MAJORANA REPRESENTATION OF HIGHER SPIN STATES

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Introduction. In a recent essay[1] I described a method—a method based upon some relatively little-known work of H. A. Kramers[2]—for constructing traceless hermitian $(2\ell+1)\times(2\ell+1)$ matrices

$$\mathbb{J}_1(\ell), \mathbb{J}_2(\ell), \mathbb{J}_3(\ell) \quad : \quad \ell = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, \ldots$$

which display the commutation properties of angular momentum operators, and serve as the generators of spin-ℓ unimodular unitary representations of the rotation group $O(3)$. I propose now to explore the question with which that essay ended: "What has any of this [the apparatus to which I have just alluded] to do with Majorana's method?"

Kramers' method dates from 1930/1931[3], Majorana's from 1932.[4] I have discovered no evidence that either ever became aware of the work of the other . . . which is a shame, for their respective creations show a methodological affinity, and for that very reason were similarly received: both proceeded algebraically (though Majorana's work had a pronounced geometrical flavor),

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[1] "Spin matrices for arbitrary spin" (August 2000)—Part A of a series that I call ASPECTS OF THE MATHEMATICS OF SPIN.

[2] For a description of Kramer's idea see §2 in "Algebraic theory of spherical harmonics" (1996).

[3] See p. 317 in M. Dresden, *H. A. Kramers: Between Tradition & Revolution* (1987) for descriptions of the soil (his "almost discovery of the Dirac equations") from which Kramer's method sprang.

[4] "Atomi orientati in campo magnetico variabile," Il Nuovo Cimento **9**, 43 (1932). This six-page note—the work of a brilliant but reclusive 25-year-old, in which only a couple of incidental paragraphs bare on the topic presently of interest to us—is widely hailed as a classic. But the best efforts of Victoria Mitchell, Reed College Science Librarian, have failed to turn up an English translation. Professor Erasmo Recami has translated three of Majorana's papers (he wrote only nine before committing suicide at 31), but not the paper in question; Recami considers this to be not a major problem, since ". . . Italian is a very easy language to learn."

and both saw their work marginalized by a contemporaries who had recently embraced group theory as the language of orthodoxy.[5]

In Part A I used methods adapted from Kramers to address a problem (the construction of certain matrices) which was of no particular interest to Kramers himself, or to Majorana—both of whom were concerned with the description and properties of the *states* of higher spin systems (atoms). Kramers and Majorana were, however, concerned with distinct aspects of that large and intricate problem area: Kramers—to judge from the account of his work published[6] by H. C. Brinkman, his former student—was interested primarily in the efficient management of states that refer to populations of spin-$\frac{1}{2}$ particles, while Majorana had interest in single particle systems of high spin. Kramers looked upon atoms as objects assembled from their parts, while Majorana found it efficient—and sufficient to his physical objective—to adopt a more wholistic point of view.

Majorana, in his short note, provides no indication of whether or not he was aware—or cared—that he worked within an analytical tradition the seed of which had been planted by Stokes in 1852, and to which Poincaré had made contributions of direct relevance in 1892. The tradition to which I refer was still lively when Majorana wrote,[7] but drew its motivation not from quantum mechanics (of which, of course, Stokes/Poincaré knew nothing) but from the physics of polarized optical beams. Recently I had occasion to review the that theory, and some of its mechanical applications, in bewildering detail.[8] I begin with review of the most relevant essentials of that tangled tale.

Antecedents in the work of Stokes & Poincaré. Look into the face of an onrushing monochromatic lightbeam; i.e., of an electromagnetic plane wave. To describe, in reference to some selected Cartesian frame, the motion of the electric vector

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[5] The latter development is due mainly to the influence of Wigner (see Zs. für Physik **40**, 883 (1926) and **43**, 624 (1927) for preliminary accounts of the work summarized in his *Gruppentheorie und irhe Anwendung auf die Quantenmechanik der Atomspektren* (1931)) and Weyl (*Gruppentheorie und Quantenmechanik* (1928)). Ironically, Wigner himself became an enthusiastic proponent of Kramer's method, which he taught in his classes (where it came to the attention of John Powell, at whose instance—Crasemann's word—an account of the method was included in their *Quantum Mechanics* (1961)). And Weyl's name will forever be linked to Majorana's in connection with the theory of the neutrino.

[6] *Applications of Spinor Invariants in Atomic Physics* (1956).

[7] P. Soleillet had made an important (but widely neglected) contribution in 1929. Thereafter the work was taken up and brought to a kind of classical completion by Hans Muller (unpublished work in the early 1940's), R. Clark Jones (1941–1947) and S. Pancharatnam (1956). The subject was then taken over and elaborated by a generation of quantum opticians.

[8] "Ellipsometry: Stokes' parameters and related constructs in optics and classical/quantum mechanics" (1999).

(in which the motion of the associated magnetic vector is implicit) we write

$$\boldsymbol{E}(t) = E_1(t)\,\boldsymbol{i} + E_2(t)\,\boldsymbol{j} \quad \text{with} \quad \begin{cases} E_1(t) = \mathcal{E}_1 \cos(\omega t + \delta_1) \\ E_2(t) = \mathcal{E}_2 \cos(\omega t + \delta_2) \end{cases} \tag{1}$$

Stokes wrote before the electromagnetic nature of light had been recognized, but had already good reason to suppose that light involved rapid transverse vibration of *some* sort: he knew that $\boldsymbol{E}(t)$—whatever its physical nature—would trace/retrace an elliptical Lissajous figure

$$\begin{gathered}\mathcal{E}_2^2 E_1^2 - 2\mathcal{E}_1\mathcal{E}_2 \cos\delta \cdot E_1 E_2 + \mathcal{E}_1^2 E_2^2 = \mathcal{E}_1^2\mathcal{E}_2^2 \sin^2\delta \\ \delta \equiv \delta_2 - \delta_1 \equiv \text{ phase difference}\end{gathered} \tag{2}$$

He possessed (as we possess) no detector able to exhibit the $\sim 10^{14}$ Hz *flight* of $\boldsymbol{E}(t)$ but—and this is a measure of the man's genius—argued that one need only equip oneself with a photometer and suitable filters to obtain a complete characterization of the elliptical *curve traced* by $\boldsymbol{E}(t)$; i.e., of the polarizational state of the lightbeam. To that end he wrote

$$\left.\begin{aligned} S_0 &= \mathcal{E}_1^2 + \mathcal{E}_2^2 \\ S_1 &= \mathcal{E}_1^2 - \mathcal{E}_2^2 &&= S_0 \cos 2\chi \cos 2\psi \\ S_2 &= 2\mathcal{E}_1\mathcal{E}_2 \cos\delta &&= S_0 \cos 2\chi \sin 2\psi \\ S_3 &= 2\mathcal{E}_1\mathcal{E}_2 \sin\delta &&= S_0 \sin 2\chi \end{aligned}\right\} \tag{3}$$

where the first set of equations define "Stokes' parameters" $\{S_0, S_1, S_2, S_3\}$ in terms of the physical variables $\mathcal{E}_1$, $\mathcal{E}_2$, δ, and the second set establishes their relation to the geometrical parameters S_0 (which—see Figure 1—sets the *scale* of the ellipse), ψ (which indicates *orientation*) and χ (which refers to the *ellipticity*).

Notice that Stokes' parameters are *quadratic in the field strengths*—are, in other words, "intensities," susceptible to direct photometric scrutiny. And that

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \tag{4}$$

which might appear to render one of the parameters redundant. We have, however, assumed perfect 100% polarization, while real lightbeams can be expected to be imperfectly or *partially polarized*; in such cases (as can be shown) one obtains

$$S_0^2 > S_1^2 + S_2^2 + S_3^2$$

It is, therefore, a further recommendation of Stokes' construction that with four measurements one can assign observational meaning to the

$$\text{"degree of polarization"}\ P \equiv \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}$$

Throughout the present discussion we will assume the polarization to be perfect.

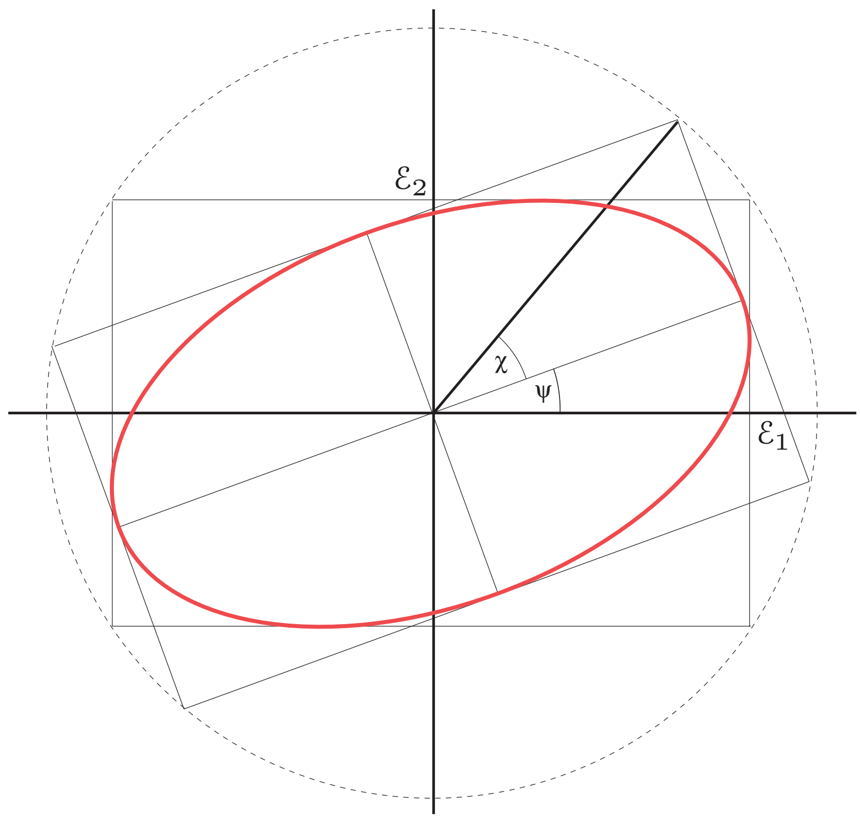

FIGURE 1: *Indication of the parameters used by Stokes to describe the orientation (ψ) and eccentricity (χ) of the ellipse traced by the flying* ***E****-vector in an idealized lightbeam. A remarkable theorem*[9] *asserts that all rectangles circumscribed about the ellipse—including in particular the two shown—have*

$$(\text{semidiagonal})^2 = \mathcal{E}_1^2 + \mathcal{E}_2^2 = S_0$$

which serves to set the scale *of the figure (intensity of the beam). Notice that $\psi \mapsto \psi + \pi$ gives back the* same *ellipse. The angle χ vanishes at the semi-major axis, and is understood to range on $\{-\frac{\pi}{2}, +\frac{\pi}{2}\}$; the adjustment $\chi \mapsto -\chi$ gives back the same ellipse,* but with reversed chirality.

[9] See Figure 2 in ELLIPSOMETRY.

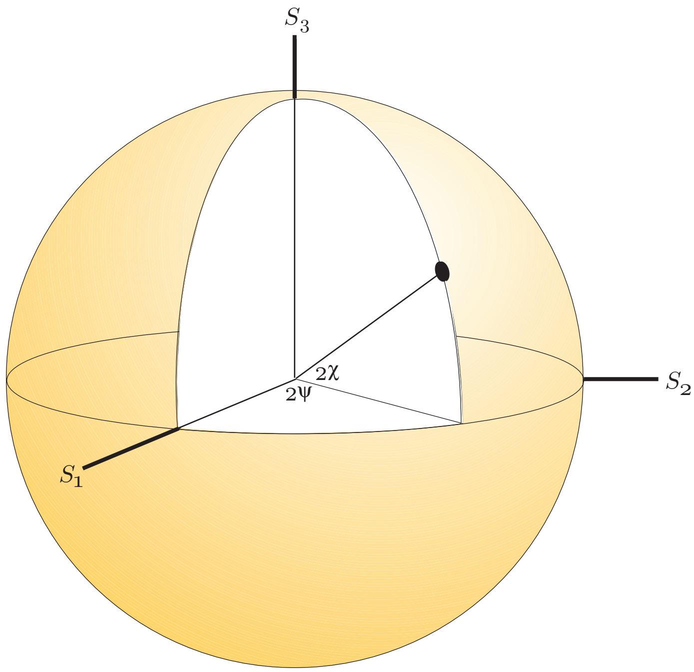

FIGURE 2: *Placement of the point on the Stokes sphere of radius S_0 which by (3) is representative of the ellipse shown in Figure 1. Note the doubled angles: as ψ advances from 0 to 2π the ellipse assumes every orientation twice, and the point shown above revolves twice around the polar axis. The sphere is, in this sense, really a* double *sphere (has an "inside" and an "outside"). Reversing the sign of χ sends the Stokes point to the opposite hemisphere (reverses the sign of S_3). Points in the Northern hemisphere represent ellipses with ↻ chirality, points in the Southern hemisphere have ↺ chirality.*

In contexts where we have interest in the figure of the ellipse (orientation, ellipticity and chirality) but not in its size it becomes natural to set $S_0 = 0$. Or—which is on dimensional grounds preferable—to introduce variables

$$\left.\begin{aligned} s_1 &\equiv \frac{S_1}{S_0} = \cos 2\chi \cos 2\psi \\ s_2 &\equiv \frac{S_2}{S_0} = \cos 2\chi \sin 2\psi \\ s_3 &\equiv \frac{S_3}{S_0} = \sin 2\chi \end{aligned}\right\} \tag{5}$$

We can, by (4), look upon these as the coordinates of a unit vector $\boldsymbol{s}$ in 3-dimensional "Poincaré Space:"

$$\boldsymbol{s}\cdot\boldsymbol{s} = s_1^2 + s_2^2 + s_3^2 = 1 \tag{6}$$

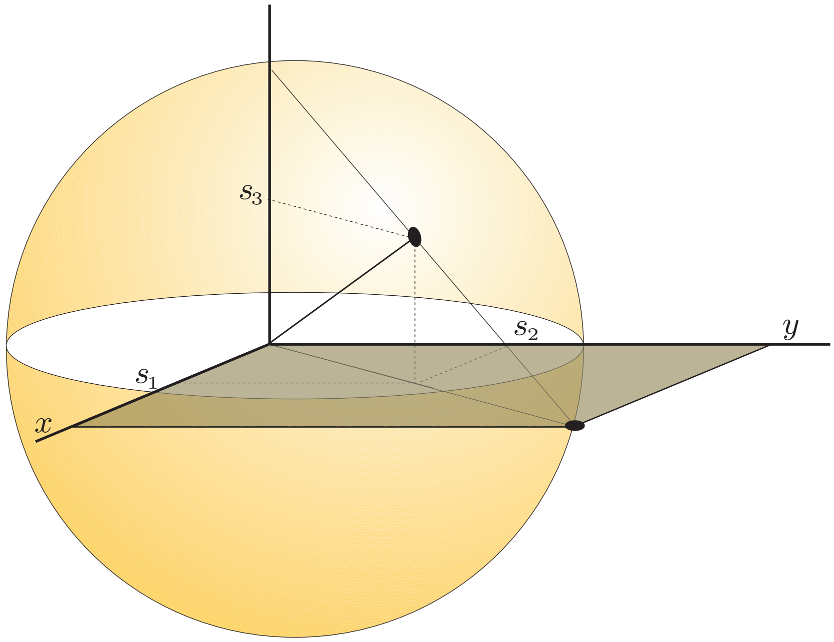

FIGURE 3: *The Stokes point sits now on a sphere of* unit *radius—the so-called “Poincaré sphere.” The figure illustrates Poincaré’s stereographic map, whereby the Stokes point is projected from the North Pole to a point* $\{x, y\}$ *on the equatorial plane. Points in the Northern hemisphere project to the exterior of the unit disk, points in the Southern hemisphere to the interior. The former have ↻ chirality, the latter have ↺ chirality, while points on equator—which associate with* linearly *polarized beams—project to the boundary of the unit disk and have undefined chirality.*

Proceeding now in Poincaré’s footsteps (which follow a trail first explored by Riemann), we (i) project points of the Poincaré sphere onto the equatorial plane. A elementary similar triangles argument based upon the preceding figure gives

$$\left.\begin{aligned} x &\equiv \frac{s_1}{1-s_3} \\ y &\equiv \frac{s_2}{1-s_3} \end{aligned}\right\} \tag{7}$$

Inversely

$$\left.\begin{aligned} s_1 &= \frac{2x}{x^2+y^2+1} \\ s_2 &= \frac{2y}{x^2+y^2+1} \\ s_3 &= \frac{x^2+y^2-1}{x^2+y^2+1} \end{aligned}\right\} \tag{8}$$

Next (ii) we identify the equatorial plane with the complex plane, writing

$$z = x + iy = \frac{s_1 + is_2}{1 - s_3} \equiv z(\boldsymbol{s}) \tag{9}$$

in terms of which (8) can be written

$$\left.\begin{aligned} s_1 + is_2 &= \frac{2z}{z^*z + 1} \\ s_1 - is_2 &= \frac{2z^*}{z^*z + 1} \\ s_3 &= \frac{z^*z - 1}{z^*z + 1} \end{aligned}\right\} \tag{10}$$

Reverting to the optical origins of this discussion: it is evident from Figure 1 that we would set $\psi = 0$ and $\chi = 0$ to describe a $\longleftrightarrow$ linearly polarized beam, and would set $\psi = \frac{\pi}{2}$ and $\chi = 0$ to describe $\updownarrow$ polarization. Poincaré, to say the same thing, would on the basis of (5) write

$$\boldsymbol{s}_{\leftrightarrow} = \begin{pmatrix} +1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{s}_{\updownarrow} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

It is physically evident even in the absence of detailed proof that the beams just described *will not interfere when superimposed*, and is obvious that $\boldsymbol{s}_{\leftrightarrow}$ and $\boldsymbol{s}_{\updownarrow}$ identify *antipodal points on the Poincaré sphere*. Stokes' construction leads by its own sweet momentum to a sweeping generalization of those elementary observations: beams with antipodal descriptors $\boldsymbol{s}$ and $-\boldsymbol{s}$ are (in Stokes' phrase) "oppositely polarized" in the sense that when physically superimposed they fail to interfere. It becomes in that light interesting to notice that

$$z(-\boldsymbol{s}) = -\frac{s_1 + is_2}{1 + s_3} = -\frac{1}{z^*(\boldsymbol{s})} \tag{12}$$

where use has been made of the fact that (6) can be expressed

$$(s_1 + is_2)(s_1 - is_2) = (1 + s_3)(1 - s_3) \tag{13}$$

Soleillet/Muller were the first to appreciate that Stokes' invention places one in position to construct an elegantly economical account of the action

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}} \xrightarrow[\text{linear transformation}]{} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{out}}$$

of linear optical devices and materials. We will have special interest here in the beam modifications achieved by the non-absorptive devices which opticians call "wave plates," "phase plates," "compensators" or "retarders":

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}} \longrightarrow \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathbb{R} & \\ 0 & & & \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}}$$

where preservation of (4) forces $\mathbb{R}$ to be a rotation matrix. We might, in such a restictive context, write

$$\boldsymbol{s}_{\text{in}} \longrightarrow \boldsymbol{s}_{\text{out}} = \mathbb{R}\,\boldsymbol{s}_{\text{in}} \tag{14}$$

to say the same thing. Thus does $O(3)$ acquire optical interest.

Back again to Poincaré, who in place of (14) writes

$$z(\boldsymbol{s}) \longrightarrow z(\,\mathbb{R}\boldsymbol{s}) \tag{15}$$

Well known to him was the remarkable fact[10] that

> The most general analytic (or conformal) transformation $z \longrightarrow z = f(z)$ which maps the plane one-to-one into itself is the "linear fractional transformation"[11]

$$z = \frac{az+b}{cz+d} \quad : \quad ad-bc \neq 0 \tag{16.1}$$

Evidently $\mathbb{R}$ and the complex numbers $\{a,b,c,d\}$—which[12] we can without loss of generality assume satisfy

$$ad-bc = 1 \tag{16.2}$$

—convey identical information.

The $\{a,b,c,d\}$ supply eight real degrees of freedom, reduced conventionally to six by (16.2). But $\mathbb{R}$ has only three real degrees of freedom, so the detailed association $\mathbb{R} \longleftrightarrow \{a,b,c,d\}$ is not quite obvious.[13] We proceed this way:

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[10] See, for example, L. R. Ford, *Automorphic Functions* (1951), p. 2.

[11] Such transformations are sometimes said to be "bilinear" or "homographic," and are sometimes called "Möbius transformations."

[12] The points to notice are that z is unaffected by

$$\{a,b,c,d\} \longrightarrow \{a,b,c,d\} \equiv \frac{\{a,b,c,d\}}{\sqrt{ad-bc}}$$

and that $ad-bc=1$.

[13] We note in this connection that equations (10) yield $\boldsymbol{s}\cdot\boldsymbol{s} = 1$ as an identity, so preservation of that condition places *no* limitation on the design of z.

From either (8) or (10) obtain

$$\left.\begin{aligned} s_1 &= \frac{z+z^*}{z^*z+1} \\ s_2 &= -i\frac{z-z^*}{z^*z+1} \\ s_3 &= \frac{z^*z-1}{z^*z+1} \end{aligned}\right\} \tag{17}$$

Insert (16) into the primed instance of those equations to obtain

$$\left.\begin{aligned} s_1 &= \frac{(az+b)(c^*z^*+d^*)+(cz+d)(a^*z^*+b^*)}{(az+b)(a^*z^*+b^*)+(cz+d)(c^*z^*+d^*)} \\ s_2 &= -i\frac{(az+b)(c^*z^*+d^*)-(cz+d)(a^*z^*+b^*)}{(az+b)(a^*z^*+b^*)+(cz+d)(c^*z^*+d^*)} \\ s_3 &= \frac{(az+b)(a^*z^*+b^*)-(cz+d)(c^*z^*+d^*)}{(az+b)(a^*z^*+b^*)+(cz+d)(c^*z^*+d^*)} \end{aligned}\right\} \tag{18}$$

Require of the shared denominator that it conform to the design of (17):

$$(az+b)(a^*z^*+b^*)+(cz+d)(c^*z^*+d^*) = z^*z+1$$

This entails

$$\begin{aligned} a^*a + c^*c &= 1 \\ b^*b + d^*d &= 1 \\ a^*b + c^*d &= 0 \quad \text{and its conjugate} \end{aligned}$$

Write $a = Ae^{i\alpha}$, $b = Be^{i\beta}$, $c = Ce^{i\gamma}$, $d = De^{i\delta}$ and consider $\{c, d\}$ to be the unknowns; then

$$\begin{aligned} C^2 &= 1 - A^2 \\ D^2 &= 1 - B^2 \\ CD &= \sqrt{1-A^2}\sqrt{1-B^2} = AB \Rightarrow A^2 + B^2 = 1 \\ \delta - \gamma &\equiv (\beta - \alpha + \pi) \bmod 2\pi \end{aligned}$$

So we have $C = B$ and $D = A$. The condition (16.2) becomes

$$A^2 e^{i(\alpha+\delta)} - (1 - A^2)e^{i(\beta+\gamma)} = 1$$

giving

$$\begin{aligned} \alpha + \delta &\equiv (\beta + \gamma + \pi) \bmod 2\pi \quad : \quad \text{already known} \\ \beta + \gamma + \pi &\equiv 0 \bmod 2\pi \end{aligned}$$

from which we obtain

$$\begin{aligned} \gamma &\equiv -(\beta + \pi) \bmod 2\pi \\ \delta &\equiv -\alpha \bmod 2\pi \end{aligned}$$

The upshot of the argument is that

$$\left.\begin{aligned} d &= a^* \\ c &= -b^* \end{aligned}\right\} \quad \text{with} \quad a^*a + b^*b = 1 \tag{19}$$

which can be phrased this way:

$$\mathbb{S} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{is unitary and unimodular!} \tag{20}$$

Returning with (19) to (18) we obtain

$$\left.\begin{aligned} s_1 &= \frac{(a^2 - b^{*2})z + (a^{*2} - b^2)z^* - (ab + a^*b^*)(z^*z - 1)}{z^*z + 1} \\ s_2 &= -i\frac{(a^2 + b^{*2})z - (a^{*2} + b^2)z^* - (ab - a^*b^*)(z^*z - 1)}{z^*z + 1} \\ s_3 &= \frac{(2\,ab^*)z + (2\,a^*b)z^* + (a^*a - b^*b)(z^*z - 1)}{z^*z + 1} \end{aligned}\right\} \tag{21}$$

which after some tedious manipulation can be written

$$\boldsymbol{s} = \mathbb{R}\,\boldsymbol{s} \tag{22.1}$$

with

$$\left.\begin{aligned} R_{11} &= \tfrac{1}{2}\big(a^2 + a^{*2} - b^2 - b^{*2}\big) \\ R_{12} &= i\tfrac{1}{2}\big(a^2 - a^{*2} + b^2 - b^{*2}\big) \\ R_{13} &= -\big(ab + a^*b^*\big) \\ R_{21} &= -i\tfrac{1}{2}\big(a^2 - a^{*2} - b^2 + b^{*2}\big) \\ R_{22} &= \tfrac{1}{2}\big(a^2 + a^{*2} + b^2 + b^{*2}\big) \\ R_{23} &= i\big(ab - a^*b^*\big) \\ R_{31} &= \big(a^*b + b^*a\big) \\ R_{32} &= -i\big(a^*b - b^*a\big) \\ R_{33} &= \big(a^*a - b^*b\big) \end{aligned}\right\} \tag{22.2}$$

Each of those matrix elements is manifestly real. *Mathematica* thinks for 0.05 second , then for another 0.08 second. . . and reports that

$$\det \mathbb{R} = 1 \quad \text{and} \quad \mathbb{R}^{\mathsf{T}}\mathbb{R} = \mathbb{I}$$

So the theory of optical polarization has led us as it led Poincaré (well in advance of the quantum mechanically inspired invention of the theory of spinors) back once again to the association $SU(2) \longleftrightarrow O(3)$.

Let us now agree—with Poincaré, in the tradition of Riemann—(*iii*) to write

$$z = \frac{u}{v} \tag{23}$$

and to consider (16.1) to have resulted from dividing the first of the following equations by the second:

$$\begin{aligned} \textcolor{red}{u} &= au + bv \\ \textcolor{red}{v} &= cu + dv \end{aligned}$$

These latter equations[14] we abbreviate

$$\textcolor{red}{\xi} = \mathbb{S}\,\xi \quad \text{with} \quad \xi \equiv \begin{pmatrix} u \\ v \end{pmatrix} \tag{24}$$

The complex numbers $\{u, v\}$ enter into (23) as what would in projective geometry be called "homogeneous coordinates," but in (24) they are assigned a different role: they are the coordinates of a complex vector (spinor) in complex 2-space (spin space). Spin space functions here as a "carrier space," created to provide a home for the matrix representation (24) of the unimodular linear fractional transformation (16). And since (23) is invariant under

$$\{u, v\} \longmapsto \{ku, kv\} \quad : \quad k \neq 0$$

the association

$$z \longleftrightarrow \text{spinor } \xi$$

is more properly described

$$z \longleftrightarrow \text{ray } k\xi \text{ in spin space} \tag{25}$$

Returning with (23) to (17) we obtain

$$\begin{aligned} s_1 &= \,\frac{u^*v + v^*u}{u^*u + v^*v} \\ s_2 &= i\,\frac{u^*v - v^*u}{u^*u + v^*v} \\ s_3 &= \,\frac{u^*u - v^*v}{u^*u + v^*v} \end{aligned}$$

which—if we define Pauli matrices in the usual way

$$\boldsymbol{\sigma}_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\sigma}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

—can be notated

$$\left.\begin{aligned} s_1 &= \frac{S_1}{S_0} = \frac{\xi^{\mathrm{t}}\boldsymbol{\sigma}_1\,\xi}{\xi^{\mathrm{t}}\boldsymbol{\sigma}_0\,\xi} \\ s_2 &= \frac{S_2}{S_0} = -\frac{\xi^{\mathrm{t}}\boldsymbol{\sigma}_2\,\xi}{\xi^{\mathrm{t}}\boldsymbol{\sigma}_0\,\xi} \\ s_3 &= \frac{S_3}{S_0} = \frac{\xi^{\mathrm{t}}\boldsymbol{\sigma}_3\,\xi}{\xi^{\mathrm{t}}\boldsymbol{\sigma}_0\,\xi} \end{aligned}\right\} \tag{26}$$

[14] My notation masks the fact that conditions (19) are still in force.

The unsightly minus sign appears to be an artifact of "colliding conventions;" it could be eliminated by any of several strategies—all of which entail cost, and none of which seem entirely satisfactory. We will encounter a similar problem at (28) below.

I conclude this review with brief indication of how Jones' accomplishment fits into the general scheme. Jones proceeds from the elementary observation that the electromagnetic information conveyed by (1) is recoverable as the real part of the complex vector

$$\begin{aligned}\boldsymbol{\mathcal{E}}(t) &= \boldsymbol{\mathcal{E}}\cdot e^{i\omega t}\\ \boldsymbol{\mathcal{E}} &\equiv \begin{pmatrix}\mathcal{E}_1 e^{i\delta_1}\\ \mathcal{E}_2 e^{i\delta_2}\end{pmatrix}\end{aligned} \tag{27}$$

and that the Stokes parameters, as first encountered at (3), can in this notation be described

$$\left.\begin{aligned}S_0 &= \mathcal{E}_1^2 + \mathcal{E}_2^2 &&= \boldsymbol{\mathcal{E}}^{\mathrm{t}}\boldsymbol{\sigma}_0\boldsymbol{\mathcal{E}}\\ S_1 &= \mathcal{E}_1^2 - \mathcal{E}_2^2 &&= \boldsymbol{\mathcal{E}}^{\mathrm{t}}\boldsymbol{\sigma}_3\boldsymbol{\mathcal{E}}\\ S_2 &= 2\mathcal{E}_1\mathcal{E}_2\cos\delta &&= \boldsymbol{\mathcal{E}}^{\mathrm{t}}\boldsymbol{\sigma}_1\boldsymbol{\mathcal{E}}\\ S_3 &= 2\mathcal{E}_1\mathcal{E}_2\sin\delta &&= \boldsymbol{\mathcal{E}}^{\mathrm{t}}\boldsymbol{\sigma}_2\boldsymbol{\mathcal{E}}\end{aligned}\right\} \tag{28}$$

He then develops a "Jones calculus" which employs equations of the form

$$\boldsymbol{\mathcal{E}}_{\text{out}} = \mathbb{J}\,\boldsymbol{\mathcal{E}}_{\text{in}}$$

to describe the action of certain classes of linear optical devices.[15] It is by now evident that the object $\boldsymbol{\mathcal{E}}$ known to the literature as the "Jones vector" should more properly be called the "Jones spinor;"Jones had stepped with undergraduate innocence onto the caboose of a train that had been chugging through the station (with Stokes/Poincaré at the controls) for nearly 90 years.[16]

[15] Jones had special interest in "compensators" and the "optical activity" of certain materials. The scope of his theory is expanded in §6.4 of L. Mandel & E. Wolf, *Optical Coherence & Quantum Optics* (1995), who follow G. B. Parrent & P. Roman, "On the matrix formulation of the theory of partial polarization in terms of observables," Nuovo Cimento **15**, 370 (1960).

[16] Jones first papers were written while he was an undergraduate at Harvard, and employed as a research apprentice by the Polaroid Corporation. A series of eight papers appeared in the pages of the Journal of the Optical Society of America between 1941 and 1956 (see Mandel & Wolf for detailed citations), by which time Jones had joined the research staff at Bell Laboratories. In only one paper (co-authored by H. Hurwitz, of the Harvard physics faculty) is the work of Poincaré cited, and Jones seems never to have become aware that he was plowing a field that by then had been elaborately cultivated by quantum physicists; he refers his readers to quantum texts by E. C. Kemble (1937) and V. Rojansky (1938), but only in connection with the definition of the Pauli matrices. "Reinvention of the wheel" appears to be an entrenched tradition in this field: see, for example, K. C. Westfold, "New analysis of the polarization of radiation and the Faraday effect," JOSA **49**, 717 (1959).

One curious detail: the Pauli matrices on the right side of (28) enter in permuted order. In §7 of ELLIPSOMETRY I trace this circumstance to the fact that built unwittingly into the design of (1) and Jones’ (27) is a preoccupation with *linear* $\longleftrightarrow\updownarrow$ polarization, while when we elected to project from the North Pole of the Poincaré sphere we tacitly assigned a central place to *circular* ↻↺ polarization. I show there how easy it is to switch from one basis to the other, and to interconvert (28) ⇔ (26).

The developments surveyed above sprang from the physics of optical polarization (but see service also in connection with description of the *statistical* properties of optical beams). One can, however, abandon the optics and keep the mathematics. . . or reassign the mathematics to other physical tasks. The mathematics pertains without change to the *quantum mechanics of 2-state systems* or—which is formally the same—to the *quantum theory of spin* $\frac{1}{2}$. Majorana’s problem: How most naturally to loosen up the mathematics so as to create apparatus that will support the *quantum theory of arbitrary spin*?

“Proto-spinors” orthogonal to a given spinor. The first of the essays in this series[1] proceeded from and culminated in the following observation: The rotational transform properties of an N-spinor

$$\xi \equiv \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_n \\ \vdots \\ \xi_{2\ell} \end{pmatrix} \quad : \quad \xi \mapsto \xi = \mathbb{S}\,\xi \quad \text{with} \quad \mathbb{S} \in \text{subgroup of } SU(N) \qquad N \equiv 2\ell + 1$$

are implicit in the statement that

$$\xi_n \quad \text{mimics the response of} \quad \sqrt{\binom{2\ell}{n}}\cdot u^{2\ell-n}\,v^n \quad : \quad n = 0, 1, \ldots, 2\,\ell$$

to

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where the Cayley-Klein parameters $\{\alpha, \beta\}$ refer to an element of $O(3)$.[17]

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[17] One comment before we proceed: It is true that $O(3)$ marks the birthplace, and remains a principal workplace, of spinor analysis (the Lorentz group being another). But spinor algebra/analysis, as fashioned by van der Waerden (1929) and others, amounts in effect to a “complexified tensor algebra/analysis” in which one builds upon not two but four categories of vector:

- contravariant vector (superscript);
- conjugated contravariant vector (dotted superscript);
- covariant vector (subscript);
- conjugated covariant vector (dotted superscript).

The objects mimiced, to be plain about it, are

$$\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} uu \\ \sqrt{2}\,uv \\ vv \end{pmatrix}, \begin{pmatrix} uuu \\ \sqrt{3}\,uuv \\ \sqrt{3}\,uvv \\ vvv \end{pmatrix}, \begin{pmatrix} u^4v^0 \\ \sqrt{4}\,u^3v^1 \\ \sqrt{6}\,u^2v^2 \\ \sqrt{4}\,u^1v^3 \\ u^0v^4 \end{pmatrix}, \ldots$$

which (except when $v = 0$, which places z "at infinity") can be notated

$$v\begin{pmatrix} z^1 \\ z^0 \end{pmatrix}, v^2\begin{pmatrix} z^2 \\ \sqrt{2}\,z^1 \\ z^0 \end{pmatrix}, v^3\begin{pmatrix} z^3 \\ \sqrt{3}\,z^2 \\ \sqrt{3}\,z^1 \\ z^0 \end{pmatrix}, v^4\begin{pmatrix} z^4 \\ \sqrt{4}\,z^3 \\ \sqrt{6}\,z^2 \\ \sqrt{4}\,z^1 \\ z^0 \end{pmatrix}, \ldots \tag{29}$$

with $z \equiv u/v$. I need to assign spinors of this specialized 2-parameter design a name; let us agree, in the absence of an established nomenclature, to call them "proto-spinors." For 2-spinors the spinor/proto-spinor distinction is empty, but not so in cases $N > 2$; look, for example, to the case $N = 5$, where one has

$$\sqrt{4}\,\frac{\xi_0}{\xi_1} = \sqrt{\tfrac{6}{4}}\,\frac{\xi_1}{\xi_2} = \sqrt{\tfrac{4}{6}}\,\frac{\xi_2}{\xi_3} = \sqrt{\tfrac{1}{4}}\,\frac{\xi_3}{\xi_4} = z$$

if ξ is a proto-spinor, but certainly not otherwise. Henceforth I will use π instead of ξ to emphasize that I have in mind a spinor with proto-structure.

We are in position now to consider the question which (to avoid distracting notational complexity) I will pose in the simplest non-trivial case: Let

$$\xi = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{pmatrix}$$

be given 3-spinor ($\ell = 1$), and let π be a proto-3-spinor. Then

$$\xi^{\mathrm{t}}\pi = v^2 \cdot \left(\xi_0^* z^2 + \sqrt{2}\,\xi_1^* z + \xi_2^*\right)$$

(continued from the preceding page) In tensor algebra one might *stipulate* the transformation group to be rotational, but cannot tacitly *assume* it to be so. The same remark pertains to spinor algebra, but in the latter setting the "rotational preoccupation" is so widely shared that the specialness of its ramifications is typically unstressed; people write as though the abandonment of $O(3)$ were unthinkable. My purpose here is to state *explicitly* that I have embraced the standard preoccupation: that the objects of interest to me are—for reasons rooted in the physical notion of "spin"—*rotational* spinors. See Élie Cartan *The Theory of Spinors* (1937, English translation 1981), R. Brauer & H. Weyl, "Spinors in n-dimensions," Amer. J. of Math. **57**, 425 (1935) and especially Chapter 2 in E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave Equations* (1953), which contains an elaborate annotated bibliography.

Require $\pi^{\text{t}}\pi \neq 0$ so as to exclude the possibility that $v = 0$. Then we will have

$$\left.\begin{array}{c} \xi \perp \pi \quad \text{if and only if } z \text{ is a root of} \\ \xi_0^* z^2 + \sqrt{2}\,\xi_1^* z + \xi_2^* = 0 \\ \text{and irrespective of the non-zero value} \\ \text{assigned to the complex multiplier } v^2 \end{array}\right\} \tag{30}$$

Looking similarly to other cases of low order, we are led to the statements

$$\xi_0^* z + \xi_1^* = 0 \quad : \quad \ell = \tfrac{1}{2} \tag{31.1}$$
$$\xi_0^* z^2 + \sqrt{2}\,\xi_1^* z + \xi_2^* = 0 \quad : \quad \ell = 1 \tag{31.2}$$
$$\xi_0^* z^3 + \sqrt{3}\,\xi_1^* z^2 + \sqrt{3}\,\xi_2^* z + \xi_3^* = 0 \quad : \quad \ell = \tfrac{3}{2} \tag{31.3}$$
$$\xi_0^* z^4 + \sqrt{4}\,\xi_1^* z^3 + \sqrt{6}\,\xi_2^* z^2 + \sqrt{4}\,\xi_3^* z + \xi_4^* = 0 \quad : \quad \ell = 2 \tag{31.4}$$
$$\vdots$$
$$\big\{\text{polynomial } p(z;\xi) \text{ of degree } 2\ell\big\} = 0 \quad : \quad \text{general case}$$

All the elements of ξ enter into the design of $p(z;\xi)$, but *linearly*: the statements

$$\xi \perp \pi, \text{ equivalently } p(z;\xi) = 0, \text{ are invariant under } \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{2\ell} \end{pmatrix} \to k \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{2\ell} \end{pmatrix}$$

so can most properly be said to refer therefore to a π-ray *normal to the ξ-ray*. The polynomial $p(z;\xi)$ can be looked upon as a “generating function” of the ξ-ray.

It is a clear implication of $\xi \perp \pi \iff \xi \perp \pi$ that equations of type (31) are rotationally invariant:

$$p(z;\xi) = 0 \quad \iff \quad p(z;\xi) = 0 \tag{32}$$

where $\xi \mapsto \xi = \mathbb{S}(\alpha,\beta)\,\xi$ and where

$$z^n \mapsto z^n = \left[\frac{\alpha\,z + \beta}{-\beta^* z + \alpha^*}\right]^n \quad : \quad n = 0, 1, \ldots, 2\ell \tag{33}$$

serves to describe the transformation not of π (since the multiplier has been abandoned) but of the π-ray. The latter point is a little confusing, so I illustrate how it works in the case $\ell = 1$:

$$\pi \sim \begin{pmatrix} z^2 \\ \sqrt{2}\,z^1 \\ z^0 \end{pmatrix} = \begin{pmatrix} \left[\frac{\alpha\,z+\beta}{-\beta^* z+\alpha^*}\right]^2 \\ \sqrt{2}\left[\frac{\alpha\,z+\beta}{-\beta^* z+\alpha^*}\right]^1 \\ \left[\frac{\alpha\,z+\beta}{-\beta^* z+\alpha^*}\right]^0 \end{pmatrix}$$
$$= \left[\frac{1}{-\beta^* z+\alpha^*}\right]^2 \cdot \mathbb{S}(1) \begin{pmatrix} z^2 \\ \sqrt{2}\,z^1 \\ z^0 \end{pmatrix} \sim \mathbb{S}(1)\,\pi$$

where

$$\mathbb{S}(1) \equiv \begin{pmatrix} \alpha^2 & \sqrt{2}\,\alpha\beta & \beta^2 \\ -\sqrt{2}\,\beta^*\alpha & (\alpha^*\alpha - \beta^*\beta) & \sqrt{2}\,\alpha^*\beta \\ (\beta^*)^2 & -\sqrt{2}\,\alpha^*\beta^* & (\alpha^*)^2 \end{pmatrix}$$

is precisely the unimodular unitary matrix which at (A-34) was found to effect the rotational transformation of 3-spinors. The argument extends to higher ℓ-values, but rapidly becomes burdensome.

If ξ has $N \equiv 2\ell+1$ components then the generating polynomial

> $p(z;\xi)$ is generally of degree 2ℓ, and has that many (not necessarily distinct) roots $\{z_1, z_2, \dots\}$

but

$$\begin{aligned} &\text{if } \xi_0 = 0 \qquad\quad\ \text{and } \xi_1 \neq 0 \text{ the degree is reduced to } 2\ell-1\\ &\text{if } \xi_0 = \xi_1 = 0 \text{ and } \xi_2 \neq 0 \text{ the degree is reduced to } 2\ell-2\\ &\qquad\qquad\vdots \end{aligned}$$

To each of the *distinct* roots $\{z_1, z_2, \dots\}$ of $p(z;\xi)$ we can associate one and only one proto-spinor orthogonal to ξ, so the latter—call them $\{\pi_1, \pi_2, \dots\}$—are precisely as numerous as the former, and can never exceed 2ℓ in number (which makes good sense: ξ lives in a $(2\ell+1)$-dimensional space, so the subspace orthogonal to ξ is 2ℓ-dimensional).

Look to the instance of a spinor with $\xi_0 \neq 0$: then

$$p(z;\xi) = \xi_0 \cdot (z-z_1)^{\mu_1} \cdot (z-z_2)^{\mu_2} \cdots$$

where the $\{z_1, z_2, \dots\}$ are distinct roots, and $\{\mu_1, \mu_2, \dots\}$ are the associated multiplicities:

$$\mu_1 + \mu_2 + \cdots = 2\ell$$

Since the polynomial refers to a *ray*, we can without loss of generality set $\xi_0 = 1$. Enlarging upon the latter remark, we will agree henceforth to scale the components of ξ in such a way

$$\frac{\{\overbrace{0, 0, \dots, 0}^{\text{k terms}}, \xi_k, \xi_{k+1}, \xi_{k+2}, \dots, \xi_{2\ell}\}}{\sqrt{\text{binomial coefficient}}\cdot \xi_k}$$

as to render the generating polynomial monic:

$$p(z;\xi) = (z-z_1)^{\mu_1} \cdot (z-z_2)^{\mu_2} \cdots \quad : \quad \mu_1 + \mu_2 + \cdots = 2\ell - k$$

To gain a sharper sense of what's going on here, we look to the case $\ell = \frac{3}{2}$, where we encounter monic polynomials of these types:

$$\begin{array}{rrcl}
i) & (z-z_1)(z-z_2)(z-z_3) & : & \xi_0 \neq 0;\ \text{non-degenerate} \\
ii) & (z-z_1)^2(z-z_2) & : & \xi_0 \neq 0;\ \text{singly degenerate} \\
iii) & (z-z_1)^3 & : & \xi_0 \neq 0;\ \text{doubly degenerate} \\
iv) & (z-z_1)(z-z_2) & : & \xi_0 = 0,\ \xi_1 \neq 0;\ \text{non-degenerate} \\
v) & (z-z_1)^2 & : & \xi_0 = 0,\ \xi_1 \neq 0;\ \text{singly degenerate} \\
vi) & (z-z_1) & : & \xi_0 = \xi_1 = 0,\ \xi_2 \neq 0;\ \text{non-degenerate} \\
 & \text{not defined} & : & \xi_0 = \xi_1 = \xi_3 = 0,\ \xi_4 \neq 0
\end{array}$$

In case (i) we have

$$\begin{aligned}
&z^3 - (z_1+z_2+z_3)z^2 + (z_1z_2+z_2z_3+z_3z_1)z - z_1z_2z_3 \\
&\quad = \frac{\xi_0 \cdot z^3 + \sqrt{3}\,\xi_1^* \cdot z^2 + \sqrt{3}\,\xi_2^* \cdot z + \xi_3^*}{\xi_0^*}
\end{aligned} \tag{34.1}$$

from which cases (ii) and (iii) can be obtained by specialization; in case (iv)

$$\begin{aligned}
&z^2 - (z_1+z_2)z + z_1z_2 \\
&\quad = \frac{0 \cdot z^3 + \sqrt{3}\xi_1^* \cdot z^2 + \sqrt{3}\,\xi_2^* \cdot z + \xi_3^*}{\sqrt{3}\,\xi_1^*}
\end{aligned} \tag{34.2}$$

which gives (v) by specialization, and in case (vi)

$$\begin{aligned}
&z - z_1 \\
&\quad = \frac{0 \cdot z^3 + 0 \cdot z^2 + \sqrt{3}\,\xi_2^* \cdot z + \xi_3^*}{\sqrt{3}\,\xi_2^*}
\end{aligned} \tag{34.3}$$

If we exercise our still-lively option to

set the leading non-zero $\xi_{\bullet}$ equal to (say) unity

and observe that

$$1 = \frac{z_1^0 + z_2^0 + \cdots}{\text{number of roots}} = \text{symmetric function of roots}$$

we are led on the evidence of (34) to the conclusion that

> the components ξ_k of ξ are expressible as *symmetric functions* of the roots of the generating polynomial.

It is instructive to look to these special instances of (34):

$$\left.\begin{aligned}\xi &= \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \iff p(z;\xi) = (z-0)^3\\ \xi &= \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \iff p(z;\xi) = (z-0)^2\\ \xi &= \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \iff p(z;\xi) = (z-0)^1\end{aligned}\right\} \tag{35.1}$$

The element of mystery is removed from the remaining case when one reinstates homogeneous coordinates:

$$\xi = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \iff 0\cdot u^3 + \sqrt{3}\,0\cdot u^2v + \sqrt{3}\,0\cdot uv^2 + 1\cdot v^3 = 0 \tag{35.2}$$

which entails $v = 0$ whence $z \equiv u/v = \infty$, which we might express

$$(z-\infty) = 0 \quad : \quad \text{root has become the "point at infinity"}$$

It was originally Riemann's idea to adjoint to the complex plane a "point ∞ at infinity," and to consider it to be the stereographic image of the North pole.[18] The unifying value of the idea will become clearer as we proceed.

Look finally to the primitive case $\ell = \frac{1}{2}$, where (see again (31.1))

$$\xi_0^* z + \xi_1^* = \xi_0^*(z - z_1) = 0$$

entails

$$z_1 = -\frac{\xi_1^*}{\xi_0^*} \quad : \quad \text{root always non-degenerate} \tag{36}$$

More particularly, we have

$$\begin{pmatrix}1\\0\end{pmatrix} \iff \text{root } z_1 \text{ at origin} \quad : \quad \begin{pmatrix}0\\1\end{pmatrix} \iff \text{root } z_1 \text{ "at infinity"}$$

If we restore to 2-spinors the plumage which—at (24)—they wore when they stepped out of the egg

$$\xi = \begin{pmatrix}\xi_0\\ \xi_1\end{pmatrix} \quad \text{reverts to} \quad \begin{pmatrix}u\\v\end{pmatrix}$$

then (36) becomes

$$\begin{aligned} z_1 &= -\frac{v^*}{u^*} \quad : \quad \text{compare} \quad z \equiv \frac{u}{v}\\ &= -\frac{1}{z^*} \end{aligned} \tag{37}$$

[18] §74.D in *Encyclopedic Dictionary of Mathematics* (1996).

While z lends structure to ξ, z_1 was designed to lend structure to a spinor π_1 *normal* to ξ

$$\pi_1 \sim \begin{pmatrix} z_1 \\ 1 \end{pmatrix} \quad \perp \quad \xi = \begin{pmatrix} u \\ v \end{pmatrix}$$

and indeed: $\pi^{\text{t}}\xi \sim z_1^* u + v = (-v/u)u + v = 0$.

We have touched here on an instance of a more general circumstance, which I illustrate in the case $\ell = \frac{3}{2}$. Let z refer to an *arbitrary proto*-spinor

$$\pi \sim \begin{pmatrix} z^3 \\ \sqrt{3}\, z^2 \\ \sqrt{3}\, z^1 \\ z^0 \end{pmatrix}$$

and let z_1 refer to a proto-spinor $\pi_1 \perp \pi$. Immediately

$$\begin{aligned} \pi^{\text{t}}\pi_1 &\sim (z^* z_1)^3 + 3(z^* z_1)^2 + 3(z^* z_1)^1 + 1 \\ &= (z^* z_1 + 1)^3 \\ &= 0 \quad \text{if and only if} \quad z_1 = -\frac{1}{z^*} \end{aligned}$$

Evidently there exists only one such π_1, and it is described—not just in the case $\ell = \frac{1}{2}$ but in every case—by (37). It is a notable fact—which we will soon want to exploit—that (37) is structurally reminiscent of Poincaré's (12):

$$z(-\boldsymbol{s}) = -\frac{1}{z^*(\boldsymbol{s})}$$

Majorana's construction. In the preceding section we looked not to[19]

$$\xi = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

itself, but to the population of proto-spinors *normal* to ξ:

$$\pi_1 \sim \begin{pmatrix} z_1^3 \\ \sqrt{3}\, z_1^2 \\ \sqrt{3}\, z_1^1 \\ z_1^0 \end{pmatrix}, \quad \pi_2 \sim \begin{pmatrix} z_2^3 \\ \sqrt{3}\, z_2^2 \\ \sqrt{3}\, z_2^1 \\ z_2^0 \end{pmatrix}, \quad \pi_3 \sim \begin{pmatrix} z_3^3 \\ \sqrt{3}\, z_3^2 \\ \sqrt{3}\, z_3^1 \\ z_3^0 \end{pmatrix}$$

[19] It is, as before, "to avoid notational distractions" that I write (as has become my habit) in language specific to the case $\ell = \frac{3}{2}$, which should make clear the pattern of statements in the general case. And I pretend there to be a "full house" of π-spinors (i.e., that they are 2ℓ in number), even though that is (as we have seen) not always the case.

We found that the π-population can be looked upon as a "property" of ξ —more properly: not of ξ itself, but of the associated ξ-ray—as can the set $\{z_1, z_2, \ldots\}$ of complex numbers from which the π-population derives. Those —together with the associated multiplicites $\{\mu_1, \mu_2, \ldots\}$—permit one to write the generating polynomial $p(z;\xi)$ of the ξ-ray, whence to reconstruct ξ itself (to within an overall complex multiplier). The ξ-ray has thus been represented by a population of (possibly coincident) *points sprinkled on the complex plane*.

North-polar stereographic projection back upon the unit sphere—which in the optical application (where only 2-spinors were encountered) deposited a single point on the "Poincaré sphere"—produces a *population of points sprinkled on the surface of the what in the present application we will agree to call the "Majorana sphere"* (see Figure 4). If we adopt the understanding that

$$[\text{degree } \# \text{ of } p(z;\xi)] + [\text{number of "points at } \infty\text{"}] = 2\ell \tag{38.1}$$

then we can assert that in the case of spin ℓ

$$\text{every } \xi\text{-ray is represented by } 2\ell \text{ points on the Majorana sphere} \tag{38.2}$$

of which $2\ell - \#$ reside at the North pole.

Transformation theory informs us that we are, in fact, *forced* to adopt the viewpoint just described, for degree-controlling conditions of the forms

$$\begin{aligned} \text{"if } \xi_0 = 0 \quad\quad\quad &\text{and } \xi_1 \neq 0 \text{ the degree is reduced to } 2\ell - 1, \\ \text{if } \xi_0 = \xi_1 = 0 \; &\text{and } \xi_2 \neq 0 \text{ the degree is reduced to } 2\ell - 2 \\ &\vdots \\ &\text{etc."} \end{aligned}$$

encountered earlier are *transformationally unstable*, which is to say: they are, in general, not preserved under $\xi \mapsto \xi = \mathbb{S}\,\xi$. To illustrate the point I revert to the example (case $\ell = 1$) used to illustrate (32/33): Suppose it to be the case that $\xi_0 = 0$ and $\xi_1 \neq 0$. Then $p(z;\xi) = \sqrt{2}\,\xi_1^* z + \xi_2^* = 0$ entails

$$\begin{aligned} z_1 &= \frac{u_1}{v_1} = -\frac{\xi_2^*}{\sqrt{2}\,\xi_1^*} \\ z_2 &= \frac{u_2}{0} = \infty \end{aligned}$$

Rotation engenders

$$\xi \mapsto \xi = \begin{pmatrix} \alpha^2 & \sqrt{2}\,\alpha\beta & \beta^2 \\ -\sqrt{2}\,\beta^*\alpha & (\alpha^*\alpha - \beta^*\beta) & \sqrt{2}\,\alpha^*\beta \\ (\beta^*)^2 & -\sqrt{2}\,\alpha^*\beta^* & (\alpha^*)^2 \end{pmatrix} \begin{pmatrix} 0 \\ \xi_1 \\ \xi_2 \end{pmatrix}$$

giving

$$\begin{aligned} p(z;\xi) = \big[\sqrt{2}\,\alpha^*\beta^*\,\xi_1^* + (\beta^*)^2\xi_2^*\big]z^2 + \sqrt{2}\big[(\alpha^*\alpha - \beta^*\beta)\,\xi_1^* + \sqrt{2}\,\alpha\beta^*\xi_2^*\big]z \\ + \big[-\sqrt{2}\,\alpha\beta\,\xi_1^* + \alpha^2\xi_2^*\big] \end{aligned}$$

and so has (in typical cases) *lifted the degree* of the generating polynomial (restored it to its generic value $2\ell = 2$)

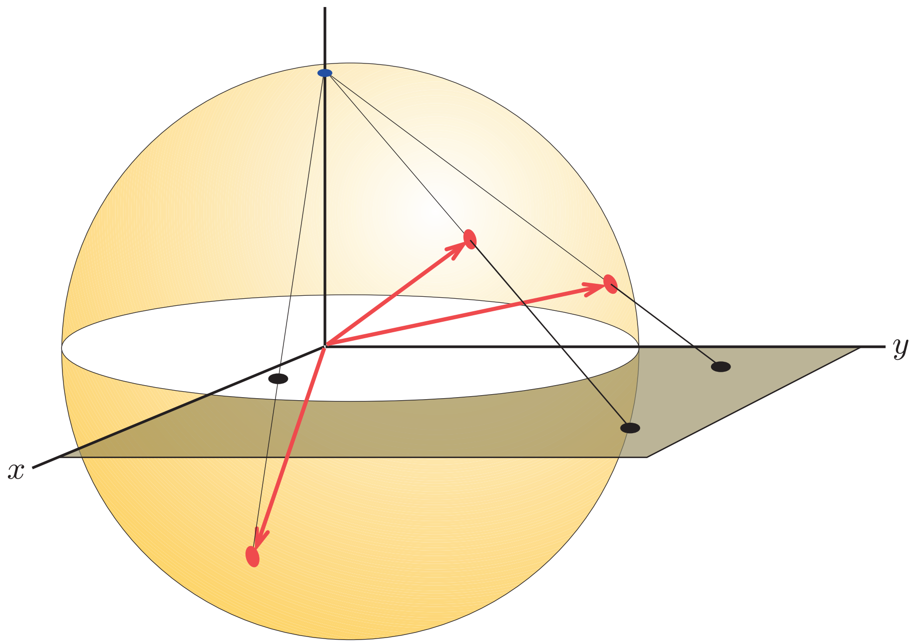

FIGURE 4: *Three points* ● *sprinked on the complex plane represent a* ξ*-ray with* $\ell = \frac{3}{2}$*. Stereographic projection yields a trio of points* ● *sprinkled on the Majorana sphere. If a point* ● *were to recede to* ∞ *the associated point* ● *would retreat to the North pole.*