In[1]:= **2 - 2**

Out[1]= 0

ENTANGLED?

Notes for a Reed College Physics Seminar presented 11 October 2006

REMARK ADDED AFTER COMPLETION OF THE NOTEBOOK: In the seminar I allowed myself, in the interest of expository simplicity, to assume that all vector components are real. In an addendum—written partly in response to a remark by Tom Wieting—I develop the quite independent separation criteria that arise when complex phase factors are assigned to the components of composite state vectors.

■ Introduction: correlations latent in the entangled states of composite quantum systems

Alice is in her laboratory, studying quantum system $\mathcal{S}_{\text{Alice}}$. Bob is in his lab, studying his quite independent system $\mathcal{S}_{\text{Bob}}$.

For expository convenience we assume both systems to be 2-state systems (which, experimentally, happens actually to be the most commonly encountered situation). Alice writes

In[2]:= $\psi_{\text{Alice}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix};$

to describe the (in this instance "pure") quantum state of $\mathcal{S}_{\text{Alice}}$, and presents 2×2 hermitian matrices

In[3]:= $\text{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix};$

to describe her Hamiltonian, meters, etc. Bob writes

In[4]:= $\psi_{\text{Bob}} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix};$

In[5]:= $\text{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix};$

Constructions like this

In[6]:= $\text{A}.\psi_{\text{Alice}}$ **// MatrixForm**
$\text{B}.\psi_{\text{Bob}}$ **// MatrixForm**

Out[6]//MatrixForm=

$$\begin{pmatrix} a_{11}\,\alpha_1 + a_{12}\,\alpha_2 \\ a_{21}\,\alpha_1 + a_{22}\,\alpha_2 \end{pmatrix}$$

Out[7]//MatrixForm=

$$\begin{pmatrix} b_{11}\,\beta_1 + b_{12}\,\beta_2 \\ b_{21}\,\beta_1 + b_{22}\,\beta_2 \end{pmatrix}$$

assume importance when they sit down to do standard quantum calculations. This simple setting is rich enough to provide a toy model of all the aspects of full-blown quantum mechanics that do not depend upon **[x,p]=iℏ**. It is, in particular, rich enough to support a theory of angular momentum (spin one-half).

God, looking down from on high, gives equal attention simultaneously to *both* Alice and Bob. How does He do it?

Introduction of the KRONECKER PRODUCT:

Let **P** and **Q** be matrices of arbitrary—and typically dissimilar—dimension:

In[8]:=
$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \\ p_{31} & p_{32} \end{pmatrix};$$

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix};$$

The Kronecker product has this meaning

$$P \otimes Q \equiv \begin{pmatrix} p_{11}\,Q & p_{12}\,Q \\ p_{21}\,Q & p_{22}\,Q \\ p_{31}\,Q & p_{32}\,Q \end{pmatrix}$$

and is produced by the following command (semi-standard to *Mathematica* is a different command, but I like this one better):

In[10]:=
```
P_⊗Q_ := Flatten[Table[
    Flatten[Table[Part[Outer[Times, P, Q], i, j, k], {j, Dimensions[P]⟦2⟧}]],
    {i, Dimensions[P]⟦1⟧}, {k, Dimensions[Q]⟦1⟧}], 1]
```

In[11]:=
```
P⊗Q // MatrixForm
```

Out[11]//MatrixForm=

$$\begin{pmatrix} p_{11}\,q_{11} & p_{11}\,q_{12} & p_{12}\,q_{11} & p_{12}\,q_{12} \\ p_{11}\,q_{21} & p_{11}\,q_{22} & p_{12}\,q_{21} & p_{12}\,q_{22} \\ p_{21}\,q_{11} & p_{21}\,q_{12} & p_{22}\,q_{11} & p_{22}\,q_{12} \\ p_{21}\,q_{21} & p_{21}\,q_{22} & p_{22}\,q_{21} & p_{22}\,q_{22} \\ p_{31}\,q_{11} & p_{31}\,q_{12} & p_{32}\,q_{11} & p_{32}\,q_{12} \\ p_{31}\,q_{21} & p_{31}\,q_{22} & p_{32}\,q_{21} & p_{32}\,q_{22} \end{pmatrix}$$

Out[11]//MatrixForm=

$$\begin{pmatrix} p_{11}\, q_{11} & p_{11}\, q_{12} & p_{12}\, q_{11} & p_{12}\, q_{12} \\ p_{11}\, q_{21} & p_{11}\, q_{22} & p_{12}\, q_{21} & p_{12}\, q_{22} \\ p_{21}\, q_{11} & p_{21}\, q_{12} & p_{22}\, q_{11} & p_{22}\, q_{12} \\ p_{21}\, q_{21} & p_{21}\, q_{22} & p_{22}\, q_{21} & p_{22}\, q_{22} \\ p_{31}\, q_{11} & p_{31}\, q_{12} & p_{32}\, q_{11} & p_{32}\, q_{12} \\ p_{31}\, q_{21} & p_{31}\, q_{22} & p_{32}\, q_{21} & p_{32}\, q_{22} \end{pmatrix}$$

The Kronecker product provides a natural mechanism for developing multiple copies of linear algebra simultaneously , and at the same time for establishing cross-talk among them.

To described the state of the **composite Alice/Bob system**, God writes

In[12]:= $\Psi = \psi_{\text{Alice}} \otimes \psi_{\text{Bob}}$;

In[13]:= **Ψ // MatrixForm**

Out[13]//MatrixForm=

$$\begin{pmatrix} \alpha_1\, \beta_1 \\ \alpha_1\, \beta_2 \\ \alpha_2\, \beta_1 \\ \alpha_2\, \beta_2 \end{pmatrix}$$

and to describe their respective Hamiltonians/meters/etc. He writes

In[14]:= $A_{\text{God}} = A \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

$B_{\text{God}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes B$;

In[16]:= **A_{God} // MatrixForm**
B_{God} // MatrixForm

Out[16]//MatrixForm=

$$\begin{pmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & a_{11} & 0 & a_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & a_{21} & 0 & a_{22} \end{pmatrix}$$

Out[17]//MatrixForm=

$$\begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix}$$

He is thus enabled to describe Alice/Bob's activity in a unified way:

In[18]:= $(A_{\text{God}}).\Psi == (A.\psi_{\text{Alice}}) \otimes \psi_{\text{Bob}}$ **// Simplify**
$(B_{\text{God}}).\Psi == \psi_{\text{Alice}} \otimes (B.\psi_{\text{Bob}})$ **// Simplify**

Out[18]= True

Out[19]= True

TYPICAL CONCRETE CASE (of, as it happens, very great special interest):

Alice selects the measurement device (or "meter") described by the hermitian matrix

In[20]:= $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$

From

In[21]:= `Eigenvalues[A]`

Out[21]= {-1, 1}

we see that her meter always announces either **+1** or **−1** (up/down, true/false, yes/no). God writes

In[22]:= $\mathbf{A}_{\mathbf{God}} = \mathbf{A} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$

In[23]:= $\mathbf{A}_{\mathbf{God}}$ `// MatrixForm`

Out[23]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

to describe Alice's meter. The "spectral resolution" of A_{God} is immediate:

In[24]:= $\mathbf{A}_{\mathbf{God}} == (+1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + (-1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Out[24]= True

where the matrices are obviously projective (project onto orthogonal 2-spaces.

We will suppose Bob to have selected a similar meter

In[25]:= $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$

which leads God to write

In[26]:= $\mathbf{B}_{\mathbf{God}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbf{B};$

In[27]:=
```
B_God // MatrixForm
```
Out[27]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Immediately

In[28]:= $$B_{God} == (+1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[28]= True

Suppose God—trickster that He is—presents Alice serially with many copies of the state

In[29]:= $$\Psi = \frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\sqrt{2}};$$

In[30]:=
```
Ψ // MatrixForm
```
Out[30]//MatrixForm=

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Alice's meter will then announce **+1** with probability

In[31]:= $$\text{Transpose}[\Psi].\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\Psi$$

Out[31]= $\left\{\left\{\frac{1}{2}\right\}\right\}$

and will on such occasions prepare the state (here the $\sqrt{2}$ is normalization factor)

In[32]:= $$\text{AliceOutUp} = \sqrt{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\Psi;$$

In[33]:= **AliceOutUp // MatrixForm**

Out[33]//MatrixForm=

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Alternatively, her meter announces **-1** with probabilty

In[34]:= **Transpose[Ψ].** $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ **.Ψ**

Out[34]= $\{\{\frac{1}{2}\}\}$

and prepares the state

In[35]:= **AliceOutDn =** $\sqrt{2}$ $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ **.Ψ;**

In[36]:= **AliceOutDn // MatrixForm**

Out[36]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Because Bob—gentleman that he is—always lets Alice go first, he with his B-meter examines states that have been altered by Alice's prior measurement activity. When he inherits an **AliceOutUp** state his meter reads **+1** with probability

In[37]:= **Transpose[AliceOutUp].** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ **.AliceOutUp**

Out[37]= {{0}}

—which is to say: never—and reads **-1** with probability

In[38]:= **Transpose[AliceOutUp].** $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ **.AliceOutUp**

Out[38]= {{1}}

—which is to say: with certainty. On the other hand, when he inherits an **AliceOutDn** state his meter reads **+1** with probability

In[39]:= **Transpose[AliceOutDn].** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ **.AliceOutDn**

Out[39]= {{1}}

—which is to say: with certainty—and never reads **-1**:

In[40]:= **Transpose[AliceOutDn].** $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ **.AliceOutDn**

Out[40]= {{0}}

Those statistical results are summarized in the following table, where the red entries are JOINT probabilities, while the blue entries are MARGINAL probabilities, got by adding all joint probabilities in a row/column.

□	**BobUp**	**BobDn**	**Alice Marginal**
AliceUp	0	$\frac{1}{2}$	$\frac{1}{2}$
AliceDn	$\frac{1}{2}$	0	$\frac{1}{2}$
Bob Marginal	$\frac{1}{2}$	$\frac{1}{2}$	□

Alice's meter reads **±** with equal frequency, and so does Bob's. But when they come together (Alice works at Stanford, Bob at Harvard) they find that their respective meter readings are **perfectly anticorrelated**!

What we have encountered here is David Bohm's sharp illustration (1951) of the essential upshot of the famous **EPR paradox** (1935). It was in this setting that John Bell (1964)—after supplying Alice and Bob with more general devices

$$A = \begin{pmatrix} a_3 & a_1 - i\, a_2 \\ a_1 + i\, a_2 & -a_3 \end{pmatrix}$$

$$B = \begin{pmatrix} b_3 & b_1 - i\, b_2 \\ b_1 + i\, b_2 & -b_3 \end{pmatrix}$$

—was led to the **Bell inequality** that sparked the on-going *experimental* inquiry into the foundations of quantum mechanics, one early result of which was to rule out the possibility that behind quantum theory lurks a deterministic local hidden variable theory.

To lend quantitative precision to our intuitive allusion to "correlation", write

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

and define

```
correlation ≡ (averaged product) - (product of averages)
```

Are led by just such argumentation—analysis of Alice/Bob statistics—to define

In[41]:= `correlation[w_, x_, y_, z_] := w^2 - x^2 - y^2 + z^2 - (w^2 + x^2 - y^2 - z^2) (w^2 - x^2 + y^2 - z^2)`

which when applied to Ψ gives

In[42]:= `correlation[ψ1, ψ2, ψ3, ψ4]`

Out[42]= $\psi_1^2 - \psi_2^2 - \psi_3^2 + \psi_4^2 - (\psi_1^2 + \psi_2^2 - \psi_3^2 - \psi_4^2)(\psi_1^2 - \psi_2^2 + \psi_3^2 - \psi_4^2)$

and in the example just studied

In[43]:= $\Psi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$

supplies

In[43]:= $\text{correlation}\left[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]$

Out[43]= -1

If, on the other hand, God had respected the "physical independence" of Alice/Bob's systems we would have had

$$\Psi = \begin{pmatrix} \alpha_1\ \beta_1 \\ \alpha_1\ \beta_2 \\ \alpha_2\ \beta_1 \\ \alpha_2\ \beta_2 \end{pmatrix}$$

giving the intuitively more comprehensible result

In[44]:= `FullSimplify[correlation[α1 β1, α1 β2, α2 β1, α2 β2]]`

Out[44]= $-(\alpha_1^2 - \alpha_2^2)(\beta_1^2 - \beta_2^2)(-1 + (\alpha_1^2 + \alpha_2^2)(\beta_1^2 + \beta_2^2))$

In[45]:= $\% \,/.\, \{(\alpha_1^2 + \alpha_2^2) \to 1,\ (\beta_1^2 + \beta_2^2) \to 1\}$

Out[45]= 0

■ Entangled states of composite quantum systems

Correlation effects have been shown to be absent when the composite state vector Ψ possesses the Kronecker factored form $\alpha\otimes\beta$. In all other ("non-separable") cases they are present, and Ψ is said to describe an **entangled composite state**. One can then not speak separately of

- the quantum physics of $\mathcal{S}_{\text{Alice}}$
- the quantum physics of $\mathcal{S}_{\text{Bob}}$

but can speak only of

- the quantum physics of $\mathcal{S}_{\text{composite}} \equiv \mathcal{S}_{\text{Alice}} \otimes \mathcal{S}_{\text{Bob}}$.

It is the entanglement phenomenon that lies at the base of

- **quantum computation**
- **quantum encryption**
- **teleportation**, etc.

It was Schrödinger—writing in response to publication of the EPR paper—who coined the term "entanglement," and who wrote

"when two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives [the quantum states] have become entangled.

…Another way of expressing the peculiar situation is: the best possible knowledge of a *whole* does not necessarily include the best possible knowledge of all its *parts,* even though they may be entirely separate and therefore virtually capable of being 'best possibly known,' i.e., of possessing, each of them, a representative of its own. The lack of knowledge is by no means due to the interaction being insufficiently known — at least not in the way that it could possibly be known more completely — it is due to the interaction itself."

■ DECISION PROBLEM and the related QUANTIFICATION PROBLEM

I quote from "The decision problem for entanglement," by Wayne C. Myrvold (1997):

In March 1995, Abner Shimony [professor of philosophy at Boston University, and an influential theorist in this area] attended a conference held in honor of hte 60th anniversary of the famous EPR paper at Technion University in Haifa, Israel. Among the lecturers at the conference was Alain Aspect [a leading experimentalist]. In Paris, immediately after the conference, Abner had a dream in which Aspect…posed the problem: is it algorithmically decidable whether a given quantum mechanical state is decidable or not? Upon returning to the United States, Abner posed the question to me [Myrvold, then a recent PhD in philosophy, who specialized in "computability theory"].

[Within a couple of weeks, Myrvold had established that] the question is almost decidable—there is an algorithmic procedure that answers correctly in all but the borderline cases, and for these cases fails to terminate… The function that gives the minimum distance from Ψ to a product state is a computable function, even though there is no effective procedure that always produces the product state [in question]."

Beyond answering "yes/no" to the question "Is Ψ separable?" one would like, in non-separable cases, to be in position to respond to the question "How nearly separable is it; *how* entangled is the state Ψ?"

The literature provides many attempted answers to these questions.

- Many are elegantly abstruse;
- Some claim to be "stronger" than others;
- Most are computationally awkward.

■ Elementary solution of the decision problem

QUESTION: Do there exist normalized vectors $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ that satisfy the non-linear coupled system of equations

$$\begin{pmatrix} a_1\, b_1 \\ a_1\, b_2 \\ a_2\, b_1 \\ a_2\, b_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

where the ψs are the specified components of a normalized 4-vector?

REMARK: From

In[46]:= $\psi_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \psi_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ // MatrixForm

Out[46]//MatrixForm=

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

we see that it is always possible to display Ψ as a linear combination of not more than four separated terms. And the elegant Schmidt decomposition theorem—which is based upon the SVD of the associated *density matrix*

$$\rho \equiv \Psi\text{.Transpose}[\Psi]$$

—asserts that it is always possible to write

$$\Psi = \sqrt{\mu_1}\; u_1 \otimes v_1 + \sqrt{\mu_2}\; u_2 \otimes v_2$$

where u_1 and u_2 (ditto v_1 and v_2) are orthogonal 2-vectors. This method is restricted, however, to bipartite systems, cannot be extended to (for example) tripartite systems, where in separable/disentangled cases one has

In[47]:= $\Psi = \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ // MatrixForm

Out[47]//MatrixForm=

$$\begin{pmatrix} a_1\, b_1\, c_1 \\ a_1\, b_1\, c_2 \\ a_1\, b_2\, c_1 \\ a_1\, b_2\, c_2 \\ a_2\, b_1\, c_1 \\ a_2\, b_1\, c_2 \\ a_2\, b_2\, c_1 \\ a_2\, b_2\, c_2 \end{pmatrix}$$

Constructive solution of the decision problem

Step One:

Take logs, get

$$\begin{pmatrix} \text{Log}[a_1] + \text{Log}[b_1] \\ \text{Log}[a_1] + \text{Log}[b_2] \\ \text{Log}[a_2] + \text{Log}[b_1] \\ \text{Log}[a_2] + \text{Log}[b_2] \end{pmatrix} = \begin{pmatrix} \text{Log}[\psi_1] \\ \text{Log}[\psi_2] \\ \text{Log}[\psi_3] \\ \text{Log}[\psi_4] \end{pmatrix}$$

Step Two:

Write

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Log}[a_1] \\ \text{Log}[a_2] \\ \text{Log}[b_1] \\ \text{Log}[b_1] \end{pmatrix} = \begin{pmatrix} \text{Log}[\psi_1] \\ \text{Log}[\psi_2] \\ \text{Log}[\psi_3] \\ \text{Log}[\psi_4] \end{pmatrix}$$

Straightforward inversion is not possible, because

In[48]:= $\text{Det}\left[\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\right]$

Out[48]= 0

But the singularity (non-invertibility) of the matrix signals not that the equations cannot be solved, but only that the <u>solutions are non-unique</u>: from

In[49]:= $\text{Transpose}\left[\text{NullSpace}\left[\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\right]\right]$ `// MatrixForm`

Out[49]//MatrixForm=

$$\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

we see that to every purported solution vector

$$\begin{pmatrix} \text{Log}[a_1] \\ \text{Log}[a_2] \\ \text{Log}[b_1] \\ \text{Log}[b_1] \end{pmatrix}$$

can be added any vector of the form

$$\begin{pmatrix} -\lambda \\ -\lambda \\ \lambda \\ \lambda \end{pmatrix}$$

since

In[50]:= $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} . \begin{pmatrix} -\lambda \\ -\lambda \\ \lambda \\ \lambda \end{pmatrix} == \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Out[50]= `True`

We are led thus to write

$$\begin{pmatrix} \text{Log}[a_1] \\ \text{Log}[a_2] \\ \text{Log}[b_1] \\ \text{Log}[b_1] \end{pmatrix} = \text{PseudoInverse}\left[\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\right] . \begin{pmatrix} \text{Log}[\psi_1] \\ \text{Log}[\psi_2] \\ \text{Log}[\psi_3] \\ \text{Log}[\psi_4] \end{pmatrix} + \begin{pmatrix} -\lambda \\ -\lambda \\ \lambda \\ \lambda \end{pmatrix}$$

That the Moore-Penrose "pseudo-inverse"—which can be considered to be a child of the SVD—is in this intance a funny-looking thing

In[51]:= $\text{PseudoInverse}\left[\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\right]$ `// MatrixForm`

Out[51]//MatrixForm=

$$\begin{pmatrix} \frac{3}{8} & \frac{3}{8} & -\frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & -\frac{1}{8} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} & -\frac{1}{8} & \frac{3}{8} \end{pmatrix}$$

need not concern us. If we set

$$\lambda = \text{Log}[k]$$

then the preceding equation becomes

$$\begin{pmatrix} \text{Log}[k\,a_1] \\ \text{Log}[k\,a_2] \\ \text{Log}[b_1/k] \\ \text{Log}[b_1/k] \end{pmatrix} = \text{PseudoInverse}\left[\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\right] . \begin{pmatrix} \text{Log}[\psi_1] \\ \text{Log}[\psi_2] \\ \text{Log}[\psi_3] \\ \text{Log}[\psi_4] \end{pmatrix}$$

DIGRESSION: It helps to notice that matrices are lists of lists, and that functions like **`Log`** and **`Exp`** are, in *Mathematica*'s opinion, "listable" in the very useful sense that I now demonstrate:

In[52]:= $\mathbf{Log}\left[\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}\right]$ **// MatrixForm**

Out[52]//MatrixForm=

$$\begin{pmatrix}\mathrm{Log}[\psi_1]\\ \mathrm{Log}[\psi_2]\\ \mathrm{Log}[\psi_3]\\ \mathrm{Log}[\psi_4]\end{pmatrix}$$

In[53]:= **Exp[%] // MatrixForm**

Out[53]//MatrixForm=

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

End of digression.

We are placed thus in position to set

In[54]:= $\mathbf{Exp}\left[\begin{pmatrix}\mathrm{Log}[k\,a_1]\\ \mathrm{Log}[k\,a_2]\\ \mathrm{Log}[b_1/k]\\ \mathrm{Log}[b_1/k]\end{pmatrix}\right]$ **// MatrixForm**

Out[54]//MatrixForm=

$$\begin{pmatrix}k\,a_1\\ k\,a_2\\ \frac{b_1}{k}\\ \frac{b_1}{k}\end{pmatrix}$$

equal to

$$\mathrm{Exp}\left[\mathrm{PseudoInverse}\left[\begin{pmatrix}1&0&1&0\\1&0&0&1\\0&1&1&0\\0&1&0&1\end{pmatrix}\right].\mathrm{Log}\left[\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}\right]\right]$$

Or, still more efficiently, to write

In[55]:= $\begin{pmatrix}a_1\\ a_2\\ b_1\\ b_2\end{pmatrix} = \begin{pmatrix}1/k_a&0&0&0\\0&1/k_a&0&0\\0&0&k_b&0\\0&0&0&k_b\end{pmatrix}.\mathbf{Exp}\left[\mathbf{PseudoInverse}\left[\begin{pmatrix}1&0&1&0\\1&0&0&1\\0&1&1&0\\0&1&0&1\end{pmatrix}\right].\mathbf{Log}\left[\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}\right]\right]$

Out[55]= $\left\{\left\{\frac{\psi_1^{3/8}\,\psi_2^{3/8}}{k_a\,\psi_3^{1/8}\,\psi_4^{1/8}}\right\},\left\{\frac{\psi_3^{3/8}\,\psi_4^{3/8}}{k_a\,\psi_1^{1/8}\,\psi_2^{1/8}}\right\},\left\{\frac{k_b\,\psi_1^{3/8}\,\psi_3^{3/8}}{\psi_2^{1/8}\,\psi_4^{1/8}}\right\},\left\{\frac{k_b\,\psi_2^{3/8}\,\psi_4^{3/8}}{\psi_1^{1/8}\,\psi_3^{1/8}}\right\}\right\}$

In[56]:= `MatrixForm[%]`

Out[56]//MatrixForm=

$$\begin{pmatrix} \frac{\psi_1^{3/8}\,\psi_2^{3/8}}{k_a\,\psi_3^{1/8}\,\psi_4^{1/8}} \\ \frac{\psi_3^{3/8}\,\psi_4^{3/8}}{k_a\,\psi_1^{1/8}\,\psi_2^{1/8}} \\ \frac{k_b\,\psi_1^{3/8}\,\psi_3^{3/8}}{\psi_2^{1/8}\,\psi_4^{1/8}} \\ \frac{k_b\,\psi_2^{3/8}\,\psi_4^{3/8}}{\psi_1^{1/8}\,\psi_3^{1/8}} \end{pmatrix}$$

The multipliers k_a and k_b acquire values from the requirements that α and β be unit vectors:

In[57]:=

$$\text{Solve}\left[\left(\frac{\psi_1^{3/8}\,\psi_2^{3/8}}{k_a\,\psi_3^{1/8}\,\psi_4^{1/8}}\right)^2+\left(\frac{\psi_3^{3/8}\,\psi_4^{3/8}}{k_a\,\psi_1^{1/8}\,\psi_2^{1/8}}\right)^2==1,\ k_a\right]$$

$$\text{Solve}\left[\left(\frac{k_b\,\psi_1^{3/8}\,\psi_3^{3/8}}{\psi_2^{1/8}\,\psi_4^{1/8}}\right)^2+\left(\frac{k_b\,\psi_2^{3/8}\,\psi_4^{3/8}}{\psi_1^{1/8}\,\psi_3^{1/8}}\right)^2==1,\ k_b\right]$$

Out[57]=

$$\left\{\left\{k_a\to-\frac{i\sqrt{-\psi_1\,\psi_2-\psi_3\,\psi_4}}{\psi_1^{1/8}\,\psi_2^{1/8}\,\psi_3^{1/8}\,\psi_4^{1/8}}\right\},\left\{k_a\to\frac{i\sqrt{-\psi_1\,\psi_2-\psi_3\,\psi_4}}{\psi_1^{1/8}\,\psi_2^{1/8}\,\psi_3^{1/8}\,\psi_4^{1/8}}\right\}\right\}$$

Out[58]=

$$\left\{\left\{k_b\to-\frac{1}{\sqrt{\frac{\psi_1\,\psi_3+\psi_2\,\psi_4}{\psi_1^{1/4}\,\psi_2^{1/4}\,\psi_3^{1/4}\,\psi_4^{1/4}}}}\right\},\left\{k_b\to\frac{1}{\sqrt{\frac{\psi_1\,\psi_3+\psi_2\,\psi_4}{\psi_1^{1/4}\,\psi_2^{1/4}\,\psi_3^{1/4}\,\psi_4^{1/4}}}}\right\}\right\}$$

The composite state vector Ψ will be separable if and only if

$$k_a = k_b$$

FIRST EXAMPLE:

Look to the case

In[59]:=

$$\Psi=\begin{pmatrix}\sqrt{3/8}\\ \sqrt{3/8}\\ \sqrt{1/8}\\ \sqrt{1/8}\end{pmatrix};$$

We in this case have

In[60]:=

$$\begin{pmatrix}1/k_a & 0 & 0 & 0\\ 0 & 1/k_a & 0 & 0\\ 0 & 0 & k_b & 0\\ 0 & 0 & 0 & k_b\end{pmatrix}.\text{Exp}\left[\text{PseudoInverse}\left[\begin{pmatrix}1&0&1&0\\1&0&0&1\\0&1&1&0\\0&1&0&1\end{pmatrix}\right].\text{Log}[\Psi]\right]$$

Out[60]=

$$\left\{\left\{\frac{3^{3/8}}{2^{3/4}\,k_a}\right\},\left\{\frac{1}{2^{3/4}\,3^{1/8}\,k_a}\right\},\left\{\frac{3^{1/8}\,k_b}{2^{3/4}}\right\},\left\{\frac{3^{1/8}\,k_b}{2^{3/4}}\right\}\right\}$$

In[61]:= $\text{Solve}\left[\left(\frac{3^{3/8}}{2^{3/4}\,k_a}\right)^2+\left(\frac{1}{2^{3/4}\,3^{1/8}\,k_a}\right)^2==1,\ k_a\right]\ //\ \text{Simplify}$

$\text{Solve}\left[\left(\frac{3^{1/8}\,k_b}{2^{3/4}}\right)^2+\left(\frac{3^{1/8}\,k_b}{2^{3/4}}\right)^2==1,\ k_b\right]\ //\ \text{Simplify}$

Out[61]= $\left\{\left\{k_a\to-\frac{2^{1/4}}{3^{1/8}}\right\},\ \left\{k_a\to\frac{2^{1/4}}{3^{1/8}}\right\}\right\}$

Out[62]= $\left\{\left\{k_b\to-\frac{2^{1/4}}{3^{1/8}}\right\},\ \left\{k_b\to\frac{2^{1/4}}{3^{1/8}}\right\}\right\}$

Here $k_a = k_b$ so the composite state Ψ is disentangled/separable. Our calculation has supplied

In[63]:= $\alpha=\left\{\left\{\frac{3^{3/8}}{2^{3/4}\,k_a}\right\},\ \left\{\frac{1}{2^{3/4}\,3^{1/8}\,k_a}\right\}\right\}\ /.\ k_a\to\frac{2^{1/4}}{3^{1/8}}$

$\beta=\left\{\left\{\frac{3^{1/8}\,k_b}{2^{3/4}}\right\},\ \left\{\frac{3^{1/8}\,k_b}{2^{3/4}}\right\}\right\}\ /.\ k_b\to\frac{2^{1/4}}{3^{1/8}}$

Out[63]= $\left\{\left\{\frac{\sqrt{3}}{2}\right\},\ \left\{\frac{1}{2}\right\}\right\}$

Out[64]= $\left\{\left\{\frac{1}{\sqrt{2}}\right\},\ \left\{\frac{1}{\sqrt{2}}\right\}\right\}$

and it is indeed the case that

In[65]:= $\alpha\otimes\beta==\Psi$

Out[65]= True

SECOND EXAMPLE:

In[66]:= $\Psi=\frac{1}{\sqrt{8^2+10^2+12^2+15^2}}\begin{pmatrix}8\\10\\12\\15\end{pmatrix};$

In[67]:= $\begin{pmatrix}1/k_a&0&0&0\\0&1/k_a&0&0\\0&0&k_b&0\\0&0&0&k_b\end{pmatrix}.\text{Exp}\left[\text{PseudoInverse}\left[\begin{pmatrix}1&0&1&0\\1&0&0&1\\0&1&1&0\\0&1&0&1\end{pmatrix}\right].\text{Log}[\Psi]\right]$

Out[67]= $\left\{\left\{\frac{2\left(\frac{10}{1599}\right)^{1/4}}{k_a}\right\},\ \left\{\frac{\left(\frac{10}{533}\right)^{1/4}3^{3/4}}{k_a}\right\},\ \left\{2\left(\frac{3}{2665}\right)^{1/4}2^{3/4}\,k_b\right\},\ \left\{\left(\frac{3}{1066}\right)^{1/4}5^{3/4}\,k_b\right\}\right\}$

In[68]:= $$\text{Solve}\left[\left(\frac{2\,(\frac{10}{1599})^{1/4}}{k_a}\right)^2 + \left(\frac{(\frac{10}{533})^{1/4}\,3^{3/4}}{k_a}\right)^2 == 1,\ k_a\right]\ //\ N$$

$$\text{Solve}\left[\left(2\left(\frac{3}{2665}\right)^{1/4} 2^{3/4}\,k_b\right)^2 + \left(\left(\frac{3}{1066}\right)^{1/4} 5^{3/4}\,k_b\right)^2 == 1,\ k_b\right]\ //\ N$$

Out[68]= $\{\{k_a \to -1.01393\}, \{k_a \to 1.01393\}\}$

Out[69]= $\{\{k_b \to -1.01393\}, \{k_b \to 1.01393\}\}$

We have come again upon an disentangled case, and proceed now to check out that claim:

In[70]:= $$\alpha = \left\{\left\{\frac{2\,(\frac{10}{1599})^{1/4}}{k_a}\right\}, \left\{\frac{(\frac{10}{533})^{1/4}\,3^{3/4}}{k_a}\right\}\right\}\ /.\ k_a \to 1.0139337054294644`$$

$$\beta = \left\{\left\{2\left(\frac{3}{2665}\right)^{1/4} 2^{3/4}\,k_b\right\}, \left\{\left(\frac{3}{1066}\right)^{1/4} 5^{3/4}\,k_b\right\}\right\}\ /.\ k_b \to 1.0139337054294646`$$

Out[70]= {{0.5547}, {0.83205}}

Out[71]= {{0.624695}, {0.780869}}

In[72]:= $\alpha \otimes \beta - \Psi$ // Chop // MatrixForm

Out[72]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

THIRD EXAMPLE:

In[73]:= $$\Psi = \frac{1}{\sqrt{23^2 + 12^2 + 42^2 + 37^2}} \begin{pmatrix} 23 \\ 12 \\ 42 \\ 37 \end{pmatrix};$$

In[74]:= $$\begin{pmatrix} 1/k_a & 0 & 0 & 0 \\ 0 & 1/k_a & 0 & 0 \\ 0 & 0 & k_b & 0 \\ 0 & 0 & 0 & k_b \end{pmatrix}.\text{Exp}\left[\text{PseudoInverse}\left[\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\right].\text{Log}[\Psi]\right]\ //\ N$$

Out[74]= $$\left\{\left\{\frac{0.418112}{k_a}\right\}, \left\{\frac{0.992118}{k_a}\right\}, \{0.782215\,k_b\}, \{0.530309\,k_b\}\right\}$$

In[75]:= $$\text{Solve}\left[\left(\frac{0.41811171195346475`}{k_a}\right)^2 + \left(\frac{0.9921179316209712`}{k_a}\right)^2 == 1,\ k_a\right]\ //\ N$$

$$\text{Solve}[(0.7822153877636896`\,k_b)^2 + (0.5303093410572127`\,k_b)^2 == 1,\ k_b]\ //\ N$$

Out[75]= $\{\{k_a \to -1.07662\}, \{k_a \to 1.07662\}\}$

Out[76]= $\{\{k_b \to -1.05816\}, \{k_b \to 1.05816\}\}$

Here k_a and k_b are distinct, so Ψ is NOT separable, describes an entangled composite state. If we nevertheless construct

In[77]:= $\alpha = \left\{\left\{\frac{0.41811171195346475`}{k_a}\right\}, \left\{\frac{0.9921179316209712`}{k_a}\right\}\right\}$ /. k_a -> 1.0766222150394869`

β = {{0.7822153877636896` $k_b$}, {0.5303093410572127` k_b}} /. k_b -> 1.0581631947198626`

Out[77]= {{0.388355}, {0.92151}}

Out[78]= {{0.827712}, {0.561154}}

then

In[79]:= $\alpha \otimes \beta$ // MatrixForm

Out[79]//MatrixForm=

$$\begin{pmatrix} 0.321446 \\ 0.217927 \\ 0.762744 \\ 0.517109 \end{pmatrix}$$

describes a disentangled "neighbor"—can we argue that it is the "closest" such neighbor?—of the entangled state

In[80]:= Ψ // N // MatrixForm

Out[80]//MatrixForm=

$$\begin{pmatrix} 0.372815 \\ 0.194512 \\ 0.680793 \\ 0.599746 \end{pmatrix}$$

FOURTH EXAMPLE:

Look finally to the state

In[81]:= $\Psi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$;

that a playful God inflicted upon Alice. *Mathematica*, when asked to execute our algorithm, complains

In[82]:= $$\begin{pmatrix} 1/k_a & 0 & 0 & 0 \\ 0 & 1/k_a & 0 & 0 \\ 0 & 0 & k_b & 0 \\ 0 & 0 & 0 & k_b \end{pmatrix}.\mathbf{Exp}\left[\mathbf{PseudoInverse}\left[\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\right].\mathbf{Log}[\Psi]\right]$$

∞::indet : Indeterminate expression $-\infty + \infty - \frac{3\,\text{Log}[2]}{16} + \frac{\text{Log}[2]}{16}$ encountered. More…

∞::indet : Indeterminate expression $-\infty + \infty - \frac{3\,\text{Log}[2]}{16} + \frac{\text{Log}[2]}{16}$ encountered. More…

∞::indet : Indeterminate expression $-\infty + \infty - \frac{3\,\text{Log}[2]}{16} + \frac{\text{Log}[2]}{16}$ encountered. More…

General::stop : Further output of ∞::indet will be suppressed during this calculation. More…

Out[82]= {{Indeterminate}, {Indeterminate}, {Indeterminate}, {Indeterminate}}

because it has encountered terms of the indeterminate form **Log[0]**. These we eliminate by setting

In[83]:= $$\Psi_{God} = \begin{pmatrix} \epsilon \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \epsilon \end{pmatrix};$$

with the intention of letting $\epsilon \downarrow 0$ at the end of the day. We get

In[84]:= $$\begin{pmatrix} 1/k_a & 0 & 0 & 0 \\ 0 & 1/k_a & 0 & 0 \\ 0 & 0 & k_b & 0 \\ 0 & 0 & 0 & k_b \end{pmatrix}.\mathbf{Exp}\left[\mathbf{PseudoInverse}\left[\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\right].\mathbf{Log}[\Psi_{God}]\right]$$

Out[84]= $$\left\{\left\{\frac{\epsilon^{1/4}}{2^{1/8}\,k_a}\right\}, \left\{\frac{\epsilon^{1/4}}{2^{1/8}\,k_a}\right\}, \left\{\frac{\epsilon^{1/4}\,k_b}{2^{1/8}}\right\}, \left\{\frac{\epsilon^{1/4}\,k_b}{2^{1/8}}\right\}\right\}$$

Proceeding in the standard way

In[85]:= $$\mathbf{Solve}\left[\left(\frac{\epsilon^{1/4}}{2^{1/8}\,k_a}\right)^2 + \left(\frac{\epsilon^{1/4}}{2^{1/8}\,k_a}\right)^2 == 1,\ k_a\right]$$

$$\mathbf{Solve}\left[\left(\frac{\epsilon^{1/4}\,k_b}{2^{1/8}}\right)^2 + \left(\frac{\epsilon^{1/4}\,k_b}{2^{1/8}}\right)^2 == 1,\ k_b\right]$$

Out[85]= $\{\{k_a \to -2^{3/8}\,\epsilon^{1/4}\}, \{k_a \to 2^{3/8}\,\epsilon^{1/4}\}\}$

Out[86]= $$\left\{\left\{k_b \to -\frac{1}{2^{3/8}\,\epsilon^{1/4}}\right\}, \left\{k_b \to \frac{1}{2^{3/8}\,\epsilon^{1/4}}\right\}\right\}$$

we find that k_a and k_b are certainly unequal (the state Ψ is certainly entangled) and that in the limit

$k_a \downarrow 0$
$k_b \uparrow \infty$

k_a and k_b **could not be more different**!

The state Ψ_{God} was actually drawn from the population of so-called "Bell states," which are commonly held to be composite **states of maximal entanglement**.

■ Concluding comments

1. Quantification

One cannot speak of "maximal entanglement" except in reference to some well-defined *measure of entanglement* (though one can speak unambituously of the *absence* of entanglement). Of those, easily half a dozen have been proposed, of which the most widely accepted is the entropy of entanglement, the definition of which may be taken up in Julia Keller's thesis seminar, and is patterned upon von Neumann's definition of the thermodynamic

```
entropy of mixture = -Tr[ρ Log[ρ]]
```

where ρ is the density matrix. I am prepared, by the pattern of the preceding argument, to suggest this

In[87]:= `NewMeasureOfEntanglement[ka_, kb_] := 1 - e^-(kb-ka)^2`

$$\text{NewMeasureOfEntanglement}[ka\_,\ kb\_] := 1 - e^{-(kb-ka)^2}$$

For unentangled states we would then have

In[88]:= `NewMeasureOfEntanglement[ka, ka]`

Out[88]= 0

while for God's Bell state

In[89]:= `NewMeasureOfEntanglement[0, ∞]`

Out[89]= 1

2. Source of the k-ambiguity

In[90]:=
```
Clear[α, β]
a1 =.
a2 =.
b1 =.
b2 =.
```

The k-ambiguity has a very simple and very general origin, which I illustrate in the simplest instance. Suppose once again that

In[95]:= $\alpha = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix};$

$\beta = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix};$

Then

In[97]:= `α⊗β // MatrixForm`

Out[97]//MatrixForm=

$$\begin{pmatrix} a_1\, b_1 \\ a_1\, b_2 \\ a_2\, b_1 \\ a_2\, b_2 \end{pmatrix}$$

renders transparent the fact that

In[98]:= `(k α) ⊗ (β / k) == α⊗β`

Out[98]= `True`

3. It takes two to entangle, but…

The realization that entanglement makes a difference, joined to the observation that every "isolated" system a detached fragment of what was once a larger system, points up the fact that in quantum mechanics the "isolated system" concept is an idealization. But so also, for that same reason, is the bipartite "Alice/Bob system." One is driven to the view that in quantum physics there is really only one system, only one state function, and that refers to the whole universe! (But what, by any semi-standard interpretation of quantum mechanics, can one mean by Ψ_{universe}?)

The question therefore arises: Why does the "isolated system" concept enjoy such success in work-a-day quantum mechanics?

That entanglement does not make a difference in classical physics—that one classically *can* contemplate systems in isolation (I set aside "Mach's Principle" and all similarly vague suggestions)—emerges therefore as "the most characteristic feature of physics in the limit $\hbar \to 0$." And a major mystery, since $\hbar$ does, in point of physical fact, *not* $\to 0$: all "classical" systems are ultimately quantum mechanical.

4. Generalization to the case of tripartite 2-component composite systems

Introduce

In[99]:= $\gamma = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix};$

construct

In[100]:= `(α⊗β)⊗γ // MatrixForm`

Out[100]//MatrixForm=

$$\begin{pmatrix} a_1\,b_1\,c_1 \\ a_1\,b_1\,c_2 \\ a_1\,b_2\,c_1 \\ a_1\,b_2\,c_2 \\ a_2\,b_1\,c_1 \\ a_2\,b_1\,c_2 \\ a_2\,b_2\,c_1 \\ a_2\,b_2\,c_2 \end{pmatrix}$$

and attempt to solve

$$\begin{pmatrix} a_1\,b_1\,c_1 \\ a_1\,b_1\,c_2 \\ a_1\,b_2\,c_1 \\ a_1\,b_2\,c_2 \\ a_2\,b_1\,c_1 \\ a_2\,b_1\,c_2 \\ a_2\,b_2\,c_1 \\ a_2\,b_2\,c_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \end{pmatrix}$$

Immediately

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Log}[a_1] \\ \text{Log}[a_2] \\ \text{Log}[b_1] \\ \text{Log}[b_2] \\ \text{Log}[c_1] \\ \text{Log}[c_2] \end{pmatrix} = \text{Log}\left[\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \end{pmatrix}\right]$$

and from

In[101]:= `NullSpace`

$$\text{NullSpace}\left[\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}\right]$$

Out[101]= `{{-1, -1, 0, 0, 1, 1}, {-1, -1, 1, 1, 0, 0}}`

we are led to write

$$\begin{pmatrix} \text{Log}[a_1] \\ \text{Log}[a_2] \\ \text{Log}[b_1] \\ \text{Log}[b_2] \\ \text{Log}[c_1] \\ \text{Log}[c_2] \end{pmatrix} + \begin{pmatrix} \lambda \\ \lambda \\ 0 \\ 0 \\ -\lambda \\ -\lambda \end{pmatrix} + \begin{pmatrix} \mu \\ \mu \\ -\mu \\ -\mu \\ 0 \\ 0 \end{pmatrix} = \text{PseudoInverse}\left[\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}\right].\text{Log}\left[\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \end{pmatrix}\right]$$

Writing

$$\begin{pmatrix} \lambda \\ \lambda \\ 0 \\ 0 \\ -\lambda \\ -\lambda \end{pmatrix} = \begin{pmatrix} \text{Log}[k] \\ \text{Log}[k] \\ \text{Log}[1] \\ \text{Log}[1] \\ \text{Log}[1/k] \\ \text{Log}[1/k] \end{pmatrix}$$

$$\begin{pmatrix} \mu \\ \mu \\ -\mu \\ -\mu \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \text{Log}[m] \\ \text{Log}[m] \\ \text{Log}[1/m] \\ \text{Log}[1/m] \\ \text{Log}[1] \\ \text{Log}[1] \end{pmatrix}$$

we would proceed essentially as before, paying no heed to the fact that the pseudo-inverse is now even goofier:

In[102]:= $\text{PseudoInverse}\left[\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}\right] \text{ // MatrixForm}$

Out[102]//MatrixForm=

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} \\ -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{12} & -\frac{1}{12} \\ -\frac{1}{12} & -\frac{1}{12} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} \\ -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} & \frac{1}{6} \end{pmatrix}$$

I am encouraged to think that the method described here works even in the n-partite case. The Schmidt decomposition method and such other methods for resolving the decision problem as are presently known to me fail if $n > 2$.

Note finally that such "ambituity" as is present in the tripartite case can be traced to the obvious facts (1) that

In[103]:= $((p\,\alpha)\otimes(q\,\beta))\otimes(r\,\gamma) == (p\,q\,r)\;(\alpha\otimes\beta)\otimes\gamma$

Out[103]= True

and (2) that the expression on the right reduces to

$$(\alpha\otimes\beta)\otimes\gamma$$

when the constants **p**, **q**, **r** are constrained to satisfy

$$p\,q\,r = 1$$

The n-partite generalization of this remark is evident. As was previously remarked, Schmidt decomposition and many of the other procedures described in the literature are special to 2-partite systems.

■ Important ADDENDUM

It was as a matter of (I thought) mere expository convenience that I have heretofore assumed all vector components to be REAL. Here I discuss the consequence of relaxing that assumption. It is (really this time!) as a matter of mere expository convenience that I return to the particulars of my **FIRST EXAMPLE**

In[104]:= $\Psi = \begin{pmatrix} \sqrt{3/8} \\ \sqrt{3/8} \\ \sqrt{1/8} \\ \sqrt{1/8} \end{pmatrix};$

in which we make now the following adjustment:

In[105]:= $\Psi = \begin{pmatrix} \sqrt{3/8}\,e^{i\,\phi_1} \\ \sqrt{3/8}\,e^{i\,\phi_2} \\ \sqrt{1/8}\,e^{i\,\phi_3} \\ \sqrt{1/8}\,e^{i\,\phi_4} \end{pmatrix};$

Our algorithm then provides

In[106]:= $\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1/k_a & 0 & 0 & 0 \\ 0 & 1/k_a & 0 & 0 \\ 0 & 0 & k_b & 0 \\ 0 & 0 & 0 & k_b \end{pmatrix}.\mathrm{Exp}\Big[\mathrm{PseudoInverse}\Big[\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\Big].\mathrm{Log}[\Psi]\Big]$

Out[106]= $\Big\{\Big\{\frac{3^{3/8}\,(e^{i\,\phi_1})^{3/8}\,(e^{i\,\phi_2})^{3/8}}{2^{3/4}\,(e^{i\,\phi_3})^{1/8}\,(e^{i\,\phi_4})^{1/8}\,k_a}\Big\}, \Big\{\frac{(e^{i\,\phi_3})^{3/8}\,(e^{i\,\phi_4})^{3/8}}{2^{3/4}\,3^{1/8}\,(e^{i\,\phi_1})^{1/8}\,(e^{i\,\phi_2})^{1/8}\,k_a}\Big\},$
$\Big\{\frac{3^{1/8}\,(e^{i\,\phi_1})^{3/8}\,(e^{i\,\phi_3})^{3/8}\,k_b}{2^{3/4}\,(e^{i\,\phi_2})^{1/8}\,(e^{i\,\phi_4})^{1/8}}\Big\}, \Big\{\frac{3^{1/8}\,(e^{i\,\phi_2})^{3/8}\,(e^{i\,\phi_4})^{3/8}\,k_b}{2^{3/4}\,(e^{i\,\phi_1})^{1/8}\,(e^{i\,\phi_3})^{1/8}}\Big\}\Big\}$

In[107]:= **PowerExpand[%]**

Out[107]= $\left\{\left\{\frac{3^{3/8}\, e^{\frac{3 i \phi_1}{8}+\frac{3 i \phi_2}{8}-\frac{i \phi_3}{8}-\frac{i \phi_4}{8}}}{2^{3/4}\, k_a}\right\}, \left\{\frac{e^{-\frac{i \phi_1}{8}-\frac{i \phi_2}{8}+\frac{3 i \phi_3}{8}+\frac{3 i \phi_4}{8}}}{2^{3/4}\, 3^{1/8}\, k_a}\right\}, \left\{\frac{3^{1/8}\, e^{\frac{3 i \phi_1}{8}-\frac{i \phi_2}{8}+\frac{3 i \phi_3}{8}-\frac{i \phi_4}{8}}\, k_b}{2^{3/4}}\right\}, \left\{\frac{3^{1/8}\, e^{-\frac{i \phi_1}{8}+\frac{3 i \phi_2}{8}-\frac{i \phi_3}{8}+\frac{3 i \phi_4}{8}}\, k_b}{2^{3/4}}\right\}\right\}$

So we have

In[108]:= $\alpha = \left\{\left\{\frac{3^{3/8}\, e^{\frac{3 i \phi_1}{8}+\frac{3 i \phi_2}{8}-\frac{i \phi_3}{8}-\frac{i \phi_4}{8}}}{2^{3/4}\, k_a}\right\}, \left\{\frac{e^{-\frac{i \phi_1}{8}-\frac{i \phi_2}{8}+\frac{3 i \phi_3}{8}+\frac{3 i \phi_4}{8}}}{2^{3/4}\, 3^{1/8}\, k_a}\right\}\right\};$

$\beta = \left\{\left\{\frac{3^{1/8}\, e^{\frac{3 i \phi_1}{8}-\frac{i \phi_2}{8}+\frac{3 i \phi_3}{8}-\frac{i \phi_4}{8}}\, k_b}{2^{3/4}}\right\}, \left\{\frac{3^{1/8}\, e^{-\frac{i \phi_1}{8}+\frac{3 i \phi_2}{8}-\frac{i \phi_3}{8}+\frac{3 i \phi_4}{8}}\, k_b}{2^{3/4}}\right\}\right\};$

To properly manage the complex numbers I introduce this construction (due to Joel Franklin):

In[110]:=
```
CompConj[A_] := A /. Complex[0, n_] -> -Complex[0, n]
```

The squared norms of α and β are given now by expressions

In[111]:=
```
Transpose[CompConj[α]].α
Transpose[CompConj[β]].β
```

Out[111]= $\left\{\left\{\frac{1}{2\sqrt{2}\, 3^{1/4}\, k_a^2} + \frac{3^{3/4}}{2\sqrt{2}\, k_a^2}\right\}\right\}$

Out[112]= $\left\{\left\{\frac{3^{1/4}\, k_b^2}{\sqrt{2}}\right\}\right\}$

to which the phase factors make no contribution, so we are not surprised todiscover that

In[113]:= $\text{Solve}\left[\frac{1}{2\sqrt{2}\, 3^{1/4}\, k_a^2} + \frac{3^{3/4}}{2\sqrt{2}\, k_a^2} == 1,\ k_a\right]$

$\text{Solve}\left[\frac{3^{1/4}\, k_b^2}{\sqrt{2}} == 1,\ k_b\right]$

Out[113]= $\left\{\left\{k_a \to -\frac{2^{1/4}}{3^{1/8}}\right\}, \left\{k_a \to \frac{2^{1/4}}{3^{1/8}}\right\}\right\}$

Out[114]= $\left\{\left\{k_b \to -\frac{2^{1/4}}{3^{1/8}}\right\}, \left\{k_b \to \frac{2^{1/4}}{3^{1/8}}\right\}\right\}$

yield exactly the results that we obtained prior to the introduction of phase factors—results that led us then to the conclusion that Ψ was in this instance separable.

We have arrived now at the normalized complex vectors

In[115]:=
$$\alpha = \left\{\left\{\frac{3^{3/8}\, e^{\frac{3 i \phi_1}{8}+\frac{3 i \phi_2}{8}-\frac{i \phi_3}{8}-\frac{i \phi_4}{8}}}{2^{3/4}\, k_a}\right\}, \left\{\frac{e^{-\frac{i \phi_1}{8}-\frac{i \phi_2}{8}+\frac{3 i \phi_3}{8}+\frac{3 i \phi_4}{8}}}{2^{3/4}\, 3^{1/8}\, k_a}\right\}\right\} \;/.\; k_a \to \frac{2^{1/4}}{3^{1/8}}$$

$$\beta = \left\{\left\{\frac{3^{1/8}\, e^{\frac{3 i \phi_1}{8}-\frac{i \phi_2}{8}+\frac{3 i \phi_3}{8}-\frac{i \phi_4}{8}}\, k_b}{2^{3/4}}\right\}, \left\{\frac{3^{1/8}\, e^{-\frac{i \phi_1}{8}+\frac{3 i \phi_2}{8}-\frac{i \phi_3}{8}+\frac{3 i \phi_4}{8}}\, k_b}{2^{3/4}}\right\}\right\} \;/.\; k_b \to \frac{2^{1/4}}{3^{1/8}}$$

Out[115]=
$$\left\{\left\{\frac{1}{2}\sqrt{3}\, e^{\frac{3 i \phi_1}{8}+\frac{3 i \phi_2}{8}-\frac{i \phi_3}{8}-\frac{i \phi_4}{8}}\right\}, \left\{\frac{1}{2}\, e^{-\frac{i \phi_1}{8}-\frac{i \phi_2}{8}+\frac{3 i \phi_3}{8}+\frac{3 i \phi_4}{8}}\right\}\right\}$$

Out[116]=
$$\left\{\left\{\frac{e^{\frac{3 i \phi_1}{8}-\frac{i \phi_2}{8}+\frac{3 i \phi_3}{8}-\frac{i \phi_4}{8}}}{\sqrt{2}}\right\}, \left\{\frac{e^{-\frac{i \phi_1}{8}+\frac{3 i \phi_2}{8}-\frac{i \phi_3}{8}+\frac{3 i \phi_4}{8}}}{\sqrt{2}}\right\}\right\}$$

We now construct

In[117]:=
```
α⊗β // MatrixForm
```

Out[117]//MatrixForm=
$$\begin{pmatrix} \frac{1}{2}\sqrt{\frac{3}{2}}\, e^{\frac{3 i \phi_1}{4}+\frac{i \phi_2}{4}+\frac{i \phi_3}{4}-\frac{i \phi_4}{4}} \\ \frac{1}{2}\sqrt{\frac{3}{2}}\, e^{\frac{i \phi_1}{4}+\frac{3 i \phi_2}{4}-\frac{i \phi_3}{4}+\frac{i \phi_4}{4}} \\ \frac{e^{\frac{i \phi_1}{4}-\frac{i \phi_2}{4}+\frac{3 i \phi_3}{4}+\frac{i \phi_4}{4}}}{2\sqrt{2}} \\ \frac{e^{-\frac{i \phi_1}{4}+\frac{i \phi_2}{4}+\frac{i \phi_3}{4}+\frac{3 i \phi_4}{4}}}{2\sqrt{2}} \end{pmatrix}$$

and find that to recover the given 4-vector

In[118]:=
```
Ψ // MatrixForm
```

Out[118]//MatrixForm=
$$\begin{pmatrix} \frac{1}{2}\sqrt{\frac{3}{2}}\, e^{i \phi_1} \\ \frac{1}{2}\sqrt{\frac{3}{2}}\, e^{i \phi_2} \\ \frac{e^{i \phi_3}}{2\sqrt{2}} \\ \frac{e^{i \phi_4}}{2\sqrt{2}} \end{pmatrix}$$

it must be the case that

$$\frac{3 i \phi_1}{4}+\frac{i \phi_2}{4}+\frac{i \phi_3}{4}-\frac{i \phi_4}{4} == i \phi_1$$
$$\frac{i \phi_1}{4}+\frac{3 i \phi_2}{4}-\frac{i \phi_3}{4}+\frac{i \phi_4}{4} == i \phi_2$$
$$\frac{i \phi_1}{4}-\frac{i \phi_2}{4}+\frac{3 i \phi_3}{4}+\frac{i \phi_4}{4} == i \phi_3$$
$$-\frac{i \phi_1}{4}+\frac{i \phi_2}{4}+\frac{i \phi_3}{4}+\frac{3 i \phi_4}{4} == i \phi_4$$

These equations can be formulated

$$\begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{pmatrix} . \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} == \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where, as it happens,

In[119]:= **Det**$\left[\begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{pmatrix}\right]$

Out[119]= 0

Looking to

In[120]:= **NullSpace**$\left[\begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{pmatrix}\right]$

Out[120]= {{-1, 0, 0, 1}, {1, 0, 1, 0}, {1, 1, 0, 0}}

we conclude that the ϕs can be described

In[121]:= $\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

Out[121]= {{x + y + z}, {x}, {y}, {-z}}

where **x**, **y** and **z** are arbitrary constants. The vector $\alpha \otimes \beta$ therefore becomes

In[122]:= $\begin{pmatrix} \frac{1}{2}\sqrt{\frac{3}{2}}\, e^{\frac{3 i \phi_1}{4} + \frac{i \phi_2}{4} + \frac{i \phi_3}{4} - \frac{i \phi_4}{4}} \\ \frac{1}{2}\sqrt{\frac{3}{2}}\, e^{\frac{i \phi_1}{4} + \frac{3 i \phi_2}{4} - \frac{i \phi_3}{4} + \frac{i \phi_4}{4}} \\ \frac{e^{\frac{i \phi_1}{4} - \frac{i \phi_2}{4} + \frac{3 i \phi_3}{4} + \frac{i \phi_4}{4}}}{2\sqrt{2}} \\ \frac{e^{-\frac{i \phi_1}{4} + \frac{i \phi_2}{4} + \frac{i \phi_3}{4} + \frac{3 i \phi_4}{4}}}{2\sqrt{2}} \end{pmatrix}$ /. $\{\phi_1 \to x + y + z,\ \phi_2 \to x,\ \phi_3 \to y,\ \phi_4 \to -z\}$

Out[122]= $\left\{\left\{\frac{1}{2}\sqrt{\frac{3}{2}}\, e^{\frac{i x}{4} + \frac{i y}{4} + \frac{i z}{4} + \frac{3}{4} i (x+y+z)}\right\}, \left\{\frac{1}{2}\sqrt{\frac{3}{2}}\, e^{\frac{3 i x}{4} - \frac{i y}{4} - \frac{i z}{4} + \frac{1}{4} i (x+y+z)}\right\}, \left\{\frac{e^{-\frac{i x}{4} + \frac{3 i y}{4} - \frac{i z}{4} + \frac{1}{4} i (x+y+z)}}{2\sqrt{2}}\right\}, \left\{\frac{e^{\frac{i x}{4} + \frac{i y}{4} - \frac{3 i z}{4} - \frac{1}{4} i (x+y+z)}}{2\sqrt{2}}\right\}\right\}$

In[123]:= `Simplify[%]`

Out[123]= $\left\{\left\{\frac{1}{2}\sqrt{\frac{3}{2}}\,e^{i\,(x+y+z)}\right\},\left\{\frac{1}{2}\sqrt{\frac{3}{2}}\,e^{i\,x}\right\},\left\{\frac{e^{i\,y}}{2\sqrt{2}}\right\},\left\{\frac{e^{-i\,z}}{2\sqrt{2}}\right\}\right\}$

which with the restoration of ϕ-notation becomes

In[149]:=

ϕ_1 =.
ϕ_2 =.
ϕ_3 =.
ϕ_4 =.

In[153]:= $\left\{\left\{\frac{1}{2}\sqrt{\frac{3}{2}}\,e^{i\,(x+y+z)}\right\},\left\{\frac{1}{2}\sqrt{\frac{3}{2}}\,e^{i\,x}\right\},\left\{\frac{e^{i\,y}}{2\sqrt{2}}\right\},\left\{\frac{e^{-i\,z}}{2\sqrt{2}}\right\}\right\}$ /.
{x → ϕ_2, y → ϕ_3, z → -ϕ_4} // MatrixForm

Out[153]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\sqrt{\frac{3}{2}}\,e^{i\,(\phi_2+\phi_3-\phi_4)} \\ \frac{1}{2}\sqrt{\frac{3}{2}}\,e^{i\,\phi_2} \\ \frac{e^{i\,\phi_3}}{2\sqrt{2}} \\ \frac{e^{i\,\phi_4}}{2\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2}\sqrt{\frac{3}{2}}\,e^{i\,(\phi_2+\phi_3-\phi_4)} \\ \frac{1}{2}\sqrt{\frac{3}{2}}\,e^{i\,\phi_2} \\ \frac{1}{2\sqrt{2}}\,e^{i\,\phi_3} \\ \frac{1}{2\sqrt{2}}\,e^{i\,\phi_4} \end{pmatrix}$$

while—by construction, but gratifyingly—from the given 4-vector Ψ we now recover the same thing:

In[154]:= Ψ /. {ϕ_1 → ϕ_2 + ϕ_3 - ϕ_4} // MatrixForm

Out[154]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\sqrt{\frac{3}{2}}\,e^{i\,(\phi_2+\phi_3-\phi_4)} \\ \frac{1}{2}\sqrt{\frac{3}{2}}\,e^{i\,\phi_2} \\ \frac{e^{i\,\phi_3}}{2\sqrt{2}} \\ \frac{e^{i\,\phi_4}}{2\sqrt{2}} \end{pmatrix}$$

We conclude that the separability decision problem splits into two independent problems:

- the **DECISION PROBLEM WITH RESPECT TO AMPLITUDES**, and
- the **DECISION PROBLEM WITH RESPECT TO PHASES**.

The former is resolved by the procedure discussed in the main body of the text, while the latter is resolved by the statement that for

$$\begin{pmatrix} \Psi_1\ e^{i\,\phi_1} \\ \Psi_2\ e^{i\,\phi_2} \\ \Psi_3\ e^{i\,\phi_3} \\ \Psi_4\ e^{i\,\phi_4} \end{pmatrix}$$

to be separable it must be the case that

$$\phi_1 = \phi_2 + \phi_3 - \phi_4$$

It then follows (I revert now to my **EXAMPLE**)

In[155]:=
$$\alpha = \left\{\left\{\frac{1}{2}\sqrt{3}\ e^{\frac{3 i \phi_1}{8}+\frac{3 i \phi_2}{8}-\frac{i \phi_3}{8}-\frac{i \phi_4}{8}}\right\}, \left\{\frac{1}{2}\ e^{-\frac{i \phi_1}{8}-\frac{i \phi_2}{8}+\frac{3 i \phi_3}{8}+\frac{3 i \phi_4}{8}}\right\}\right\} \text{ /. } \phi_1 \to \phi_2 + \phi_3 - \phi_4 \text{ // Simplify}$$

$$\beta = \left\{\left\{\frac{e^{\frac{3 i \phi_1}{8}-\frac{i \phi_2}{8}+\frac{3 i \phi_3}{8}-\frac{i \phi_4}{8}}}{\sqrt{2}}\right\}, \left\{\frac{e^{-\frac{i \phi_1}{8}+\frac{3 i \phi_2}{8}-\frac{i \phi_3}{8}+\frac{3 i \phi_4}{8}}}{\sqrt{2}}\right\}\right\} \text{ /. } \phi_1 \to \phi_2 + \phi_3 - \phi_4 \text{ // Simplify}$$

Out[155]=
$$\left\{\left\{\frac{1}{2}\sqrt{3}\ e^{\frac{1}{4} i (3 \phi_2+\phi_3-2 \phi_4)}\right\}, \left\{\frac{1}{2}\ e^{-\frac{1}{4} i (\phi_2-\phi_3-2 \phi_4)}\right\}\right\}$$

Out[156]=
$$\left\{\left\{\frac{e^{\frac{1}{4} i (\phi_2+3 \phi_3-2 \phi_4)}}{\sqrt{2}}\right\}, \left\{\frac{e^{\frac{1}{4} i (\phi_2-\phi_3+2 \phi_4)}}{\sqrt{2}}\right\}\right\}$$

In[157]:=
```
α // MatrixForm
β // MatrixForm
```

Out[157]//MatrixForm=
$$\begin{pmatrix} \frac{1}{2}\sqrt{3}\ e^{\frac{1}{4} i (3 \phi_2+\phi_3-2 \phi_4)} \\ \frac{1}{2}\ e^{-\frac{1}{4} i (\phi_2-\phi_3-2 \phi_4)} \end{pmatrix}$$

Out[158]//MatrixForm=
$$\begin{pmatrix} \frac{e^{\frac{1}{4} i (\phi_2+3 \phi_3-2 \phi_4)}}{\sqrt{2}} \\ \frac{e^{\frac{1}{4} i (\phi_2-\phi_3+2 \phi_4)}}{\sqrt{2}} \end{pmatrix}$$

Notice finally that from

In[159]:=
```
ϕ1[ϕ2_, ϕ3_, ϕ4_] := ϕ2 + ϕ3 - ϕ4
```

it follows that if

$$\phi_2 \to \phi_2 + \chi$$
$$\phi_3 \to \phi_3 + \chi$$
$$\phi_4 \to \phi_4 + \chi$$

then by

In[160]:= $\phi_1[\phi_2 + \chi, \phi_3 + \chi, \phi_4 + \chi]$

Out[160]= $\chi + \phi_2 + \phi_3 - \phi_4$

it is automatic that

$$\phi_1 \to \phi_1 + \chi$$

and in net effect we have the unphysical phase adjustment

$$\Psi \to \Psi\, e^{i\chi}$$

In[129]:=

Looking to the induced response of the phases that enter into the descriptions of α and β, we have

$$\frac{1}{4}(3\phi_2 + \phi_3 - 2\phi_4) \to \frac{1}{4}(3\phi_2 + \phi_3 - 2\phi_4) + \frac{1}{2}\chi$$
$$-\frac{1}{4}(\phi_2 - \phi_3 - 2\phi_4) \to -\frac{1}{4}(\phi_2 - \phi_3 - 2\phi_4) + \frac{1}{2}\chi$$

$$\frac{1}{4}(\phi_2 + 3\phi_3 - 2\phi_4) \to \frac{1}{4}(\phi_2 + 3\phi_3 - 2\phi_4) + \frac{1}{2}\chi$$
$$\frac{1}{4}(\phi_2 - \phi_3 + 2\phi_4) \to \frac{1}{4}(\phi_2 - \phi_3 + 2\phi_4) + \frac{1}{2}\chi$$

We have here traced $\Psi \to \Psi\, e^{i\chi}$ back to the statements

$$\alpha \to \alpha\, e^{i\chi/2}$$
$$\beta \to \beta\, e^{i\chi/2}$$

We note belatedly that the multipliers k_a and k_b should, in the complex formalism, be endowed with phase factors, which (as an examination of the argument would show) remain subject to no restriction. Thus are we led to the unphysical phase adjustments

$$\alpha \to \alpha\, e^{i\chi_a/2}$$
$$\beta \to \beta\, e^{i\chi_b/2}$$

which induce this equally unphysical composite companion:

$$\Psi \to \Psi\, e^{i(\chi_a + \chi_b)}$$