sin\omega u}{m\omega}\\
p_2(t,u) &= \cos\omega t\big\{q_2\cos\omega u + q_1\sin\omega u\big\} - m\omega\sin\omega t\big\{y_2\cos\omega u + y_1\sin\omega u\big\}
\end{aligned}$$

In each of the preceding cases we recover S-generated motion

$$\left.\begin{aligned} x_1 &= y_1\cos\omega t + \tfrac{1}{m\omega}q_1\sin\omega t\\ p_1 &= q_1\cos\omega t - m\omega\, y_1\sin\omega t\\ x_2 &= y_2\cos\omega t + \tfrac{1}{m\omega}q_2\sin\omega t\\ p_2 &= q_2\cos\omega t - m\omega\, y_2\sin\omega t \end{aligned}\right\}\text{ in the limit } u\downarrow 0$$

and each transformation serves to map orbits $\mapsto$ orbits. But the principal lesson appears to be that *finitistic analysis is not the language of choice if one's objective is to render transparent the underlying group theory*. We have gained new appreciation of the force of Sophus Lie's fundamental insight: to see what's going on one should—as at (28)—*look to the algebra of the generators of the transformations*.

And yet. . . I have many times advanced the proposition that "to know the two-point action $S(\boldsymbol{x},t;\boldsymbol{y},0)$ is to know everything," while by Noether we are encouraged to suppose that (consistently with the preceding proposition) "conservation laws arise from and refer to symmetries of the action." We may therefore imagine ourselves to be in position to *demonstrate the validity of that claim as it refers to the isotropic oscillator*, and thus to clarify where within the structure of

$$S(x_1,x_2,t;y_1,y_2,0) = \tfrac{m\,\omega}{2\sin\omega t}\Big\{(x_1^2+x_1^2+y_1^2+y_2^2)\cos\omega t - 2(x_1y_1+x_2y_2)\Big\}$$

lurks the celebrated $O(3)$-symmetry of the isotropic oscillator. The effort to do so meets, however, with only limited success, and exposes a deep but seldom remarked fact:

Hidden symmetry eludes Noether's principle. At the top of the preceding page we obtained a description of the compose (same in either order) of the canonical transformations generated by H and A_1. *Mathematica* informs us that

$$\begin{aligned} &S\big(x_1(t,u),x_2(t,u),t;x_1(0,u),x_2(0,u),0\big)\\ &\qquad - S\big(x_1(t,0),x_2(t,0),t;x_1(0,0),x_2(0,0),0\big) = \text{non-vanishing mess} \end{aligned}$$

A similar result is obtained when we work from the description (middle of the page) of the canonical transformations generated by H and A_3. But when we work from the description (bottom of the page) of the canonical transformations generated by H and A_2 we find that indeed

$$\begin{aligned} &S\big(x_1(t,u),x_2(t,u),t;x_1(0,u),x_2(0,u),0\big)\\ &\qquad - S\big(x_1(t,0),x_2(t,0),t;x_1(0,0),x_2(0,0),0\big) = 0 \end{aligned}\tag{34}$$

Noether herself—working within the framework provided by the calculus of variations—proceeded from a statement of the form

$$\tfrac{\partial}{\partial u}S\big(x_1(t,u),x_2(t,u),t;x_1(0,u),x_2(0,u),0\big)\Big|_{u\to 0} = 0$$

which is correct in this instance, but fails to capture the full force of the remarkable circumstance that, according to (34),

$$S\big(x_1(t,u),x_2(t,u),t;x_1(0,u),x_2(0,u),0\big)\;\begin{cases}\text{is in fact } \underline{u\text{-independent}};\\ \text{it vanishes under } \big(\frac{\partial}{\partial u}\big)^n\\ \text{for all } n, \text{ at all } u\end{cases}$$

Why do we have "symmetry of the action" in this case, but not in the other two?

Noether's principle, as it applies to classical mechanics (as opposed to classical field theory), gives rise to local conservation laws of the form $\frac{d}{dt}J_r = 0$ with

$$J_r = \sum_\nu p_\nu \varPhi_r^\nu + \Big\{L - \sum_\nu p_\nu \dot q^\nu\Big\}\mathfrak{X}_r + \Lambda_r \quad : \quad r = 1,2,\ldots,\alpha$$

where $\varPhi_r^\nu$, $\mathfrak{X}_r$ and Λ_r are structure functions that serve to describe some $\delta\omega_r$-parameterized map that acts on $\{\boldsymbol{q},t\}$-space, but may contain also a gauge component.[25] In Hamiltonian language we have

$$J_r(p,q,t) = \sum_\nu p_\nu \varPhi_r^\nu(q,t) - H(\boldsymbol{p},\boldsymbol{q})\mathfrak{X}_r(q,t) + \Lambda_r(q,t)$$

into which *momenta enter <u>linearly</u>*, except for such non-linear p-dependence as may be incorporated into the design of $H(\boldsymbol{p},\boldsymbol{q})$.[26]

Look back now to (27.2) and notice that

- A_2, which we found *does* generate a symmetry of the oscillator action S, *is* linear in the momenta, and therefore *could* arise from Noether's principle; in point of fact,

$$A_2 \sim \text{angular momentum}$$

which is known to reflect the *rotational invariance* of S.

- A_1 and A_3 display such quadratic dependence upon momenta that they *could not* have arisen from applications of Noether's principle: the statements

$$\tfrac{d}{dt}A_1 = \tfrac{d}{dt}A_3 = 0$$

are <u>non-Noetherean conservation laws</u>. And, as we have seen, A_1 and A_3 generate transformations which are *not symmetries* of S; i.e., with respect to which S is not invariant.

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[25] See CLASSICAL MECHANICS (1983), page 169.

[26] See page 18 in "Kepler problem by descent from the Euler problem," Notes for a Reed College Physics Seminar (6 October 1996), where the same point comes up in another context.

We are brought thus to this insight: "Hidden symmetries" live in phase space, not in configuration space, where they would become susceptible to Noetherean analysis. Hidden symmetries give rise to what it now becomes natural to call "non-Noetherean conservation laws." Maybe it is possible to devise an "extended Noether's principle" which would embrace such conservation laws, but until that has been accomplished it seems important to recognize the *existence* of such things.[27]

Quantum mechanical expression of the Noetherean/non-Noetherean distinction requires at the very least that one adopt a formalism within which Noether's principle is a player. The obvious thing to do is to look upon quantum mechanics as a classical field theory, which in the case of immediate interest (isotropic oscillator) proceeds from the Lagrange density[28]

$$\mathcal{L} = -\tfrac{\hbar^2}{2m}\Big\{\tfrac{\partial\psi^*}{\partial x_1}\tfrac{\partial\psi}{\partial x_1} + \tfrac{\partial\psi^*}{\partial x_2}\tfrac{\partial\psi}{\partial x_2}\Big\} - \psi^*\Big\{\tfrac{1}{2}m\omega^2\big(x_1^2 + x_2^2\big)\Big\}\psi + \tfrac{1}{2}i\hbar\big(\psi^*\psi_t - \psi_t^*\psi\big)$$

but alternative launch pads would appear to be provided by Schwinger's variational principle and by a little known but lovely formalism devised by E. T. Whittaker in the early 1940's.[29] The topic merits closer study, but this is not the occasion; for the moment it is sufficient to observe that

> *It is futile to search for evidence of hidden symmetry written into the design of the quantum mechanical Green's function; it is in precisely that sense that such symmetry is "hidden."*

Note, however, that in the phase space formalism

$$\psi(\boldsymbol{x},t) = \iint G(\boldsymbol{x},t;\boldsymbol{y},0)\psi(\boldsymbol{y},0)\,dy_1dy_2$$

becomes[30]

$$P(\boldsymbol{x},\boldsymbol{p},t) = \iiiint K(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{q},0)P(\boldsymbol{y},\boldsymbol{q},0)\,dy_1dy_2dq_1dq_2$$

and that it just might be sensible to search for evidence of hidden symmetry in the design of the "phase space propagator" $K(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{q},0)$. As a preliminary step one might want to expose the classical counterpart to K—the object $\mathcal{S}(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{q},0)$ that stands to K as $S(\boldsymbol{x},t;\boldsymbol{y},0)$ stands to $G(\boldsymbol{x},t;\boldsymbol{y},0)$. Note also that upon elimination of the t-variable the classical theory of isotropic

[27] Preceding remarks contribute nothing toward clarification of the *relation between hidden symmetry and multiple separability*. See page 9 in "Classical/quantum theory of 2-dimensional hydrogen," Notes for a Reed College Physics Seminar (3 February 1996).

[28] See CLASSICAL FIELD THEORY (1979), p.130.

[29] See QUANTUM MECHANICS (1967), Chapter 3, pp. 68–83.

[30] See page 113 in the notes just cited, or page 18 in ADVANCED QUANTUM TOPICS, Chapter 2, "Weyl transform and the phase space formalism" (2000).

oscillators becomes a "theory of centered ellipses," which $A_{1,2,3}$ serve to map one to another in a manner elegantly described by Stokes and Poincaré, and elaborately reviewed in my ELLIPSOMETRY (1999); one might expect that formalism to be latent in the design of $K(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{q},0)$, but to be eclipsed by the process that proceeds $K\to G$.

A small point concerning "equivalent but gauge-inequivalent" Lagrangians. At (29.2) we were led from $A_1(x_1,x_2,p_1,p_2)$ to a "Lagrangian" of the form

$$B_1(x_1,x_2,\dot x_1,\dot x_2)=m\big\{\dot x_1\dot x_2-\omega^2x_1x_2\big\}$$

which gives

$$\begin{aligned}\ddot x_2+\omega^2x_2&=0\\ \ddot x_1+\omega^2x_1&=0\end{aligned}$$

At (31.2) we were led similarly from $A_3(x_1,x_2,p_1,p_2)$ to a "Lagrangian"

$$B_3(x_1,x_2,\dot x_1,\dot x_2)=\tfrac12m(\dot x_1^2-\dot x_2^2)+\tfrac12m\omega^2(x_1^2-x_2^2)$$

which (i) is evidently *not gauge-equivalent to* B_1

$$B_3\text{ cannot be written }B_1+\tfrac{d}{dt}\Lambda(x_1,x_2)$$

though (ii) it leads to the *same system of differential equations*

$$\begin{aligned}\ddot x_1+\omega^2x_1&=0\\ \ddot x_2+\omega^2x_2&=0\end{aligned}$$

but a system in which the *members are presented in reversed order*. The small point to which I draw attention is that "gauge equivalence" and—what to call it?—"plain equivalence" are, on this evidence, distinct concepts.

Conclusions. In my introductory remarks I drew attention to common features shared by "Jacobi's theta transformation" and "Mehler's formula" asked (not for the first time in my career): Does there exist a sense in which all such statements are instances of the *same* over-arching abstract statement? Can such statements be unified/generalized? The short answer: Beats me! We have seen that the former is a discrete analog of, and the latter a corollary of . . . the elementary Gaussian identity (1.2), which in the multivariate case reads (12). In that sense we have established "commonality," but unification/generalization have eluded us.

I remarked also that "a bivariate [version of Mehler's formula] stands central to the quantum theory of the . . . isotropic 2-dimensional oscillator" and speculated that it "must, therefore, lurk somewhere within the quantum theory of angular momentum." The discussion has established that $O(3)$ lurks in a truly well-hidden place (phase space); that it is futile to attempt to read $O(3)$ in the design (Mehler's design) of the oscillator propagator.