Causality violation by localized wavepackets

It can be easily shown that localized wavepackets spread instantaneously to arbitrary distances. The evolution Hamiltonian can be relativistic, implying that this is not just an approximate non-relativistic result and may be testable. It has been argued that one way out of this apparent paradox is the claim that QFT prevents precise localization of wavefunctions due to vacuum fluctuations. However, there are low-energy systems (e.g. quantum dots in graphene) which admit of ultra-localized wavefunctions and for which QFT effects can be safely ignored. The evolution operator for an exact relativistic Hamiltonian would then spread the wavefunction instantaneously and acausally. We present here a particular example, that of a wavefunction which is initially localized. The analysis can be extended to exact wavefunctions of particles in various boxes.

Consider the following wavefunction evolution

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp_x \, e^{i[p_x x - E(p_x)t]/\hbar} \, \phi(p_x) \; , \tag{1}$$

$$\text{where} \quad \phi(p_x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \, e^{-ip_x x/\hbar} \, \psi(x,0) \; . \tag{2}$$

$$\text{Therefore} \quad \psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp_x \, e^{i[p_x x - E(p_x)t]/\hbar} \, \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx' \, e^{-ip_x x'/\hbar} \, \psi(x') \; . \tag{3}$$

From Eq.(3) one can see that $\psi(x,t) \neq 0 \; \forall x \neq 0$ at any $t > 0$.

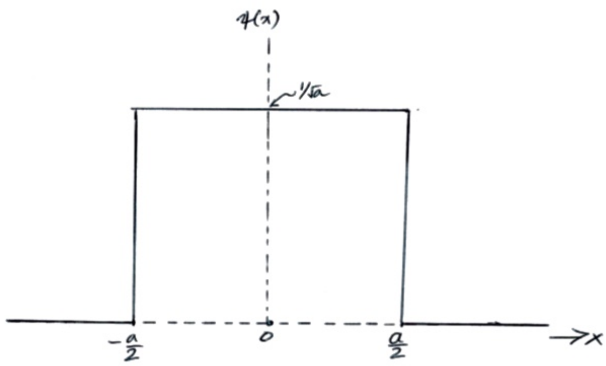
Choose as $\psi(x,0)$ the localized rectangular function, as seen in the accompanying figure

$$\psi(x,0) \;=\; \frac{1}{\sqrt{a}} \;,\; |x| \leq \frac{a}{2} \;, \tag{4}$$

$$\;=\; 0 \;,\; |x| > \frac{a}{2} \;, \tag{5}$$

$$\int_{-\infty}^{\infty} dx \, |\psi(x,0)|^2 \;=\; 1 \;, \tag{6}$$

for example for a localized quantum dot.



Then

$$\begin{aligned} \phi(p_x) &= -\sqrt{\frac{2\hbar a}{\pi}}\ \frac{\sin(p_x a/2\hbar)}{p_x a} && (7)\\ \psi(x,t) &= -\frac{\sqrt{a}}{\pi}\int_{-\infty}^{\infty} dp_x\, e^{i(p_x x - E(p_x)t)/\hbar}\ \frac{\sin(p_x a/2\hbar)}{p_x a} && (8)\\ &= -\frac{\sqrt{a}}{\pi}\int_{-\infty}^{\infty} dp_x\, e^{i(p_x x - \sqrt{(pc)^2+(mc^2)^2}\,t)/\hbar}\ \frac{\sin(p_x a/2\hbar)}{p_x a} && (9)\\ \text{Using,}\ \ E &= \sqrt{(pc)^2+(mc^2)^2}\ . && (10) \end{aligned}$$

Next, assume

$$\begin{aligned} x &= n_1 a && (11)\\ ct &= n_2 a, && (12)\\ N &\equiv \frac{n_1}{n_2} \geq 1 && (13)\\ \lambda_c &= \frac{\hbar}{mc} = n_3 a\ \ \text{(Compton wavelength)} && (14) \end{aligned}$$

Then defining the dimensionless quantity $z \equiv p_x a/\hbar$, we have from (9)

$$\psi(z,t) = \psi(n_1, n_2, n_3) = -\frac{1}{\pi\sqrt{a}}\int_{-\infty}^{\infty} dz\, e^{i\left(n_1\, z - n_2\ ,\sqrt{\frac{1}{n_3^2}+z^2}\ \right)}\ \frac{\sin(p_x a/2\hbar)}{p_x a} \tag{15}$$

Furthermore, for simplicity we assume

$$n_2 = n_3 = 1\ . \tag{16}$$

Then from (9), defining the dimensionless quantity $z \equiv p_x a/\hbar$ we get

$$\psi(N) = -\frac{1}{\pi\sqrt{a}}\int_{-\infty}^{\infty} dz\, e^{i(Nz - \sqrt{1+z^2})}\ \frac{\sin(z/2)}{z}\ . \tag{17}$$

$N = 1$ signifies the light cone centred at at $x = 0$ and starting at $t = 0$. $N < 1$ is inside the lightcone and $N > 1$ outside the lightcone. Therefore for any $N \geq 1$, $\psi(N) \neq 0$ would imply a violation of causality. In fact, we have

$$\begin{aligned} |\psi(N)|^2 \times a &= 0.25\ ,\ N = 1 && (18)\\ &= 5.2\times 10^{-6}\ ,\ N = 10\ . && (19) \end{aligned}$$

In other words, there can be as large as 25% probability of finding the particle within one Compton wavelength, but outside its initial lightcone, and hence that of causality violation under very simple assumptions! Question is, if it can be tested in the laboratory.