Remarks concerning the explicit construction of

SPIN MATRICES FOR ARBITRARY SPIN

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Introduction. This subject lies a bit skew to my main areas of interest. I feel a need, therefore, to explain why, somewhat to my surprise, I find myself writing about it today—lacing my boots, as it were, in preparation for the hike that will take me to places in which I do have an established interest.

On 3 November 1999, Thomas Wieting presented at Reed College a physics seminar entitled "The Penrose Dodecahedron." That talk was inspired by the recent publication of a paper by Jordan E. Massad & P. K. Arvind,[1] which in turn derived from an idea sketched in Roger Penrose's *Shadows of the Mind* (1994) and developed in greater detail in a paper written collaboratively by Penrose and J. Zamba.[2] Massad & Aravind speculate that Penrose drew his inspiration from Asher Peres, but there is no need to speculate in this regard; a little essay entitled "The Artist, the Physicist and the Waterfall" which appears on page 30 of the February 1993 issue of Scientific American describes the entangled source of Penrose's involvement in fascinating detail.[3] Not at all speculative is their observation that Penrose/Zamba drew critically upon an idea incidental to work reported by the then 25-year-old Etorre Majorana in a classic paper having to do with the spin-reversal of polarized particles/atoms in time-dependent magnetic fields.[4] Massad/Aravind remark that "whatever the genesis of [Penrose/Zamba's line of argument, they] deftly manipulated a *geometrical picture of quantum spins due to Majorana* to deduce all the properties of [certain] rays they needed in their proofs of the Bell and Bell-Kochen-Specker theorems."

[1] "The Penrose dodecahedron revisited," AJP **67**, 631 (1999).

[2] "On Bell non-locality without probabilities: More curious geometry," Stud. Hist. Phil. Sci. **24**, 697 (1993). The paper is based upon Zamba's thesis.

[3] The story involves Maurits C. Escher in an unexpected way, but hinges on the circumstance that Penrose was present at a seminar given by Peres in 1991, the upshot of which was published as "Two simple proofs of the Kochen-Specker theorem," J. Phys. A: Math. Gen. **24**, L175 (1991). See also Peres' *Quantum Theory: Concepts & Methods* (1995).

[4] "Atomi orientati in campo magnetico variabile," Nuovo Cimento **9**, 43–50 (1932).

Massad & Avarind suggest that "an even bigger virtue of the Majorana approach [bigger, that is to say, than its computational efficiency] is that it seems to have hinted at the existence of the Penrose dodecahedron in the first place," but observe that "the Majorana picture of spin, while very elegant and also remarkably economical for the problem at hand, is still unfamiliar to the vast majority of physicists," so have undertaken to construct a "poor man's version" of the Penrose/Zamba paper which proceeds entirely without reference to Majorana's techniques.

In their §2, Massad & Aravind draw upon certain results which they consider "standard" to the quantum theory of angular momentum, citing such works as the text by J. J. Sakuri[5] and the monograph by A. R. Edmonds.[6] But I myself have never had specific need of that "standard" material, and have always found it—at least as it is presented in the sources most familiar to me[7]—to be so offputtingly ugly that I have been happy to avoid it; what may indeed be "too familiar for comment" to many/most of the world's quantum physicists remains, I admit, only distantly familiar to me. So I confronted afresh this elementary question:

What *are* the angular momentum matrices
characteristic of a particle of (say) spin $\frac{3}{2}$?

My business here is to describe a non-standard approach to the solution of such questions, and to make some comments.

SU(2), O(3) and all that. I begin with brief review of some standard material, partly to underscore how elementary are the essential ideas, and partly to establish some notational conventions. It is upon this elementary foundation that we will build.

Let $\mathbb{M}$ be any 2×2 matrix. From $\det(\mathbb{M}-\lambda\mathbb{I}) = \lambda^2 - (\mathrm{tr}\,\mathbb{M})\lambda + \det\mathbb{M}$ we by the Cayley-Hamilton theorem have

$$\mathbb{M}^{-1} = \frac{(\mathrm{tr}\,\mathbb{M})\,\mathbb{I} - \mathbb{M}}{\det\mathbb{M}} \quad \text{unless } \mathbb{M} \text{ is singular}$$

which gives

$$\mathbb{M}^{-1} = (\mathrm{tr}\,\mathbb{M})\mathbb{I} - \mathbb{M} \quad \text{if } \mathbb{M} \text{ is "unimodular"} \;:\; \det\mathbb{M} = 1$$

Trivially, any 2×2 matrix $\mathbb{M}$ can be written

$$\mathbb{M} \quad = \tfrac{1}{2}(\mathrm{tr}\,\mathbb{M})\,\mathbb{I} + \underbrace{\big\{\mathbb{M} - \tfrac{1}{2}(\mathrm{tr}\,\mathbb{M})\,\mathbb{I}\big\}}_{\text{traceless}} \tag{1.1}$$

while we have just established that in unimodular cases

$$\mathbb{M}^{-1} = \tfrac{1}{2}(\mathrm{tr}\,\mathbb{M})\,\mathbb{I} - \big\{\mathbb{M} - \tfrac{1}{2}(\mathrm{tr}\,\mathbb{M})\,\mathbb{I}\big\} \tag{1.2}$$

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[5] *Modern Quantum Mechanics* (1994).

[6] *Angular Momentum in Quantum Mechanics* (1960).

[7] Leonard Schiff, *Quantum Mechanics* (3rd edition 1968), pages 202–203; J. L. Powell & B. Crasemann, *Quantum Mechanics* (1961), §10–4.

Evidently such a matrix will be *unitary* if and only if

- $a_0 \equiv \frac{1}{2}\mathrm{tr}\mathbb{M}$ is real, and
- $\mathbb{A} \equiv \{\mathbb{M} - \frac{1}{2}(\mathrm{tr}\mathbb{M})\mathbb{I}\}$ is antihermitian: $\mathbb{A}^{\mathrm{t}} = -\mathbb{A}$.

The Pauli matrices

$$\boldsymbol{\sigma}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

are traceless hermitian, and permit the most general such $\mathbb{A}$-matrix to be notated

$$\mathbb{A} = i\{a_1\boldsymbol{\sigma}_1 + a_2\boldsymbol{\sigma}_2 + a_3\boldsymbol{\sigma}_3\}$$

We then have

$$\mathbb{M} = \begin{pmatrix} a_0 + ia_3 & a_2 + ia_1 \\ -a_2 + ia_1 & a_0 - ia_3 \end{pmatrix} \tag{3.1}$$

and recover unimodularity by imposing upon the real numbers $\{a_0, a_1, a_2, a_3\}$ the requirement that

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 \tag{3.2}$$

Such matrices with henceforth be denoted $\mathbb{S}$, to emphasize that they have been $\mathbb{S}$pecialized; they provide the natural representation of the *special unimodular group* $SU(2)$. In terms of the so-called *Cayley-Klein parameters*[8]

$$\left.\begin{aligned} \alpha &\equiv a_0 + ia_3 \\ \beta &\equiv a_2 + ia_1 \end{aligned}\right\} \tag{4}$$

we have

$$\mathbb{S} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad \text{with} \quad \alpha^*\alpha + \beta^*\beta = 1 \tag{5}$$

Note the several points at which the argument has hinged on circumstances special to the 2-dimensional case.

Swifter and more familiar is the line of argument that proceeds from the observation that

$$\mathbb{S} = e^{i\mathbb{H}}$$

is unitary if and only if $\mathbb{H}$ is hermitian, and by $\det\mathbb{S} = e^{\mathrm{tr}(i\mathbb{H})}$ will be unimodular if and only if $\mathbb{H}$ is also traceless. The most general such $\mathbb{H}$ can in 2-dimensions be written

$$\mathbb{H} = h_1\boldsymbol{\sigma}_1 + h_2\boldsymbol{\sigma}_2 + h_3\boldsymbol{\sigma}_3 = \begin{pmatrix} h_3 & h_1 - ih_2 \\ h_1 + ih_2 & -h_3 \end{pmatrix}$$

Then

$$\mathbb{H}^2 = (h_1^2 + h_2^2 + h_3^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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[8] See §4–5 in H. Goldstein, *Classical Mechanics* (2^{nd} edition 1980).

which implies and is implied by the statements

$$\boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_2^2 = \boldsymbol{\sigma}_3^2 = \mathbb{I} \tag{6.1}$$

$$\left.\begin{aligned} \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 &= -\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 &= -\boldsymbol{\sigma}_3\boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 &= -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 \end{aligned}\right\} \tag{6.2}$$

Also demonstrably true—but *not* implicit in the preceding line of argument—are the statements

$$\left.\begin{aligned} \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 &= i\,\boldsymbol{\sigma}_3 \\ \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 &= i\,\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 &= i\,\boldsymbol{\sigma}_2 \end{aligned}\right\} \tag{6.3}$$

Results now in hand lead us to write

$$\boldsymbol{h} = \theta\,\boldsymbol{k} \quad : \quad \boldsymbol{k} \text{ a real unit 3-vector}$$

and obtain

$$\begin{aligned} \mathbb{S} &= \exp\Big[i\theta\big\{k_1\boldsymbol{\sigma}_1 + k_2\boldsymbol{\sigma}_2 + k_3\boldsymbol{\sigma}_3\big\}\Big] \\ &= \cos\theta\cdot\mathbb{I} + i\sin\theta\cdot\big\{k_1\boldsymbol{\sigma}_1 + k_2\boldsymbol{\sigma}_2 + k_3\boldsymbol{\sigma}_3\big\} \qquad (7.1) \\ &= \begin{pmatrix} \cos\theta + ik_3\sin\theta & (k_2+ik_1)\sin\theta \\ -(k_2-ik_1)\sin\theta & \cos\theta - ik_3\sin\theta \end{pmatrix} \qquad (7.2) \end{aligned}$$

in which notation the Cayley-Klein parameters (4) become

$$\left.\begin{aligned} \alpha &= \cos\theta + ik_3\sin\theta \\ \beta &= (k_2+ik_1)\sin\theta \end{aligned}\right\} \tag{8}$$

The Cayley-Klein parameters were introduced by Felix Klein in connection with the theory of tops; i.e., as aids to the description of the 3-dimensional rotation group $O(3)$, with which we have now to establish contact. This we will accomplish in two different ways. The more standard approach is to introduce the traceless hermitian matrix

$$\mathbb{X} \equiv x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + x_3\boldsymbol{\sigma}_3 = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \tag{9}$$

and to notice that $\mathbb{X} \longmapsto \mathbb{X} \equiv \mathbb{S}^{-1}\mathbb{X}\mathbb{S}$ sends $\mathbb{X}$ into a matrix which is again traceless hermitian, and which has the same determinant:

$$\det\mathbb{X} = \det\mathbb{X} = -(x_1^2 + x_2^2 + x_3^2)$$

Evidently

$$\mathbb{X} \longmapsto \mathbb{S}^{-1}\mathbb{X}\mathbb{S} \equiv \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \quad : \quad \mathbb{S} \in SU(2) \tag{10}$$

and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \mathbb{R}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad : \quad \mathbb{R} \in O(3) \tag{11}$$

say substantially the same thing, but in different ways.

The simplest way to render explicit the $SU(2) \leftrightarrow O(3)$ connection is to work in the neighborhood of the identity; i.e., assume θ to be small and to retain only terms of first order. Thus

$$\mathbb{S} = \mathbb{I} + i\delta\theta \cdot \big(k_1\boldsymbol{\sigma}_1 + k_2\boldsymbol{\sigma}_2 + k_3\boldsymbol{\sigma}_3\big) + \cdots \tag{12}$$

and $\mathbb{X} \equiv \mathbb{S}^{-1}\mathbb{X}\mathbb{S}$ becomes

$$\begin{aligned}
\mathbb{X} &= \Big\{\mathbb{I} - i\delta\theta \cdot \big(k_1\boldsymbol{\sigma}_1 + k_2\boldsymbol{\sigma}_2 + k_3\boldsymbol{\sigma}_3\big)\Big\} \\
&\qquad\qquad \cdot \big(x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + x_3\boldsymbol{\sigma}_3\big)\Big\{\mathbb{I} + i\delta\theta \cdot \big(k_1\boldsymbol{\sigma}_1 + k_2\boldsymbol{\sigma}_2 + k_3\boldsymbol{\sigma}_3\big)\Big\} \\
&= \mathbb{X} - i\delta\theta\big[\big(k_1\boldsymbol{\sigma}_1 + k_2\boldsymbol{\sigma}_2 + k_3\boldsymbol{\sigma}_3\big), \big(x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + x_3\boldsymbol{\sigma}_3\big)\big] + \cdots \\
&= \mathbb{X} + \delta\,\mathbb{X}
\end{aligned}$$

But by (6)

$$\begin{aligned}
\big[\text{etc.}\big] &= (k_1x_2 - k_2x_1)\big[\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\big] + (k_2x_3 - k_3x_2)\big[\boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\big] + (k_3x_1 - k_1x_3)\big[\boldsymbol{\sigma}_3, \boldsymbol{\sigma}_1\big] \\
&= 2i\Big\{(k_1x_2 - k_2x_1)\,\boldsymbol{\sigma}_3 + (k_2x_3 - k_3x_2)\,\boldsymbol{\sigma}_1 + (k_3x_1 - k_1x_3)\,\boldsymbol{\sigma}_2\Big\}
\end{aligned}$$

so we have

$$\delta\,\mathbb{X} = \delta x_1\boldsymbol{\sigma}_1 + \delta x_2\boldsymbol{\sigma}_2 + \delta x_3\boldsymbol{\sigma}_3 = 2\delta\theta\Big\{\text{etc.}\Big\}$$

which can be expressed

$$\begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix} = 2\delta\theta \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

More compactly $\delta\boldsymbol{x} = 2\delta\theta \cdot \mathbb{K}\,\boldsymbol{x}$ with $\mathbb{K} = k_1\mathbb{T}_1 + k_2\mathbb{T}_2 + k_3\mathbb{T}_3$, which entails

$$\mathbb{T}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}, \; \mathbb{T}_2 = \begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; \mathbb{T}_2 = \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$

The real antisymmetric matrices $\mathbb{T}$ are—compare (6.3)—not multiplicatively closed, but are closed under commutation:

$$\left.\begin{aligned}
\big[\mathbb{T}_1, \mathbb{T}_2\big] &= \mathbb{T}_3 \\
\big[\mathbb{T}_2, \mathbb{T}_3\big] &= \mathbb{T}_1 \\
\big[\mathbb{T}_3, \mathbb{T}_1\big] &= \mathbb{T}_2
\end{aligned}\right\} \tag{14}$$

Straightforward iteration leads now to the conclusion that

$$\begin{aligned}
\mathbb{S} &= \exp\Big[\,i\theta\big\{k_1\boldsymbol{\sigma}_1 + k_2\boldsymbol{\sigma}_2 + k_3\boldsymbol{\sigma}_3\big\}\Big] \\
&\;\updownarrow \\
\mathbb{R} &= \exp\Big[2\theta\big\{k_1\mathbb{T}_1 + k_2\mathbb{T}_2 + k_3\mathbb{T}_3\big\}\Big]
\end{aligned} \tag{15}$$

If $\{x_1, x_2, x_3\}$ refer to a right hand frame, then $\mathbb{R}$ describes a right handed rotation through angle 2θ about the unit vector $\boldsymbol{k}$. Notice that—and how elementary is the analytical origin of the fact that—the association (15) is biunique: reading from (7.1) we see that

$$\mathbb{S} \rightarrow -\mathbb{S} \text{ as } \theta \text{ advances from } 0 \text{ to } \pi$$

and that $\mathbb{S}$ returns to its initial value at $\theta = 2\pi$. But from the quadratic structure of $\mathbb{X} \equiv \mathbb{S}^{-1}\mathbb{X}\mathbb{S}$ we see that $\pm\mathbb{S}$ both yield the same $\mathbb{X}$; i.e., that $\mathbb{R}$ returns to its *initial* value as θ advances from 0 to π, and goes twice around as $\mathbb{S}$ goes once around. One speaks in this connection of the "double-valuedness of the spin representations of the rotation group," and it is partly to place us in position to do so—but mainly to prepare for things to come—that I turn now to an alternative approach to the issue just discussed.

Introduce the

$$\text{2-component spinor } \xi \equiv \begin{pmatrix} u \\ v \end{pmatrix} \tag{16}$$

(which is by nature just a complex 2-vector) and consider transformations of the form

$$\xi \longmapsto \xi \equiv \mathbb{S}\,\xi \tag{17}$$

From the unitarity of $\mathbb{S}$ (unimodularity is in this respect not needed) it follows that

$$\xi^{\mathsf{t}}\xi = \xi^{\mathsf{t}}\xi \tag{18}$$

In more explicit notation we have

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{19}$$

and

$$u^*u + v^*v = u^*u + v^*v \tag{20}$$

At (8) we acquired descriptions of α and β to which we can now assign rotational interpretations in Euclidean 3-space. But how to extract $O(3)$ from (19) without simply hiking backward along the path we have just traveled?

Kramer's method for recovering O(3). Hendrik Anthony Kramers (1894–1952) was very closely associated with many/most of the persons and events that contributed to the invention of quantum mechanics, but his memory lives in the shadow of such giants as Bohr, Pauli, Ehrenfest, Born, Dirac. Max Dresden, in a fascinating recent biography,[9] has considered why Kramers career fell somewhat short of his potential, and has concluded that a contributing factor was his excessive interest in mathematical elegance and trickery. It is upon one (widely uncelebrated) manifestation of that personality trait that I base much of what follows.

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[9] *H. A. Kramers: Between Tradition and Revolution* (1987).

In 1930/31 Kramers developed an algebraic approach to the quantum theory of spin/angular momentum which—though clever, and computationally powerful—gained few adherents.[10] A posthumous account of Kramers' method was presented in a thin monograph by one of his students,[11] and I myself have written about the subject on several occasions.[12]

We proceed from this question:

$$\text{What is the } \begin{pmatrix} uu \\ uv \\ vv \end{pmatrix} \to \begin{pmatrix} \color{red}{uu} \\ \color{red}{uv} \\ \color{red}{vv} \end{pmatrix} \text{ induced by (19)?}$$

Immediately

$$\begin{pmatrix} \color{red}{uu} \\ \color{red}{uv} \\ \color{red}{vv} \end{pmatrix} = \begin{pmatrix} (\alpha u + \beta v)(\alpha u + \beta v) \\ (\alpha u + \beta v)(-\beta^* u + \alpha^* v) \\ (-\beta^* u + \alpha^* v)(-\beta^* u + \alpha^* v) \end{pmatrix}$$

Entrusting the computational labor to *Mathematica*, we find

$$= \begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ -\alpha\beta^* & \alpha\alpha^* - \beta\beta^* & \beta\alpha^* \\ (\beta^*)^2 & -2\alpha^*\beta^* & (\alpha^*)^2 \end{pmatrix} \begin{pmatrix} uu \\ uv \\ vv \end{pmatrix} \tag{21.1}$$

which we agree to abbreviate

$$\color{red}{\boldsymbol{\zeta}} = \mathbb{Q}\boldsymbol{\zeta} \tag{21.2}$$

where $\mathbb{Q}$ is intended to suggest "quadratic."

At an early point in his argument Kramers develops an interest in null 3-vectors. Such a vector, if we are to avoid triviality, must necessarily be complex. He writes

$$\boldsymbol{a} = \boldsymbol{b} + i\boldsymbol{c} \quad : \quad \text{require } b^2 = c^2 \text{ and } \boldsymbol{b}\cdot\boldsymbol{c} = 0 \tag{22}$$

Since $a_1^2 + a_2^2 + a_3^2 = 0$ entails $a_3 = \pm i\sqrt{(a_1 + ia_2)(a_1 - ia_2)}$ it becomes fairly natural to introduce complex variables

$$\begin{aligned} u &\equiv \sqrt{a_1 - ia_2} \\ v &\equiv i\sqrt{a_1 + ia_2} \end{aligned}$$

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[10] Among those few was E. P. Wigner. Another was John L. Powell who, after graduating from Reed College in 1943, became a student of Wigner, learned Kramers' method from him, and provided an account of the subject in §7–2 of his beautiful text.[7] Powell cites the Appendix to Chapter VII in R. Courant & D. Hilbert, *Methods of Mathematical Physics, Volume I*, where W. Magnus has taken some similar ideas from "unpublished notes by the late G. Herglotz."

[11] H. Brinkman, *Applications of Spinor Invariants in Atomic Physics* (1956).

[12] See, for example, "Algebraic theory of spherical harmonics" (1996).

Then

$$\left.\begin{aligned} a_1 &= \ \tfrac{1}{2}(u^2 - v^2) \\ a_2 &= i\tfrac{1}{2}(u^2 + v^2) \\ a_3 &\equiv -iuv \end{aligned}\right\} \tag{23}$$

The $\boldsymbol{a}(u,v)$ thus defined has the property that

$$\boldsymbol{a}(u,v) = \boldsymbol{a}(-u,-v) \tag{24}$$

We conclude that, as (u,v) ranges over complex 2-space, $\boldsymbol{a}(u,v)$ ranges twice over the set of null 3-vectors, of which (23) provides a parameterized description.

Thus prepared, we ask: What is the $\boldsymbol{a} \longmapsto \boldsymbol{a}$ that results when $\mathbb{S}$ acts by (17) on the parameter space? We saw at (21) that (17) induces

$$\boldsymbol{\zeta} \longmapsto \boldsymbol{\zeta} = \mathbb{Q}\boldsymbol{\zeta} \tag{25}$$

while at (23) we obtained

$$\boldsymbol{a} = \mathbb{C}\boldsymbol{\zeta} \quad \text{with} \quad \mathbb{C} \equiv \tfrac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ i & 0 & i \\ 0 & -2 & 0 \end{pmatrix} \tag{26}$$

So we have

$$\boldsymbol{a} \longmapsto \boldsymbol{a} = \mathbb{R}\boldsymbol{a} \tag{27.1}$$

$$\mathbb{R} \equiv \mathbb{C}\mathbb{Q}\mathbb{C}^{-1} \tag{27.2}$$

Mathematica is quick to inform us that, while $\mathbb{R}$ is a bit of a mess, it is a mess with the property that every element is manifestly conjugation-invariant (meaning *real*). And, moreover, that $\mathbb{R}^{\mathsf{T}}\mathbb{R} = \mathbb{I}$. In short,

$\mathbb{R}$ is a real rotation matrix, an element of $O(3)$

To sharpen that statement we again (in the tradition of Sophus Lie) retreat to the neighborhood of the identity and work in first order; from (8) we have

$$\left.\begin{aligned} \alpha &= \ \ 1 \ + ik_3 \ \delta\theta \\ \beta &= (k_2 + ik_1)\delta\theta \end{aligned}\right\} \tag{28}$$

Substitute into $\mathbb{Q}$, abandon terms of second order and obtain

$$\mathbb{Q} = \mathbb{I} + \begin{pmatrix} 2ik_3 & 2(k_2 + ik_1) & 0 \\ -(k_2 - ik_1) & 0 & (k_2 + ik_1) \\ 0 & -2(k_2 - ik_1) & -2ik_3 \end{pmatrix} \delta\theta + \cdots$$

whence (by calculation which I have entrusted to *Mathematica*)

$$\mathbb{R} = \mathbb{I} \underset{\substack{\uparrow \\ \text{Note!}}}{-} \begin{pmatrix} 0 & -k_3 & +k_1 \\ +k_3 & 0 & -k_2 \\ -k_1 & +k_2 & 0 \end{pmatrix} 2\,\delta\theta + \cdots \tag{29}$$

This describes a doubled-angle rotation about $\boldsymbol{k}$ which is, however, *retrograde*.[13]

The preceding argument has served—redundantly, but by different means —to establish

$$SU(2) \twoheadleftarrow\!\!\longrightarrow O(3)$$

The main point of the discussion lies, however, elsewhere. We now back up to (21) and head off in a new direction.

Spin matrices by Kramer's method. For reasons that will soon acquire a high degree of naturalness, we agree at this point in place of (17) to write

$$\xi(\tfrac{1}{2}) \longmapsto \xi(\tfrac{1}{2}) \equiv \mathbb{S}(\tfrac{1}{2})\,\xi(\tfrac{1}{2}) \tag{30}$$

and to adopt this abbreviation of (21):

$$\zeta(1) \longmapsto \zeta(1) \equiv \mathbb{Q}(1)\,\zeta(1)$$

Mathematica informs us that the 3×3 matrix $\mathbb{Q}(1)$ is unimodular

$$\det \mathbb{Q}(1) = 1$$

but not unitary. . . nor do we expect it to be: the invariance of $u^*u + v^*v$ implies that not of $(uu)^*(uu) + (uv)^*(uv) + (vv)^*(vv)$ but of

$$\begin{aligned}(u^*u + v^*v)^2 &= (uu)^*(uu) + 2(uv)^*(uv) + (vv)^*(vv)\\ &= \zeta^{\mathsf{t}}(1)\,\mathbb{G}(1)\,\zeta(1)\\ \mathbb{G}(1) &\equiv \begin{pmatrix} 1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}\end{aligned}$$

Define

$$\xi(1) \equiv \sqrt{\mathbb{G}}\,\zeta(1) \quad \text{with} \quad \sqrt{\mathbb{G}} = \begin{pmatrix} 1 & 0 & 0\\ 0 & \sqrt{2} & 0\\ 0 & 0 & 1\end{pmatrix} \tag{31}$$

Then the left side of

$$\xi^{\mathsf{t}}(1)\xi(1) = (u^*u + v^*v)^2 = \big[\xi^{\mathsf{t}}(\tfrac{1}{2})\xi(\tfrac{1}{2})\big]^2 \tag{32}$$

is invariant under under the induced transformation

$$\xi(1) \longmapsto \xi(1) \equiv \mathbb{S}(1)\,\xi(1) \tag{33.1}$$

$$\mathbb{S}(1) \equiv \mathbb{G}^{+\frac{1}{2}} \cdot \mathbb{Q}(1) \cdot \mathbb{G}^{-\frac{1}{2}} \tag{33.2}$$

because the right side is invariant under $\xi(\tfrac{1}{2}) \longmapsto \xi(\tfrac{1}{2}) \equiv \mathbb{S}(\tfrac{1}{2})\,\xi(\tfrac{1}{2})$. We are

[13] To achieve prograde rotation in the essay just cited I tacitly abandoned Pauli's conventions. I had reason at (2) to adopt those conventions, and here pay the price. It was to make things work out as nicely as they have that at (23) I departed slightly from the corresponding equations (27) in "Algebraic theory. . . "[12]

not surprised to discover that the 3×3 matrix

$$\mathbb{S}(1) = \begin{pmatrix} \alpha^2 & \sqrt{2}\,\alpha\beta & \beta^2 \\ -\sqrt{2}\,\beta^*\alpha & (\alpha^*\alpha - \beta^*\beta) & \sqrt{2}\,\alpha^*\beta \\ (\beta^*)^2 & -\sqrt{2}\,\alpha^*\beta^* & (\alpha^*)^2 \end{pmatrix} \tag{34}$$

is unitary and unimodular:

$$\mathbb{S}^{\mathsf{t}}(1)\,\mathbb{S}(1) = \mathbb{I} \qquad \text{and} \qquad \det \mathbb{S}(1) = 1 \tag{35}$$

In the infinitesimal case[14] we have

$$\mathbb{S}(1) = \mathbb{I} + \begin{pmatrix} +2ik_3 & +\sqrt{2}(k_2 + ik_1) & 0 \\ -\sqrt{2}(k_2 - ik_1) & 0 & +\sqrt{2}(k_2 + ik_1) \\ 0 & -\sqrt{2}(k_2 - ik_1) & -2ik_3 \end{pmatrix} \delta\theta + \cdots$$

which (compare (12)) can be written

$$\mathbb{S}(1) = \mathbb{I} + i\,\delta\theta \cdot \big(k_1\boldsymbol{\sigma}_1(1) + k_2\boldsymbol{\sigma}_2(1) + k_3\boldsymbol{\sigma}_3(1)\big) + \cdots \tag{36}$$

with

$$\left.\begin{aligned} \boldsymbol{\sigma}_1(1) &\equiv \begin{pmatrix} 0 & +\sqrt{2} & 0 \\ \sqrt{2} & 0 & +\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \\ \boldsymbol{\sigma}_2(1) &\equiv i\begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \\ \boldsymbol{\sigma}_3(1) &\equiv \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned}\right\} \tag{37}$$

The $3{\times}3$ matrices $\boldsymbol{\sigma}_j(1)$ are manifestly traceless hermitian—the generators, evidently, of a representation of $SU(2)$ contained within the 8-parameter group $SU(3)$. They are, unlike the Pauli matrices (see again (6.3)), not closed under multiplication,[15] but are closed under commutation. In the latter respect the precisely mimic the 2×2 Pauli matrices:

$$\left.\begin{aligned} \big[\boldsymbol{\sigma}_1(\tfrac{1}{2}), \boldsymbol{\sigma}_2(\tfrac{1}{2})\big] &= 2i\,\boldsymbol{\sigma}_3(\tfrac{1}{2}) & & \big[\boldsymbol{\sigma}_1(1), \boldsymbol{\sigma}_2(1)\big] = 2i\,\boldsymbol{\sigma}_3(1) \\ \big[\boldsymbol{\sigma}_2(\tfrac{1}{2}), \boldsymbol{\sigma}_3(\tfrac{1}{2})\big] &= 2i\,\boldsymbol{\sigma}_1(\tfrac{1}{2}) & :\text{ similarly } & \big[\boldsymbol{\sigma}_2(1), \boldsymbol{\sigma}_3(1)\big] = 2i\,\boldsymbol{\sigma}_1(1) \\ \big[\boldsymbol{\sigma}_3(\tfrac{1}{2}), \boldsymbol{\sigma}_1(\tfrac{1}{2})\big] &= 2i\,\boldsymbol{\sigma}_2(\tfrac{1}{2}) & & \big[\boldsymbol{\sigma}_3(1), \boldsymbol{\sigma}_1(1)\big] = 2i\,\boldsymbol{\sigma}_2(1) \end{aligned}\right\} \tag{38}$$

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[14] The retreat to the neighborhood of the identity—which we have done already several times, and will have occasion to do again—is accomplished by this familiar routine: install (27), abandon terms of second order and simplify.

[15] Quick computation shows, for example, that

$$\boldsymbol{\sigma}_1(1)\,\boldsymbol{\sigma}_1(1) = 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{\sigma}_2(1)\,\boldsymbol{\sigma}_3(1) = 2\begin{pmatrix} 0 & 0 & 0 \\ i\sqrt{2} & 0 & i\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

There are, however, some notable differences, which will acquire importance:

$$\boldsymbol{\sigma}_1^2(\tfrac{1}{2})+\boldsymbol{\sigma}_2^2(\tfrac{1}{2})+\boldsymbol{\sigma}_3^2(\tfrac{1}{2}) = 3\begin{pmatrix}1&0\\0&1\end{pmatrix} \tag{39.05}$$

$$\boldsymbol{\sigma}_1^2(1)+\boldsymbol{\sigma}_2^2(1)+\boldsymbol{\sigma}_3^2(1) = 8\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \tag{39.10}$$

We are within sight now of our objective, which is to say: we are in possession of technique sufficient to construct $(2\ell+1)$-dimensional traceless hermitian triples

$$\big\{\boldsymbol{\sigma}_1(\ell),\boldsymbol{\sigma}_2(\ell),\boldsymbol{\sigma}_3(\ell)\big\}\quad:\quad \ell=\tfrac{3}{2},2,\tfrac{5}{2},\ldots \tag{40}$$

with properties that represent natural extensions of those just encountered. But before looking to the details, I digress to assemble the...

Bare bones of the quantum theory of angular momentum. Classical mechanics engenders interest in the construction $\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p}$; i.e., in

$$\begin{aligned}L_1 &= x_2p_3 - x_3p_2\\ L_2 &= x_3p_1 - x_1p_3\\ L_3 &= x_1p_2 - x_2p_1\end{aligned}$$

The primitive Poisson bracket relations

$$[x_m,x_n]=[p_m,p_n]=0\quad\text{and}\quad[x_m,p_n]=\delta_{mn}$$

are readily found to entail

$$\begin{aligned}[L_1,L_2]&=L_3\\ [L_2,L_3]&=L_1\\ [L_3,L_1]&=L_2\end{aligned}$$

and

$$[L_1,L^2]=[L_2,L^2]=[L_3,L^2]=0$$

where $L^2\equiv\boldsymbol{L}\cdot\boldsymbol{L}=L_1^2+L_2^2+L_3^2$. In quantum theory those classical observables become hermitian operators L_1, L_2, L_3 and $\mathsf{L}^2\equiv\mathsf{L}_1^2+\mathsf{L}_2^2+\mathsf{L}_3^2$ and Dirac's principle $[x_m,p_n]=\delta_{mn}\longrightarrow[\mathsf{x}_m,\mathsf{p}_n]=i\hbar\mathsf{I}$ gives rise to commutation relations that mimic the preceding Poisson bracket relations:

$$\left.\begin{aligned}[\mathsf{L}_1,\mathsf{L}_2]&=i\hbar\mathsf{L}_3\\ [\mathsf{L}_2,\mathsf{L}_3]&=i\hbar\mathsf{L}_1\\ [\mathsf{L}_3,\mathsf{L}_1]&=i\hbar\mathsf{L}_2\end{aligned}\right\} \tag{41}$$

$$[\mathsf{L}_1,\mathsf{L}^2]=[\mathsf{L}_2,\mathsf{L}^2]=[\mathsf{L}_3,\mathsf{L}^2]=\mathbf{0} \tag{42}$$

The function-theoretic approach to the subject[16] proceeds from

$$\mathsf{x} \to \boldsymbol{x}\cdot \quad \text{and} \quad \mathsf{p} \to \tfrac{\hbar}{i}\boldsymbol{\nabla}$$

and leads (one works actually in spherical coordinates) to the theory of spherical harmonics

$$Y_\ell^m(\boldsymbol{x}) \quad : \quad \begin{cases} \ell \ = 0, 1, 2, \dots \\ m = -\ell, -(\ell-1), \dots, -1, 0, +1, \dots, +(\ell-1), +\ell \end{cases}$$

$$\left.\begin{aligned} \mathsf{L}^2 Y_\ell^m &= \ell(\ell+1)\cdot Y_\ell^m \\ \mathsf{L}_3 Y_\ell^m &= m\hbar \cdot Y_\ell^m \end{aligned}\right\} \tag{43}$$

But the subject admits also of algebraic development, and that approach, while it serves to reproduce the function-theoretic results just summarized, leads also to a physically important *algebraic extension*

$$\ell = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}\dots \tag{44}$$

of the preceding theory.

One makes a notational adjustment

$$\mathsf{L} \to \mathsf{J}$$

to emphasize that one is talking now about an expanded subject (fusion of the "theory of orbital angular momentum" and a "theory of intrinsic spin") and introduces non-hermitian "ladder operators"

$$\left.\begin{aligned} \mathsf{J}_+ &= \mathsf{J}_1 + i\,\mathsf{J}_2 \\ \mathsf{J}_- &= \mathsf{J}_1 - i\,\mathsf{J}_2 \end{aligned}\right\} \tag{45}$$

One finds the ℓ^{th} eigenspace of J^2 to be $(2\ell+1)$-dimensional, and uses J_3 to resolve the degeneracy. Let the normalized simultaneous eigenvectors of J^2 and J_3 be denoted $|\ell, m)$:

$$\left.\begin{aligned} \mathsf{J}^2\,|\ell,m) &= \ell(\ell+1)\hbar^2\cdot|\ell,m) \\ \mathsf{J}_3\,|\ell,m) &= m\hbar\cdot|\ell,m) \\ (\ell,m|\ell',m') &= \delta_{\ell\ell'}\delta_{mm'} \end{aligned}\right\} \tag{46}$$

One finds that

$$\left.\begin{aligned} &\mathsf{J}_+|\ell,m) \sim |\ell,m+1) &&: \quad m < \ell \\ &\mathsf{J}_+|\ell,m) = 0 &&: \quad m = \ell \end{aligned}\right\} \tag{47.1}$$

$$\left.\begin{aligned} &\mathsf{J}_-|\ell,m) \sim |\ell,m-1) &&: \quad m > -\ell \\ &\mathsf{J}_+|\ell,m) = 0 &&: \quad m = -\ell \end{aligned}\right\} \tag{47.2}$$

[16] See D. Griffiths' *Introduction to Quantum Mechanics* (1995), §§4.1.2 and 4.3 or, indeed, any quantum text. For an approach (Kramers') more in keeping with the spirit of the present discussion see again my "Algebraic approach…"[12]

and that one has to introduce ugly renormalization factors

$$\frac{1}{\hbar\sqrt{\ell(\ell+1)-m(m\pm 1)}} \tag{48}$$

to turn the $\sim$'s into $=$'s.

My plan is not to make active *use* of this standard material, but to show that the results we achieve by other means *exemplify* it. We note in that connection that if matrices $\{\boldsymbol{\varpi}_1, \boldsymbol{\varpi}_2, \boldsymbol{\varpi}_3\}$ satisfy commutation relations of the form (38), then the matrices

$$\mathbb{J}_n \equiv \tfrac{1}{2}\hbar\boldsymbol{\varpi}_n \quad : \quad n = 1, 2, 3 \tag{49}$$

bear the physical dimensionality of $\hbar$ (i.e., of angular momentum) and satisfy (41); such matrices are called "angular momentum matrices" (more pointedly: *spin matrices*).[17]

To reduce notational clutter we agree at this point to adopt units in which

$\hbar$ is numerically equal to unity

Simple dimensional analysis would serve to restore all missing $\hbar$-factors.

Look now to particulars of the case $\ell = \frac{1}{2}$. We have

$$\mathbb{J}_1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbb{J}_2 = \tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbb{J}_3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{50.1}$$

whence

$$\mathbb{J}^2 \equiv \mathbb{J}_1^2 + \mathbb{J}_2^2 + \mathbb{J}_3^2 = \tfrac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{50.2}$$

$$\tfrac{3}{4} = \tfrac{1}{2}\big(\tfrac{1}{2}+1\big) \tag{50.3}$$

All 2-vectors are eigenvectors of $\mathbb{J}^2$. To resolve that universal degeneracy we select $\mathbb{J}_3$ which, because it is diagonal, has obvious spectral properties:

$$\left.\begin{array}{l} \text{eigenvalue } +\tfrac{1}{2} \text{ with normalized eigenvector } |\tfrac{1}{2}, +\tfrac{1}{2}) \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \text{eigenvalue } -\tfrac{1}{2} \text{ with normalized eigenvector } |\tfrac{1}{2}, -\tfrac{1}{2}) \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array}\right\} \tag{50.4}$$

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[17] The commutators of the generators $\mathbb{T}_n$ of O(3) were found at (14) to mimic the Poisson bracket relations $[L_1, L_2] = L_3$, etc. But if, in the spirit of the preceding discussion, we construct $\mathbb{J}_n \equiv i\hbar\mathbb{T}_n$ we obtain $[\mathbb{J}_1, \mathbb{J}_2] = -i\hbar\mathbb{J}_3$, etc. which have *reverse chirality*.

$\mathbb{J}_1$ and $\mathbb{J}_2$ have spectra identical to that of $\mathbb{J}_3$, but distinct sets of eigenvectors. The ladder operators (45) become non-hermitian ladder matrices

$$\mathbb{J}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{J}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{50.5}$$

By inspection we have

$$\left.\begin{aligned} &\text{action of } \mathbb{J}_+ \;:\; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \nearrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \nearrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\text{action of } \mathbb{J}_- \;:\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \searrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \searrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}\right. \tag{50.6}$$

which are illustrative of (47). Note the absence in this case of normalization factors; i.e., that the $\sim$'s have become equalities.

A better glimpse of the situation-in-general is provided by the case $\ell = 1$. Reading from (37) we have

$$\left.\begin{aligned} \mathbb{J}_1(1) &\equiv \tfrac{1}{2}\begin{pmatrix} 0 & +\sqrt{2} & 0 \\ \sqrt{2} & 0 & +\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \\ \mathbb{J}_2(1) &\equiv i\,\tfrac{1}{2}\begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \\ \mathbb{J}_3(1) &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}\right\} \tag{51.1}$$

whence

$$\mathbb{J}^2 \equiv \mathbb{J}_1^2 + \mathbb{J}_2^2 + \mathbb{J}_3^2 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{51.2}$$

$$2 = 1(1+1) \tag{51.3}$$

All 3-vectors are eigenvectors of $\mathbb{J}^2$. To resolve that universal degeneracy we select $\mathbb{J}_3$ which, because it is diagonal, has obvious spectral properties:

$$\left.\begin{aligned} &\text{eigenvalue } +1 \text{ with normalized eigenvector } |1,+1) \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &\text{eigenvalue } 0 \text{ with normalized eigenvector } |1,0) \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &\text{eigenvalue } -1 \text{ with normalized eigenvector } |1,-1) \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}\right\} \tag{51.4}$$

The ladder matrices become

$$\mathbb{J}_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \quad\text{and}\quad \mathbb{J}_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{51.5}$$

which by quick calculation give

$$\begin{aligned} \mathbb{J}_+\,|1,-1) &= \sqrt{2}\,|1,\ \ 0) \\ \mathbb{J}_+\,|1,\ \ 0) &= \sqrt{2}\,|1,+1) \\ \mathbb{J}_+\,|1,+1) &= \ \ 0 \end{aligned} \tag{51.6 $\uparrow$}$$

$$\begin{aligned} \mathbb{J}_-\,|1,+1) &= \sqrt{2}\,|1,\ \ 0) \\ \mathbb{J}_-\,|1,\ \ 0) &= \sqrt{2}\,|1,-1) \\ \mathbb{J}_-\,|1,-1) &= \ \ 0 \end{aligned} \tag{51.6 $\downarrow$}$$

Normalization now *is* necessary, and the factor

$$N_\pm(\ell,m) \equiv \frac{1}{\sqrt{\ell(\ell+1)-m(m\pm 1)}} \tag{52}$$

advertised at (48) does in fact do the job, since

$$\begin{aligned} N_+(1,-1) &= 1/\sqrt{2} \\ N_+(1,\ \ 0) &= 1/\sqrt{2} \\ N_-(1,+1) &= 1/\sqrt{2} \\ N_-(1,\ \ 0) &= 1/\sqrt{2} \end{aligned}$$

Spin 3/2. We look now to the case $\ell = \frac{3}{2}$, which was of special interest to Penrose (as—for other reasons—it has been to quantum field theorists[18]), and is of special interest therefore also to us. The simplest extension of the argument that gave (21) now gives

$$\begin{pmatrix} uuu \\ uuv \\ uvv \\ vvv \end{pmatrix} = \begin{pmatrix} \alpha^3 & 3\alpha^2\beta & 3\alpha\beta^2 & \beta^3 \\ -\alpha^2\beta^* & \alpha^2\alpha^*-2\alpha\beta\beta^* & 2\alpha\alpha^*\beta-\beta^2\beta^* & \alpha^*\beta^2 \\ \alpha(\beta^*)^2 & -2\alpha\alpha^*\beta^*+\beta(\beta^*)^2 & \alpha(\alpha^*)^2-2\alpha^*\beta\beta^* & (\alpha^*)^2\beta \\ -(\beta^*)^3 & 3\alpha^*(\beta^*)^2 & -3(\alpha^*)^2\beta^* & (\alpha^*)^3 \end{pmatrix} \begin{pmatrix} uuu \\ uuv \\ uvv \\ vvv \end{pmatrix}$$

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[18] See H. Umezawa, *Quantum Field Theory* (1956), p. 71 for discussion of what the "Rarita– Schwinger formalism" (Phys. Rev. **60**, 61 (1941)) has to say about the case $\ell = \frac{3}{2}$. Relatedly: on p. 454 in I. Duck & E. C. G. Sudarshan, *Pauli and the Spin–Statistics Theorem* (1997) it is remarked that "…The kinematics thus depends on the dynamics! As a result, for a charged spin-$\frac{3}{2}$ field the anticommutator…depends on the external field in such a manner that quantization becomes inconsistent."

which we agree to abbreviate

$$\zeta(\tfrac{3}{2}) = \mathbb{Q}(\tfrac{3}{2})\zeta(\tfrac{3}{2})$$

Write

$$\begin{aligned}(u^*u+v^*v)^3 &= (uuu)^*(uuu)+3(uuv)^*(uuv)+3(uvv)^*(uvv)+(vvv)^*(vvv)\\ &= \zeta^{\mathsf{t}}(\tfrac{3}{2})\,\mathbb{G}(\tfrac{3}{2})\,\zeta(\tfrac{3}{2})\\ \mathbb{G}(\tfrac{3}{2}) &\equiv \begin{pmatrix}1&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&1\end{pmatrix}\end{aligned}$$

Define

$$\xi(\tfrac{3}{2}) \equiv \sqrt{\mathbb{G}}\,\zeta(\tfrac{3}{2}) \quad \text{with} \quad \sqrt{\mathbb{G}} = \begin{pmatrix}1&0&0&0\\0&\sqrt{3}&0&0\\0&0&\sqrt{3}&0\\0&0&0&1\end{pmatrix}$$

Then the left side of

$$\xi^{\mathsf{t}}(\tfrac{3}{2})\xi(\tfrac{3}{2}) = (u^*u+v^*v)^3 = \big[\xi^{\mathsf{t}}(\tfrac{1}{2})\xi(\tfrac{1}{2})\big]^3$$

is invariant under under the induced transformation

$$\xi(\tfrac{3}{2}) \longmapsto \xi(\tfrac{3}{2}) \equiv \mathbb{S}(\tfrac{3}{2})\,\xi(\tfrac{3}{2}) \tag{53.1}$$

$$\mathbb{S}(\tfrac{3}{2}) \equiv \mathbb{G}^{+\frac{1}{2}}\cdot\mathbb{Q}(\tfrac{3}{2})\cdot\mathbb{G}^{-\frac{1}{2}} \tag{53.2}$$

Mathematica informs us—now not at all to our surprise—that the 4×4 matrix

$$\mathbb{S}(\tfrac{3}{2}) = \begin{pmatrix}\alpha^3 & \sqrt{3}\alpha^2\beta & \sqrt{3}\alpha\beta^2 & \beta^3\\ -\sqrt{3}\alpha^2\beta^* & \alpha^2\alpha^*-2\alpha\beta\beta^* & 2\alpha\alpha^*\beta-\beta^2\beta^* & \sqrt{3}\alpha^*\beta^2\\ \sqrt{3}\alpha(\beta^*)^2 & -2\alpha\alpha^*\beta^*+\beta(\beta^*)^2 & \alpha(\alpha^*)^2-2\alpha^*\beta\beta^* & \sqrt{3}(\alpha^*)^2\beta\\ -(\beta^*)^3 & \sqrt{3}\alpha^*(\beta^*)^2 & -\sqrt{3}(\alpha^*)^2\beta^* & (\alpha^*)^3\end{pmatrix} \tag{54}$$

is unitary and unimodular:

$$\mathbb{S}^{\mathsf{t}}(\tfrac{3}{2})\,\mathbb{S}(\tfrac{3}{2}) = \mathbb{I} \qquad \text{and} \qquad \det\mathbb{S}(\tfrac{3}{2}) = 1 \tag{55}$$

Drawing again upon (28) we in leading order have

$$\mathbb{S}(\tfrac{3}{2}) = \begin{pmatrix}\alpha^3 & \sqrt{3}\beta & 0 & 0\\ -\sqrt{3}\beta^* & \alpha^2\alpha^* & 2\beta & 0\\ 0 & -2\beta^* & \alpha(\alpha^*)^2 & \sqrt{3}\beta\\ 0 & 0 & -\sqrt{3}\beta^* & (\alpha^*)^3\end{pmatrix} + \cdots$$

which becomes

$$\mathbb{S}(\tfrac{3}{2}) = \mathbb{I} + i\,\delta\theta\cdot\big(k_1\boldsymbol{\sigma}_1(\tfrac{3}{2}) + k_2\boldsymbol{\sigma}_2(\tfrac{3}{2}) + k_3\boldsymbol{\sigma}_3(\tfrac{3}{2})\big) + \cdots \tag{56}$$

with

$$\left.\begin{aligned}
\boldsymbol{\sigma}_1(\tfrac{3}{2}) &\equiv \begin{pmatrix} 0 & +\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & +2 & 0 \\ 0 & 2 & 0 & +\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \\
\boldsymbol{\sigma}_2(\tfrac{3}{2}) &\equiv i\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \\
\boldsymbol{\sigma}_3(\tfrac{3}{2}) &\equiv \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}
\end{aligned}\right\} \tag{57}$$

By quick computation we verify that (compare (38))

$$\left.\begin{aligned}
\big[\boldsymbol{\sigma}_1(\tfrac{3}{2}), \boldsymbol{\sigma}_2(\tfrac{3}{2})\big] &= 2i\,\boldsymbol{\sigma}_3(\tfrac{3}{2}) \\
\big[\boldsymbol{\sigma}_2(\tfrac{3}{2}), \boldsymbol{\sigma}_3(\tfrac{3}{2})\big] &= 2i\,\boldsymbol{\sigma}_1(\tfrac{3}{2}) \\
\big[\boldsymbol{\sigma}_3(\tfrac{3}{2}), \boldsymbol{\sigma}_1(\tfrac{3}{2})\big] &= 2i\,\boldsymbol{\sigma}_2(\tfrac{3}{2})
\end{aligned}\right\} \tag{58}$$

and find that (compare (39))

$$\boldsymbol{\sigma}_1^2(\tfrac{3}{2}) + \boldsymbol{\sigma}_2^2(\tfrac{3}{2}) + \boldsymbol{\sigma}_3^2(\tfrac{3}{2}) = 15\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{59}$$

Thus are we led by (49: $\hbar = 1$) to the spin matrices

$$\left.\begin{aligned}
\mathbb{J}_1(\tfrac{3}{2}) &\equiv \tfrac{1}{2}\begin{pmatrix} 0 & +\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & +2 & 0 \\ 0 & 2 & 0 & +\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \\
\mathbb{J}_2(\tfrac{3}{2}) &\equiv i\tfrac{1}{2}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \\
\mathbb{J}_3(\tfrac{3}{2}) &\equiv \begin{pmatrix} \tfrac{3}{2} & 0 & 0 & 0 \\ 0 & \tfrac{1}{2} & 0 & 0 \\ 0 & 0 & -\tfrac{1}{2} & 0 \\ 0 & 0 & 0 & -\tfrac{3}{2} \end{pmatrix}
\end{aligned}\right\} \tag{60.1}$$

which give

$$\mathbb{J}^2(\tfrac{3}{2}) \equiv \mathbb{J}_1^2(\tfrac{3}{2}) + \mathbb{J}_2^2(\tfrac{3}{2}) + \mathbb{J}_3^2(\tfrac{3}{2}) = \tfrac{15}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{60.2}$$

$$\tfrac{15}{4} = \tfrac{3}{2}\big(\tfrac{3}{2} + 1\big) \tag{60.3}$$

All 4-vectors are eigenvectors of $\mathbb{J}^2(\frac{3}{2})$. To resolve that universal degeneracy we select $\mathbb{J}_3(\frac{3}{2})$ which, because it is diagonal, has obvious spectral properties:

$$\left.\begin{array}{l}\text{eigenvalue } +\tfrac{3}{2} \text{ with normalized eigenvector } |\tfrac{3}{2},+\tfrac{3}{2}) \equiv \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\\ \text{eigenvalue } +\tfrac{1}{2} \text{ with normalized eigenvector } |\tfrac{3}{2},+\tfrac{1}{2}) \equiv \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\\ \text{eigenvalue } -\tfrac{1}{2} \text{ with normalized eigenvector } |\tfrac{3}{2},-\tfrac{1}{2}) \equiv \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\\ \text{eigenvalue } -\tfrac{3}{2} \text{ with normalized eigenvector } |\tfrac{3}{2},-\tfrac{3}{2}) \equiv \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\end{array}\right\} \tag{60.4}$$

$\mathbb{J}_1(\frac{3}{2})$ and $\mathbb{J}_2(\frac{3}{2})$ have spectra identical to that of $\mathbb{J}_3(\frac{3}{2})$, but distinct sets of eigenvectors. The ladder operators (45) become non-hermitian ladder matrices

$$\mathbb{J}_+(\tfrac{3}{2}) = \begin{pmatrix}0 & \sqrt{3} & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & \sqrt{3}\\ 0 & 0 & 0 & 0\end{pmatrix} \quad\text{and}\quad \mathbb{J}_-(\tfrac{3}{2}) = \begin{pmatrix}0 & 0 & 0 & 0\\ \sqrt{3} & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & \sqrt{3} & 0\end{pmatrix} \tag{60.5}$$

By inspection we have

$$\left.\begin{array}{lll}\mathbb{J}_+(\tfrac{3}{2})\,|\tfrac{3}{2},-\tfrac{3}{2}) = \sqrt{3}\,|\tfrac{3}{2},-\tfrac{1}{2}) & \text{with} & N(\tfrac{3}{2},-\tfrac{3}{2}) = 1/\sqrt{3}\\ \mathbb{J}_+(\tfrac{3}{2})\,|\tfrac{3}{2},-\tfrac{1}{2}) = \quad 2\,|\tfrac{3}{2},+\tfrac{1}{2}) & \text{with} & N(\tfrac{3}{2},-\tfrac{1}{2}) = 1/2\\ \mathbb{J}_+(\tfrac{3}{2})\,|\tfrac{3}{2},+\tfrac{1}{2}) = \sqrt{3}\,|\tfrac{3}{2},+\tfrac{3}{2}) & \text{with} & N(\tfrac{3}{2},+\tfrac{1}{2}) = 1/\sqrt{3}\end{array}\right\} \tag{60.6 $\uparrow$}$$
$$\mathbb{J}_+(\tfrac{3}{2})\,|\tfrac{3}{2},+\tfrac{3}{2}) = \quad 0$$

and similar equations describing the step-down action of $\mathbb{J}_-(\frac{3}{2})$. Matrices identical to (60.1) and (60.5) appear on p. 203 of Schiff and p. 345 of Powell & Crasemann.[7]

Spin 2. The *pattern* of the argument—which I have reviewed in plodding detail—remains unchanged, so I will be content to record only the most salient particulars. We look to the (19)-induced transformation of

$$\zeta(2) \equiv \begin{pmatrix}uuuu\\uuuv\\uuvv\\uvvv\\vvvv\end{pmatrix}$$

and obtain $\zeta(2) = \mathbb{Q}(2)\zeta(2)$ where $\mathbb{Q}(2)$ is a 5×5 mess. From

$$(u^*u + v^*v)^4 = \zeta^{\mathrm{t}}(2)\,\mathbb{G}(2)\,\zeta(2)$$

$$\mathbb{G}(2) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

we are led to write

$$\xi(2) \equiv \sqrt{\mathbb{G}}\,\zeta(2) \quad \text{with} \quad \sqrt{\mathbb{G}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

giving

$$\xi(2) \longmapsto \xi(2) \equiv \mathbb{S}(2)\,\xi(2)$$

$$\mathbb{S}(2) \equiv \mathbb{G}^{+\frac{1}{2}} \cdot \mathbb{Q}(2) \cdot \mathbb{G}^{-\frac{1}{2}}$$

where $\mathbb{S}(2)$ is again a patterned mess (*Mathematica* assures us that $\mathbb{S}(2)$ is unitary and unimodular) which, however, simplifies markedly in leading order:

$$\begin{aligned}
\mathbb{S}(2) &= \begin{pmatrix} \alpha^4 & 2\beta & 0 & 0 & 0 \\ -2\beta^* & \alpha^3\alpha^* & \sqrt{6}\beta & 0 & 0 \\ 0 & -\sqrt{6}\beta^* & \alpha^2(\alpha^*)^2 & \sqrt{6}\beta & 0 \\ 0 & 0 & -\sqrt{6}\beta^* & \alpha(\alpha^*)^3 & 2\beta \\ 0 & 0 & 0 & -2\beta^* & (\alpha^*)^4 \end{pmatrix} + \cdots \\
&= \mathbb{I} + i \begin{pmatrix} 4k_3 & 2(k_1 - ik_2) & 0 & 0 & 0 \\ 2(k_1 + ik_2) & 2k_3 & \sqrt{6}(k_1 - ik_2) & 0 & 0 \\ 0 & \sqrt{6}(k_1 + ik_2) & 0 & \sqrt{6}(k_1 - ik_2) & 0 \\ 0 & 0 & \sqrt{6}(k_1 + ik_2) & -2k_3 & 2(k_1 - ik_2) \\ 0 & 0 & 0 & 2(k_1 + ik_2) & -4k_3 \end{pmatrix} + \cdots
\end{aligned}$$

From this information we extract the information reported on the next page:

$$\left.\begin{aligned}
\mathbb{J}_1(2) &= \tfrac{1}{2}\begin{pmatrix} 0 & +2 & 0 & 0 & 0 \\ 2 & 0 & +\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & +\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & +2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \\
\mathbb{J}_2(2) &= i\,\tfrac{1}{2}\begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \\
\mathbb{J}_3(2) &= \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}
\end{aligned}\right\} \tag{61.1}$$

$$\mathbb{J}_1^2(2) + \mathbb{J}_2^2(2) + \mathbb{J}_3^2(2) = 6\,\mathbb{I} \tag{61.2}$$

$$6 = 2(2+1) \tag{61.3}$$

The ladder matrices become

$$\mathbb{J}_+(2) = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{J}_-(2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \tag{61.5}$$

in which connection we notice that

$$\begin{aligned}
N_+(2,+2) &= \infty & N_-(2,+2) &= 1/2 \\
N_+(2,+1) &= 1/2 & N_-(2,+1) &= 1/\sqrt{6} \\
N_+(2,\;\;\,0) &= 1/\sqrt{6} & N_-(2,\;\;\,0) &= 1/\sqrt{6} \\
N_+(2,-1) &= 1/\sqrt{6} & N_-(2,-1) &= 1/2 \\
N_+(2,-2) &= 1/2 & N_-(2,-2) &= \infty
\end{aligned}$$

Extrapolation to the general case. The method just reviewed, as it relates to the case $\ell = 2$, can in principle be extended to arbitrary ℓ, but the labor increases as ℓ^2. The results in hand are, however, so highly patterned and so simple that one can readily *guess* the design of $\{\mathbb{J}_1(\ell), \mathbb{J}_2(\ell), \mathbb{J}_3(\ell)\}$, and then busy oneself demonstrating that one's guess is actually correct. I illustrate how one might proceed in the case $\ell = \frac{5}{2}$.

We expect in any event to have

$$\mathbb{J}_3(\tfrac{5}{2}) = \begin{pmatrix} \frac{5}{2} &&&&& \\ & \frac{3}{2} &&&& \\ && \frac{1}{2} &&& \\ &&& -\frac{1}{2} && \\ &&&& -\frac{3}{2} & \\ &&&&& -\frac{5}{2} \end{pmatrix} \tag{62.1}$$

where (as henceforth) the omitted matrix elements are 0's. And we guess that $\mathbb{J}_1(\frac{5}{2})$ and $\mathbb{J}_2(\frac{5}{2})$ are of the designs

$$\mathbb{J}_1(\tfrac{5}{2}) = \ \tfrac{1}{2}\begin{pmatrix} 0 & +a &&&& \\ a & 0 & +b &&& \\ & b & 0 & +c && \\ && c & 0 & +d & \\ &&& d & 0 & +e \\ &&&& e & 0 \end{pmatrix}$$

$$\mathbb{J}_2(\tfrac{5}{2}) = i\,\tfrac{1}{2}\begin{pmatrix} 0 & -a &&&& \\ a & 0 & -b &&& \\ & b & 0 & -c && \\ && c & 0 & -d & \\ &&& d & 0 & -e \\ &&&& e & 0 \end{pmatrix}$$

To achieve $\mathbb{J}_1\mathbb{J}_2 - \mathbb{J}_2\mathbb{J}_1 = i\,\mathbb{J}_3$ we ask *Mathematica* to solve the equations

$$\begin{aligned} \tfrac{1}{2}\big(a^2 \qquad\big) &= +\tfrac{5}{2} \\ \tfrac{1}{2}\big(b^2 - a^2\big) &= +\tfrac{3}{2} \\ \tfrac{1}{2}\big(c^2 - b^2\big) &= +\tfrac{1}{2} \\ \tfrac{1}{2}\big(d^2 - c^2\big) &= -\tfrac{1}{2} \\ \tfrac{1}{2}\big(e^2 - d^2\big) &= -\tfrac{3}{2} \\ \tfrac{1}{2}\big(\qquad - e^2\big) &= -\tfrac{5}{2} \end{aligned}$$

and are informed that

$$\begin{aligned} a &= \pm\sqrt{5} \\ b &= \pm\sqrt{8} \\ c &= \pm\sqrt{9} \\ d &= \pm\sqrt{8} \\ e &= \pm\sqrt{5} \end{aligned}$$

where the signs are independently specifiable. Take all signs to be positive (any of the $2^5 - 1 = 31$ alternative sign assignments would, however, work just as well) and obtain

$$\left.\begin{aligned}
\mathbb{J}_1(\tfrac{5}{2}) &= \tfrac{1}{2}\begin{pmatrix} 0 & +\sqrt{5} & & & & \\ \sqrt{5} & 0 & +\sqrt{8} & & & \\ & \sqrt{8} & 0 & +\sqrt{9} & & \\ & & \sqrt{9} & 0 & +\sqrt{8} & \\ & & & \sqrt{8} & 0 & +\sqrt{5} \\ & & & & \sqrt{5} & 0 \end{pmatrix}\\
\mathbb{J}_2(\tfrac{5}{2}) &= i\,\tfrac{1}{2}\begin{pmatrix} 0 & -\sqrt{5} & & & & \\ \sqrt{5} & 0 & -\sqrt{8} & & & \\ & \sqrt{8} & 0 & -\sqrt{9} & & \\ & & \sqrt{9} & 0 & -\sqrt{8} & \\ & & & \sqrt{8} & 0 & -\sqrt{5} \\ & & & & \sqrt{5} & 0 \end{pmatrix}
\end{aligned}\right\} \tag{62.2}$$

The commutation relations check out, and we have

$$\begin{aligned}
\mathbb{J}_1^2(\tfrac{5}{2}) + \mathbb{J}_2^2(\tfrac{5}{2}) + \mathbb{J}_3^2(\tfrac{5}{2}) &= \tfrac{35}{4}\,\mathbb{I}\\
\tfrac{35}{4} &= \tfrac{5}{2}(\tfrac{5}{2}+1)
\end{aligned}$$

We observe that

$$\begin{aligned}
N_+(\tfrac{5}{2},+\tfrac{5}{2}) &= \infty & N_+(\tfrac{5}{2},+\tfrac{5}{2}) &= 1/\sqrt{5}\\
N_+(\tfrac{5}{2},+\tfrac{3}{2}) &= 1/\sqrt{5} & N_+(\tfrac{5}{2},+\tfrac{3}{2}) &= 1/\sqrt{8}\\
N_+(\tfrac{5}{2},+\tfrac{1}{2}) &= 1/\sqrt{8} & N_+(\tfrac{5}{2},+\tfrac{1}{2}) &= 1/\sqrt{9}\\
N_+(\tfrac{5}{2},-\tfrac{1}{2}) &= 1/\sqrt{9} & N_+(\tfrac{5}{2},-\tfrac{1}{2}) &= 1/\sqrt{8}\\
N_+(\tfrac{5}{2},-\tfrac{3}{2}) &= 1/\sqrt{8} & N_+(\tfrac{5}{2},-\tfrac{3}{2}) &= 1/\sqrt{5}\\
N_+(\tfrac{5}{2},-\tfrac{5}{2}) &= 1/\sqrt{5} & N_+(\tfrac{5}{2},-\tfrac{5}{2}) &= \infty
\end{aligned}$$

In the *general* case we would (i) compute the numbers

$$1/N_+(\ell,m) \quad : \quad m = -\ell,\ldots,+(\ell-1) \tag{63}$$

(ii) insert them into off-diagonal positions in the now obvious way to construct $\mathbb{J}_1(\ell)$ and $\mathbb{J}_2(\ell)$, (iii) adjoin

$$\mathbb{J}_3 = \begin{pmatrix} \ell & & & & \\ & (\ell-1) & & & \\ & & \ddots & & \\ & & & -(\ell-1) & \\ & & & & -\ell \end{pmatrix} \tag{64}$$

and be done.

Concluding remarks. We find ourselves—now that the dust has settled—doing pretty much what Schiff advocates on his (notationally dense) pages 200–203.

David Griffiths reports in conversation that he himself proceeds in a manner that he considers to be "non-standard" (though it is the method advocated by Powell & Crasemann in their §10–4): he takes the descriptions (64) and (52) of $\mathbb{J}_3$ and $N_\pm(\ell, m)$ to be given; constructs $\mathbb{J}_\pm$ matrices that, on the pattern of (51.6) and (60.6), do their intended work; assembles

$$\begin{aligned}\mathbb{J}_1 &= \quad \tfrac{1}{2}\big(\mathbb{J}_+ + \mathbb{J}_-\big)\\ \mathbb{J}_2 &= -i\,\tfrac{1}{2}\big(\mathbb{J}_+ - \mathbb{J}_-\big)\end{aligned}$$

The merit (such as it is) of my own approach lies in the fact that it is "constructive;" it draws explicit attention to the circumstance that the $(2\ell+1)\times(2\ell+1)$ unimodular unitary matrices $\mathbb{S}(\ell)$—elements of $SU(2\ell+1)$—which supply the $(2\ell+1)$-spinor representation of $O(3)$ act upon

$$\xi = \begin{pmatrix}\xi_0\\ \xi_1\\ \vdots\\ \xi_n\\ \vdots\\ \xi_{2\ell}\end{pmatrix}$$

in such a way that

$$\xi_n \quad \text{transforms like} \quad \sqrt{\binom{2\ell}{n}}\cdot u^{2\ell-n}\,v^n \quad : \quad n = 0, 1, \ldots, 2\ell$$

under (17).

What has any of this to do with "Majorana's method"? That is a question to which I turn in Part B of this short series of essays, to which I have given the collective title ASPECTS OF THE MATHEMATICS OF SPIN.