Comments concerning Julian Schwinger's

"ON ANGULAR MOMENTUM"

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Introduction. It is to Thomas Wieting[1] that I owe my first exposure to some characteristically elegant work by Roger Penrose[2] that bears tellingly upon a nest of foundational issues in quantum mechanics. Penrose draws upon certain ideas that Etorre Majorana—incidentally to a classic note[3] concerned with the magnetic manipulation of atomic beams—had injected into the quantum theory of angular momentum. Penrose remarks that [Majorana's] "description of general spin states is not very familiar to physicists. . . [though] it provides a useful and geometrically illuminating picture." It was in recognition of the undeniable fact that "the Majorana picture of spin, while very elegant and also remarkably economical. . . is still [after nearly seventy years!] unfamiliar to the vast majority of physicists" that J. E. Massad & P. K. Aravind undertook to construct an "elementary account" of Penrose's work, an account that proceeds entirely without reference to Majorana.[4] Thus was I motivated to make an effort to gain some understanding of what it was that Majorana had accomplished . . . and the world had elected to ignore, and why. Quite recently it has been brought to my attention[5] that Schwinger's relatively little-known contributions to the quantum theory of angular momentum were similarly motivated.

The work to which I have just referred (most of which was published only belatedly, and in obscure places) is the subject of a retrospective essay which Julian Schwinger contributed to a festschrift honoring the 80$^{\text{th}}$ birthday

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[1] "The Penrose Dodecahedron," Reed College Physics Seminar of 3 November 1999.

[2] *Shadows of the Mind* (1994); see especially Appendix C to Chapter 5. Additional bibilographic information can be found in the introduction to my "Spin matrices for arbitrary spin," (2000), to which I will refer here as Part A.

[3] "Atomi orientati in campo magnetico variabile," Nuovo Cimento **9**, 43–50 (1932)

[4] "The Penrose dodecahedron revisited," AJP **67**, 631 (1999).

[5] I am indebted to Victoria Mitchell for this news.

of Isidore Rabi.[6] Schwinger (1918–1988) graduated from Columbia University—which, but for the intervention of Rabi, would have expelled him for gross neglect of his non-technical classes—in 1936 at the age of seventeen, and stayed on to take his doctorate in 1939. Schwinger considered it to be "a natural consequence of [his] early association with Rabi and his atomic beam school[7] [that he developed] a protracted fascination with atomic and molecular moments and, more generally, the quantum theory of angular momentum." It was from Rabi that Schwinger became acquainted with Majorana's work, which at first he found "baffling ['. . . ogni stato sarà rappresentato da $2j$ punti sulla sfera unitaria. . . ' Where did that come from?]." Schwinger promptly resolved that question to his own satisfaction, but remarks that "my answer was only hinted at in a 1937 paper." It is from that paper,[8] and Schwinger's remarks concerning it, that I now quote:

> " 'The evaluation of [a certain matrix element] may be carried out, for arbitrary j, by a method which will not be given here. The results are in complete agreement with those obtained from Majorana's general theorem' (by which I meant formula).' That method was, of course, the explicit construction of an arbitrary angular momentum j as a superposition of $2j$ spin $\frac{1}{2}$ systems. What was thus left implicit (did I then think it of so little consequence?) was actually the most important result in this work. . . "

Schwinger returned to the subject in 1945 under stimulus of a review article[9] in which Block and Rabi "remarked on the derivation of Majorana's formula from the spin $\frac{1}{2}$ representation." He undertook to write a paper "supporting the thesis that the expression of symmetry concepts in quantum

[6] *A Festschrift for I. I. Rabi* (Transactions of the New York Academy of Sciences **38**, 1977). Schwinger's "The Majorana formula" appears on pages 170–172. Two previously unpublished papers ("A note on group theory and quantum mechanics" (1945) and "The Majorana formula" (1959)) are attached as appendices.

[7] Rabi (1898–1988), who had been a student of Otto Stern, was to receive the Nobel Prize in 1944 for "the resonance method for recording the magnetic properties of the atomic nucleus," work which led to the development of NMR.

[8] "On nonadiabatic processes in homogeneous fields," Phys. Rev. **51**, 648 (1937). The 19-year-old Schwinger states that "the purpose of this paper is to point out that Güttinger equations are incorrect and lead to erroneous results in any case other than that of the rotating field, which he considered. The corrected equations are applied in the calculation of transition probabilities between the various magnetic states of a field precessing with constant angular velocity." The Güttinger reference is to P. Güttinger, "Das Verhalten von Atomen im magnetischen Drehfeld," Zeits. f. Physik **73**, 169 (1931), a reference (one of only two) cited also by Majorana. Schwinger's only other references are to Majorana[3] and a recent paper by Rabi.

[9] Rev. Mod Phys. **17**, 237 (1945).

mechanics does not require the injection of group theory as an independent mathematical discipline." Specifically, he proposed to use "elementary quantal operator methods" to obtain "a number of results that have thus far been considered striking examples of the power of group theoretic methods." All of the results in question had to do with rotations/angular momentum, and first on Schwinger's list was Majorana's theorem. But that paper (reproduced in its incomplete state as APPENDIX 1 in the *Rabi Festschrift*) was never completed—a casualty, perhaps, of the quantum electrodynamical work which Schwinger was soon to take up (the Shelter Island conference took place in June 1947) and for which he was to share a Nobel Prize in 1965.

In 1951 Schwinger wrote "On angular momentum," to which he prepended this

> ABSTRACT: *The commutation relations of an arbitrary angular momentum vector can be reduced to those of the harmonic oscillator. This provides a powerful method for constructing & developing the properties of angular momentum eigenvectors. In this paper many known theorems are derived in this way, and some new results obtained. Among the topics treated are the properties of the rotation matrices; the addition of two, three, and four angular momenta; and the theory of tensor operators.*

This 88-page monograph—the subject of the present commentary—might be looked upon as a detailed realization (if stripped—except by implication—of the group-theoretic polemics) of the intention Schwinger had formed in 1945. It comprises, in any event, Schwinger's most comprehensive account of a point of view and a body of technique he had been cultivating and advocating since adolescence. But this monograph was never published.[10] And though it contains references to works by Weyl, Güttinger, van der Waerden, Racah and Wigner, it surprisingly does not cite Majorana, nor does it contain any mention of Majorana's theorem/formula.

Schwinger was "made... aware, to my chagrin" of the latter oversight by the publication of an elegant little paper by one Alvin Meckler.[11] In response, he wrote a short paper which he submitted to The Physical Review in 1959, but that paper was "rejected by [the famously crusty] Editor S. Goudsmit for reasons that I then found so incomprehensible that I cannot now recall them." The paper appears as APPENDIX 2 in the *Rabi Festschrift*.

In his brief *Festschrift* essay Schwinger has one further thing to say about his 1951 monograph: citing the footnote that spans pages 242–245 in his

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[10] . . . except as USAEC Document NYO-3071, distributed by the Department of Commerce Office of Technical Services, from which I obtained my copy in 1958 for 60 cents. "On angular momentum" was reprinted in L. C. Biedenhorn & H. van Dam (editors), *Quantum Theory of Angular Momentum* (1965), which has long been out of print.

[11] "Majorana Formula," Phys. Rev. **111**, 1447 (1958). Meckler was attached to the National Security Agency's Division of Physical Sciences.

Quantum Kinematics and Dynamics,[12] he observes that "the operator construction used in [the] angular momentum representation [of the 1951 work can be shown to appear] naturally, at a more elementary level than the multiparticle viewpoint of second quantization."

In his ABSTRACT Schwinger speaks of "oscillators" but not of "second quantization;" here, in reference to that same work, he speaks of second quantization but not of oscillators. It is, in my experience, a *general* feature of Schwinger's expository style that he achieves swift elegance—and has acquired the reputation of a "difficult" author—by neglecting to inform his readers. . . in so many plain words. . . what he has assumed, what his symbols are intended to mean. Such information the reader is expected to glean inferentially, from context, from details of the elegant dances in which those symbols indulge: "Schwinger must mean this, else what he writes would be meaningless."

Such, in brief, is the context in which I read "On angular momentum," and the reason that I consider the reading to carry with it an obligation to write. One wonders why Schwinger's work in this sphere is so *little* read. Difficulty—part stylistic, part because the monograph is so crudely typed as to be almost unreadable—may account for some if it. But perhaps Schwinger was a victim also—as was Kramers before him[13]—of what might be called the "group-theoretic hegemony," which especially in the quantum theory of angular momentum has been entrenched throughout virtually the entire history of quantum mechanics.

It is interesting to reflect that the body of work discussed above radiates from a "Where did that come from?" that the 17-year-old Schwinger asked of a remark which the 26-year-old Majorana had considered too obvious to require detailed explanation.

BACKGROUND

Classical isotropic oscillator. Look to the isotropic 2-dimensional classical harmonic oscillator

$$H(p_1,p_2,x_1,x_2) \equiv \tfrac{1}{2m}\Big\{(p_1^2+p_2^2)+m^2\omega^2(x_1^2+x_2^2)\Big\} \tag{1.1}$$

$$= \tfrac{1}{2}\hbar\omega\Big\{q_1^2+q_2^2+y_1^2+y_2^2\Big\} \tag{1.2}$$

$$= \hbar\omega\Big\{a_1^*a_1+a_2^*a_2\Big\} \tag{1.3}$$

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[12] This project was started and abandoned in 1955. The typed notes, of which I have treasured a copy since about 1957, were published in their incomplete state in 1970.

[13] See pages 7–8 in "Spin matrices for arbitrary spin" (2000) for remarks derived from Max Dresden's *H. A. Kramers: Betweeen Tradition and Revolution* (1987).

where[14]

$$y \equiv \sqrt{\tfrac{m\omega^2}{\hbar\omega}}\,x \quad\text{and}\quad q \equiv \sqrt{\tfrac{1}{m\cdot\hbar\omega}}\,p \tag{2.1}$$

are dimensionless (subscripts surpressed), and

$$a \equiv \tfrac{1}{\sqrt{2}}\big\{y + iq\big\} \quad ; \quad a^* \equiv \tfrac{1}{\sqrt{2}}\big\{y - iq\big\} \tag{2.2}$$

are, it should be noted, complex-valued.

From $\big[x_1, p_1\big] = \big[x_2, p_2\big] = 1$ (all other Poisson brackets constructable from x_1, p_1, x_2 and p_2 vanish) obtain

$$\big[y, q\big] = \hbar^{-1} \quad : \quad \begin{cases}\text{identical subscripts surpressed,}\\ \text{other brackets vanish}\end{cases}$$

whence

$$\big[a_1^*, a_1\big] = \big[a_2^*, a_2\big] = \tfrac{i}{\hbar} \quad : \quad \text{all other } a\text{-brackets vanish}$$

of which

$$\big[a_r^*, a_s\big] = \tfrac{i}{\hbar}\delta_{rs} \tag{3}$$

provides a useful summary.

Construct the complex 2-vector

$$\boldsymbol{a} \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{4}$$

Adopt the usual definitions of the traceless hermitian Pauli matrices

$$\boldsymbol{\sigma}_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \boldsymbol{\sigma}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \boldsymbol{\sigma}_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \boldsymbol{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and form the real-valued observables

$$Q_\mu \equiv \boldsymbol{a}^{\mathrm{t}}\boldsymbol{\sigma}_\mu\boldsymbol{a} \quad : \quad \mu = 0, 1, 2, 3 \tag{5}$$

Explicitly[15]

$$\left.\begin{aligned} Q_0 &= a_1^*a_1 + a_2^*a_2 \\ Q_1 &= a_1^*a_2 + a_2^*a_1 \\ Q_2 &= -i(a_1^*a_2 - a_2^*a_1) \\ Q_3 &= a_1^*a_1 - a_2^*a_2 \end{aligned}\right\} \tag{6}$$

It is a quick implication of the commutation relations (3) that

$$\big[\boldsymbol{a}^{\mathrm{t}}\mathbb{M}\boldsymbol{a}, \boldsymbol{a}^{\mathrm{t}}\mathbb{N}\boldsymbol{a}\big] = \tfrac{1}{i\hbar}\boldsymbol{a}^{\mathrm{t}}\big[\mathbb{M}, \mathbb{N}\big]\boldsymbol{a} \ : \ \text{any } 2\times 2 \text{ matrices } \mathbb{M} \text{ and } \mathbb{N} \tag{7}$$

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[14] For more detailed discussion of what is, after all, standard material, see "Ellipsometry" (1999) p. 63, from which I take my notation.

[15] Beware: in "Ellipsometry" I had reason to adopt Stokes' conventions rather than Pauli's, so the Q's described there at (163) are cyclic permutations of those described below.

so from $H = \hbar\omega Q_0 = \hbar\omega\boldsymbol{a}^{\mathsf{t}}\mathbb{I}\,\boldsymbol{a}$ it follows that each of the Q_μ is a constant of the motion

$$\big[H, Q_\mu\big] = 0 \quad : \quad \mu = 0, 1, 2, 3 \tag{8}$$

while from a familiar commutation property of the Pauli matrices

$$\big[\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\big] = 2i\,\boldsymbol{\sigma}_3\,, \quad \big[\boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\big] = 2i\,\boldsymbol{\sigma}_1\,, \quad \big[\boldsymbol{\sigma}_3, \boldsymbol{\sigma}_1\big] = 2i\,\boldsymbol{\sigma}_2$$

we obtain Poisson bracket relations

$$\left.\begin{aligned} \big[Q_1, Q_2\big] &= \tfrac{2}{\hbar}Q_3 \\ \big[Q_2, Q_3\big] &= \tfrac{2}{\hbar}Q_1 \\ \big[Q_3, Q_1\big] &= \tfrac{2}{\hbar}Q_2 \end{aligned}\right\} \tag{9}$$

The observables

$$J_k \equiv \tfrac{\hbar}{2}Q_k \quad : \quad k = 1, 2, 3 \tag{10}$$

have (as the Lie generators of canonical transformations are in all cases obligated to have) the physical dimensions of angular momentum (i.e., of action). They satisfy

$$\left.\begin{aligned} \big[J_1, J_2\big] &= J_3 \\ \big[J_2, J_3\big] &= J_1 \\ \big[J_3, J_1\big] &= J_2 \end{aligned}\right\} \tag{11}$$

from which they emerge as the generators, within 4-dimensional phase space, of a representation of $O(3)$, the "accidental symmetry group"[16] of the isotropic oscillator—a system which on its face possesses only the $O(2)$ symmetry generated by

$$J_2 = \tfrac{1}{2}(x_1 p_2 - x_2 p_1)$$

We observe finally that

$$Q_0^2 - Q_1^2 - Q_2^2 - Q_3^2 = 0 \tag{12}$$

Evidently the observables Q_μ, since subject to that constraint, cannot be specified independently.[17]

[16] For a good short introduction to this concept see H. V. McIntosh, "On accidental degeneracy in classical & quantum mechanics," AJP **27**, 620 (1959). See also footnote 20 in "Classical/quantum theory of 2-dimensional hydrogen," (February 1999).

[17] Classical harmonic oscillators trace elliptical curves in configuration space. Identical curves are—for other physical reasons—traced by the flying $\boldsymbol{E}$-vector of an on-rushing electromagnetic plane wave. We are not surprised, therefore, to discover that Stokes' theory of optical polarization makes tacit use of some mathematical machinery quite similar to that just reviewed. . . and that an equation of the form (12) plays a prominent role in Stokes' theory. Reversing the traffic of ideas, we can look upon the Q_μ as mechanical analogs of the optical Stokes parameters S_μ, and use Stokes' theory to answer mechanical questions. It becomes in this light clear that/how the conserved expressions Q_μ serve to describe the size/orientation/ellipticity/circulatory sense of the mechanical trajectory.

Isotropic quantum oscillator. Look now to the parallel *quantum* theory of an isotropic oscillator. We have

$$\begin{aligned}\mathsf{H} &\equiv \tfrac{1}{2m}\Big\{(\mathsf{p}_1^2+\mathsf{p}_2^2)+m^2\omega^2(\mathsf{x}_1^2+\mathsf{x}_2^2)\Big\} &(13.1)\\ &=\tfrac{1}{2}\hbar\omega\Big\{\mathsf{q}_1^2+\mathsf{q}_2^2+\mathsf{y}_1^2+\mathsf{y}_2^2\Big\} &(13.2)\\ &=\hbar\omega\Big\{\mathsf{a}_1^+\mathsf{a}_1+\mathsf{a}_2^+\mathsf{a}_2+\big(\tfrac{1}{2}+\tfrac{1}{2}\big)\mathsf{I}\Big\} &(13.3)\end{aligned}$$

where

$$\mathsf{y}\equiv\sqrt{\tfrac{m\omega^2}{\hbar\omega}}\,\mathsf{x}\quad\text{and}\quad\mathsf{q}\equiv\sqrt{\tfrac{1}{m\cdot\hbar\omega}}\,\mathsf{p} \tag{14.1}$$

are dimensionless self-adjoint operators (again subscripts have been surpressed), and the operators

$$\mathsf{a}\equiv\tfrac{1}{\sqrt{2}}\big\{\mathsf{y}+i\mathsf{q}\big\}\quad;\quad\mathsf{a}^+\equiv\tfrac{1}{\sqrt{2}}\big\{\mathsf{y}-i\mathsf{q}\big\} \tag{14.2}$$

are *not* self-adjoint, and therefore do not represent "observables:" they are the familiar step-down/up "ladder operators," which were first introduced into oscillator theory by Dirac,[18] were generalized by Schrödinger in 1940,[19] and are encountered today in most elementary texts.[20]

From $\big[\mathsf{x}_1,\mathsf{p}_1\big]=\big[\mathsf{x}_2,\mathsf{p}_2\big]=1$ (all other commutators constructable from x_1, p_1, x_2 and p_2 vanish) obtain

$$\big[\mathsf{y},\mathsf{q}\big]=\hbar^{-1}\big[\mathsf{x},\mathsf{p}\big]=i\mathsf{I}\quad:\quad\begin{cases}\text{identical subscripts surpressed,}\\ \text{other commutators vanish}\end{cases}$$

whence

$$\big[\mathsf{a}_1^+,\mathsf{a}_1\big]=\big[\mathsf{a}_2^+,\mathsf{a}_2\big]=-\mathsf{I}\quad:\quad\text{all other }\mathsf{a}\text{-commutators vanish}$$

of which

$$\big[\mathsf{a}_r^+,\mathsf{a}_s\big]=-\,\delta_{rs}\mathsf{I} \tag{15}$$

provides a useful summary.

Construct (simply as a notational device) vectors with operator-valued elements

$$\boldsymbol{a}\equiv\begin{pmatrix}\mathsf{a}_1\\ \mathsf{a}_2\end{pmatrix}\quad\text{and}\quad\boldsymbol{a}^{\mathsf{t}}\equiv\begin{pmatrix}\mathsf{a}_1^+ & \mathsf{a}_2^+\end{pmatrix} \tag{16}$$

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[18] *Principles of Quantum Mechanics* (2nd edition 1935), §34.

[19] See L. Infeld & T. E. Hull, "The factorization method," Rev. Mod. Phys. **23**, 21 (1951) for additional references and indication of what became of Dirac's pretty idea. For its more recent adventures, see Chapter I in Christopher Lee's thesis, "Supersymmetric Quantum Mechanics" (Reed College, 1999).

[20] See, for example, David Griffiths' *Introduction to Quantum Mechanics* (1994), §2.3.1.

and from them construct the dimensionless hermitian operators (observables)

$$\mathsf{Q}_\mu \equiv \boldsymbol{a}^{\mathrm{t}}\boldsymbol{\sigma}_\mu \boldsymbol{a} \quad : \quad \mu = 0,1,2,3 \tag{17}$$

Explicitly

$$\left.\begin{aligned} \mathsf{Q}_0 &= \quad\ \mathsf{a}_1^+\mathsf{a}_1 + \mathsf{a}_2^+\mathsf{a}_2 \\ \mathsf{Q}_1 &= \quad\ \mathsf{a}_1^+\mathsf{a}_2 + \mathsf{a}_2^+\mathsf{a}_1 \\ \mathsf{Q}_2 &= -i(\mathsf{a}_1^+\mathsf{a}_2 - \mathsf{a}_2^+\mathsf{a}_1) \\ \mathsf{Q}_3 &= \quad\ \mathsf{a}_1^+\mathsf{a}_1 - \mathsf{a}_2^+\mathsf{a}_2 \end{aligned}\right\} \tag{18}$$

The quantum analog of the pretty identity (7) reads[21]

$$\big[\boldsymbol{a}^{\mathrm{t}}\mathbb{M}\boldsymbol{a}, \boldsymbol{a}^{\mathrm{t}}\mathbb{N}\boldsymbol{a}\big] = \boldsymbol{a}^{\mathrm{t}}\big[\mathbb{M},\mathbb{N}\big]\boldsymbol{a} \tag{19}$$

where $\mathbb{M}$ and $\mathbb{N}$ are any 2×2 matrices with number-valued elements. Therefore

$$\big[\mathsf{Q}_0, \mathsf{Q}_1\big] = \big[\mathsf{Q}_0, \mathsf{Q}_2\big] = \big[\mathsf{Q}_0, \mathsf{Q}_3\big] = \mathbf{0} \tag{20.1}$$

$$\left.\begin{aligned} \big[\mathsf{Q}_1, \mathsf{Q}_2\big] &= 2i\mathsf{Q}_3 \\ \big[\mathsf{Q}_2, \mathsf{Q}_3\big] &= 2i\mathsf{Q}_1 \\ \big[\mathsf{Q}_3, \mathsf{Q}_1\big] &= 2i\mathsf{Q}_2 \end{aligned}\right\} \tag{20.2}$$

The associated observables

$$\mathsf{J}_k \equiv \tfrac{\hbar}{2}\mathsf{Q}_k \quad : \quad k = 1,2,3 \tag{21}$$

possess the physical dimension of—and the commutation properties

$$\left.\begin{aligned} \big[\mathsf{J}_1, \mathsf{J}_2\big] &= i\hbar\mathsf{J}_3 \\ \big[\mathsf{J}_2, \mathsf{J}_3\big] &= i\hbar\mathsf{J}_1 \\ \big[\mathsf{J}_3, \mathsf{J}_1\big] &= i\hbar\mathsf{J}_2 \end{aligned}\right\} \tag{22}$$

quantum mechanically characteristic of—angular momentum. The "accidental" intrusion of $O(3)$ into the dynamics of a centrally-symmetric 2-dimensional system is, of course, no less surprising when encountered in quantum theory than it was when encountered in the classical oscillator theory.

Working from (18) with the aid of (15) we find

$$\mathsf{Q}_1^2 + \mathsf{Q}_2^2 + \mathsf{Q}_3^2 = \mathsf{Q}_0^2 + 2\mathsf{Q}_0 \tag{23}$$

where the second term on the right, which has no counterpart in (12), is an artifact of non-commutativity; in J-language the preceding equation becomes

$$\mathsf{J}^2 \equiv \mathsf{J}_1^2 + \mathsf{J}_2^2 + \mathsf{J}_3^2 = \mathsf{J}_0^2 + \hbar\mathsf{J}_0 \tag{24}$$

[21] The short argument: use $i\hbar\big[a^*, a\big] = -1 \longmapsto \big[\mathsf{a}^+, \mathsf{a}\big] = -\mathsf{I}$.

where the second term on the right vanishes in the classical limit $\hbar \downarrow 0$. It follows now from (20.1) that

$$\big[\,\mathsf{J}^2, \mathsf{J}_1\big] = \big[\,\mathsf{J}^2, \mathsf{J}_2\big] = \big[\,\mathsf{J}^2, \mathsf{J}_3\big] = \mathsf{0} \tag{25}$$

Equations (22) and (25) are familiar from the quantum theory of angular momentum, though they pertain here to the theory of 2-dimensional isotropic oscillators. Quoting now from McIntosh:[16]

> *Perhaps one of the most interesting aspects of the theory of the two-dimensional harmonic oscillator has been the use made of it by Schwinger to develop the theory of angular momentum,*[10] *by exploiting the appearance of the three-dimensional rotation group as the symmetry group of the plane isotropic harmonic oscillator. In Schwinger's paper the language is that of second-quantized field theory, and of course no connection is made with the theory of accidental degeneracy. Nevertheless the two theories are intimately related, and a very far-reaching theory of angular momentum may be created by the use of... properties of the harmonic oscillator.*

At this point in angular momentum theory it is standard to introduce (as computational aids) non-hermitian operators

$$\left.\begin{aligned} \mathsf{J}_+ &\equiv \mathsf{J}_1 + i\,\mathsf{J}_2 = \hbar\,\mathsf{a}_1^+\mathsf{a}_2 \\ \mathsf{J}_- &\equiv \mathsf{J}_1 - i\,\mathsf{J}_2 = \hbar\,\mathsf{a}_2^+\mathsf{a}_1 \end{aligned}\right\} \tag{26}$$

and to observe that[22]

$$\big[\,\mathsf{J}^2, \mathsf{J}_+\big] = \big[\,\mathsf{J}^2, \mathsf{J}_-\big] = \mathsf{0} \tag{27}$$

$$\left.\begin{aligned} \big[\,\mathsf{J}_3, \mathsf{J}_+\big] &= +\hbar\,\mathsf{J}_+ \\ \big[\,\mathsf{J}_3, \mathsf{J}_-\big] &= -\hbar\,\mathsf{J}_- \end{aligned}\right\} \tag{28}$$

It is from (28) that (a few steps down the road) J_+ and J_- acquire significance as "step-up/down operators" (in Griffiths' terminology: "raising and lowering operators"). Notice, however, that at (26) those operators are presented now in a "factored" form *not* standard to textbook treatments of angular momentum.

Bosonic populations of 2-state systems. Introduce "indistinguishability" into the quantum theory of many-particle systems. The resulting "quantum field theory" comes in two flavors, according as the particles are

$$\begin{aligned} \text{BOSONS} &: \ \Psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) \text{ is totally symmetric} \\ \text{FERMIONS} &: \ \Psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) \text{ is totally antisymmetric} \end{aligned}$$

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[22] Compare Griffiths' §4.3.1,[20] and notice how relatively simple is the present line of argument (the details of which I have omitted). That simplicity traces to our ability to "factor" J_+ and J_-, which we acquired at the =’s in (26). This remark exposes the algebraic secret of Schwinger's success, the force of McIntosh's observation.

It is assured in either case that $\Psi^*\Psi$ is permutationally invariant. In both cases one encounters populations of "annihilation operators" $\mathbf{a}_r$ and of associated "creation operators" $\mathbf{a}_s^+$ which are found satisfy

$$\mathbf{a}_r\mathbf{a}_s^+ - \mathbf{a}_s^+\mathbf{a}_r = \delta_{rs}\mathbf{I} \quad : \quad \text{bosonic case} \tag{29.1}$$
$$\mathbf{a}_r\mathbf{a}_s^+ + \mathbf{a}_s^+\mathbf{a}_r = \delta_{rs}\mathbf{I} \quad : \quad \text{fermionic case} \tag{29.2}$$

I digress to review the *origin* of those statements in order to emphasize that, while (29.1) is formally identical to (15), it *has an entirely different meaning*.

Let $\mathfrak{S}$ be a quantum system, and write $\mathfrak{S}^n \equiv \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_n$ to signify the composite system constructed from n individually labled copies of $\mathfrak{S}$. The question before us—How to describe the *state* of $\mathfrak{S}^n$?—is "pre-dynamical;" we have, therefore, no immediate interest in the Hamiltonian that impells motion with $\mathfrak{S}$, or in the distinction between "mental" composites and "physically interactive" composites.

Let us assume $\mathfrak{S}$ to be an N-state system. It is this assumption that separates the following material from more standard accounts[23] of our subject. The state space of $\mathfrak{S}$ has, by force of this assumption, become an N-dimensional complex vector space $\mathcal{V}$. As an expository convenience I will set $N = 3$, though in the end we will have special interest in the case $N = 2$; to describe the state of $\mathfrak{S}$ we write

$$|\psi) = \psi_1|1) + \psi_2|2) + \psi_3|3) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{30.1}$$

where

$$|1) \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2) \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3) \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{30.2}$$

refer to some orthonormal basis in $\mathcal{V}$.

Look not to the simplest composite system, $\mathfrak{S}^2 \equiv \mathfrak{S}_1 \times \mathfrak{S}_2$. If $\mathfrak{S}_1$ is in state $|\alpha)$ and $\mathfrak{S}_2$ is in state $|\beta)$, then to describe the state of $\mathfrak{S}^2$ we construct

$$|\psi) \equiv |\alpha) \otimes |\beta) = \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \alpha_1\beta_3 \\ \alpha_2\beta_1 \\ \alpha_2\beta_2 \\ \alpha_2\beta_3 \\ \alpha_3\beta_1 \\ \alpha_3\beta_2 \\ \alpha_3\beta_3 \end{pmatrix} = \sum_{r,s} \alpha_r\beta_s|r,s) \quad \text{with} \quad |r,s) \equiv |r) \otimes |s)$$

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[23] See, for example, Chapter 6 in S. S. Schweber, *Introduction to Relativistic Quantum Field Theory* (1961).

where $\otimes$ signifies the Kronecker (also called the outer/direct/tensor) product. Notice that $(\psi|\psi) = (\alpha|\alpha)\cdot(\beta|\beta) = 1$; i.e., that the $|r,s)$ comprise an "induced orthonormal basis" in the $N^2 = 3^2 = 9$-dimensional statespace $\mathcal{V}^2$ of $\mathfrak{S}^2$. And that the extension of this procedure to $\mathfrak{S}^{n>2}$ is straightforward.

If, however, the systems are *indistinguishable*, then we have to
- achieve $\alpha\beta$-symmetrization in the bosonic case;
- achieve $\alpha\beta$-antisymmetrization in the fermionic case

which are easily accomplished. Look first to the bosonic case. Construct

$$|\psi)_{\text{bosonic}} \equiv \frac{|\alpha)\otimes|\beta) + |\beta)\otimes|\alpha)}{\text{normalization factor}} \sim \begin{pmatrix} 2\alpha_1\beta_1 \\ \alpha_1\beta_2+\beta_1\alpha_2 \\ \alpha_1\beta_3+\beta_1\alpha_3 \\ \alpha_2\beta_1+\beta_2\alpha_1 \\ 2\alpha_2\beta_2 \\ \alpha_2\beta_3+\beta_2\alpha_3 \\ \alpha_3\beta_1+\beta_3\alpha_1 \\ \alpha_3\beta_2+\beta_3\alpha_2 \\ 2\alpha_3\beta_3 \end{pmatrix} = \begin{pmatrix} \text{type 1} \\ \text{type 2} \\ \text{type 3} \\ \text{type 2} \\ \text{type 4} \\ \text{type 5} \\ \text{type 3} \\ \text{type 5} \\ \text{type 6} \end{pmatrix}$$

and notice that the 9-vector contains terms of only 6 types. This is because only six of the objects

$$|r)\otimes|s) + |s)\otimes|r) \quad : \quad r,s = 1,2,3$$

are distinct. They are

$$\left.\begin{aligned} |2,0,0) &\equiv \tfrac{1}{2}\big\{|1)\otimes|1) + |1)\otimes|1)\big\} \\ |1,1,0) &\equiv \tfrac{1}{\sqrt{2}}\big\{|1)\otimes|2) + |1)\otimes|2)\big\} \\ |1,0,1) &\equiv \tfrac{1}{\sqrt{2}}\big\{|1)\otimes|3) + |1)\otimes|3)\big\} \\ |0,2,0) &\equiv \tfrac{1}{2}\big\{|2)\otimes|2) + |2)\otimes|2)\big\} \\ |0,1,1) &\equiv \tfrac{1}{\sqrt{2}}\big\{|2)\otimes|3) + |3)\otimes|2)\big\} \\ |0,0,2) &\equiv \tfrac{1}{2}\big\{|3)\otimes|3) + |3)\otimes|3)\big\} \end{aligned}\right\} \tag{31.1}$$

of which I give now explicit descriptions:

$$|2,0,0) = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}, \qquad |1,1,0) = \tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\\0\\0\\0\\0\\0\end{pmatrix}, \qquad |1,0,1) = \tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\0\\0\\0\\1\\0\\0\end{pmatrix}$$

$$|0,2,0) = \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\\0\end{pmatrix}, \qquad |0,1,1) = \tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\1\\0\end{pmatrix}, \qquad |0,0,2) = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}$$

These vectors are readily seen to be orthonormal. They span a 6-dimensional subspace

$$\mathcal{V}^2_{\text{bosons}} \equiv (\mathcal{V}\otimes\mathcal{V})_{\text{sym}} \in \mathcal{V}\otimes\mathcal{V}$$

in $(3^2 = 9)$-dimensional vector space $\mathcal{V}\otimes\mathcal{V}$.

In the fermionic case the situation is similar but simpler: we have

$$|\psi)_{\text{fermionic}} \equiv \frac{|\alpha)\otimes|\beta) - |\beta)\otimes|\alpha)}{\text{normalization factor}} \sim \begin{pmatrix}0\\ \alpha_1\beta_2-\beta_1\alpha_2\\ \alpha_1\beta_3-\beta_1\alpha_3\\ \alpha_2\beta_1-\beta_2\alpha_1\\ 0\\ \alpha_2\beta_3-\beta_2\alpha_3\\ \alpha_3\beta_1-\beta_3\alpha_1\\ \alpha_3\beta_2-\beta_3\alpha_2\\ 0\end{pmatrix} = \begin{pmatrix}0\\ +\text{type }3\\ +\text{type }2\\ -\text{type }3\\ 0\\ +\text{type }1\\ -\text{type }2\\ -\text{type }1\\ 0\end{pmatrix}$$

There are now only 3 term types because only three of the objects

$$|r)\otimes|s) - |s)\otimes|r) \quad : \quad r,s = 1,2,3$$

are distinct. They are

$$\left.\begin{aligned}|1,1,0) &\equiv \tfrac{1}{\sqrt{2}}\big\{|1)\otimes|2) - |2)\otimes|1)\big\}\\ |1,0,1) &\equiv \tfrac{1}{\sqrt{2}}\big\{|1)\otimes|3) - |3)\otimes|1)\big\}\\ |0,1,1) &\equiv \tfrac{1}{\sqrt{2}}\big\{|2)\otimes|3) + |3)\otimes|2)\big\}\end{aligned}\right\} \tag{31.2}$$

Explicitly

$$|1,1,0) = \tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\+1\\0\\-1\\0\\0\\0\\0\\0\end{pmatrix}, \qquad |0,1,1) = \tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\+1\\0\\0\\0\\-1\\0\\0\end{pmatrix}, \qquad |0,1,1) = \tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\0\\0\\+1\\0\\-1\\0\end{pmatrix}$$

These manifestly orthonormal vectors span a 3-dimensional subspace

$$\mathcal{V}^2_{\text{fermions}} \equiv (\mathcal{V}\otimes\mathcal{V})_{\text{antisym}} \in \mathcal{V}\otimes\mathcal{V}$$

Bosons and fermions are seldom discussed in the same breath, so the fact that

$$|1,1,0)_{\text{boson}} \neq |1,1,0)_{\text{fermion}}$$

should be noted, but is seldom a source of notational confusion.

To describe the state of $\mathfrak{S}^2_{\text{bosonic}} \equiv \mathfrak{S}\circledcirc\mathfrak{S}$ we are in position now to write

$$\begin{aligned}|\psi) &= \psi_{2,0,0}|2,0,0) + \psi_{1,1,0}|1,1,0) + \psi_{1,0,1}|1,0,1)\\ &\qquad\qquad + \psi_{0,2,0}|0,2,0) + \psi_{0,1,1}|0,1,1) + \psi_{0,0,2}|0,0,2)\\ &= \sum_{\substack{n_1,n_2,n_3\\ n_1+n_2+n_3=2}} \psi_{n_1,n_2,n_3}|n_1,n_2,n_3)\end{aligned} \tag{32.2}$$

while the state of $\mathfrak{S}^2_{\text{fermionic}} \equiv \mathfrak{S}\circledast\mathfrak{S}$ becomes

$$|\psi) = \psi_{1,1,0}|1,1,0) + \psi_{1,0,1}|1,0,1) + \psi_{0,1,1}|0,1,1)$$

The systems $\mathfrak{S}^3_{\text{bosonic}} \equiv \mathfrak{S}\circledcirc\mathfrak{S}\circledcirc\mathfrak{S}$ and $\mathfrak{S}^3_{\text{fermionic}} \equiv \mathfrak{S}\circledast\mathfrak{S}\circledast\mathfrak{S}$ lead to this variant of (32)

$$|\psi) = \sum_{\substack{n_1,n_2,n_3\\ n_1+n_2+n_3=3}} \psi_{n_1,n_2,n_3}|n_1,n_2,n_3) \tag{32.3}$$

So it goes…the extension to $\mathfrak{S}^n$ being (apart from a combinatorial problem to which I return in a moment) entirely straightforward. So, in fact, it was in the beginning, since (30) can be written

$$|\psi) = \sum_{\substack{n_1,n_2,n_3\\ n_1+n_2+n_3=1}} \psi_{n_1,n_2,n_3}|n_1,n_2,n_3) \tag{32.1}$$

with

$$\begin{aligned}|1,0,0) &\equiv |1)\\ |0,1,0) &\equiv |2)\\ |0,0,1) &\equiv |3)\end{aligned}$$

The integers $\{n_1,n_2,n_3\}$ are called "occupation numbers," and equations (32) display $|\psi)$ in the "occupation number representation." It is futile to attempt to state *which* of the indistinguishable systems $\mathfrak{S}$ are in the r^{th} state, but remains meaningful to state *how many* are, which is precisely what n_r does. For bosonic systems n_r can assume any value $\leqslant n$ (if the composite system is $\mathfrak{S}^n$), but for fermionic systems $n_r \in \{0,1\}$ (by implication of the antisymmetry condition).

At cost of some repetition, I turn now to a summary of results which will serve as our launch pad for developments that soon follow. All computation was performed by *Mathematica*; details can be found in a detached APPENDIX.

SOME DETAILS PERTAINING TO BOSONIC 3-STATE SYSTEMS

The state space $\mathcal{V}^1_{\text{bosonic}} \equiv \mathcal{V}$ of $\mathfrak{S}^1_{\text{bosonic}}$ is 3-dimensional, and is spanned by

$$\begin{aligned}
(1,0,0| &\equiv (\,1 \quad 0 \quad 0\,)\\
(0,1,0| &\equiv (\,0 \quad 1 \quad 0\,)\\
(0,0,1| &\equiv (\,0 \quad 0 \quad 1\,)
\end{aligned}$$

The state space $\mathcal{V}^2_{\text{bosonic}} \equiv (\mathcal{V}\otimes\mathcal{V})_{\text{sym}}$ of $\mathfrak{S}^2_{\text{bosonic}}$ is 6-dimensional, and is spanned by

$$\begin{aligned}
(2,0,0| &= \tfrac{1}{\sqrt{4}}(\,2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,)\\
(1,1,0| &= \tfrac{1}{\sqrt{2}}(\,0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,)\\
(1,0,1| &= \tfrac{1}{\sqrt{2}}(\,0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0\,)\\
(0,2,0| &= \tfrac{1}{\sqrt{4}}(\,0 \quad 0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \quad 0\,)\\
(0,1,1| &= \tfrac{1}{\sqrt{2}}(\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0\,)\\
(0,0,2| &= \tfrac{1}{\sqrt{4}}(\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 2\,)
\end{aligned}$$

The state space $\mathcal{V}^3_{\text{bosonic}} \equiv (\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V})_{\text{sym}}$ of $\mathfrak{S}^3_{\text{bosonic}}$ is 10-dimensional, and is spanned by

$$\begin{aligned}
(3,0,0| &= \tfrac{1}{\sqrt{36}}\begin{array}{l}(\,6 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0\\ \;\; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0\,)\end{array}\\
(2,1,0| &= \tfrac{1}{\sqrt{12}}\begin{array}{l}(\,0 \; 2 \; 0 \; 2 \; 0 \; 0 \; 0 \; 0 \; 0 \; 2 \; 0 \; 0\\ \;\; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0\,)\end{array}\\
(2,0,1| &= \tfrac{1}{\sqrt{12}}\begin{array}{l}(\,0 \; 0 \; 2 \; 0 \; 0 \; 0 \; 2 \; 0 \; 0 \; 0 \; 0 \; 0\\ \;\; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 2 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0\,)\end{array}\\
(1,2,0| &= \tfrac{1}{\sqrt{12}}\begin{array}{l}(\,0 \; 0 \; 0 \; 0 \; 2 \; 0 \; 0 \; 0 \; 0 \; 0 \; 2 \; 0\\ \;\; 2 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0\,)\end{array}\\
(1,1,1| &= \tfrac{1}{\sqrt{6}}\begin{array}{l}(\,0 \; 0 \; 0 \; 0 \; 0 \; 1 \; 0 \; 1 \; 0 \; 0 \; 0 \; 1\\ \;\; 0 \; 0 \; 0 \; 1 \; 0 \; 0 \; 0 \; 1 \; 0 \; 1 \; 0 \; 0 \; 0 \; 0 \; 0\,)\end{array}\\
(1,0,2| &= \tfrac{1}{\sqrt{12}}\begin{array}{l}(\,0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 2 \; 0 \; 0 \; 0\\ \;\; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 2 \; 0 \; 0 \; 0 \; 2 \; 0 \; 0\,)\end{array}\\
(0,3,0| &= \tfrac{1}{\sqrt{36}}\begin{array}{l}(\,0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0\\ \;\; 0 \; 6 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0\,)\end{array}\\
(0,2,1| &= \tfrac{1}{\sqrt{12}}\begin{array}{l}(\,0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0\\ \;\; 0 \; 0 \; 2 \; 0 \; 2 \; 0 \; 0 \; 0 \; 0 \; 0 \; 2 \; 0 \; 0 \; 0 \; 0\,)\end{array}\\
(0,1,2| &= \tfrac{1}{\sqrt{12}}\begin{array}{l}(\,0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0\\ \;\; 0 \; 0 \; 0 \; 0 \; 0 \; 2 \; 0 \; 0 \; 0 \; 0 \; 0 \; 2 \; 0 \; 2 \; 0\,)\end{array}\\
(0,0,3| &= \tfrac{1}{\sqrt{36}}\begin{array}{l}(\,0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0\\ \;\; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 6\,)\end{array}
\end{aligned}$$

Generally,

$$
\begin{aligned}
[N,n]_{\text{bosonic}} &\equiv \text{dimension of } \mathcal{V}^n_{\text{bosonic }N\text{-state}}\\
&= \text{number of terms in } (x_1+x_2+\cdots+x_N)^n\\
&= \begin{cases}\text{number of ``words'' } **\,|\,*\,||\,*\cdots*\,|* \\ \text{constructable from } n\ *\text{'s and } (N-1)\ |\text{'s}\end{cases}\\
&= \frac{(N+n-1)!}{(N-1)!\,n!}
\end{aligned}
$$

which the following (symmetric) table serves to illustrate:

	1	2	3	4	5	6	7	$\cdots$
2	2	3	4	5	6	7	8	
3	3	6	10	15	21	28	36	
4	4	10	20	35	56	84	120	
5	5	15	35	70	126	210	330	
6	6	21	56	126	252	462	792	
7	7	28	84	210	462	924	1716	
8	8	36	120	330	792	1716	3432	
$\vdots$								

TABLE 1: *The number n at the top of each column indicates how many copies of $\mathfrak{S}$ make up the bosonic population. The number N at the left of each row indicates how many linearly independent states each $\mathfrak{S}$ is assumed to possess. The values of $[N,n]_{\text{bosonic}}$ are tabulated. The red entries are detailed on the preceding page. Notice that the data in a row reappears as differences in the next row: $[N,n]_{\text{bosonic}} - [N,n-1]_{\text{bosonic}} = [N-1,n]_{\text{bosonic}}$.*

SOME DETAILS PERTAINING TO FERMIONIC 3-STATE SYSTEMS

The state space $\mathcal{V}^1_{\text{fermionic}} \equiv \mathcal{V}$ of $\mathfrak{S}^1_{\text{fermionic}}$ is 3-dimensional, and is spanned by

$$
\begin{aligned}
(1,0,0| &\equiv (\,1\quad 0\quad 0\,)\\
(0,1,0| &\equiv (\,0\quad 1\quad 0\,)\\
(0,0,1| &\equiv (\,0\quad 0\quad 1\,)
\end{aligned}
$$

NOTE: There might appear to be no way for a solitary system to declare whether it is latently bosonic or fermionic. . . but—surprisingly—in contexts to which the "spin-statistics theorem" pertains it *does* make sense to speak of (say) a solitary fermion.

The state space $\mathcal{V}^2_{\text{fermionic}} \equiv (\mathcal{V}\otimes\mathcal{V})_{\text{antisym}}$ of $\mathfrak{S}^2_{\text{fermionic}}$ is 3-dimensional, and is spanned by

$$\begin{aligned}
(1,1,0| &= \tfrac{1}{\sqrt{2}}\big(\,0\quad p\quad 0\quad q\quad 0\quad 0\quad 0\quad 0\quad 0\,\big)\\
(1,0,1| &= \tfrac{1}{\sqrt{2}}\big(\,0\quad 0\quad p\quad 0\quad 0\quad 0\quad q\quad 0\quad 0\,\big)\quad:\quad p=+1,\ q=-1\\
(0,1,1| &= \tfrac{1}{\sqrt{2}}\big(\,0\quad 0\quad 0\quad 0\quad 0\quad p\quad 0\quad q\quad 0\,\big)
\end{aligned}$$

The state space $\mathcal{V}^3_{\text{fermionic}} \equiv (\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V})_{\text{antisym}}$ of $\mathfrak{S}^2_{\text{fermionic}}$ is 1-dimensional, and its solitary element is

$$\begin{aligned}
(1,1,1| = \tfrac{1}{\sqrt{6}}\big(\,&0\quad 0\quad 0\quad 0\quad 0\quad p\quad 0\quad q\quad 0\quad 0\quad 0\quad q\\
&0\quad 0\quad 0\quad p\quad 0\quad 0\quad 0\quad p\quad 0\quad q\quad 0\quad 0\quad 0\quad 0\quad 0\,\big)
\end{aligned}$$

The state spaces $\mathcal{V}^{n>3}_{\text{fermionic}}$ of $\mathfrak{S}^{n>3}_{\text{fermionic}}$ are empty. Generally

$$\begin{aligned}
[N,n]_{\text{fermionic}} &\equiv \text{dimension of } \mathcal{V}^n_{\text{fermionic }N\text{-state}}\\
&= \begin{cases}\text{number of ``words'' } **\,|\,*\,||\,*\cdots*\,|*\\ \text{constructable from } n\ *\text{'s and } (N-1)\ |\text{'s}\\ \underline{\text{if adjacent } *\text{'s are disallowed}}\end{cases}\\
&= \begin{cases}\binom{N}{n} & \text{if } n=1,2,\ldots,N\\[1ex] 0 & \text{if } n>N\end{cases}
\end{aligned}$$

as illustrated below:

	1	2	3	4	5	6	7	$\cdots$
2	2	1	0	0	0	0	0	
3	3	3	1	0	0	0	0	
4	4	6	4	1	0	0	0	
5	5	10	10	5	1	0	0	
6	6	15	20	15	6	1	0	
7	7	21	35	35	21	7	1	
8	8	28	56	70	56	28	8	
$\vdots$								

TABLE 2: *Fermionic companion of* TABLE 1 *The values of* $[N,n]_{\text{fermionic}}$ *are tabulated. Entries in a row are related now by "Pascal's principle" to entries in the preceding row. The zeros reflect the fact that* *<u>it is impossible to construct a fermionic composite containing more than N copies of an N-state system</u>*.

It is by now clear that algebra appropriate to the fermionic composition of N-state systems comes to us ready-made: is precisely the *exterior algebra*,[24] which provides precursors also of some of the ideas to which we will turn after

[24] For an account of this subject, see Chapter 2 in H. Flanders, *Differential Forms, with Applications to the Physical Sciences* (1963).

we have clarified one dangling detail:

On recent pages I have provided explicit descriptions of vectors $(n_1, n_2, n_3|$ with $n \equiv n_1 + n_2 + n_3 = 1$ else 2 else 3 in both the bosonic and fermionic cases. Normalization factors $\frac{1}{\sqrt{\text{etc.}}}$ were introduced "by hand;" orthogonality could be verified by similarly direct means. We need to be in position to establish orthonormality *in general*, and to that end must acquire some apparatus. Reading from the list of Kronecker product properties presented at (63) in Chapter 1 of *Advanced Quantum Topics* (2000), we have[25]

$$(\mathbb{A} \otimes \mathbb{B})^{\mathsf{T}} = \mathbb{A}^{\mathsf{T}} \otimes \mathbb{B}^{\mathsf{T}} \tag{33.1}$$

$$(\mathbb{A} \otimes \mathbb{B})(\mathbb{C} \otimes \mathbb{D}) = \mathbb{AC} \otimes \mathbb{BD} \quad \text{if} \quad \begin{cases} \mathbb{A} \text{ is } m \times p \\ \mathbb{B} \text{ is } n \times q \\ \mathbb{C} \text{ is } p \times u \\ \mathbb{D} \text{ is } q \times v \end{cases} \tag{33.2}$$

Suppose, in particular, that $\boldsymbol{a}$ and $\boldsymbol{c}$ are p-vectors and that $\boldsymbol{b}$ and $\boldsymbol{d}$ are q-vectors. Then $\boldsymbol{u} \equiv \boldsymbol{a} \otimes \boldsymbol{b}$ and $\boldsymbol{v} \equiv \boldsymbol{c} \otimes \boldsymbol{d}$ are pq-vectors, and their inner product

$$\begin{aligned} \boldsymbol{u}^{\mathsf{T}}\boldsymbol{v} &= (\boldsymbol{a} \otimes \boldsymbol{b})^{\mathsf{T}}(\boldsymbol{c} \otimes \boldsymbol{d}) \\ &= (\boldsymbol{a}^{\mathsf{T}} \otimes \boldsymbol{b}^{\mathsf{T}})(\boldsymbol{c} \otimes \boldsymbol{d}) \\ &= \boldsymbol{a}^{\mathsf{T}}\boldsymbol{c} \cdot \boldsymbol{b}^{\mathsf{T}}\boldsymbol{d} \end{aligned}$$

It follows by extension that if $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n\}$ is an n-tuple of N-vectors, and if $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ is another, then

$$\begin{aligned} &(\boldsymbol{f}_1 \otimes \boldsymbol{f}_2 \otimes \ldots \otimes \boldsymbol{f}_n)^{\mathsf{T}}(\boldsymbol{g}_1 \otimes \boldsymbol{g}_2 \otimes \ldots \otimes \boldsymbol{g}_n) \\ &\qquad = (\boldsymbol{f}_1^{\mathsf{T}}\boldsymbol{g}_1) \cdot (\boldsymbol{f}_2^{\mathsf{T}}\boldsymbol{g}_2) \cdots (\boldsymbol{f}_n^{\mathsf{T}}\boldsymbol{g}_n) \end{aligned} \tag{34}$$

Suppose every $\boldsymbol{f}$ and every $\boldsymbol{g}$ has been associated with one or another of the elements of an *orthonormal* set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N\}$ of N-vectors; then

$$= \begin{cases} 1 & \text{if } \boldsymbol{f}_1 = \boldsymbol{g}_1,\ \boldsymbol{f}_2 = \boldsymbol{g}_2, \text{ etc.} \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

Reinstate (in the interest of notational simplicity) the assumption that $N = 3$. Define

$$\boldsymbol{E} \equiv \underbrace{\boldsymbol{e}_1 \otimes \cdots \otimes \boldsymbol{e}_1}_{n_1 \text{ factors}} \otimes \underbrace{\boldsymbol{e}_2 \otimes \cdots \otimes \boldsymbol{e}_2}_{n_2 \text{ factors}} \otimes \underbrace{\boldsymbol{e}_3 \otimes \cdots \otimes \boldsymbol{e}_3}_{n_3 \text{ factors}}$$

---

[25] The second of the following identities is actually a vast generalization of the identity quoted, which arises in the case $m = p = u$, $n = q = v$. My assertion is that the identity holds *whenever all the matrix products make sense*. In "Kronecker products with Mathematica" (October 2000) I indicate how *Mathematica* can be used to generate evidence in support of that assertion.

which is a 3^n-vector with $n = n_1 + n_2 + n_3$; it is by convention that the factors have been placed in dictionary order.

Symmetrize; i.e., construct the sum-over-permutations

$$\begin{aligned} \boldsymbol{S}_{n_1,n_2,n_3} &\equiv \sum_{\wp} \boldsymbol{e}_1 \otimes \cdots \otimes \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \otimes \cdots \otimes \boldsymbol{e}_2 \otimes \boldsymbol{e}_3 \otimes \cdots \otimes \boldsymbol{e}_3 \\ &= \begin{cases} n!\text{-fold sum of } 3^n\text{-vectors, in which} \\ \text{each term appears } n_1!n_2!n_3! \text{ times} \end{cases} \end{aligned} \tag{36}$$

From (35) obtain

$$\boldsymbol{S}^{\mathsf{T}}\boldsymbol{S} = (n_1 + n_2 + n_3)! \cdot n_1!n_2!n_3! \tag{37}$$

and this more sweeping statement:

$$\boldsymbol{S}^{\mathsf{T}}_{m_1,m_2,m_3}\boldsymbol{S}_{n_1,n_2,n_3} = \big\{(n_1 + n_2 + n_3)! \cdot n_1!n_2!n_3!\big\} \cdot \delta_{m_1n_1}\delta_{m_2n_2}\delta_{m_3n_3}$$

Make the notational adjustments $\boldsymbol{e}_1 \mapsto |1)$, $\boldsymbol{e}_2 \mapsto |2)$, $\boldsymbol{e}_3 \mapsto |3)$ and discover that we have reproduced all previous bosonic normalization factors and orthogonality relations. Alternatively...

Antisymmetrize; i.e., construct the sum-over-*signed*-permutations[26]

$$\begin{aligned} \boldsymbol{A}_{n_1,n_2,n_3} &\equiv \sum_{\wp} (-)^{\wp} \boldsymbol{e}_1 \otimes \cdots \otimes \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \otimes \cdots \otimes \boldsymbol{e}_2 \otimes \boldsymbol{e}_3 \otimes \cdots \otimes \boldsymbol{e}_3 \\ &= \begin{cases} n!\text{-fold sum of } 3^n\text{-vectors if } n_1!n_2!n_3! = 1 \\ \mathbf{0} \quad \text{otherwise} \end{cases} \end{aligned} \tag{38}$$

From (34) obtain

$$\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A} = (n_1 + n_2 + n_3)! \tag{39}$$

and discover that we have reproduced all previous fermionic normalization factors and orthogonality relations. Notice that

$$(n_1 + n_2 + n_3)! = (n_1 + n_2 + n_3)!\, n_1!n_2!n_3! \quad \text{by} \quad 0! = 1! = 1$$

so the bosonic and fermionic normalization factors are formally *identical*.

We come at last to the point of these preparatory remarks:

BOSONIC CREATION & ANNIHILATION OPERATORS

Let b_1 denote an operator designed to achieve *symmetrized admixture of an* $\boldsymbol{e}_1$, and let b_2, b_3, ... be defined similarly. Evidently

$$\mathsf{b}_1|n_1, n_2, n_3) \sim |n_1 + 1, n_2, n_3)$$

[26] One might reasonably call these "determinantal" or "exterior" sums.

We will in fact adopt equations of the form

$$\mathsf{b}_1|n_1,n_2,n_3) = \sqrt{n_1+1}\cdot|n_1+1,n_2,n_3) \tag{40.1}$$

which by

$$(m_1,m_2,m_3|\overset{\text{acts to the right}\downarrow}{\mathsf{b}_1}|n_1,n_2,n_3) \equiv (m_1,m_2,m_3|\overset{\downarrow\text{acts to the left}}{\mathsf{b}_1^{\,\mathsf{t}}}|n_1,n_2,n_3)$$

entail

$$\mathsf{b}_1^{\,\mathsf{t}}|n_1,n_2,n_3) = \sqrt{n_1}\cdot|n_1-1,n_2,n_3) \tag{40.2}$$

and serve perfectly well to *define* b_1 and $\mathsf{b}_1^{\,\mathsf{t}}$, but might seem unmotivated. The following remarks are intended to remove that defect:

The normalized 3^n-vector $|n_1,n_2,n_3)$ is presented to us as a sum of $n!$ terms:

$$|n_1,n_2,n_3) = \frac{1}{\sqrt{n_1!\,n_2!\,n_3!}}\sum_{\wp}\frac{\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\cdots\otimes\boldsymbol{e}_{i_n}}{\sqrt{n!}}$$

"Symmetrized admixture of an $\boldsymbol{e}$" turns each term into $(n+1)$ terms, by the following mechanism:

$$\begin{array}{l}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\\ \downarrow\\ \underbrace{\boldsymbol{e}\otimes\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k+\boldsymbol{e}_i\otimes\boldsymbol{e}\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k+\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}\otimes\boldsymbol{e}_k+\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}}_{\text{"symmetrized admixture," denoted }\boldsymbol{e}\,Ⓢ\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k}\end{array}$$

It becomes in this light natural to define

$$\begin{aligned}\mathsf{b}_1|n_1,n_2,n_3) &\equiv \frac{1}{\sqrt{n+1}}\cdot\boldsymbol{e}_1\,Ⓢ\,|n_1,n_2,n_3)\\ &= \frac{1}{\sqrt{n_1!\,n_2!\,n_3!}}\sum_{\wp}\frac{\boldsymbol{e}_1\,Ⓢ\,\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\cdots\otimes\boldsymbol{e}_{i_n}}{\sqrt{n+1}\sqrt{n!}}\\ &= \sqrt{n_1+1}\cdot\underbrace{\frac{1}{\sqrt{(n_1+1)!\,n_2!\,n_3!}}\sum_{\wp}\frac{\boldsymbol{e}_1\,Ⓢ\,\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\cdots\otimes\boldsymbol{e}_{i_n}}{\sqrt{n+1}\sqrt{n!}}}_{|n_1+1,n_2,n_3)}\end{aligned}$$

which brings us back to (40.1). One curious detail: normally we might expect to define

$$\text{symmetrization} = \frac{\text{sum of permuted terms}}{\text{number of terms}}$$

but have here adopted a convention

$$\text{state vector symmetrization} = \frac{\text{sum of permuted terms}}{\sqrt{\text{number of terms}}}$$

which is justified on grounds that it leads to the most natural expression of the theory, but might be understood as a reflection of the circumstance that state vectors enter *quadratically* into the construction of expectation values.

Look next to the synthesis of $\mathsf{b}_1^{\mathsf{t}}$. We begin by isolating an implication of (33). Assume as before that $\boldsymbol{c}$ is a p-vector, $\boldsymbol{b}$ and $\boldsymbol{d}$ are q-vectors. Then $\boldsymbol{c}\otimes\boldsymbol{d}$ is a pq-vector. Let

$$\mathbb{I}_p \equiv \text{the } p\times p \text{ identity matrix}$$

Then $\mathbb{I}_p\otimes\boldsymbol{b}$ is a $pq\times p$ matrix, its transpose is $p\times pq$ and by (33) we have

$$\begin{aligned}(\,\mathbb{I}_p\otimes\boldsymbol{b})^{\mathsf{T}}(\boldsymbol{c}\otimes\boldsymbol{d}\,) &= (\,\mathbb{I}_p\otimes\boldsymbol{b}^{\mathsf{T}})(\boldsymbol{c}\otimes\boldsymbol{d}\,)\\ &= \boldsymbol{c}\cdot(\boldsymbol{b}^{\mathsf{T}}\boldsymbol{d}\,)\end{aligned}\tag{41}$$

The rectangular matrix $(\,\mathbb{I}_p\otimes\boldsymbol{b})^{\mathsf{T}}$ has successfully "devoured" the second of the factors in the Kronecker product $\boldsymbol{c}\otimes\boldsymbol{d}$; we might reasonably call this process "Kronecker division."[27] To see the process in action, look at

$$(\,\mathbb{I}_9\otimes\boldsymbol{e}_1)^{\mathsf{T}}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k) = (\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\cdot(\boldsymbol{e}_1{}^{\mathsf{T}}\boldsymbol{e}_k) = \begin{cases}\boldsymbol{e}_i\otimes\boldsymbol{e}_j & \text{if } k=1\\ \quad\mathbf{0} & \text{otherwise}\end{cases}$$

Or look at the following (in which we retain the 3-state assumption):

$$\mathsf{b}_1^{\mathsf{t}}|n_1,n_2,n_3) \equiv \sqrt{n}(\mathbb{I}_{3^{n-1}}\otimes\boldsymbol{e}_1)^{\mathsf{T}}\cdot\frac{1}{\sqrt{n_1!\,n_2!\,n_3!}}\sum_{\wp}\frac{\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\cdots\otimes\boldsymbol{e}_{i_n}}{\sqrt{n!}}$$

Notice that the $n!$ terms presented by $\sum_{\wp}\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\cdots\otimes\boldsymbol{e}_{i_n}$ can be grouped

$$\begin{aligned}\sum_{\wp}\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\cdots\otimes\boldsymbol{e}_{i_n} &= \text{sum of terms with no terminal } \boldsymbol{e}_1 && : \quad n_1=0\\ &= \text{such stuff} + \Big\{\sum_{\wp}\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\cdots\otimes\boldsymbol{e}_{i_{\,n-1}}\Big\}\otimes\boldsymbol{e}_1 && : \quad n_1=1\\ &= \text{such stuff} + 2\Big\{\sum_{\wp}\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\cdots\otimes\boldsymbol{e}_{i_{\,n-1}}\Big\}\otimes\boldsymbol{e}_1 && : \quad n_1=2\\ &\ \ \vdots\end{aligned}$$

so by (41) we have

$$\mathsf{b}_1^{\mathsf{t}}|n_1,n_2,n_3) = \sqrt{n}\bigg\{\frac{1}{\sqrt{n_1!\,n_2!\,n_3!}}\,n_1\sum_{\wp}\frac{\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\cdots\otimes\boldsymbol{e}_{i_{\,n-1}}}{\sqrt{n!}}\bigg\}$$

(an $\boldsymbol{e}_1$ has been removed from the $\boldsymbol{e}$ population, and "stuff" has been killed)

$$= \sqrt{n_1}\bigg\{\frac{1}{\sqrt{(n_1-1)!\,n_2!\,n_3!}}\sum_{\wp}\frac{\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\cdots\otimes\boldsymbol{e}_{i_{\,n-1}}}{\sqrt{(n-1)!}}\bigg\}$$

and have recovered (40.2).

[27] This happy idea occurred to me while I stood bedrizzled in the parking lot of Tualatin Valley Builders Supply, where I had gone to get materials to repair my garage door, and took much of the pain out of that experience.

FERMIONIC CREATION & ANNIHILATION OPERATORS

The general pattern of the argument runs as in the bosonic case, but to make theorems possible we must *establish a sign convention*, and to prove them we must accept the tedium of some careful sign juggling.[28] Agree at the outset to abandon our former 3-state assumption $N = 3$ (which proves now too restrictive to serve expositor clarity), and fix in mind the facts that in the fermionic case

- all n's are 0 or 1 (so $|n_1, n_2, \ldots, n_N)$ looks in every case rather like a binary number);
- the subscripts in $\boldsymbol{e}_{i_1}\otimes \boldsymbol{e}_{i_2}\otimes \cdots \otimes \boldsymbol{e}_{i_n}$ are always distinct.

Adopt the

> CONVENTION: The term $\boldsymbol{e}_{i_1}\otimes \boldsymbol{e}_{i_2}\otimes \cdots \otimes \boldsymbol{e}_{i_n}$ with
> $$i_1 < i_2 < \cdots < i_n \quad : \quad \text{dictionary order}$$
> enters *positively* into the design of $|n_1, , n_2, \ldots, n_N)$.

The normalized N^n-vector $|n_1, n_2, \ldots, n_N)$ is presented to us as a sum of $n!$ terms:

$$|n_1, n_2, \ldots, n_N) = \sum_{\wp} (-)^{\wp} \frac{\boldsymbol{e}_{i_1}\otimes \boldsymbol{e}_{i_2}\otimes \cdots \otimes \boldsymbol{e}_{i_n}}{\sqrt{n!}}$$

"Antisymmetrized admixture of an $\boldsymbol{e}$" turns each term into $(n+1)$ terms, by the following mechanism:

$$\begin{array}{l}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \\ \downarrow \\ \underbrace{\boldsymbol{e} \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k - \boldsymbol{e}_i \otimes \boldsymbol{e} \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k + \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e} \otimes \boldsymbol{e}_k - \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}}_{\text{“antisymmetrized admixture,” denoted } \boldsymbol{e}\,Ⓐ\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k}\end{array}$$

It becomes in this light natural to define

$$\begin{aligned}\mathsf{b}_j|n_1, n_2, \ldots, n_N) &\equiv \frac{1}{\sqrt{n+1}} \cdot \boldsymbol{e}_j \;Ⓐ\; |n_1, n_2, n_3) \\ &= \begin{cases} \mathbf{0} & \text{if } k \text{ present in the ordered } i\text{-list} \\ \displaystyle\sum_{\wp} (-)^{\wp} \frac{\boldsymbol{e}_j \;Ⓐ\; \boldsymbol{e}_{i_1}\otimes \boldsymbol{e}_{i_2}\otimes \cdots \otimes \boldsymbol{e}_{i_n}}{\sqrt{n+1}\sqrt{n!}} & \text{if } k \text{ absent} \end{cases} \\ &= \begin{cases} \mathbf{0} \\ \displaystyle(-)^s \sum_{\wp} (-)^{\wp} \frac{\boldsymbol{e}_{i_1}\otimes \cdots \otimes \boldsymbol{e}_{i_s}\otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_{k_1}\otimes \cdots \otimes \boldsymbol{e}_{k_{n-s}}}{\sqrt{(n+1)!}} \end{cases}\end{aligned}$$

according as $n_j = 1$ or $n_j = 0$. Here we have moved $\boldsymbol{e}_j$ to its canonical place

$$\text{all } i\text{'s} < j < \text{all } k\text{'s}$$

[28] An identical tedium bedevils the exterior calculus.

and

$$\begin{aligned} s \equiv s_j &\equiv \text{number of } \boldsymbol{e}_i \text{ that conventionally stand left of } \boldsymbol{e}_j \\ &= \sum_{i<j} n_i \end{aligned}$$

So we have

$$\begin{aligned} \mathsf{b}_j|n_1, n_2, \dots, n_j, \dots, n_N) \\ = (-)^{s_j}(1-n_j)\cdot|n_1, n_2, \dots, n_j+1, \dots, n_N) \end{aligned} \tag{42.1}$$

where we have installed

$$1 - n_j = \begin{cases} 0 & \text{if} \quad n_j = 1 \\ 1 & \text{if} \quad n_j = 0 \end{cases}$$

as a nifty "switch" to distinguish one case from the other.[29] We are in possession now of apparatus sufficient to produce a "constructive" account of $\mathsf{b}_j^{\mathsf{t}}$, but I won't: we have actual need only of a statement

$$\begin{aligned} \mathsf{b}_j^{\mathsf{t}}\,|n_1, n_2, \dots, n_j, \dots, n_N) \\ = (-)^{s_j}\, n_j \cdot|n_1, n_2, \dots, n_j-1, \dots, n_N) \end{aligned} \tag{42.2}$$

that follows already from (42.1) by the *meaning* of the adjoint.

I have found it easier, from a constructive standpoint, to discuss the **b**irth of an $\boldsymbol{e}$-factor than its demise, and have elected to assign the simpler name to the simpler object. But now that all the work is behind us, we can revert

$$\begin{aligned} \mathsf{b}_j &\longmapsto \mathsf{a}_j^{\mathsf{t}} \quad : \quad \text{creation operators} \\ \mathsf{b}_j^{\mathsf{t}} &\longmapsto \mathsf{a}_j \quad : \quad \text{annihilation operators} \end{aligned}$$

to the notation which has long been standard. We have

$$\begin{aligned} \mathsf{a}_j^{\mathsf{t}}|n_1, n_2, \dots, n_j, \dots, n_N) \\ = \sqrt{n_j+1}\cdot|n_1, n_2, \dots, n_j+1, \dots, n_N) \end{aligned} \tag{43.11}$$

$$\begin{aligned} \mathsf{a}_j\,|n_1, n_2, \dots, n_j, \dots, n_N) \\ = \sqrt{\quad n_j \quad}\cdot|n_1, n_2, \dots, n_j-1, \dots, n_N) \end{aligned} \tag{43.12}$$

$$\begin{aligned} \mathsf{a}_j^{\mathsf{t}}|n_1, n_2, \dots, n_j, \dots, n_N) \\ = (-)^{s_j}(1-n_j)\cdot|n_1, n_2, \dots, n_j+1, \dots, n_N) \end{aligned} \tag{43.21}$$

$$\begin{aligned} \mathsf{a}_j\,|n_1, n_2, \dots, n_j, \dots, n_N) \\ = (-)^{s_j}(\quad n_j \quad)\cdot|n_1, n_2, \dots, n_j-1, \dots, n_N) \end{aligned} \tag{43.22}$$

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[29] The complementary switch—needed in a moment—is even simpler:

$$n_k = \begin{cases} 1 & \text{if} \quad n_k = 1 \\ 0 & \text{if} \quad n_k = 0 \end{cases}$$

in the bosonic/fermionic cases, respectively. In the former (bosonic) case it is evident that

$$\big[\mathbf{a}_i\,,\mathbf{a}_j\big] = \big[\mathbf{a}_i^{\mathsf{t}}\,,\mathbf{a}_j^{\mathsf{t}}\,\big] = \mathbf{0} \quad : \quad \text{all } i,\, j \tag{44.1}$$

and

$$\big[\mathbf{a}_i\,,\mathbf{a}_j^{\mathsf{t}}\,\big] = \mathbf{0} \quad : \quad i \neq j \tag{44.2}$$

while from

$$\begin{aligned}
\mathbf{a}_i\mathbf{a}_i^{\mathsf{t}}\,|n_1, n_2, \ldots, n_i\,, \ldots, n_N) &= (n_i + 1)|n_1, n_2, \ldots, n_i\,, \ldots, n_N)\\
\mathbf{a}_i^{\mathsf{t}}\mathbf{a}_i\,|n_1, n_2, \ldots, n_i\,, \ldots, n_N) &= (\quad n_i \quad)|n_1, n_2, \ldots, n_i\,, \ldots, n_N)
\end{aligned}$$

we obtain

$$\big[\mathbf{a}_i\,,\mathbf{a}_i^{\mathsf{t}}\,\big] = \mathbf{I} \quad : \quad \text{all } i \tag{44.3}$$

But to develop fermionic analogs of those statements we must take into account the sign factors, which have some curious consequences. Look first to $\mathbf{a}_i\mathbf{a}_j|\text{etc.})$ on the assumption that $i < j$. In $\mathbf{a}_i\mathbf{a}_j|\text{etc.})$ we encounter $(-)^{S_i+S_j}$ while the $\mathbf{a}_j$ in $\mathbf{a}_j\mathbf{a}_i|\text{etc.})$ *sees an extra term*, so presents $(-)^{S_i+S_j+1} = -\,(-)^{S_i+S_j}$. This upshot of this line of argument is that

$$\big[\mathbf{a}_i\,,\mathbf{a}_j\big]_+ = \big[\mathbf{a}_i^{\mathsf{t}}\,,\mathbf{a}_j^{\mathsf{t}}\,\big]_+ = \mathbf{0} \quad : \quad \text{all } i,\, j \tag{45.1}$$

$$\big[\mathbf{a}_i\,,\mathbf{a}_j^{\mathsf{t}}\,\big]_+ = \mathbf{0} \quad : \quad i \neq j \tag{45.2}$$

where the $_+$ signifies *anti*commutation:

$$\big[\mathbf{A}, \mathbf{B}\big]_+ \equiv \mathbf{A}\,\mathbf{B} + \mathbf{B}\,\mathbf{A}$$

It follows more simply from

$$\begin{aligned}
\mathbf{a}_i\mathbf{a}_i^{\mathsf{t}}\,|\ldots, n_i, \ldots) &= \begin{cases} (-)^{2S_i}\,|\ldots, n_i, \ldots) & \text{if} \quad n_i = 0\\ \mathbf{0} & \text{if} \quad n_i = 1 \end{cases}\\
\mathbf{a}_i^{\mathsf{t}}\mathbf{a}_i\,|\ldots, n_i, \ldots) &= \begin{cases} \mathbf{0} & \text{if} \quad n_i = 0\\ (-)^{2S_i}\,|\ldots, n_i, \ldots) & \text{if} \quad n_i = 1 \end{cases}
\end{aligned}$$

and $(-)^{2S_i} = 1$ (all cases) that

$$\big[\mathbf{a}_i\,,\mathbf{a}_i^{\mathsf{t}}\,\big]_+ = \mathbf{I} \quad : \quad \text{all } i \tag{45.3}$$

It is clear from the preceding discussion that

$$\begin{aligned}
&\mathbf{a}_i \text{ acts on elements of } \mathcal{V}_N^n \text{ to yield elements of } \mathcal{V}_N^{n-1}\\
&\mathbf{a}_i^{\mathsf{t}} \text{ acts on elements of } \mathcal{V}_N^n \text{ to yield elements of } \mathcal{V}_N^{n+1}
\end{aligned}$$

Which is to say: creation/annihilation operators have as their sphere of activity not the state spaces of individual composite systems, but the formal *union* of such spaces—a place called "Fock space"[30]

$$\mathcal{V}_{\text{Fock}} \equiv \begin{cases} \mathcal{V}_N^0 \oplus \mathcal{V}_N^1 \oplus \cdots \oplus \mathcal{V}_N^N \oplus \mathcal{V}_N^{N+1} \oplus \cdots & : \quad \text{bosonic case}\\ \mathcal{V}_N^0 \oplus \mathcal{V}_N^1 \oplus \cdots \oplus \mathcal{V}_N^N & : \quad \text{fermionic case} \end{cases}$$

[30] V. Fock, "Konfigurationsraum und zweite Quantelung," Zeit. f. Physik **75**, 622 (1932).

which is

∞-dimensional in the bosonic case, but only
2^N-dimensional in the fermionic case.

Combinations of creation and annihilation operators, on the other hand, *may* be meaningful on individual $\mathcal{V}$'s: the so-called "number operators"

$$\mathsf{N}_i \equiv \mathsf{a}_i^{\mathsf{t}}\mathsf{a}_i$$

$$\mathsf{N}_i\,|\ldots,n_i,\ldots) = n_i\,|\ldots,n_i,\ldots) \quad : \quad \text{bosonic or fermionic}$$

and the "total number operator"

$$\mathsf{N} \equiv \sum_{i=1}^{N} \mathsf{N}_i$$

provide (hermitian) examples. We note in passing that

$$\left.\begin{aligned} \big[\mathsf{N},\mathsf{a}_i\big] &= \big[\mathsf{N}_i,\mathsf{a}_i\big] = -\mathsf{a}_i \\ \big[\mathsf{N},\mathsf{a}_i^{\mathsf{t}}\big] &= \big[\mathsf{N}_i,\mathsf{a}_i^{\mathsf{t}}\big] = +\mathsf{a}_i^{\mathsf{t}} \end{aligned}\right\} \quad : \quad \text{bosonic case}$$

$$\left.\begin{aligned} \big[\mathsf{N},\mathsf{a}_i\big]_+ &= \big[\mathsf{N}_i,\mathsf{a}_i\big]_+ = +\mathsf{a}_i \\ \big[\mathsf{N},\mathsf{a}_i^{\mathsf{t}}\big]_+ &= \big[\mathsf{N}_i,\mathsf{a}_i^{\mathsf{t}}\big]_+ = +\mathsf{a}_i^{\mathsf{t}} \end{aligned}\right\} \quad : \quad \text{fermionic case}$$

and that from

$$\mathsf{a}_i\mathsf{a}_i = \mathsf{a}_i^{\mathsf{t}}\mathsf{a}_i^{\mathsf{t}} = \mathbf{0} \quad : \quad \text{fermionic case}$$

it follows quickly that the operators N_i are projective:

$$\mathsf{N}_i^2 = \mathsf{N}_i \quad : \quad \text{fermionic case}$$

. . . which is another way of saying what we already knew: the eigenvalues of N_i are, in teh latter case, 0's else 1's.

We come now to the profound development that (upon abandonment of our characteristic finite-state assumption) earns for this subject the name "quantum field theory," that gives rise to the concept of "second quantization." For the purposes of this discussion we confine our explicit attention to bosons, and reinstate our assumption that $N = 3$. Let H be a hermitian operator defined on the state space $\mathcal{V} \equiv \mathcal{V}_3^1$ of the solitary 3-state system $\mathfrak{S}$. To describe H relative to a selected orthonormal basis, Dirac would write

$$\mathsf{H} = \sum_{i,j}|i)(i|\mathsf{H}|j)(j| \equiv \sum_{i,j}H_{ij}|i)(j|$$

When presented to $|\psi) = \sum_k |k)(k|\psi) \equiv \sum_k |k)\,\psi_k$ the operator $H_{ij}|i)(j|$ plucks out the $|j)$-component and turns it into an $|i)$-component, to which it assigns weight H_{ij}. To express the same idea, we might write

$$|\psi) = |1,0,0)(1,0,0|\psi) + |0,1,0)(0,1,0|\psi) + |0,0,1)(0,0,1|\psi)$$

$$|i)(j| = \mathsf{a}_j^{\mathsf{t}}\mathsf{a}_i$$

$$H_{11} = (1,0,0|\mathsf{H}|1,0,0), \quad H_{12} = (1,0,0|\mathsf{H}|0,1,0), \quad \text{etc.}$$

It becomes in this light natural to introduce

$$\boldsymbol{\Psi} \equiv |1,0,0)\,\mathsf{a}_1 + |0,1,0)\,\mathsf{a}_2 + |0,0,1)\,\mathsf{a}_3 \tag{46}$$

and to write

$$\underbrace{\mathsf{H}}_{\text{has become meaningful on } \mathcal{V}_{\text{Fock}}} = \boldsymbol{\Psi}^{\mathsf{t}}\overbrace{\mathsf{H}}^{\text{was initially defined only on } \mathcal{V}^1}\boldsymbol{\Psi} \tag{47}$$

I will underscore the point just made by notational adjustment:

$$\mathcal{H} = \boldsymbol{\Psi}^{\mathsf{t}}\overset{\downarrow}{\mathsf{H}}\boldsymbol{\Psi}$$

In the ordinary quantum theory of $\mathfrak{S}$ the expression $(\psi|\mathsf{H}|\psi)$ carries the interpretation of a number-valued "expectation value." But the right side of (47) is *operator*-valued: it acts simultaneously on $\mathcal{V}_3^1$, $\mathcal{V}_3^2$, $\mathcal{V}_3^3, \ldots$; it acts, in short, on the elements

$$|\psi)_{\text{Fock}} \in \mathcal{V}_{\text{Fock}}$$

and provides accounts simultaneously of

quantum physics on $\mathfrak{S}$
quantum physics on $\mathfrak{S} \circledcirc \mathfrak{S}$
quantum physics on $\mathfrak{S} \circledcirc \mathfrak{S} \circledcirc \mathfrak{S}$
$\vdots$

We expect to have ${}_{\text{Fock}}(\psi|\psi)_{\text{Fock}} = 1$, but $\boldsymbol{\Psi}^{\mathsf{t}}\boldsymbol{\Psi}$ is a different kind of beast: we find

$$\boldsymbol{\Psi}^{\mathsf{t}}\boldsymbol{\Psi} = \sum_i \mathsf{a}_i^{\mathsf{t}}\mathsf{a}_i = \sum_i \mathsf{N}_i = \mathsf{N}$$

It is clear by this point that we can abandon our initial "3-state bosons" assumption.

The finite-state Schrödinger equation $\mathbb{H}\boldsymbol{\psi} = i\hbar\partial_t\boldsymbol{\psi}$ can—together with its conjugate— be obtained from the classical (!) Lagrangian

$$L = \boldsymbol{\psi}^{\mathsf{t}}(i\hbar\,\boldsymbol{\psi}_t - \mathbb{H}\,\boldsymbol{\psi})$$

on the strength of which we are led to define[31]

$$\boldsymbol{\pi} \equiv \frac{\partial L}{\partial \boldsymbol{\psi}_t} = i\hbar\,\boldsymbol{\psi}^{\mathsf{t}} \quad : \quad \text{“conjugate momentum”}$$

[31] Note that the gauge-equivalent Lagrangian $L = \frac{1}{2}i\hbar(\boldsymbol{\psi}^{\mathsf{t}}\boldsymbol{\psi}_t - \boldsymbol{\psi}_t^{\mathsf{t}}\boldsymbol{\psi}) - \boldsymbol{\psi}^{\mathsf{t}}\,\mathbb{H}\,\boldsymbol{\psi}$ leads to different results.

Imitating that result, we define

$$\mathbf{\Pi} \equiv i\hbar\,\mathbf{\Psi}^{\mathsf{t}}$$

and obtain

$$\left.\begin{array}{lll} \text{bosonic case} & : & \big[\mathbf{\Psi},\mathbf{\Pi}\big] \;\;= i\hbar \sum\limits_k \big[\mathsf{a}_k\,,\mathsf{a}_k^{\mathsf{t}}\big] \\ \text{fermionic case} & : & \big[\mathbf{\Psi},\mathbf{\Pi}\big]_+ = i\hbar \sum\limits_k \big[\mathsf{a}_k\,,\mathsf{a}_k^{\mathsf{t}}\big]_+ \end{array}\right\} = i\hbar\cdot(\text{number } N \text{ of states})$$

In field theory the "number of states" (degrees of freedom) becomes infinite, and one is led (in the non-relativistic theory) to statements of the form

$$\big[\mathbf{\Psi}(\boldsymbol{x}),\mathbf{\Pi}(\boldsymbol{y})\big]_\pm = i\hbar\delta(\boldsymbol{x}-\boldsymbol{y})$$

"Quantum field theory" is the name given to the quantum theory of indefinitely many indistinguishable bosonic/fermionic subsystems (usually understood to be "particles"), and can be considered to arise by formal "second quantization" of the ψ-field that serves to describe the state of a solitary subsystem. But having illustrated the grounds for such a statement, I abandon the topic. . . in order to return to my motivating problem; i.e., to the clarification of Schwinger's point of departure:

Look now to a *bosonic population of 2-state systems*. Such systems are commonly called "spin systems," and (since spin $\frac{1}{2}$ is known to imply fermionic physics) the notion of a "bosonic population of spins" might appear to be physically absurd; it poses, however, no *formal* problem for the theory sketched in recent pages. The theory presents creation/annihilation operators of only two flavors: call them $\mathsf{a}_1^{\mathsf{t}},\ \mathsf{a}_1,\ \mathsf{a}_2^{\mathsf{t}},\ \mathsf{a}_2$. They satisfy commutation relations which by (44) are identical to those supplied at (15) by the isotropic oscillator—operators which we used in (18)–(28) to establish contact with the quantum theory of angular momentum.

So we have now in hand two distinct quartets of operators

$$\begin{array}{lll} \big\{\mathsf{a}_1^{\mathsf{t}},\mathsf{a}_1,\mathsf{a}_2^{\mathsf{t}},\mathsf{a}_2\big\} & : & \text{refer to an isotropic oscillator} \\ \big\{\mathsf{a}_1^{\mathsf{t}},\mathsf{a}_1,\mathsf{a}_2^{\mathsf{t}},\mathsf{a}_2\big\} & : & \text{refer to a bosonic population of spin } \tfrac{1}{2} \text{ systems} \end{array}$$

which, though algebraically identical (and therefore equally able to support formal imitations of the quantum theory of angular momentum), have physically quite different meanings—not least because they operate upon entirely different kinds of objects.

All of which was understood—taken for granted—by Schwinger before he wrote a line of "On angular momentum."

ELEMENTS OF SCHWINGER'S ARGUMENT

Raw materials. I collect together, for hand reference, the algebraic material now at our disposal as we—following in the footsteps of Schwinger[32]—undertake to reproduce (i.e., to construct a theory *formally identical* to) the quantum theory of angular momentum. I highlight statements that are available to Schwinger but not available to authors who pursue the standard algebraic line of argument.

$$\big[\mathsf{a}_1, \mathsf{a}_1^{\mathsf{t}}\big] = \big[\mathsf{a}_2, \mathsf{a}_2^{\mathsf{t}}\big] = \mathsf{I} \;\;:\;\; \text{all other } \mathsf{a}\text{-commutators vanish} \tag{48.1}$$

$$\left.\begin{aligned} \mathsf{N}_1 &\equiv \mathsf{a}_1^{\mathsf{t}}\mathsf{a}_1 \\ \mathsf{N}_2 &\equiv \mathsf{a}_2^{\mathsf{t}}\mathsf{a}_2 \end{aligned}\right\} \tag{48.2}$$

$$\left.\begin{aligned} \mathsf{N} &\equiv \mathsf{N}_1 + \mathsf{N}_2 \\ \mathsf{M} &\equiv \mathsf{N}_1 - \mathsf{N}_2 \end{aligned}\right\} \tag{48.3}$$

$$\left.\begin{aligned} \mathsf{J}_0 &= \tfrac{\hbar}{2}(\mathsf{a}_1^{\mathsf{t}}\mathsf{a}_1 + \mathsf{a}_2^{\mathsf{t}}\mathsf{a}_2) = \tfrac{\hbar}{2}\mathsf{N} \\ \mathsf{J}_1 &= \tfrac{\hbar}{2}(\mathsf{a}_1^{\mathsf{t}}\mathsf{a}_2 + \mathsf{a}_2^{\mathsf{t}}\mathsf{a}_1) \\ \mathsf{J}_2 &= -i\tfrac{\hbar}{2}(\mathsf{a}_1^{\mathsf{t}}\mathsf{a}_2 - \mathsf{a}_2^{\mathsf{t}}\mathsf{a}_1) \\ \mathsf{J}_3 &= \tfrac{\hbar}{2}(\mathsf{a}_1^{\mathsf{t}}\mathsf{a}_1 - \mathsf{a}_2^{\mathsf{t}}\mathsf{a}_2) = \tfrac{\hbar}{2}\mathsf{M} \end{aligned}\right\} \tag{48.4}$$

$$\big[\mathsf{J}_0, \mathsf{J}_1\big] = \big[\mathsf{J}_0, \mathsf{J}_2\big] = \big[\mathsf{J}_0, \mathsf{J}_3\big] = \mathbf{0} \tag{48.5}$$

$$\left.\begin{aligned} \big[\mathsf{J}_1, \mathsf{J}_2\big] &= i\hbar\mathsf{J}_3 \\ \big[\mathsf{J}_2, \mathsf{J}_3\big] &= i\hbar\mathsf{J}_1 \\ \big[\mathsf{J}_3, \mathsf{J}_1\big] &= i\hbar\mathsf{J}_2 \end{aligned}\right\} \tag{48.6}$$

$$\begin{aligned} \mathsf{J}^2 \equiv \mathsf{J}_1^2 + \mathsf{J}_2^2 + \mathsf{J}_3^2 &= \mathsf{J}_0^2 + \hbar\mathsf{J}_0 \\ &= \hbar^2\Big\{\tfrac{1}{4}\mathsf{N}^2 + \tfrac{1}{2}\mathsf{N}\Big\} \end{aligned} \tag{48.7}$$

$$\big[\mathsf{J}^2, \mathsf{J}_1\big] = \big[\mathsf{J}^2, \mathsf{J}_2\big] = \big[\mathsf{J}^2, \mathsf{J}_3\big] = \mathbf{0} \tag{48.8}$$

$$\left.\begin{aligned} \mathsf{J}_+ &\equiv \mathsf{J}_1 + i\,\mathsf{J}_2 = \hbar\,\mathsf{a}_1^{\mathsf{t}}\mathsf{a}_2 \\ \mathsf{J}_- &\equiv \mathsf{J}_1 - i\,\mathsf{J}_2 = \hbar\,\mathsf{a}_2^{\mathsf{t}}\mathsf{a}_1 \end{aligned}\right\} \tag{48.9}$$

$$\big[\mathsf{J}^2, \mathsf{J}_+\big] = \big[\mathsf{J}^2, \mathsf{J}_-\big] = \mathbf{0} \tag{48.10}$$

$$\left.\begin{aligned} \big[\mathsf{J}_3, \mathsf{J}_+\big] &= +\hbar\mathsf{J}_+ \\ \big[\mathsf{J}_3, \mathsf{J}_-\big] &= -\hbar\mathsf{J}_- \end{aligned}\right\} \tag{48.11}$$

[32] See again (page 3) Schwinger's ABSTRACT, where he promises only to "derive many known theorems" and to obtain "some new results."

Construction & properties of eigenstates. One takes (48.8) as a license to search for simultaneous eigenfunctions of J^2 and J_3, which Schwinger interprets to be a search for simultaneous eigenfunctions of N and M.[33]

Let us agree to write $|0)$ with these alternative intentions:

$$|0) \equiv \begin{cases} \underline{\text{ground state}} \text{ in the oscillator realization of the algebra} \\ \underline{\text{vacuum state}} \text{ in the bosonic spin field realization} \end{cases}$$

In either realization we have

$$|n_1, n_2) = \frac{1}{\sqrt{n_1!\, n_2!}}(\mathsf{a}_1^{\mathsf{t}})^{n_1}\,(\mathsf{a}_2^{\mathsf{t}})^{n_2}\,|0) \tag{49}$$

and

$$\left.\begin{aligned} \mathsf{N}_1|n_1, n_2) &= n_1|n_1, n_2) \\ \mathsf{N}_2|n_1, n_2) &= n_2|n_1, n_2) \end{aligned}\right\} \tag{50}$$

Immediately

$$\begin{aligned} \mathsf{N}|n_1, n_2) &= (n_1 + n_2)\,|n_1, n_2) \\ \mathsf{M}|n_1, n_2) &= (n_1 - n_2)\,|n_1, n_2) \end{aligned}$$

which give

$$\begin{aligned} \mathsf{J}^2\,|n_1, n_2) &= \hbar^2\Big\{\tfrac{1}{4}\mathsf{N}^2 + \tfrac{1}{2}\mathsf{N}\Big\}|n_1, n_2) \\ &= \hbar^2 j(j+1)|n_1, n_2) \end{aligned} \tag{51.11}$$

$$j \equiv \tfrac{1}{2}(n_1 + n_2) \tag{51.12}$$

$$\begin{aligned} \mathsf{J}_3|n_1, n_2) &= \tfrac{\hbar}{2}\mathsf{M}|n_1, n_2) \\ &= \hbar m\,|n_1, n_2) \end{aligned} \tag{51.21}$$

$$m \equiv \tfrac{1}{2}(n_1 - n_2) \tag{51.22}$$

From

$$\left.\begin{aligned} n_1 &= j + m \\ n_2 &= j - m \end{aligned}\right\} \tag{52}$$

we therefore have simultaneous eigenstates

$$\begin{aligned} |j, m\rangle &\equiv |j+m, j-m) \\ &= \frac{1}{\sqrt{(j+m)!(j-m)!}}(\mathsf{a}_1^{\mathsf{t}})^{j+m}\,(\mathsf{a}_2^{\mathsf{t}})^{j-m}\,|0) \end{aligned} \tag{53}$$

In a footnote, Schwinger remarks that an expression of very nearly the same design as appears on the right side of (53) can be found on page 189 of the English translation (1931) of H. Weyl's *The Theory of Groups and Quantum*

[33] Notice that

$$\big[\mathsf{N}, \mathsf{M}\big] = \big[\mathsf{N}_1 + \mathsf{N}_2, \mathsf{N}_1 - \mathsf{N}_2\big] = 2\big[\mathsf{N}_2, \mathsf{N}_1\big] = 2\big[\mathsf{a}_2^{\mathsf{t}}\mathsf{a}_2, \mathsf{a}_1^{\mathsf{t}}\mathsf{a}_1\big] = \mathbf{0}$$

by (48.1).

Mechanics, but that it refers there to the *transformation properties* of the eigenstates, while (53) refers to their explicit *construction*.

Reading from (43) we have

$$\left.\begin{aligned} \mathsf{a}_1^{\mathsf{t}}\,|n_1,n_2) &= \sqrt{n_1+1}\,|n_1+1,n_2) \\ \mathsf{a}_1\,|n_1,n_2) &= \sqrt{n_1}\,|n_1-1,n_2) \end{aligned}\right\} \tag{54}$$

which (in the company of its $_2$ companion) enable one to take "hikes" of the form

$$|n_1,n_2)_{\text{initial}} \qquad |n_1,n_2)_{\text{final}}$$

These are the elemental steps in the jm notation standard to the quantum theory of angular momentum:

$$\left.\begin{aligned} \mathsf{a}_1^{\mathsf{t}}\,|j,m\rangle &= \sqrt{j+m+1}\,|j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle &&: \nearrow \\ \mathsf{a}_1\,|j,m\rangle &= \sqrt{j+m}\,|j-\tfrac{1}{2},m-\tfrac{1}{2}\rangle &&: \swarrow \\ \mathsf{a}_2^{\mathsf{t}}\,|j,m\rangle &= \sqrt{j-m+1}\,|j+\tfrac{1}{2},m-\tfrac{1}{2}\rangle &&: \nwarrow \\ \mathsf{a}_2\,|j,m\rangle &= \sqrt{j-m}\,|j-\tfrac{1}{2},m+\tfrac{1}{2}\rangle &&: \searrow \end{aligned}\right\} \tag{55}$$

The arrows on the right refer to the following diagram:

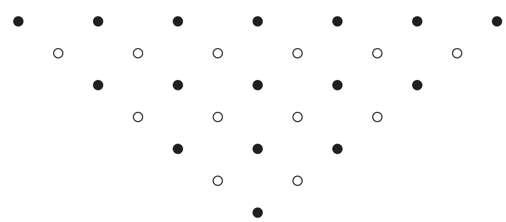

FIGURE 1: *Diagram of the jm values that enter into the description of angular momentum states. The value of j is constant on a row, and augments by $\frac{1}{2}$ when one ascends from one row to the next. The value of m augments by 1 when one steps to the right on any row. One has $m=-j$ on the left border, $m=+j$ on the right border, with $j=0,\frac{1}{2},1,\frac{3}{2},\ldots$ Integral point are indicated $\bullet$, half-integral points marked with a $\circ$.*

It follows that the action of $\mathsf{a}_1^{\mathsf{t}}\mathsf{a}_2$

$$\mathsf{a}_1^{\mathsf{t}}\mathsf{a}_2\,|j,m\rangle = \sqrt{(j+m+1)(j-m)}\,|j,m+1\rangle \quad : \searrow\!\nearrow$$

preserves j but augments m: $m \mapsto m+1$. In short, it executes a *step to the right* along the j-row. At the right end of the row one has the state

$$|j,+j\rangle = |2j,0) = \frac{1}{\sqrt{(2j)!(0)!}}(\mathsf{a}_1^{\mathsf{t}})^{2j}\,|0)$$

which is *killed* by $\mathsf{a}_1^{\mathsf{t}}\mathsf{a}_2$:

$$\mathsf{a}_1^{\mathsf{t}}\mathsf{a}_2\,|j,+j\rangle = 0 \quad\text{because}\quad \mathsf{a}_2|0) = 0$$

Similarly,

$$\mathsf{a}_2^{\mathsf{t}}\mathsf{a}_1\,|j,m\rangle = \sqrt{(j-m+1)(j+m)}\,|j,m-1\rangle \quad :$$

preserves j but diminishes m: $m \mapsto m-1$. In short, it executes a step to the left along the j-row. At the left end of the row one has the state

$$|j,-j\rangle = |0,2j) = \frac{1}{\sqrt{(0)!(2j)!}}(\mathsf{a}_2^{\mathsf{t}})^{2j}\,|0)$$

which is killed by $\mathsf{a}_2^{\mathsf{t}}\mathsf{a}_1$:

$$\mathsf{a}_2^{\mathsf{t}}\mathsf{a}_1\,|j,-j\rangle = 0 \quad\text{because}\quad \mathsf{a}_1|0) = 0$$

These remarks account for the action of the "ladder operators" $\mathsf{J}_\pm$, which Griffiths[20] (§4.3.1) calls "raising/lowering" operators, but which I would find it more natural to call "step right/left" operators.

Extending the preceding discussion now in a direction *not* encountered in the textbooks... we note first that such right/left stepping is achieved also by

$$\begin{aligned} &\mathsf{a}_2\mathsf{a}_1^{\mathsf{t}} \quad : \\ &\mathsf{a}_1\mathsf{a}_2^{\mathsf{t}} \quad : \end{aligned}$$

Climbing (straight) up/down can be accomplished

$$\begin{array}{lcl} \mathsf{a}_1^{\mathsf{t}}\mathsf{a}_2^{\mathsf{t}} \quad : & \text{or} & \mathsf{a}_2^{\mathsf{t}}\mathsf{a}_1^{\mathsf{t}} \quad : \\ \mathsf{a}_1\mathsf{a}_2 \quad : & \text{or} & \mathsf{a}_2\mathsf{a}_1 \quad : \end{array}$$

... which is to say: by appeal to resources that are *not available to the standard theory*.

Generating function techniques. ... were always favored by Schwinger, often used by him to good advantage. Let us, with him, look upon (53) as an invitation to construct the operator-valued object

$$\mathsf{G}(u) \equiv \sum_{j\triangledown m}\frac{1}{(j+m)!(j-m)!}(u_1\mathsf{a}_1^{\mathsf{t}})^{j+m}\,(u_2\mathsf{a}_2^{\mathsf{t}})^{j-m} \tag{56.1}$$

where the sum ranges over the $\triangledown$-population of jm-points shown in Figure 1. We have

$$= \sum_j \frac{1}{(2j)!}\big(u_1\mathsf{a}_1^{\mathsf{t}} + u_2\mathsf{a}_2^{\mathsf{t}}\big)^{2j}$$

which—because $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$—becomes

$$= e^{(u_1\mathsf{a}_1^{\mathsf{t}} + u_2\mathsf{a}_2^{\mathsf{t}})} \equiv e^{u\mathsf{a}^{\mathsf{t}}} \tag{56.2}$$

From

$$\mathsf{G}(u)|0) = \sum_{j\triangledown m} \frac{1}{\sqrt{(j+m)!(j-m)!}}(u_1)^{j+m}(u_2)^{j-m}|j,m\rangle \tag{57}$$

we discover the sense in which $\mathsf{G}(u)|0)$ is an "eigenstate generating function."

Look now to the inner product of $\mathsf{G}(u)|0)$ and $\mathsf{G}(v)|0)$:

$$(0|\mathsf{G}^{\mathsf{t}}(v)\mathsf{G}(u)|0) = (0|e^{v^*\mathsf{a}}e^{u\mathsf{a}^{\mathsf{t}}}|0) \tag{58}$$

The operators a and a^{t} *commute with their commutator*(s), so by a celebrated Campbell-Baker-Hausdorff identity due to Kermack & McCrae[34] we have

$$\begin{aligned} e^{v^*\mathsf{a}}e^{u\mathsf{a}^{\mathsf{t}}} &= e^{[v^*\mathsf{a},\,u\mathsf{a}^{\mathsf{t}}]}\cdot e^{u\mathsf{a}^{\mathsf{t}}}e^{v^*\mathsf{a}} \\ &= e^{(v_1^*u_1+v_2^*u_2)}\cdot e^{u\mathsf{a}^{\mathsf{t}}}e^{v^*\mathsf{a}} \end{aligned} \tag{59}$$

The operators are now ordered in such a way that we can use the facts that $|0)$ is (i) killed by annihilation operators and (ii) normalized...to obtain

$$\begin{aligned} (0|e^{v^*\mathsf{a}}e^{u\mathsf{a}^{\mathsf{t}}}|0) &= e^{(v_1^*u_1+v_2^*u_2)} \\ &= \sum_k \frac{1}{k!}(v_1^*u_1+v_2^*u_2)^k \\ &= \sum_j \frac{1}{(2j)!}(v_1^*u_1+v_2^*u_2)^{2j} \quad : \quad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \\ &= \sum_j \sum_{m=-j}^{m=+j} \frac{1}{(j+m)!(j-m)!}(v_1^*u_1)^{j+m}(v_2^*u_2)^{j-m} \\ &= \sum_{j\triangledown m}\sum_{j\triangledown m} \frac{(v_1^*)^{j+m}(v_2^*)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\delta_{jj}\delta_{mm}\frac{(u_1)^{j+m}(u_2)^{j-m}}{\sqrt{(j+m)!(j-m)!}} \end{aligned}$$

Bringing (57) to the left side of (58) we on the other hand have

$$(0|\mathsf{G}^{\mathsf{t}}(v)\mathsf{G}(u)|0) = \sum_{j\triangledown m}\sum_{j\triangledown m} \frac{(v_1^*)^{j+m}(v_2^*)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\langle jm|jm\rangle\frac{(u_1)^{j+m}(u_2)^{j-m}}{\sqrt{(j+m)!(j-m)!}}$$

The implication is that

$$\langle jm|jm\rangle = \delta_{jj}\delta_{mm} \tag{60}$$

Note that Schwinger's argument (of which I have presented a somewhat streamlined version) makes critical use of resources—the a-operators—that again are *not available to the standard theory*. And that it makes *no* use of the Kronecker product technology which, within the field-theoretic setting, led us to the entirely equivalent statements

$$(n_1,n_2|n_1,n_2) = \delta_{n_1n_1}\delta_{n_2n_2} \tag{61}$$

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[34] See equation (73.6) in Chapter 0 of ADVANCED QUANTUM TOPICS (2000).

In the text of an ancient seminar[35] I have described a method—taken from some unpublished Schwinger class notes—for constructing *simultaneously* the eigenvalues and eigenstates of $\mathsf{H}_{\text{oscillator}}$. One expects the technique to be adaptable to isotropic 2-dimensional oscillators, therefore to the quantum theory of angular momentum... so is not surprised to find discussion of precisely this idea in Schwinger's APPENDIX A. Look, by way of preparation, to the pattern of the argument in the simplest instance: one has, quite generally,

$$\mathsf{H} = \sum |n)E_n(n| \quad\Longrightarrow\quad \mathsf{U}(t) \equiv e^{-(i/\hbar)\mathsf{H}t} = \sum |n)e^{-(i/\hbar)E_n t}(n| \tag{62}$$

which for an oscillator becomes[36]

$$\begin{aligned}\mathsf{U}_{\text{osc}}(t) &= e^{-i\,\omega(\mathsf{a}^+\mathsf{a}+\frac{1}{2}\mathsf{I})\,t}\\ &= e^{-i\frac{1}{2}\omega t}\cdot e^{-i\,\omega(\mathsf{a}^+\mathsf{a})\,t}\end{aligned}$$

One can, with sufficient cleverness,[37] use $\big[\mathsf{a}, \mathsf{a}^+\big] = \mathsf{I}$ to bring operator on the right to *ordered* form

$$\begin{aligned}&= e^{-i\frac{1}{2}\omega t}\cdot {}_{\mathsf{a}^+}\Big[\exp\Big\{\big(e^{-i\omega t}-1\big)\,\mathsf{a}^+\mathsf{a}\Big\}\Big]_{\mathsf{a}}\\ &= e^{-i\frac{1}{2}\omega t}\cdot\sum_n e^{-in\,\omega t}\frac{(\mathsf{a}^+)^n}{\sqrt{n!}}\,\mathsf{P}_0\,\frac{(\mathsf{a})^n}{\sqrt{n!}}\end{aligned} \tag{63}$$

and to establish that

$$\mathsf{P}_0 \equiv {}_{\mathsf{a}^+}\Big[e^{-\mathsf{a}^+\mathsf{a}}\Big]_{\mathsf{a}}$$

- is a projection operator: $\mathsf{P}_0^2 = \mathsf{P}_0$;
- has unit trace (so projects onto a single state);
- projects onto a state killed by a: $\mathsf{a}\,\mathsf{P}_0 = \mathbf{0}$.

So Schwinger—magician that he is—writes $\mathsf{P}_0 \equiv |0)(0|$ and

$$|n) \equiv \frac{(\mathsf{a}^+)^n}{\sqrt{n!}}\,|0) \tag{64.1}$$

and observes that in (63) he has reproduced the "spectral resolution" design of (62), with

$$E_n = \hbar\omega\big(n+\tfrac{1}{2}\big) \tag{64.2}$$

By a modification of the generating function technique he has produced—at a single blow—*both the eigenvalues & the eigenfunctions* of $\mathsf{H}_{\text{oscillator}}$. His similar but more ambitious objective in APPENDIX A is to construct simultaneously the eigenvalues and eigenfunctions <u>both of J^2 and of J_3</u>.

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[35] "An operator ordering technique with quantum mechanical applications," (Reed College Physics Seminar of 12 October 1966), which can be found in COLLECTED SEMINARS 1963–1970. See more recently (92) in the notes[34] just cited.

[36] To avoid confusion of $^{\text{t}}$ with t we will, throughout this discussion, use $^+$ to denote the adjoint.

[37] Relevant tricks were the subject of that "ancient seminar."

Note first that we can—by (48.7), and very much to our advantage—look instead for the eigenvalues/eigenvectors of

$$\mathsf{N} = \mathsf{a}_1^+\mathsf{a}_1 + \mathsf{a}_2^+\mathsf{a}_2 \qquad \text{and} \qquad \mathsf{M} = \mathsf{a}_1^+\mathsf{a}_1 - \mathsf{a}_2^+\mathsf{a}_2$$

i.e., of $\mathsf{N}_1 = \mathsf{a}_1^+\mathsf{a}_1$ and $\mathsf{N}_2 = \mathsf{a}_2^+\mathsf{a}_2$; by this point we have, in effect, converted the angular momentum problem *into* the isotropic oscillator problem. It becomes therefore entirely natural, in the light of the preceding discussion, to introduce the unitary operator

$$\begin{aligned} \mathsf{V}(r,s) &= e^{i(r\,\mathsf{a}_1^+\mathsf{a}_1 + s\,\mathsf{a}_2^+\mathsf{a}_2)} &&: \quad r \text{ and } s \text{ real} \\ &= e^{ir\,\mathsf{a}_1^+\mathsf{a}_1}\cdot e^{is\,\mathsf{a}_2^+\mathsf{a}_2} &&: \quad \mathsf{a}_1\text{'s commute with } \mathsf{a}_2\text{'s} \end{aligned} \tag{65}$$

Drawing again upon the operator-ordering theorem that gave (63) we find

$$\mathsf{V}(r,s) = \sum_n e^{i(rn_1+sn_2)}\frac{(\mathsf{a}_1^+)^{n_1}(\mathsf{a}_2^+)^{n_2}}{\sqrt{n_1!\,n_2!}}\,\mathsf{P}_0\,\frac{(\mathsf{a}_1)^{n_1}(\mathsf{a}_2)^{n_2}}{\sqrt{n_1!\,n_2!}} \tag{66}$$
$$\mathsf{P}_0 \equiv |0,0)(0,0|$$

So we write

$$|n_1,n_2) \equiv \frac{(\mathsf{a}_1^+)^{n_1}(\mathsf{a}_2^+)^{n_2}}{\sqrt{n_1!\,n_2!}}\,|0,0) \quad : \quad \text{eigenvectors}$$

and have

$$\left.\begin{aligned} \mathsf{N}_1|n_1,n_2) &= n_1|n_1,n_2) \\ \mathsf{N}_2|n_1,n_2) &= n_2|n_1,n_2) \end{aligned}\right\} \quad : \quad \text{associated eigenvalues}$$

Therefore

$$\begin{aligned} \mathsf{N}\,|n_1,n_2) &= (n_1+n_2)\,|n_1,n_2) \\ \mathsf{M}\,|n_1,n_2) &= (n_1-n_2)\,|n_1,n_2) \end{aligned}$$

These statements place us again in position to reproduce equations (51)–(53), which provide

$$\mathsf{J}^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle \qquad \text{and} \qquad \mathsf{J}_3|jm\rangle = \hbar m|jm\rangle$$

with $\{j,m\}$ ranging on the $\triangledown$ of Figure 1.

The argument has supplied no information we did not already possess, but is of some *methodological* interest. We note that it has, once again, drawn upon resources that are *not available to the standard quantum theory of angular momentum*. And that the argument extends straightforwardly to the theory of N-dimensional oscillators; i.e., to the bosonic field theory of N-state systems. In the latter applications the symmetry group is $SU(N)$: the theory of angular momentum has been left behind.