ISOTROPIC OSCILLATOR & 2-DIMENSIONAL KEPLER PROBLEM IN THE PHASE SPACE FORMULATION OF QUANTUM MECHANICS

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Introduction. Let a mass point m be bound to the origin by an isotropic harmonic force:

$$\text{Lagrangian} = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \tfrac{1}{2}m\omega^2(x_1^2 + x_2^2)$$

As is well known, such an particle will trace/retrace an elliptical path. The specific figure of the ellipse (size, eccentricity, orientation, helicity) depends upon launch data, but all such ellipses are "concentric" in the sense that they have coincident centers. Similar remarks pertain to the $\boldsymbol{E}$-vector in an onrushing monochromatic lightbeam. An elegant train of thought, initiated by Stokes and brought to perfection by Poincaré (who were concerned with the phenomenon of optical polarization), leads to the realization that the population of such concentric ellipses (of given/fixed "size") can be identified with the points on the surface of a certain abstract sphere.[1] One is therefore not surprised to learn that, while $O(2)$ is an overt symmetry of the isotropic oscillator, $O(3)$ is present as a "hidden symmetry."

The situation is somewhat clarified when one passes to the Hamiltonian formalism

$$H = \tfrac{1}{2m}(p_1^2 + p_2^2) + \tfrac{1}{2}m\omega^2(x_1^2 + x_2^2) \tag{1}$$

where it emerges readily that

$$A_1 \equiv \tfrac{1}{m}\,p_1p_2 + m\omega^2\,x_1x_2 \tag{2.1}$$

$$A_2 \equiv \omega(x_1p_2 - x_2p_1) \tag{2.2}$$

$$A_3 \equiv \tfrac{1}{2m}(p_1^2 - p_2^2) + \tfrac{1}{2}m\omega^2(x_1^2 - x_2^2) \tag{2.3}$$

are constants of the motion

$$\big[H, A_1\big] = \big[H, A_2\big] = \big[H, A_3\big] = 0$$

[1] These developments are reviewed fairly exhaustively in "Ellipsometry" (1999).

and that

$$\begin{aligned}[A_1, A_2] &= 2\omega A_3\\ [A_2, A_3] &= 2\omega A_1\\ [A_3, A_1] &= 2\omega A_2\end{aligned}$$

which is to say: $\boldsymbol{L} \equiv \frac{1}{2\omega}\boldsymbol{A}$ possesses Poisson bracket properties that mimic those of angular momentum.[2]

The line of argument just sketched extends straightforwardly to quantum mechanics, and shows $O(3)$ to be a hidden symmetry also of the quantum isotropic oscillator . . . though in the latter context the notion of "concentric elliptical orbits" has dropped away (or at any rate retreated into the shadows).

In §5 of "Classical/quantum theory of 2-dimensional hydrogen" (February 1999) I note and exploit an extremely close connection between the isotropic oscillator and the 2-dimensional Kepler problem. The work of a thesis student, whose attempt "to expose the classical orbits that hide in the quantum shadows" provides one of my principal lines of motivation, has revived my interest in that connection, which will figure in these pages. But I have also a second line of motivation:

In work already cited[2] I establish that

> *It is futile to search for evidence of hidden symmetry written into the design of the quantum mechanical Green's function; it is in precisely that sense that such symmetry is "hidden."*

Hidden symmetry lives (classically) not in configuration space, but in phase space. I speculated that it would come more naturally into view if one elected to work within the Wigner/Weyl/Moyal "phase space formulation" of quantum mechanics.[3] My primary intent here is to explore the merit of that idea.

Review of the essentials of the phase space formalism. Weyl proposed a simple Fourier-analytic rule

$$\begin{aligned} A(x,p) &= \iint a(\alpha,\beta)\, e^{\frac{i}{\hbar}(\alpha p+\beta x)}\, d\alpha d\beta \\ &\quad\Big\downarrow \text{Weyl} \\ \mathbf{A} &= \iint a(\alpha,\beta)\, e^{\frac{i}{\hbar}(\alpha \mathbf{p}+\beta \mathbf{x})}\, d\alpha d\beta \end{aligned} \tag{3}$$

for associating classical observables (real-valued functions defined on phase space) with their quantum counterparts (self-adjoint operators linear that act

[2] I have borrowed here from (27/28) in "Jacobi's theta transformation and Mehler's formula: their interrelation, and their role in the quantum theory of angular momentum" (November 2000).

[3] The subject is reviewed in Chapter 2 (Weyl Transform and the Phase Space Formalism) of ADVANCED QUANTUM TOPICS (2000).

on the space of states). The "Weyl correspondence" is recommended by its many attractive properties . . . especially by this one:

$$\mathrm{tr}\,\mathbf{AB} = \tfrac{1}{h}\iint A(x,p)B(x,p)\,dxdp \tag{4}$$

The expected value $\mathbf{A}$-measurements performed on a system in state $|\psi)$ is standardly described

$$\langle\mathbf{A}\rangle = (\psi|\mathbf{A}|\psi) = \mathrm{tr}\,\mathbf{A}\boldsymbol{\rho}$$
$$\boldsymbol{\rho} \equiv |\psi)(\psi| \text{ is the "density operator"}$$

which by (4) means that we can write

$$\langle\mathbf{A}\rangle = \iint A(x,p)P_\psi(x,p)\,dxdp \tag{5}$$

with

$$hP_\psi(x,p) \xleftrightarrow[\text{Weyl}]{\hspace{2cm}} \boldsymbol{\rho} \tag{6}$$

In the case $\mathbf{A} \mapsto \mathbf{I}$ we then have

$$\iint P_\psi(x,p)\,dxdp = 1 \tag{7}$$

Equations (5) and (7) resemble equations that arise in *classical* statistical mechanics. But computation leads to the "Wigner distribution function "

$$\begin{aligned} P_\psi(x,p) &= \tfrac{1}{h}\int \psi^*(x-\tfrac{1}{2}\alpha)\,e^{-\frac{i}{\hbar}p\alpha}\,\psi(x+\tfrac{1}{2}\alpha)\,d\alpha \\ &= \tfrac{2}{h}\int \psi^*(x+\xi)\,e^{2\frac{i}{\hbar}p\xi}\,\psi(x-\xi)\,d\xi \end{aligned} \tag{8}$$

which can be shown to be distribution-like in all respects but one: it may assume *negative values* (so is sometimes called a "quasi-distribution"), and in this fact resides much—but by no means all!—that is most characteristically "strange" about quantum mechanics.[4]

Launch $|\psi)$ into dynamical motion: $|\psi)_0 \longrightarrow |\psi)_t = \mathbf{U}(t)|\psi)_0$. We have interest in the induced motion $P_\psi(x,p;0) \longrightarrow P_\psi(x,p;t)$. Working from

$$\psi(x,t) = \int G(x,t;x',0)\,\psi(x',0)\,dx'$$

we obtain

$$\begin{aligned} &P_\psi(x,p;t) \\ &\quad = \tfrac{2}{h}\iiint G^*(x+\xi,t;x',0)\,e^{2\frac{i}{\hbar}p\xi}\,G(x-\xi,t;x'',0)\psi^*(x',0)\psi(x'',0)\,d\xi dx'dx'' \end{aligned}$$

[4] In the "classical limit" (whatever that means) one expects to be able to demonstrate that "regions of negativity" evaporate—spontaneously.

which we bring to the form

$$P_\psi(x,p;t) = \iint K(x,p,t;y,q,0)P_\psi(y,q;0)\,dydq \tag{9}$$

by the following manipulations: write $x' = y+\zeta$ and $x'' = y-\zeta$ and, noting that $dx'dx'' = 2\,dyd\zeta$, obtain

$$\begin{aligned}
&P_\psi(x,p;t)\\
&\quad= \tfrac{2}{h}\iiint G^*(x+\xi,t;y+\zeta,0)\,e^{2\frac{i}{\hbar}p\,\xi}\,G(x-\xi,t;y-\zeta,0)\\
&\qquad\qquad\qquad\qquad\cdot 2\,\psi^*(y+\zeta,0)\psi(y-\zeta,0)\,d\xi dyd\zeta\\
&\quad= \tfrac{2}{h}\iiiint G^*(x+\xi,t;y+\zeta,0)\,e^{2\frac{i}{\hbar}p\,\xi}\,G(x-\xi,t;y-\zeta,0)\delta(\zeta-\zeta)\\
&\qquad\qquad\qquad\qquad\cdot 2\,\psi^*(y+\zeta,0)\psi(y-\zeta,0)\,d\xi dyd\zeta d\zeta
\end{aligned}$$

Use

$$\delta(\zeta-\zeta) = \tfrac{2}{h}\int e^{2\frac{i}{\hbar}q(\zeta-\zeta)}dq$$

to obtain

$$\begin{aligned}
&P_\psi(x,p;t)\\
&\quad= \tfrac{2}{h}\iiiint G^*(x+\xi,t;y+\zeta,0)\,e^{2\frac{i}{\hbar}(p\,\xi-q\zeta)}\,G(x-\xi,t;y-\zeta,0)\\
&\qquad\qquad\cdot 2\Big\{\tfrac{2}{h}\int \psi^*(y+\zeta,0)\,e^{2\frac{i}{\hbar}q\zeta}\,\psi(y-\zeta,0)\,d\zeta\Big\}\,d\xi d\zeta\cdot dydq
\end{aligned}$$

The implication is that

$$\begin{aligned}
&K(x,p,t;y,q,0)\\
&\quad= \tfrac{4}{h}\iint G^*(x+\xi,t;y+\xi,0)\,e^{2\frac{i}{\hbar}(p\,\xi-q\,\xi)}\,G(x-\xi,t;y-\xi,0)\,d\xi d\xi
\end{aligned} \tag{10}$$

which as $t\downarrow 0$ becomes

$$\begin{aligned}
&K(x,p,0;y,q,0)\\
&\quad= \tfrac{4}{h}\iint \delta(x+\xi-y-\xi)\,e^{2\frac{i}{\hbar}(p\,\xi-q\,\xi)}\,\delta(x-\xi-y+\xi)\,d\xi d\xi\\
&\quad= e^{-2\frac{i}{\hbar}q(x-y)}2\delta(2[x-y])\cdot\tfrac{2}{h}\int e^{2\frac{i}{\hbar}(p-q)\xi}\;d\xi\\
&\quad= \delta(x-y)\delta(p-q)
\end{aligned} \tag{11}$$

This result inspires confidence in the accuracy of (10).

The phase space analog of the Schrödinger equation arises as the Weyl transform of $i\hbar\partial_t\boldsymbol{\rho} = \big[\mathbf{H},\boldsymbol{\rho}\big]$ and is a fairly complicated affair:

$$\begin{aligned}\tfrac{\partial}{\partial t}P(x,p;t) &= \tfrac{2}{\hbar}\sin\Big\{\tfrac{\hbar}{2}\Big[\big(\tfrac{\partial}{\partial x}\big)_H\big(\tfrac{\partial}{\partial p}\big)_P - \big(\tfrac{\partial}{\partial x}\big)_P\big(\tfrac{\partial}{\partial p}\big)_H\Big]\Big\}H(x,p)P(x,p;t) \qquad (12.1)\\ &= \big\{H_xP_p - H_pP_x\big\}\\ &\quad -\tfrac{1}{3!}\big(\tfrac{\hbar}{2}\big)^2\big\{H_{xxx}P_{ppp}-3H_{xxp}P_{ppx}+3H_{xpp}P_{pxx}-H_{ppp}P_{xxx}\big\}+\cdots\\ &= \underbrace{[H,P]}_{\text{Poisson bracket}} + \text{terms of order }\hbar^2 \qquad (12.2)\\ &= \iint \mathcal{K}(x,p,t;y,q,0)P_\psi(y,q;t)\,dydq \qquad (12.3)\\ &\qquad \mathcal{K}(x,p,t;y,q,0) \equiv \tfrac{\partial}{\partial t}K(x,p,t;y,q,0)\end{aligned}$$

Major simplifications do, however, arise in cases where $H(x,p)$ is too simple to support high-order differentiation. We turn now to just such an instance.

Simple harmonic oscillator.[5] The normalized oscillator eigenfunctions can be described

$$\begin{aligned}\psi_n(x) &= \big(\tfrac{2m\omega}{h}\big)^{\frac{1}{4}}\tfrac{1}{\sqrt{2^n n!}}\,e^{-\frac{1}{2}(m\omega/\hbar)x^2}H_n\big(\sqrt{\tfrac{m\omega}{\hbar}}\,x\big)\\ &= \big(\tfrac{2m\omega}{h}\big)^{\frac{1}{4}}\tfrac{1}{\sqrt{2^n n!}}\,e^{-\frac{1}{2}\mathrm{x}^2}H_n(\mathrm{x}) \qquad (13)\end{aligned}$$

where $\mathrm{x} \equiv \sqrt{\frac{m\omega}{\hbar}}\,x$ is a "dimensionless length" variable, and where

$$H_n(y) \equiv e^{+y^2}\big(-\tfrac{d}{dy}\big)^n e^{-y^2}$$

are Hermite polynomials:

$$\begin{aligned}H_0(y) &= 1\\ H_1(y) &= 2y\\ H_2(y) &= 4y^2-2\\ H_3(y) &= 8y^3-12y\\ H_4(y) &= 16y^4-48y^2+12\\ &\vdots\end{aligned}$$

[5] For all missing details see pages 6–9 in an essay previously cited.[2] My notational conventions there and here are those of Chapter 24 in Spanier & Oldham's *Atlas of Functions*. Closely related material can be found on pages 11 and 19–22 of the class notes already mentioned,[3] but beware: there I adopt the other (monic) definition of the Hermite polynomials.

From (8) we now obtain

$$
\begin{aligned}
P_n(x,p) &= \tfrac{2}{\hbar}\int \left(\tfrac{2m\omega}{h}\right)^{\frac{1}{4}} \tfrac{1}{\sqrt{2^n n!}}\, e^{-\frac{1}{2}(x+\xi)^2} H_n(x+\xi)\, e^{2ix\,\xi} \\
&\qquad\cdot \left(\tfrac{2m\omega}{h}\right)^{\frac{1}{4}} \tfrac{1}{\sqrt{2^n n!}}\, e^{-\frac{1}{2}(x-\xi)^2} H_n(x-\xi)\cdot\sqrt{\tfrac{\hbar}{m\omega}}\,d\xi \\
&= \tfrac{2}{\hbar}\cdot\tfrac{1}{2^n n!\sqrt{\pi}}\int e^{-\frac{1}{2}(x+\xi)^2-\frac{1}{2}(x-\xi)^2+2ip\,\xi} H_n(x+\xi)H_n(x-\xi)d\xi
\end{aligned}
$$

where $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}\,\xi$ is again a "dimensionless length" and $p \equiv \sqrt{\frac{1}{\hbar m\omega}}\,p$ is a "dimensionless momentum;" We will have immediate need also of the "dimensionless energy" $\mathcal{E} \equiv 2(p^2+x^2)$. To achieve $P(x,p)dxdp = P(x,p)dxdp$ we write

$$
\begin{aligned}
P_n(x,p) &= \hbar P_n(x,p) \\
&= \tfrac{1}{\pi}\int \psi_n(x+\xi)e^{2ip\,\xi}\psi_n(x-\xi)\,d\xi
\end{aligned}
\tag{14.1}
$$

where the "dimensionless eigenfunctions"

$$
\psi_n(x) \equiv \tfrac{1}{\sqrt{2^n n!\sqrt{\pi}}}e^{-\frac{1}{2}x^2}H_n(x) \tag{14.2}
$$

are orthonormal in the sense that $\int \psi_m(x)\psi_n(x)\,dx = \delta_{mn}$. These dimensionless variants of (8) and (13) are computationally much more convenient. We note in passing (in preparation for work to come) that from $\frac{1}{\pi}\int e^{2ip\,\xi}\,dp = \delta(\xi)$ it follows that if we write

$$
P_{mn}(x,p) \equiv \tfrac{1}{\pi}\int \psi_m(x+\xi)e^{2ip\,\xi}\psi_n(x-\xi)\,d\xi \tag{15.1}
$$

then

$$
\int P_{mn}(x,p)\,dp = \psi_m(x)\,\psi_n(x) \tag{15.2}
$$

$$
\iint P_{mn}(x,p)\,dpdx = \delta_{mn} \tag{15.3}
$$

With the assistance of *Mathematica* we now compute

$$
\begin{aligned}
P_0(x,p) &= \tfrac{1}{\pi}e^{-\frac{1}{2}\mathcal{E}} \\
P_1(x,p) &= \tfrac{1}{\pi}e^{-\frac{1}{2}\mathcal{E}}(\mathcal{E}-1) \\
P_2(x,p) &= \tfrac{1}{\pi}e^{-\frac{1}{2}\mathcal{E}}(\tfrac{1}{2}\mathcal{E}^2-2\mathcal{E}+1) \\
P_3(x,p) &= \tfrac{1}{\pi}e^{-\frac{1}{2}\mathcal{E}}(\tfrac{1}{6}\mathcal{E}^3-\tfrac{3}{2}\mathcal{E}^2+3\mathcal{E}-1) \\
P_4(x,p) &= \tfrac{1}{\pi}e^{-\frac{1}{2}\mathcal{E}}(\tfrac{1}{24}\mathcal{E}^4-\tfrac{2}{3}\mathcal{E}^3+3\mathcal{E}^2-4\mathcal{E}+1) \\
&\;\;\vdots
\end{aligned}
$$

without difficulty, but to develop the general formula we need more firepower. Mehler's formula, pressed into service as a generating function,[6] comes to our rescue: into

$$\begin{aligned}\sum_{n=0}^{\infty}\tau^n P_n(x,p)\\ &= \int e^{-\frac{1}{2}(x+\xi)^2-\frac{1}{2}(x-\xi)^2+2ip\xi}\Big\{\tfrac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\tfrac{\tau^n}{2^n n!}H_n(x+\xi)H_n(x-\xi)\Big\}d\xi\end{aligned}$$

we introduce Mehler's formula

$$\sum_{n=0}^{\infty}\tfrac{\tau^n}{2^n n!}H_n(x)H_n(x) = \frac{1}{\sqrt{1-\tau^2}}\exp\left\{\frac{2xy\tau-(x^2+y^2)\tau^2}{1-\tau^2}\right\} \tag{16}$$

and ask *Mathematica* to perform the integral: we get

$$\begin{aligned}&= \tfrac{1}{\pi}\cdot\tfrac{1}{1+\tau}\exp\left\{\tfrac{\tau-1}{\tau+1}(p^2+x^2)\right\}\\ &= \tfrac{1}{\pi}\cdot e^{-(p^2+x^2)}\tfrac{1}{1+\tau}\exp\left\{\tfrac{2\tau}{\tau+1}(p^2+x^2)\right\}\\ &= \tfrac{1}{\pi}\cdot e^{-\frac{1}{2}\mathcal{E}}\cdot\tfrac{1}{1+\tau}\exp\left\{\tfrac{\tau}{\tau+1}\mathcal{E}\right\}\end{aligned}$$

But

$$\tfrac{1}{1-\tau}\exp\left\{-\tfrac{\tau}{1-\tau}\,y\right\} = \sum_{n=0}^{\infty}\tau^n L_n(y)$$

serves[7] to generate the Laguerre polynomials

$$\begin{aligned}L_0(y) &= +1\\ L_1(y) &= -(y-1)\\ L_2(y) &= +(\tfrac{1}{2}y^2-2y+1)\\ L_3(y) &= -(\tfrac{1}{6}y^3-\tfrac{3}{2}y^2+3y-1)\\ L_4(y) &= +(\tfrac{1}{24}y^4-\tfrac{2}{3}y^3+3y^2-4y+1)\\ &\ \vdots\end{aligned}$$

So we have

$$P_n(x,p) = \tfrac{1}{\pi}(-)^n e^{-\frac{1}{2}\mathcal{E}}L_n(\mathcal{E}) \tag{17}$$

It was, by the way, to achieve notational simplicity at this point that $\mathcal{E}$ was defined to be four times bigger than might have seemed natural:

$$\mathcal{E} \equiv \mathbf{4}\cdot\tfrac{1}{\hbar\omega}\left\{\tfrac{1}{2m}p^2+\tfrac{1}{2}m\omega^2x^2\right\}$$

[6] See (1.1) in the essay[2] already twice cited. We are reminded in a footnote on page 11 that it was, in fact, *as a generating function* that Watson *preferred to regard* Mehler's formula.

[7] See, for example, Chapter 23 in Spanier & Oldham.

The "time-independent phase space Schrödinger equation" can be made the basis of an alternative derivation of (17).[8]

Mehler's formula is more commonly used not as a generating function, but to construct a description of the oscillator Green's function

$$G(x,t;y,0)=\sqrt{\tfrac{m\,\omega}{ih\sin\omega\,t}}\;\exp\Big\{\tfrac{i}{\hbar}\tfrac{m\,\omega}{2\sin\omega\,t}\big[(x^2+y^2)\cos\omega\,t-2xy\big]\Big\}\tag{18}$$

Returning with this information to (10), we observe that

$$\begin{aligned}G^*(x+\xi,t;y+\xi,0)\,e^{2\frac{i}{\hbar}(p\,\xi-q\,\xi)}\,G(x-\xi,t;y-\xi,0)&\\ =\tfrac{m\omega}{h\sin\omega t}\exp\Big\{2\tfrac{i}{\hbar}\Big[\tfrac{m\omega}{\sin\omega t}\big\{(x\,\xi+y\,\xi)-(x\,\xi+y\,\xi)\cos\omega t\big\}+(p\,\xi-q\,\xi)\Big]\Big\}&\end{aligned}$$

and obtain

$$\begin{aligned}K(x,p,t;y,q,0)&=\tfrac{4m\omega}{h^2\sin\omega t}\int\exp\Big\{2\tfrac{i}{\hbar}\Big[\tfrac{m\omega}{\sin\omega t}\big(y-x\cos\omega t\big)+p\Big]\xi\Big\}\,d\xi\\&\qquad\cdot\int\exp\Big\{2\tfrac{i}{\hbar}\Big[\tfrac{m\omega}{\sin\omega t}\big(x-y\cos\omega t\big)-q\Big]\xi\Big\}\,d\xi\\&=\tfrac{4m\omega}{h^2\sin\omega t}\big(\tfrac{h}{2}\big)^2\delta\big(\tfrac{m\omega}{\sin\omega t}[y-x\cos\omega t]+p\big)\delta\big(\tfrac{m\omega}{\sin\omega t}[x-y\cos\omega t]-q\big)\\&=\delta\big(p+\tfrac{m\omega}{\sin\omega t}[y-x\cos\omega t]\big)\delta\big(x-y\cos\omega t-\tfrac{1}{m\omega}q\,\sin\omega t\big)\\&=\delta\big(x-y\cos\omega t-\tfrac{1}{m\omega}q\,\sin\omega t\big)\delta\big(p-q\cos\omega t+m\omega\,y\sin\omega t\big)\end{aligned}\tag{19}$$

This is a charming result, for it describes a *moving spike which precisely tracks (on the $\{x,p\}$-plane) the classical motion*

$$\left.\begin{aligned}x(t)&=y\cos\omega t+\tfrac{1}{m\omega}\,q\,\sin\omega t\\p(t)&=q\cos\omega t-\;m\omega\,y\sin\omega t\end{aligned}\right\}\tag{20}$$

of an oscillator which at $t=0$ stood at the phase point $\{y,q\}$. That point traces the iso-energetic ellipse on which

$$\mathcal{E}\equiv 4\cdot\tfrac{1}{\hbar\omega}\big\{\tfrac{1}{2m}q^2+\tfrac{1}{2}m\omega^2y^2\big\}\quad:\quad\text{constant}$$

It becomes now elegantly clear why the Wigner distributions associated with oscillator eigenstates *do not move*:

$$\begin{aligned}P_n(x,p;t)&=\iint K(x,p,t;y,q,0)P_n(y,q)\,dydq\\&=P_n\big(y\cos\omega t+\tfrac{1}{m\omega}\,q\,\sin\omega t,q\cos\omega t-\;m\omega\,y\sin\omega t\big)\\&=P_n(y,q)\qquad\begin{cases}\text{because } P_n \text{ is a function of } \mathcal{E}(y,q)\\\text{and } \mathcal{E} \text{ is a constant of the motion}\end{cases}\end{aligned}$$

---

[8] See pages 19–20 in the class notes previously mentioned.[3]

Similarly static—and for the same reason—are the Wigner distributions

$$P(x,p)=\sum_n p_n P_n(x,p)\quad:\quad \text{all } p_n\geqslant 0 \text{ and } \textstyle\sum_n p_n=1$$

that describe "mixtures" of eigenstates. In orthodox quantum mechanics even energy eigenstates "move" in the sense that they "buzz harmonically"

$$|\psi_n)_0\longrightarrow|\psi_n)_t=e^{-\frac{i}{\hbar}E_nt}|\psi_n)_0$$

Notice that such buzzing is, by the design (8) of $P_\psi(x,p)$, surpressed in the phase space formalism.

To illustrate the quantum *dynamics* of a harmonic oscillator it has become traditional to look to the back-and-forth sloshing of a Gaussian wavepacket.[9] Let

$$\begin{aligned}\psi(x,0)&=\psi_0(x-a)\ :\ \text{displaced groundstate}\\&=\pi^{-\frac{1}{4}}e^{-\frac{1}{2}(x-a)^2}\\&=\sum_n e^{-\frac{1}{4}a^2}\frac{a^n}{\sqrt{2^nn!}}\psi_n(x)\end{aligned}\tag{21.1}$$

and note in passing that $\sum_n\big(e^{-\frac{1}{4}a^2}\frac{a^n}{\sqrt{2^nn!}}\big)^2=1$. *Mathematica* supplies

$$P(x,p)=\tfrac{1}{\pi}e^{-(x-a)^2-p^2}\tag{21.2}$$

Had we proceeded not from a displaced groundstate but from some more general initial Gaussian

$$\psi(x,0)=\pi^{-\frac{1}{4}}\frac{1}{\sqrt{\sigma}}e^{-\frac{1}{2}(\frac{x-a}{\sigma})^2}\tag{22.1}$$

we would have obtained

$$P(x,p)=\tfrac{1}{\pi}e^{-(\frac{x-a}{\sigma})^2-\sigma^2p^2}\tag{22.2}$$

which give back (13) in the case $\sigma=1$. In dimensionless variables the equations (20) read

$$\left.\begin{aligned}x(t)&=y\cos\omega t+q\sin\omega t\\p(t)&=q\cos\omega t-y\sin\omega t\end{aligned}\right\}\ :\ (x+ip)=e^{-i\omega t}(y+iq)\tag{23}$$

which in collaboration with (11) tell us that—in these cases as in all cases—when we "turn on dynamical time" (See Figure 1)

$$P(x,p;t)\text{ rotates }\circlearrowright\text{ rigidly}\tag{24}$$

[9] The tradition was inaugurated by Schrödinger himself, whose short note on the subject appeared in Die Naturwissenschaften **28**, 664 (1926). For an English translation see "The continuous transition from micro- to macro-mechanics," *Collected Papers on Wave Mechanics* (1982), pages 41–44.

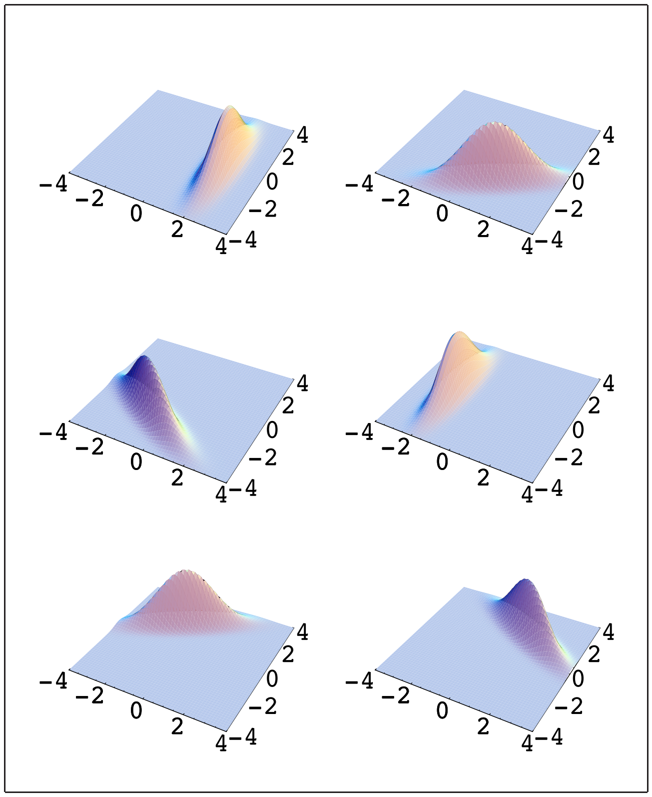

FIGURE 1: *Frames from a film (to be read like a book) illustrating the dynamical rigid ↻ rotation (24) of* $P(x,p;t)$. *The surface plotted is taken from (22.2) with* $a=2$ *and* $\sigma=\frac{1}{2}$.

In the textbooks,[10] as in Schrödinger's original paper, one usually encounters only the "projective shadow"

$$|\psi(x,t)|^2 = \int P(x,p;t)\,dp \quad : \quad \text{marginal distribution}$$

of this pretty result: it is, of course, the shadow that does the "sloshing," while $P(x,p;t)$ itself "twirls." We have been brought back again to the "reference circle" met in introductory physics courses.[11] It is clear from the figure that the marginal distribution will slosh *rigidly* only in the case $\sigma=1$ (displaced ground state) when the Gaussian mountain has circular cross-section; in other cases it progresses through phases thin/fat/thin/fat each cycle (animation makes the

[10] See, for example, Schiff, *Quantum Mechanics* (3rd edition 1968), p.74.
[11] See Halliday, Resnik & Walker, *Fundamentals of Physics* (5th edition 1997), §16-7.

point vivid). In the case shown, *Mathematica* informs us that

$$|\psi(x,t)|^2 = \sqrt{8/\pi}\,\frac{1}{\sqrt{17-15\cos 2\omega t}}\exp\Big\{-\frac{4(x-2\cos\omega t)^2}{\cos^2\omega t+16\sin^2\omega t}\Big\}$$

and that the *time-averaged* marginal distribution, though an object of physical interest, is difficult to compute (except numerically).

The preceding discussion is susceptible to this criticism: it treats a situation so specialized as to mask a fact which the phase space formalism serves to make plainly evident:

Quantum motion is an interference effect

The displaced ground state was seen at (21.1) to present a very particular superposition of *all* eigenstates; to expose the simple essence of the point at issue, let us look instead to an *arbitrary* superposition of *only two* eigenstates. Taking those (orthogonal) energy eigenstates to be $|m)$ and $|n)$, form

$$|\psi) \equiv \cos\alpha\cdot|m) + \sin\alpha\cdot|n)$$

where α is a "mixing angle," introduced to insure that

$$(\psi|\psi) = \cos^2\alpha + \sin^2\alpha = 1 \quad : \quad \text{all } \alpha$$

What—in the case of an oscillator—can we say of the P_ψ associated with such a superposition? Working from (8) we readily obtain

$$\begin{aligned} P_\psi(x,p) &= \cos^2\alpha\cdot P_m(x,p) + \sin^2\alpha\cdot P_n(x,p) \\ &\quad + \underbrace{2\cos\alpha\sin\alpha\cdot\int\psi_m(x+\xi)\,\cos\big\{2p\xi\big\}\,\psi_n(x-\xi)\,d\xi}_{\text{interference term, the only term that moves}} \end{aligned} \tag{25}$$

Note in this connection that the motion of the interference term does not place the normalization of $P_m(x,p)$ at risk, since by (15.3)

$$\iint P_m(x,p)\,dxdp = \cos^2\alpha\cdot 1 + \sin^2\alpha\cdot 1 + 2\cos\alpha\sin\alpha\cdot 0 = 1$$

One could probably use generating function techniques to work out a general description of $\big\{\text{interference term}\big\}_{mn} = \big\{\text{interference term}\big\}_{nm}$, but for the moment I am content to look at a single illustrate case: set $m = 2$, $n = 3$ and obtain

$$\begin{aligned} P_\psi(x,p) &= \tfrac{1}{\pi}e^{-\frac{1}{2}\mathcal{E}}\Big\{\cos^2\alpha\cdot(\tfrac{1}{2}\mathcal{E}^2 - 2\mathcal{E}+1) + \sin^2\alpha\cdot(\tfrac{1}{6}\mathcal{E}^3 - \tfrac{3}{2}\mathcal{E}^2 + 3\mathcal{E} - 1)\Big\} \\ &\quad + 2\cos\alpha\sin\alpha\cdot\tfrac{1}{\pi}e^{-\frac{1}{2}\mathcal{E}}\Big\{\sqrt{\tfrac{8}{3}}\,\underset{\uparrow\,\text{symmetry-breaking factor}}{x}\,(\tfrac{1}{2}\mathcal{E}^2 - 3\mathcal{E} + 3)\Big\} \end{aligned} \tag{26}$$

which is illustrated in Figure 2. Projection yields a marginal distribution

$$\int P(x,p;t)\,dp = |\cos\alpha\cdot\psi_2(x) + \sin\alpha\cdot\psi_3(x)\,e^{-i\omega t}|^2$$

which has evidently acquired a complicated undulatory motion.

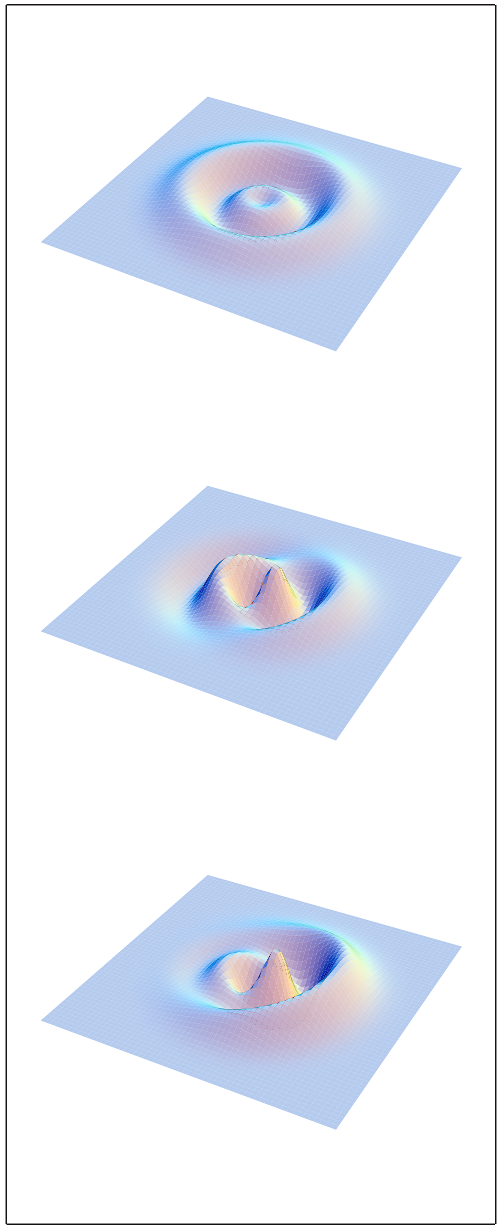

FIGURE 2: *At top is the rotationally symmetric (therefore static) component of the* $P_\psi(x,p)$ *of (26); in the middle is the "interference term" which, because of its lateral asymmetry, does sense the ↻ circulation of phase points; at bottom is their superposition. Equal weighting has here been assumed:* $\cos\alpha = \sin\alpha = 1/\sqrt{2}$.

Completing an analogy. In view of the formal similarity of

$$\psi(x,t)=\int G(x,t;y,0)\,\psi(y,0)\,dy$$

and

$$P_\psi(x,p;t)=\iint K(x,p,t;y,q,0)P_\psi(y,q;0)\,dydq$$

—which in their different ways say the same thing—it might seem reasonable to ask for a function $\mathcal{S}(x,p,t;y,q,0)$ that stands to $K(x,p,t;y,q,0)$ more or less as the classical action

$$S(x,t;y,0)=\tfrac{m\omega}{2\sin\omega t}\big[(x^2+y^2)\cos\omega t-2xy\big]$$

was seen at (18) to stand to

$$G(x,t;y,0)=\sqrt{\tfrac{i}{\hbar}\tfrac{\partial^2 S}{\partial x\partial y}}\,e^{\frac{i}{\hbar}S(x,t;y,0)}$$

This expectation is strengthened by the description at (10) of a very close interrelationship between K and G, but somewhat dimmed by the observation at (19) that K has a very singular design:

$$\begin{aligned}K(x,p,t;y,q,0)&=\delta\big(x-y\cos\omega t-\tfrac{1}{m\omega}q\sin\omega t\big)\delta\big(p-q\cos\omega t+m\omega\,y\sin\omega t\big)\\&=\tfrac{1}{\hbar}\,\delta\big(x-y\cos\omega t-q\sin\omega t\big)\delta\big(p-q\cos\omega t+y\sin\omega t\big)\\&\equiv\tfrac{1}{\hbar}\,K(x,p,t;y,q,0)\end{aligned}\tag{27}$$

But in Gaussian representation the δ-function reads

$$\delta(x-a)=\lim_{\epsilon\downarrow 0}\tfrac{1}{\sqrt{\pi\epsilon}}\,e^{-(x-a)^2/\epsilon}\tag{28}$$

and it becomes in this light semi-natural to write

$$K(x,p,t;y,q,0)=\lim_{\epsilon\downarrow 0}\tfrac{1}{\pi\epsilon}\,e^{-\mathcal{S}/\epsilon}\tag{29}$$

with

$$\begin{aligned}\mathcal{S}(x,p,t;y,q,0)&\equiv\big(x-y\cos\omega t-q\sin\omega t\big)^2+\big(p-q\cos\omega t+y\sin\omega t\big)^2\\&=\tfrac{1}{\hbar}\cdot\tfrac{2}{\omega}\Big\{\tfrac{1}{2m}\big(p-q\cos\omega t+m\omega\,q\sin\omega t\big)^2\\&\qquad\qquad+\tfrac{1}{2}m\omega^2\big(x-y\cos\omega t-\tfrac{1}{m\omega}q\sin\omega t\big)^2\Big\}\\&\equiv\tfrac{1}{\hbar}\cdot\mathcal{S}(x,p,t;y,q,0)\end{aligned}\tag{30}$$

In rough mimicry of Hamilton's principle we require

$$\mathcal{S}=\text{extremum}\tag{31}$$

(the extremum is here necessarily a minimum) and recover (23):

$$\begin{aligned} x(t) &= y\cos\omega t + q\sin\omega t \\ p(t) &= q\cos\omega t - y\sin\omega t \end{aligned}$$

Richard Crandall, in conversation many years ago, has reported to me that Richard Feynman (of whom Crandall was then a student) once remarked (semi-facetiously?) that *any* physical statement of the form

$$A_1 = B_1 \text{ and } A_2 = B_2 \text{ and } \ldots \quad : \quad A\text{'s and } B\text{'s real}$$

can be cast as a "variational principle:" one has only to write

$$(A_1 - B_1)^2 + (A_2 - B_2)^2 + \cdots = \text{minimum}$$

The variational principle (31) is a swindle in precisely that sense: one must *"know the answer" before one can even write it down*, and the writing is then almost pointless. Almost, but not quite . . . for the introduction of the classical "answer" into $\mathcal{S}$ does lead *via* (29) back to the *quantum* physics of the oscillator. I discuss this point more thoroughly in a moment.

A question natural to ask—but which I will not at this point digress to explore—is this: Is there a sense in which $\mathcal{S}(x, p, t; y, q, 0)$ enters as a "natural object" into the classical mechanics of the oscillator?

Recovery of Green's function from its phase space counterpart. Wigner/Weyl gave us

$$\psi(x) \xrightarrow[\text{Wigner/Weyl}]{} P_\psi(x,p)$$

but the reverse procedure

$$\psi(x) \xleftarrow[\text{Beck}]{} P_\psi(x,p)$$

was, so far as I am aware, first described in unpublished work by Mark Beck. Beck's construction[12] yields

$$\begin{aligned} \psi(x) &= [\psi^*(a)]^{-1} \cdot \int P_\psi(\tfrac{x+a}{2}, p)e^{\frac{i}{\hbar}p(x-a)}dp \\ &\downarrow \\ &= [\psi^*(0)]^{-1} \cdot \int P_\psi(\tfrac{x}{2}, p)e^{\frac{i}{\hbar}p\,x}dp \quad \text{in the special case } a = 0 \end{aligned}$$

on the assumption that

$$\int P(a,p)\,dp = \psi^*(a)\psi(a) \neq 0$$

[12] See page 13 in the class notes previously cited.[3]

Look back again, in this light, to (10):

$$
\begin{aligned}
&K(x,p,t;y,q,0)\\
&\quad=\tfrac{4}{h}\iint G^*(x+\xi,t;y+\xi,0)\,e^{2\frac{i}{\hbar}(p\,\xi-q\,\xi)}\,G(x-\xi,t;y-\xi,0)\,d\xi d\xi
\end{aligned}
$$

By Fourier transformation we have

$$
\begin{aligned}
&\iint K(x,p,t;y,q,0)\,e^{-2\frac{i}{\hbar}(p\,\zeta-q\,\zeta)}\,dpdq\\
&\quad= h\iint G^*(x+\xi,t;y+\xi,0)\,\delta(\xi-\zeta)\delta(\xi-\zeta)\,G(x-\xi,t;y-\xi,0)\,d\xi d\xi\\
&\quad= h\,G^*(x+\zeta,t;y+\zeta,0)\,G(x-\zeta,t;y-\zeta,0)
\end{aligned}
$$

Select points a and b at which $G^*(a,t;b,0)G(a,t;b,0)\neq 0$. Set $\zeta=a-x$ and $\zeta=b-y$ to obtain

$$
\begin{aligned}
&h\,G^*(a,t;b,0)\,G(2x-a,t;2y-b,0)\\
&\qquad\qquad=\iint K(x,p,t;y,q,0)\,e^{-2\frac{i}{\hbar}[p\,(a-x)-q(b-y)]}\,dpdq
\end{aligned}
$$

which by notational adjustment $2x-a\mapsto x$, $2y-b\mapsto y$ becomes

$$
\begin{aligned}
G(x,t;y,0)&=[hG^*(a,t;b,0)]^{-1}\cdot\iint K\big(\tfrac{x+a}{2},p,t;\tfrac{y+b}{2},q,0\big)\,e^{\frac{i}{\hbar}[p\,(x-a)-q(y-b)]}\,dpdq\\
&\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (32.1)\\
&=[hG^*(0,t;0,0)]^{-1}\cdot\iint K\big(\tfrac{x}{2},p,t;\tfrac{y}{2},q,0\big)\,e^{\frac{i}{\hbar}[p\,x-q\,y]}\,dpdq
\end{aligned}
$$

The prefactors are determined to within phase factors by the statements

$$
\begin{aligned}
|hG(a,t;b,0)|^2&=\iint K\big(a,p,t;b,q,0\big)\,dpdq\\
&\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (32.2)\\
|hG(0,t;0,0)|^2&=\iint K\big(0,p,t;0,q,0\big)\,dpdq\quad\text{in the special case } a=b=0
\end{aligned}
$$

and the phase factors can be extracted from the requirement that

$$
\lim_{t\downarrow 0} G(x,t;y,0)=\delta(x-y) \tag{32.3}
$$

where the prefactors are, in effect, normalization constants, fixed to within an arbitrary phase factor.

If the $K(x,p,t;y,q,0)$ appropriate to a quantum system $\mathfrak{S}$ were available as a point of departure, then one could in principle use (32) to construct

$G(x,t;y,0)$, and if additionally one could bring the Green's function to the form

$$G(x,t;y,0) = \sum_n e^{-\frac{i}{\hbar}E_n t}\psi_n(x)\psi_n^*(y)$$

then one could simply *read off* the solutions of the associated time-independent Schrödinger equation. In favorable cases such a program can actually be carried to completion . . . as I now demonstrate:

Uncommon approach to the quantization of an oscillator. Quantization of the oscillator—first accomplished by Planck almost exactly a century ago (I write on 8 December 2000; Planck's radiation formula and its derivation were announced on 14 December 1900)—marks the birthplace of quantum mechanics. The oscillator, for reasons that can be attributed mainly to the quadraticity of its Hamiltonian, is an exceptionally accommodating system, and its quantization can be/has been approached in a great variety of ways. To that long list I add now another.

We were led at (27) to a quantum mechanical statement

$$\begin{aligned} &K_{\text{osc}}(x,p,t;y,q,0) \\ &\quad = \delta\big(x - y\cos\omega t - \tfrac{1}{m\omega}q\sin\omega t\big)\delta\big(p - q\cos\omega t + m\omega\, y\sin\omega t\big) \end{aligned} \tag{33}$$

that—remarkably—contains no $\hbar$, and that can be read as a description of the *classical* motion of the harmonically driven phase point which at time $t = 0$ resided at $\{y, q\}$. Let us suppose that (33)—though a statement which we obtained as a result of long quantum mechanical analysis (involving steps which I propose now, in effect, to reverse)—has been <u>given</u>. Returning with (33) to (32.1) we have

$$\begin{aligned} &hG_{\text{osc}}(x,t;y,0)G^*_{\text{osc}}(0,t;0,0) \\ &\quad = \iint \delta\big(\tfrac{x}{2} - \tfrac{y}{2}\cos\omega t - \tfrac{1}{m\omega}q\sin\omega t\big) \\ &\qquad\qquad \cdot\,\delta\big(p - q\cos\omega t + m\omega\,\tfrac{y}{2}\sin\omega t\big)\, e^{\frac{i}{\hbar}[px-qy]}\,dpdq \\ &\quad = \int \tfrac{m\omega}{\sin\omega t}\delta\Big(q - \tfrac{m\omega(x-y\cos\omega t)}{2\sin\omega t}\Big)\exp\Big\{\tfrac{i}{\hbar}\big[x(q\cos\omega t - m\omega\tfrac{y}{2}\sin\omega t) - qy\big]\Big\}\,dq \\ &\quad = \tfrac{m\omega}{\sin\omega t}\exp\Big\{\tfrac{i}{\hbar}\tfrac{m\omega}{2\sin\omega t}\big[(x^2+y^2)\cos\omega t - 2xy\big]\Big\} \end{aligned}$$

Therefore

$$h|G_{\text{osc}}(0,t;0,0)|^2 = \tfrac{m\omega}{\sin\omega t} \quad\Longrightarrow\quad [G^*_{\text{osc}}(0,t;0,0)]^{-1} = \sqrt{\tfrac{h\sin\omega t}{m\omega}}\cdot e^{i\alpha}$$

and we have

$$\begin{aligned} G_{\text{osc}}(x,t;y,0) &= e^{i\alpha}\sqrt{\tfrac{m\omega}{h\sin\omega t}}\exp\Big\{\tfrac{i}{\hbar}\tfrac{m\omega}{2\sin\omega t}\big[(x^2+y^2)\cos\omega t - 2xy\big]\Big\} \\ &\Downarrow \\ &= e^{i\alpha}\sqrt{i\tfrac{1}{\pi}\tfrac{m}{2i\hbar t}}\exp\Big\{-\tfrac{m}{2i\hbar t}(x-y)^2\Big\} \quad \text{as } t\downarrow 0 \end{aligned}$$

which in view of (28) requires that to achieve (32.3) we must set $e^{i\alpha} = 1/\sqrt{i}$. We then have

$$G_{\text{osc}}(x,t;y,0) = \sqrt{\tfrac{m\omega}{ih\sin\omega t}}\exp\Big\{\tfrac{i}{\hbar}\tfrac{m\omega}{2\sin\omega t}\big[(x^2+y^2)\cos\omega t - 2xy\big]\Big\}$$

which exactly reproduces (18). So we have, in this instance, managed to carry out STEP ONE

$$G_{\text{osc}}(x,t;y,0) \xleftarrow[\text{extended Beck}]{\qquad\qquad} K_{\text{osc}}(x,p,t;y,q,0)$$

of our 2-step program. The labor of STEP TWO was, as it happens, performed by Ferdinand Mehler in (1866), who established[13] that the expression on the right side of (18)

$$\sqrt{\text{etc.}}\;\exp\Big\{\tfrac{i}{\hbar}\,\text{etc.}\Big\} = \sum_n e^{-i\omega(n+\frac{1}{2})t}\psi_n(x)\psi_n(y)$$

This equation provides an explicit summary of the information that enters into the solution of

$$\mathsf{H}_{\text{osc}}\psi_n(x) = E_n\psi_n(x)$$

It would be of great interest to know what the "phase space propagator" looks like in some other exactly soluable cases, and in *those* cases to test the utility of the program (33).

2-dimensional isotropic oscillator. In Cartesian coordinates the system presents itself

$$L = \tfrac{1}{2}m(\dot x_1^2 - \omega^2 x_1^2) + \tfrac{1}{2}m(\dot x_2^2 - \omega^2 x_2^2) \tag{34.1}$$

as such a simple generalization of its 1-dimensional counterpart

$$L = \tfrac{1}{2}m(\dot x_1^2 - \omega^2 x_1^2)$$

that it might seem a little difficult to suppose that it is of any special interest, or displays any novel features. The first hint that it does so springs from the realization that the orbits have suddenly become relatively complicated figures —ellipses (simple Lissajous figures) inscribed on the coordinate plane. And that it has become possible to contemplate "symmetry adapted" coordinates: write

$$\left.\begin{aligned} x_1 &= r\cos\varphi = ae^\rho\cos\varphi \\ x_2 &= r\sin\varphi = ae^\rho\sin\varphi \end{aligned}\right\} \;:\; a \text{ an arbitrary "length"}$$

and obtain

$$\begin{aligned} L &= \tfrac{1}{2}m\big\{\dot r^2 + r^2\dot\varphi^2 - \omega^2 r^2\big\} && (34.2)\\ &= \tfrac{1}{2}ma^2 e^{2\rho}\big\{\dot\rho^2 + \dot\varphi^2 - \omega^2\big\} && (34.3)\end{aligned}$$

[13] See again equation (9) in "Jacobi's theta transformation and Mehler's formula: their interrelation, and their role in the quantum theory of angular momentum" (November 2000).

Let us agree to work initially in Cartesian coordinates, one grounds that it is the phase space formalism with which we desire to establish contact, and that formalism, as standardly presented, works *only* in Cartesian coordinates. From the symmetrically bipartite design of the Hamiltonian

$$\mathsf{H} = \tfrac{1}{2m}\mathsf{p}_1^2 + \tfrac{1}{2}m\omega^2\mathsf{x}_1^2 + \tfrac{1}{2m}\mathsf{p}_2^2 + \tfrac{1}{2}m\omega^2\mathsf{x}_2^2 \tag{35}$$

it follows that the resulting quantum mechanics is in many respects simply a duplex version of the quantum mechanics of a simple oscillator. The Schrödinger equation separates: the eigenfunctions are products

$$\psi_{n_1n_2}(x_1,x_2) = \psi_{n_1}(x_1)\cdot\psi_{n_2}(x_2) \tag{36.1}$$

of the eigenfunctions of a simple oscillator: the associated eigenvalues

$$\begin{aligned} E_{n_1n_2} &= (n_1+n_2+1)\hbar\omega \quad : \quad n_1,n_2 = 0,1,2,\ldots \\ &\downarrow \\ E_n &= (n+1)\hbar\omega \quad \text{with} \quad n \equiv n_1+n_2 \end{aligned} \tag{36.2}$$

are $(n+1)$-fold degenerate, being shared by $\psi_{0,n}\,,\ \psi_{1,n-1}\,,\ \psi_{2,n-2}\,,\ \ldots,\ \psi_{n,0}\,.$ One has

$$\psi(x_1,x_2,t) = \iint G(x_1,x_2,t;y_1,y_2,0)\,\psi(y_1,y_2,0)\,dy_1dy_2$$

with

$$G(x_1,x_2,t;y_1,y_2,0) = G_{\text{osc}}(x_1,t;y_1,0)\cdot G_{\text{osc}}(x_2,t;y_2,0) \tag{37}$$

In two dimensions the Wigner transform of $\psi(x_1,x_2)$ is—compare (8)—defined

$$\begin{aligned} &P_\psi(x_1,x_2,p_1,p_2) \\ &\quad= \big(\tfrac{2}{h}\big)^2\iint \psi^*(x_1+\xi_1,x_2+\xi_2)e^{2\frac{i}{\hbar}(p_1\xi_1+p_2\xi_2)}\psi(x_1-\xi_1,x_2-\xi_2)\,d\xi_1d\xi_2 \end{aligned}$$

which will be abbreviated

$$P_\psi(\boldsymbol{x},\boldsymbol{p}) = \big(\tfrac{2}{h}\big)^2\iint \psi^*(\boldsymbol{x}+\boldsymbol{\xi})\,e^{2\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{\xi}}\,\psi(\boldsymbol{x}-\boldsymbol{\xi})\;d\xi_1d\xi_2 \tag{38}$$

We are powerless to *plot* such functions, but do have available to us such devices as the display of 2-dimensional *sections*—such, for example, as $P_\psi(x_1,x_2,0,0)$—or of associated *marginal* distributions:

$$|\psi(x_1,x_2)|^2 = \iint P_\psi(\boldsymbol{x},\boldsymbol{p})\,dp_1dp_2 \quad : \quad \text{typical marginal distribution}$$

The Wigner transforms of the eigenfunctions have product structure

$$P_{n_1n_2}(x_1,x_2,p_1,p_2) = P_{n_1}(x_1,p_1)\cdot P_{n_2}(x_2,p_2) \tag{39}$$

where the factors were found at (17) to depend upon their respective arguments through

$$\begin{aligned}\mathcal{E}_1 &\equiv 2(p_1^2 + x_1^2)\\ \mathcal{E}_2 &\equiv 2(p_2^2 + x_2^2)\end{aligned} \tag{40}$$

Product structure attaches also to the "phase space propagator," where by enlargement upon (19) we have

$$\begin{aligned}&K(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{q},0) \qquad (40)\\ &\quad = \delta\big(x_1 - y_1\cos\omega t - \tfrac{1}{m\omega}q_1\sin\omega t\big)\delta\big(p_1 - q_1\cos\omega t + m\omega\, y_1\sin\omega t\big)\\ &\qquad\cdot\,\delta\big(x_2 - y_2\cos\omega t - \tfrac{1}{m\omega}q_2\sin\omega t\big)\delta\big(p_2 - q_2\cos\omega t + m\omega\, y_2\sin\omega t\big)\end{aligned}$$

while its hypothetical logarithmic companion grows additively: in dimensionless variables we expect to have (compare (30))

$$\begin{aligned}&\mathcal{S}(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{q},0) \qquad (41)\\ &\quad = \big(x_1 - y_1\cos\omega t - q_1\sin\omega t\big)^2 + \big(p_1 - q_1\cos\omega t + y_1\sin\omega t\big)^2\\ &\qquad + \big(x_2 - y_2\cos\omega t - q_2\sin\omega t\big)^2 + \big(p_2 - q_2\cos\omega t + y_2\sin\omega t\big)^2\end{aligned}$$

We are in position now to consider the first of the issues that motivated this discussion:

Hidden symmetry revealed. Equations (40) and (41) allude—in their distinct ways, and for distinct "reasons" (surprisingly in the first instance, since K is a quantum mechanical object)—to "dynamical phase flow" in 4-dimensional phase space Γ_4:

$$t\text{-parameterized action } \mathcal{H}[t] \text{ of } H \;:\; \begin{cases} y_1 \longmapsto x_1(t) = y_1\cos\omega t + q_1\sin\omega t\\ q_1 \longmapsto p_1(t) = q_1\cos\omega t - y_1\sin\omega t\\ y_2 \longmapsto x_2(t) = y_2\cos\omega t + q_2\sin\omega t\\ q_2 \longmapsto p_2(t) = q_2\cos\omega t - y_2\sin\omega t\end{cases}$$

They allude, that is to say, to the t-parameterized canonical transformation generated by the Hamiltonian. From the overt rotational symmetry of the Lagrangian (34) one is led *via* Noether's theorem to the identification of

$$L_3 \equiv x_1p_2 - x_2p_1 \tag{42}$$

as a conserved observable:[14]

$$[H, L_3] = 0 \tag{43}$$

[14] To facilitate work with dimensionless variables I define

$$[A,B] \equiv \frac{\partial A}{\partial x_1}\frac{\partial B}{\partial p_1} + \frac{\partial A}{\partial x_2}\frac{\partial B}{\partial p_2} - \frac{\partial B}{\partial x_1}\frac{\partial A}{\partial p_1} - \frac{\partial B}{\partial x_2}\frac{\partial A}{\partial p_2} = \hbar\cdot[A,B]$$

Also $H \equiv \frac{1}{2}\big\{p_1^2 + x_1^2 + p_2^2 + x_2^2\big\} = \frac{1}{\hbar\omega}H$ is the "dimensionless Hamiltonian" for purposes of this discussion.

Look upon $L_3(\boldsymbol{x},\boldsymbol{p})$ as the generator of a u-parameterized family of canonical transformations: specifically

$$u\text{-parameterized action } \mathcal{L}_3[u] \text{ of } L_3 \;:\; \begin{cases} x_1 \longmapsto \hat{x}_1 = x_1 \cos u - x_2 \sin u \\ p_1 \longmapsto \hat{p}_1 = p_1 \cos u - p_2 \sin u \\ x_2 \longmapsto \hat{x}_2 = x_2 \cos u + x_1 \sin u \\ p_2 \longmapsto \hat{p}_2 = p_2 \cos u + p_1 \sin u \end{cases}$$

which was obtained by integration of

$$\begin{aligned} \tfrac{d}{du}x_1 &= -[L_3, x_1] = +\partial L_3/\partial p_1 = -x_2 \\ \tfrac{d}{du}p_1 &= -[L_3, p_1] = -\partial L_3/\partial x_1 = -p_2 \\ \tfrac{d}{du}x_2 &= -[L_3, x_2] = +\partial L_3/\partial p_2 = +x_1 \\ \tfrac{d}{du}p_2 &= -[L_3, p_2] = -\partial L_3/\partial x_2 = +p_1 \end{aligned}$$

The statement (43) that L_3 is a constant of the motion can be phrases this way: $\mathcal{L}_3$ maps dynamical trajectories to dynamical trajectories. Which is to say: the following diagram is "commutative" in the sense that the red sequence of

$$\begin{tikzcd} \{\boldsymbol{y},\boldsymbol{q}\} \arrow[r, "\mathcal{H}[t]"'] \arrow[d, "\mathcal{L}_3[u]"] & \{\boldsymbol{x},\boldsymbol{p}\} \arrow[d, "\mathcal{L}_3[u]"] \\ \{\hat{\boldsymbol{y}},\hat{\boldsymbol{q}}\} \arrow[r, "\mathcal{H}[t]"'] & \{\hat{\boldsymbol{x}},\hat{\boldsymbol{p}}\} \end{tikzcd} \tag{44}$$

operations has the same effect as the black sequence. The claim is established by computation of a sort which I am content generally to entrust to *Mathematica*, but which I illustrate in a single instance:

$$\begin{aligned} y_1 &\rightarrow (y_1 \cos\omega t + q_1 \sin\omega t)\cos u - (y_2 \cos\omega t + q_2 \sin\omega t)\sin u \\ y_1 &\rightarrow (y_1 \cos u - y_2 \sin u)\cos\omega t + (q_1 \cos u - q_2 \sin u)\sin\omega t \end{aligned}$$

the expressions on the right are clearly equal

More to the immediate point, we look with the assistance of *Mathematica* to the "classical phase action" $\mathcal{S}(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{q},0)$ introduced at (41) and discover that

$$\mathcal{S}(\hat{\boldsymbol{x}},\hat{\boldsymbol{p}},t;\hat{\boldsymbol{y}},\hat{\boldsymbol{q}},0) = \mathcal{S}(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{q},0) \tag{45}$$

"Classical phase action" is <u>invariant under the action of $\mathcal{L}_3[u]$</u>.

Look finally to the (quantum mechanical) phase space propagator (40), which in dimensionless variables becomes

$$\begin{aligned} &K(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{q},0) \qquad (46) \\ &\quad = \delta\big(x_1 - y_1 \cos\omega t - q_1 \sin\omega t\big)\delta\big(p_1 - q_1 \cos\omega t + y_1 \sin\omega t\big) \\ &\quad\quad \cdot \delta\big(x_2 - y_2 \cos\omega t - q_2 \sin\omega t\big)\delta\big(p_2 - q_2 \cos\omega t + y_2 \sin\omega t\big) \end{aligned}$$

and under action of $\mathcal{L}_3$ gives

$$\begin{aligned}
&K(\hat{\boldsymbol{x}},\hat{\boldsymbol{p}},t;\hat{\boldsymbol{y}},\hat{\boldsymbol{q}},0) = \qquad (47)\\
&\delta\big([x_1\cos u - x_2\sin u] - [y_1\cos u - y_2\sin u]\cos\omega t - [q_1\cos u - q_2\sin u]\sin\omega t\big)\\
&\cdot\,\delta\big([p_1\cos u - p_2\sin u] - [q_1\cos u - q_2\sin u]\cos\omega t + [y_1\cos u - y_2\sin u]\sin\omega t\big)\\
&\cdot\,\delta\big([x_2\cos u + x_1\sin u] - [y_2\cos u + y_1\sin u]\cos\omega t - [q_2\cos u + q_1\sin u]\sin\omega t\big)\\
&\cdot\,\delta\big([p_2\cos u + p_1\sin u] - [q_2\cos u + q_1\sin u]\cos\omega t + [y_2\cos u + y_1\sin u]\sin\omega t\big)
\end{aligned}$$

It is to facilitate interpretation of this result that I digress to review

SOME PROPERTIES OF MULTIVARIATE DELTA FUNCTIONS

Adopt the notational convention

$$\delta(a_{11}x_1 + a_{12}x_2 - b_1)\delta(a_{21}x_1 + a_{22}x_2 - b_2) = \delta(\mathbb{A}\,\boldsymbol{x} - \boldsymbol{b})$$

which extends straightforwardly from the 2-dimensional to the N-dimensional case. The multivariable calculus supplies

$$\begin{aligned}
\iint f(\boldsymbol{x})\delta(\mathbb{A}\,\boldsymbol{x} - \boldsymbol{b})\,dx_1dx_2 &= \iint f\big((\mathbb{A}^{-1}(\boldsymbol{y}+\boldsymbol{b})\big)\delta(\boldsymbol{y})\,|\det\mathbb{A}\,|^{-1}dy_1dy_2\\
&= f(\mathbb{A}^{-1}\boldsymbol{b})|\det\mathbb{A}\,|^{-1}
\end{aligned}$$

where $\boldsymbol{y}\equiv\mathbb{A}\,\boldsymbol{x}-\boldsymbol{b}$ and $|\det\mathbb{A}\,|^{-1}$ is the Jacobian of the invertible transformation $\boldsymbol{x}\mapsto\boldsymbol{y}$. So we have

$$= \iint f(\boldsymbol{x})\,\delta(x - \mathbb{A}^{-1}\boldsymbol{b})|\det\mathbb{A}\,|^{-1}dx_1dx_2$$

of which

$$\delta(\mathbb{A}\,\boldsymbol{x} - \boldsymbol{b}) = |\det\mathbb{A}\,|^{-1}\cdot\delta(\boldsymbol{x} - \mathbb{A}^{-1}\boldsymbol{b}) \tag{48}$$

provides formal expression. In the 1-dimensional case we recover

$$\delta(ax-b) = |a|^{-1}\cdot\delta(x - b/a)$$

as reported by Dirac himself.

Returning now to (47), we have

$$\begin{aligned}
K(\hat{\boldsymbol{x}},\hat{\boldsymbol{p}},t;\hat{\boldsymbol{y}},\hat{\boldsymbol{q}},0) &= \delta(\mathbb{R}\,\boldsymbol{\mathfrak{X}} - \mathbb{B}\,\boldsymbol{\mathfrak{Y}})\\
&= |\det\mathbb{R}\,|^{-1}\cdot\delta(\boldsymbol{\mathfrak{X}} - \mathbb{R}^{-1}\mathbb{B}\,\boldsymbol{\mathfrak{Y}})
\end{aligned}$$

where

$$\boldsymbol{\mathfrak{X}}\equiv\begin{pmatrix}x_1\\p_1\\x_2\\p_2\end{pmatrix},\quad \boldsymbol{\mathfrak{Y}}\equiv\begin{pmatrix}y_1\\q_1\\y_2\\q_2\end{pmatrix},\quad \mathbb{R}\equiv\begin{pmatrix}\cos u & 0 & -\sin u & 0\\ 0 & \cos u & 0 & -\sin u\\ \sin u & 0 & \cos u & 0\\ 0 & \sin u & 0 & \cos u\end{pmatrix}$$

and

$$\mathbb{B} \equiv \begin{pmatrix} \cos u \cos \omega t & \cos u \sin \omega t & -\sin u \cos \omega t & -\sin u \sin \omega t \\ -\cos u \sin \omega t & \cos u \cos \omega t & \sin u \sin \omega t & -\sin u \cos \omega t \\ \sin u \cos \omega t & \sin u \sin \omega t & \cos u \cos \omega t & \cos u \sin \omega t \\ -\sin u \sin \omega t & \sin u \cos \omega t & -\cos u \sin \omega t & \cos u \cos \omega t \end{pmatrix}$$

Mathematica supplies

$$\mathbb{R}^{-1}\mathbb{B} = \begin{pmatrix} \cos \omega t & \sin \omega t & 0 & 0 \\ -\sin \omega t & \cos \omega t & 0 & 0 \\ 0 & 0 & \cos \omega t & \sin \omega t \\ 0 & 0 & -\sin \omega t & \cos \omega t \end{pmatrix} \quad \text{and} \quad \det \mathbb{R} = 1$$

Therefore

$$\begin{aligned} K(\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}, t; \hat{\boldsymbol{y}}, \hat{\boldsymbol{q}}, 0) = \delta(\boldsymbol{\mathcal{Z}}) \quad \text{with} \quad \boldsymbol{\mathcal{Z}} &\equiv \begin{pmatrix} x_1 - y_1 \cos \omega t - q_1 \sin \omega t \\ p_1 - q_1 \cos \omega t + y_1 \sin \omega t \\ x_2 - y_2 \cos \omega t - q_2 \sin \omega t \\ p_2 - q_2 \cos \omega t + y_2 \sin \omega t \end{pmatrix} \\ = K(\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{y}, \boldsymbol{q}, 0) \end{aligned} \tag{49}$$

In short: the phase space propagator is—like the "classical phase action" was found to be at (45)—<u>invariant under the action of $\mathcal{L}_3[u]$</u>.

This result is gratifying but hardly surprising, for in standard isotropic oscillator theory rotational invariance is a property already of the ordinary action/propagator, a symmetry known to lie unproblematically at the base of L_3-conservation.

In isotropic oscillator theory, as was remarked at the outset (see again (2)), L_3 is presented as the companion of two less familiar conservation laws

$$A_1 \equiv p_1 p_2 + x_1 x_2 \quad : \quad \text{non-Noetherean } p\text{-dependence} \tag{50.1}$$

$$L_3 = A_2 \equiv x_1 p_2 - x_2 p_1 \quad : \quad \text{Noetherean (overt rotational symmetry)} \tag{50.2}$$

$$A_3 \equiv \tfrac{1}{2}(p_1^2 + x_1^2 - p_2^2 - x_2^2) \quad : \quad \text{non-Noetherean } p\text{-dependence} \tag{50.3}$$

Our former line of argument gives

$$u\text{-parameterized action } \mathcal{A}_1[u] \text{ of } A_1 \quad : \quad \begin{cases} x_1 \longmapsto \hat{x}_1 = x_1 \cos u + p_2 \sin u \\ p_1 \longmapsto \hat{p}_1 = p_1 \cos u - x_2 \sin u \\ x_2 \longmapsto \hat{x}_2 = x_2 \cos u + p_1 \sin u \\ p_2 \longmapsto \hat{p}_2 = p_2 \cos u - x_1 \sin u \end{cases}$$

$$u\text{-parameterized action } \mathcal{A}_3[u] \text{ of } A_3 \quad : \quad \begin{cases} x_1 \longmapsto \hat{x}_1 = x_1 \cos u + p_1 \sin u \\ p_1 \longmapsto \hat{p}_1 = p_1 \cos u - x_1 \sin u \\ x_2 \longmapsto \hat{x}_2 = x_2 \cos u - p_2 \sin u \\ p_2 \longmapsto \hat{p}_2 = p_2 \cos u + x_2 \sin u \end{cases}$$

Computation shows that $\mathcal{S}(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{q},0)$ is invariant not just under $\mathcal{L}_3[u]$, as was established at (45), but under the action of *each* of the transformations $\mathcal{A}_1[u]$, $\mathcal{A}_2[u]$ and $\mathcal{A}_3[u]$.

Turning from the classical to the quantum physics of the system, we have

$$K(\hat{\boldsymbol{x}},\hat{\boldsymbol{p}},t;\hat{\boldsymbol{y}},\hat{\boldsymbol{q}},0)=\delta(\mathbb{R}_1\boldsymbol{\mathfrak{X}}-\mathbb{B}_1\boldsymbol{\mathfrak{Y}})=|\det\mathbb{R}_1|^{-1}\cdot\delta(\boldsymbol{\mathfrak{X}}-\mathbb{R}_1^{-1}\mathbb{B}_1\boldsymbol{\mathfrak{Y}})$$

with

$$\mathbb{R}_1\equiv\begin{pmatrix}\cos u & 0 & 0 & \sin u\\ 0 & \cos u & -\sin u & 0\\ 0 & \sin u & \cos u & 0\\ -\sin u & 0 & 0 & \cos u\end{pmatrix}$$

$$\mathbb{B}_1\equiv\begin{pmatrix}\cos u\cos\omega t & \cos u\sin\omega t & -\sin u\sin\omega t & \sin u\cos\omega t\\ -\cos u\sin\omega t & \cos u\cos\omega t & -\sin u\cos\omega t & -\sin u\sin\omega t\\ -\sin u\sin\omega t & \sin u\cos\omega t & \cos u\cos\omega t & \cos u\sin\omega t\\ -\sin u\cos\omega t & -\sin u\sin\omega t & -\cos u\sin\omega t & \cos u\cos\omega t\end{pmatrix}$$

Computation gives results already encountered

$$\mathbb{R}_1^{-1}\mathbb{B}_1=\mathbb{R}^{-1}\mathbb{B}\quad\text{(see above) and}\quad\det\mathbb{R}_1=\det\mathbb{R}$$

Similarly

$$K(\hat{\boldsymbol{x}},\hat{\boldsymbol{p}},t;\hat{\boldsymbol{y}},\hat{\boldsymbol{q}},0)=\delta(\mathbb{R}_3\boldsymbol{\mathfrak{X}}-\mathbb{B}_3\boldsymbol{\mathfrak{Y}})=|\det\mathbb{R}_3|^{-1}\cdot\delta(\boldsymbol{\mathfrak{X}}-\mathbb{R}_3^{-1}\mathbb{B}_3\boldsymbol{\mathfrak{Y}})$$

with

$$\mathbb{R}_3\equiv\begin{pmatrix}\cos u & \sin u & 0 & 0\\ -\sin u & \cos u & 0 & 0\\ 0 & 0 & \cos u & -\sin u\\ 0 & 0 & \sin u & \cos u\end{pmatrix}$$

$$\mathbb{B}_3\equiv\begin{pmatrix}\cos(\omega t+u) & \sin(\omega t+u) & 0 & 0\\ -\sin(\omega t+u) & \cos(\omega t+u) & 0 & 0\\ 0 & 0 & \cos(\omega t-u) & \sin(\omega t-u)\\ 0 & 0 & -\sin(\omega t-u) & \cos(\omega t-u)\end{pmatrix}$$

and

$$\mathbb{R}_3^{-1}\mathbb{B}_3=\mathbb{R}^{-1}\mathbb{B}\quad\text{(see above) and}\quad\det\mathbb{R}_3=\det\mathbb{R}=1$$

The wonderful implication is that the "hidden symmetries" $\mathcal{A}_1[u]$ and $\mathcal{A}_3[u]$ of the isotropic oscillator are express/explicit/revealed symmetries of the phase space propagator $K(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{q},0)$.

The $\mathbb{R}$-matrices are proper rotation matrices,[15] generated by

$$\mathbb{A}_1\equiv\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\quad \mathbb{A}_2\equiv\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\quad \mathbb{A}_3\equiv\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

[15] So, for that matter, are the $\mathbb{B}$-matrices.

respectively.[16] These matrices satisfy

$$\mathbb{A}_1^2 = \mathbb{A}_2^2 = \mathbb{A}_3^2 = -\mathbb{I}$$

and

$$\begin{aligned}\big[\mathbb{A}_1, \mathbb{A}_2\big] &= 2\,\mathbb{A}_3\\ \big[\mathbb{A}_2, \mathbb{A}_3\big] &= 2\,\mathbb{A}_1\\ \big[\mathbb{A}_3, \mathbb{A}_1\big] &= 2\,\mathbb{A}_2\end{aligned}$$

The latter equations mimic the Poisson bracket relations satisfied by the A-observables (50):

$$\begin{aligned}[A_1, A_2] &= 2A_3\\ [A_2, A_3] &= 2A_1\\ [A_3, A_1] &= 2A_2\end{aligned}$$

Which I have written to underscore once again the many-times-repeated fact that, while $O(2)$ describes the overt geometrical symmetry of the isotropic oscillator, $O(3)$ describes the "hidden symmetry," which we have found to be "overt" in classical/quantum phase space formalisms, but is eclipsed by processes—for example

$$G(\boldsymbol{x}, t; \boldsymbol{y}, 0) \xleftarrow[\text{Beck}]{\qquad\qquad} K(\boldsymbol{x}, \boldsymbol{p}, t; \boldsymbol{y}, \boldsymbol{q}, 0)$$

—which are designed to achieve pull-back to configuration space. The question arises: Why $O(3)$ instead of $O(4)$, since isoenergetic surfaces

$$\tfrac{1}{2}(p_1^2 + x_1^2 + p_2^2 + x_2^2) = \text{constant}$$

are hyperspherical in Γ_4? The short answer: It is the business of constants of motion to generate not canonical transformations that map isoenergetic surfaces each onto itself, but to map

$$\text{dynamical orbits} \;\longmapsto\; \text{dynamical orbits}$$

Dynamical orbits are *specialized decorations* inscribed on isoenergetic surfaces, whence the retraction $O(3) \leftarrow O(4)$.

Classical/quantum orbits and their elliptical projections. The Wigner function associated with the ground state of an isotropic oscillator was found at (39/17) to be given by

$$\begin{aligned}P_{00}(\boldsymbol{x}, \boldsymbol{p}) &= \tfrac{1}{\pi}(-)^0 e^{-\frac{1}{2}\mathcal{E}_1} L_0(\mathcal{E}_1)\cdot \tfrac{1}{\pi}(-)^0 e^{-\frac{1}{2}\mathcal{E}_2} L_0(\mathcal{E}_2)\\ &= \big(\tfrac{1}{\pi}\big)^2 e^{-(p_1^2 + x_1^2 + p_2^2 + x_2^2)}\end{aligned}$$

so

$$P(\boldsymbol{x}, \boldsymbol{p}) \equiv P_{00}(\boldsymbol{x} - \boldsymbol{a}, \boldsymbol{p} - \boldsymbol{b}) \tag{51}$$

[16] For typographic reasons I have here written $\mathbb{1}$ for -1.

describes a copy of the ground state that has been displaced in phase space. What can we in this instance say about

$$\psi(\boldsymbol{x}) \xleftarrow[\text{Beck}]{\hspace{2cm}} P(\boldsymbol{x},\boldsymbol{p})$$

Look to the 1-dimensional case, where we have

$$\begin{aligned}\psi(x)\,\psi^*(0) &= \int P(\tfrac{x}{2},p)e^{ip\,x}\,dp\\ &= \int \tfrac{1}{\pi}\exp\bigl\{-(\tfrac{x}{2}-a)^2-(p-b)^2\bigr\}e^{ip\,x}\,dp\\ &= \tfrac{1}{\sqrt{\pi}}e^{-\frac{1}{2}x^2+ax-a^2}e^{ib\,x}\end{aligned}$$

which by $\psi(x)\,\psi^*(0) = \frac{1}{\sqrt{\pi}}e^{-a^2} \Rightarrow \psi^*(0) = e^{i(\text{arbitrary phase})}\cdot\pi^{-\frac{1}{4}}e^{-\frac{1}{2}a^2}$ gives (after we abandon the phase factor)

$$\psi(x) = \pi^{-\frac{1}{4}}e^{-\frac{1}{2}(x-a)^2}\cdot e^{ib\,x} \tag{52.1}$$

We have here conducted in reverse the argument that gave (21.2), but have gained something for our pains: at (21) we were preparing ourselves to displace the oscillator and then simply release it; here—by a process very easy to comprehend in phase space—we have declared our intention to launch the oscillator with a flick (non-zero initial momentum), and recognize the $e^{ib\,x}$ to possess the otherwise semi-obscure significance of a "flick factor." In two dimensions we have

$$\psi(\boldsymbol{x}) = \pi^{-\frac{1}{2}}e^{-\frac{1}{2}[(x_1-a_1)^2+(x_2-a_2)^2]}\cdot e^{i(b_1x_1+b_2x_2)} \tag{52.2}$$

and the "flick" becomes vectorial.

To launch $P(\boldsymbol{x},\boldsymbol{p})$ into motion we introduce (40)—actually its dimensionless variant

$$\begin{aligned}&K(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{q},0)\\ &\quad= \delta\bigl(x_1-y_1\cos\omega t-q_1\sin\omega t\bigr)\delta\bigl(p_1-q_1\cos\omega t+y_1\sin\omega t\bigr)\\ &\quad\quad\cdot\delta\bigl(x_2-y_2\cos\omega t-q_2\sin\omega t\bigr)\delta\bigl(p_2-q_2\cos\omega t+y_2\sin\omega t\bigr)\\ &\quad= \delta(\boldsymbol{\mathfrak{X}}-\mathbb{R}\boldsymbol{\mathfrak{Y}})\\ &\qquad\qquad \mathbb{R}\equiv\begin{pmatrix}\cos\omega t & \sin\omega t & 0 & 0\\ -\sin\omega t & \cos\omega t & 0 & 0\\ 0 & 0 & \cos\omega t & \sin\omega t\\ 0 & 0 & -\sin\omega t & \cos\omega t\end{pmatrix}\\ &\quad= \delta(\boldsymbol{\mathfrak{Y}}-\mathbb{R}^{-1}\boldsymbol{\mathfrak{X}})\end{aligned}$$

—into

$$P(\boldsymbol{x},\boldsymbol{p};t) = \iiiint K(\boldsymbol{x},\boldsymbol{p},t;\boldsymbol{y},\boldsymbol{q},0)P(\boldsymbol{y},\boldsymbol{q})\,dy_1dq_1dy_2dq_2$$

and obtain

$$\begin{aligned} P(\boldsymbol{x},\boldsymbol{p};t) = \tfrac{1}{\pi}\exp\Big\{&-(x_1\cos\omega t - p_1\sin\omega t - a_1)^2 \\ &-(p_1\cos\omega t + x_1\sin\omega t - b_1)^2\Big\} \\ &\cdot(\text{similar factor with subscripts advanced: }_{1\rightarrow 2}) \end{aligned} \tag{53}$$

which describes a Gaussian in rigid 4-dimensional motion. The "moving max" —got by solving

$$(x_1\cos\omega t - p_1\sin\omega t - a_1) = (\text{etc.}) = (\text{etc.}) = (\text{etc.}) = 0$$

—is situated at

$$\begin{aligned} x_1 &= a_1\cos\omega t + b_1\sin\omega t \\ p_1 &= b_1\cos\omega t - a_1\sin\omega t \\ x_2 &= a_2\cos\omega t + b_2\sin\omega t \\ p_2 &= b_2\cos\omega t - a_2\sin\omega t \end{aligned}$$

which (by a kind of "duplex reference circle" construction) projects

$$\left.\begin{aligned} x_1 &= a_1\cos\omega t + b_1\sin\omega t \\ x_2 &= a_2\cos\omega t + b_2\sin\omega t \end{aligned}\right\} \tag{54.1}$$

onto the physical space of the oscillator. In polar notation[17] the preceding equations become

$$\left.\begin{aligned} x_1 &= \mathcal{X}_1\cos(\omega t + \delta_1) \\ x_2 &= \mathcal{X}_2\cos(\omega t + \delta_2) \end{aligned}\right\} \tag{54.2}$$

Equations (54) provide parametric description of an ellipse. Elimination of the parameter[18] gives

$$\mathcal{X}_2^2\cdot x_1^2 - 2\mathcal{X}_1\mathcal{X}_2\cos\underset{\substack{\uparrow\\ \delta\,\equiv\,\delta_2-\delta_1}}{\delta}\cdot x_1x_2 + \mathcal{X}_1^2\cdot x_2^2 = \mathcal{X}_1^2\mathcal{X}_2^2\sin^2\delta \tag{55}$$

---

[17] Write, in each case,

$$a = \mathcal{X}\cos\delta \quad\text{and}\quad b = \mathcal{X}\sin\delta$$

which entail

$$\mathcal{X} = \sqrt{a^2+b^2} \quad\text{and}\quad \delta = \arctan(b/a)$$

[18] On the basis of (54.1) write

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad : \quad \text{abbreviated} \quad \mathbb{M}^{-1}\begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix} = \boldsymbol{x}$$

which gives

$$\boldsymbol{x}^{\mathsf{T}}\mathbb{M}^{\mathsf{T}}\mathbb{M}\,\boldsymbol{x} = 1$$

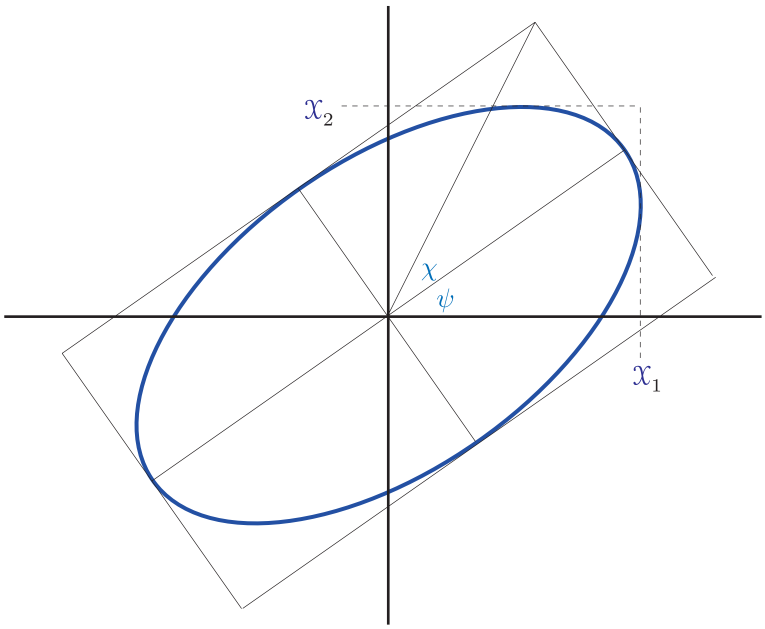

FIGURE 3: *Geometrical meaning of the parameters that enter into Stokes' definitions (56.1).*

It is to Stokes that we are indebted for the introduction of notations

$$\left.\begin{aligned} \mathcal{S}_0 &\equiv \mathcal{X}_1^2 + \mathcal{X}_2^2 \\ \mathcal{S}_1 &\equiv \mathcal{X}_1^2 - \mathcal{X}_2^2 &&= \mathcal{S}_0 \cos 2\chi \cos 2\psi \\ \mathcal{S}_2 &\equiv 2\mathcal{X}_1\mathcal{X}_2 \cos\delta &&= \mathcal{S}_0 \cos 2\chi \sin 2\psi \\ \mathcal{S}_3 &\equiv 2\mathcal{X}_1\mathcal{X}_2 \sin\delta &&= \mathcal{S}_0 \sin 2\chi \end{aligned}\right\} \tag{56.1}$$

that in conjunction with the preceding figure permit direct apprehension of the *figure* of the ellipse implicitly described by (55). The "flying spot" that at (54) traced the ellipse has at this point been expunged, but the *rotational sense* of its motion does survive: its progress was ↺ or ↻ according as the "chirality" is plus or minus, where

$$\begin{aligned} \text{chirality} &\equiv \text{sign of } \underbrace{a_1 b_2 - a_2 b_1}_{=\,\mathcal{X}_1\mathcal{X}_2 \sin\delta}, \text{ which is "angular momentum-like"} \\ &= \text{sign of } \delta \end{aligned} \tag{56.2}$$

In ordinary quantum mechanics (where one has no reason to examine $P(\boldsymbol{x}, \boldsymbol{p}; t)$) one finds it most natural to look—with Schrödinger himself—to the "marginal" distribution

$$
\begin{aligned}
|\psi(\boldsymbol{x},t)|^2 &= \iint P(\boldsymbol{x},\boldsymbol{p};t)\,dp_1 dp_2 \\
&= \tfrac{1}{\sqrt{\pi}}\exp\Big\{-(x_1 - a_1\cos\omega t - b_1\sin\omega t)^2\Big\} \\
&\qquad \cdot(\text{similar factor with subscripts advanced: }{}_{1\to2})
\end{aligned}
\tag{57}
$$

so (by-passing the preceding quartet of equations) is led *directly* to (54.1), and to the elliptical remarks we extracted therefrom.

With *Mathematica*'s ready assistance we look to the *moving expectation values* of the observables $A_0 \equiv \frac{1}{2}(p_1^2 + x_1^2 + p_2^2 + x_2^2)$ and A_1, A_2, A_3 introduced at (50), and find

$$
\begin{aligned}
\langle A_0\rangle &= \iiiint A_0(\boldsymbol{x},\boldsymbol{p})P(\boldsymbol{x},\boldsymbol{p};t)\,dx_1dp_1dx_2dp_2 \\
&= \tfrac{1}{2}(a_1^2 + b_1^2 + a_2^2 + b_2^2) + 1 \\
\langle A_1\rangle &= b_1b_2 + a_1a_2 \\
\langle A_2\rangle &= a_1b_2 - a_2b_1 \\
\langle A_3\rangle &= \tfrac{1}{2}(a_1^2 + b_1^2 - a_2^2 - b_2^2)
\end{aligned}
$$

that in fact they *do not move*; all the t-dependence drops away . . . as, indeed, it must: the observables in question are known to be <u>constants of the motion</u>! Moreover, we have

$$
\left.
\begin{aligned}
\langle A_0\rangle &= \tfrac{1}{2}(\mathcal{X}_1^2 + \mathcal{X}_2^2) + 1 = \tfrac{1}{2}\mathcal{S}_0 + 1 \\
\langle A_1\rangle &= \mathcal{X}_1\mathcal{X}_2\cos\delta \qquad\quad\;\; = \tfrac{1}{2}\mathcal{S}_2 \\
\langle A_2\rangle &= \mathcal{X}_1\mathcal{X}_2\sin\delta \qquad\quad\;\; = \tfrac{1}{2}\mathcal{S}_3 \\
\langle A_3\rangle &= \tfrac{1}{2}(\mathcal{X}_1^2 - \mathcal{X}_2^2) \qquad\;\; = \tfrac{1}{2}\mathcal{S}_1
\end{aligned}
\right\}
\tag{58}
$$

which serve (i) to associate the conserved observables A with constant features of the orbital geometry, and (ii) to establish contact with mathematical aspects of the profound innovations which Stokes/Poincaré brought to the theory of optical polarization . . . down which well-travelled road are known to lie the spinor representations of $O(3)$ and all that good stuff.

The displaced ground state $P(\boldsymbol{x},\boldsymbol{p};t)$ moves, I emphasize once again, for the same reasons that Wigner distributions in general move: because it is assembled from *superimposed* energy eigenfunctions. But it is a very special superposition, and moves in a way that is in some respects atypical. It is "minimally dispersive" and—more to the point—$P(\boldsymbol{x},\boldsymbol{p};0)$ displays $O(4)$ symmetry with respect to the phase point $\{\boldsymbol{a},\boldsymbol{b}\}$, with the consequence that its projections on all phase planes move "rigidly," i.e., in such a way as to *preserve their shape*. To say the same thing another way: its centered moments of all orders $n \geqslant 2$ are constant. In less specialized cases one expects expressions of (say) the form

$$
\iiiint (x_1 - \langle x_1\rangle_t)^n P(\boldsymbol{x},\boldsymbol{p};t)\,dx_1dp_1dx_2dp_2
$$

to display *harmonics* of the natural frequency ω. It is, however, easy to show (is the upshot, essentially, of Ehrenfest's theorem) that in *all* cases the motion of the first moments can be described[19]

$$\begin{aligned}\langle x_1\rangle &= a_1\cos\omega t + b_1\sin\omega t\\ \langle p_1\rangle &= b_1\cos\omega t - a_1\sin\omega t\\ \langle x_2\rangle &= a_2\cos\omega t + b_2\sin\omega t\\ \langle p_2\rangle &= b_2\cos\omega t - a_2\sin\omega t\end{aligned}$$

where the a's and b's refer now to the initial values of those moments. In that limited sense, elliptical "Lissajous motion" of the sort discussed above pertains *universally* to the quantum dynamics of isotropic oscillators ... quite as expected, since it pertains universally to the classical dynamics, and the phase space propagator (40/46) displays no non-classical feature.

From isotropic oscillator to the 2-dimensional Kepler problem. My objective is the 2-dimensional Kepler problem

$$H(p_x,p_y,x,y) = \tfrac{1}{2m}(p_x^2+p_y^2) - \tfrac{k}{\sqrt{x^2+y^2}} \tag{59}$$

More specifically, I have interest in what might be called the "orbital theory of 2-dimensional hydrogen," which I propose to study in phase space the better to expose the origin of the "hidden symmetry" (accidental degeneracy) which the system is known to display. To get there I intend to exploit the fact[20] that in confocal parabolic coordinates

$$\left.\begin{aligned} x &= \tfrac{1}{2}(\mu^2-\nu^2)\\ y &= \mu\nu\end{aligned}\right\} \tag{60}$$

the Lagrangian becomes "separable in the sense of Liouville"

$$L = \tfrac{1}{2}m(\mu^2+\nu^2)(\dot\mu^2+\dot\nu^2) - \tfrac{2k}{\mu^2+\nu^2}$$

and supplies

$$\begin{aligned}p_\mu &= m(\mu^2+\nu^2)\dot\mu\\ p_\nu &= m(\mu^2+\nu^2)\dot\nu\\ H(p_\mu,p_\nu,\mu,\nu) &= \tfrac{1}{\mu^2+\nu^2}\big\{\tfrac{1}{2m}(\dot\mu^2+\dot\nu^2)+2k\big\}\end{aligned} \tag{61}$$

and leads to a Schrödinger equation that separates:

$$\begin{aligned}\big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{d}{d\mu}\big)^2 - E\,\mu^2 - k_1 - \epsilon_1\big\}M(\mu) &= 0\\ \big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{d}{d\nu}\big)^2 - E\,\nu^2 - k_2 - \epsilon_2\big\}N(\nu) &= 0\end{aligned}$$

[19] For detailed discussion of "quantum mechanics as a theory of coupled moments" (with special reference to the phase space formalism) see pages 51–60 in ADVANCED QUANTUM TOPICS (2000), Chapter 2.

[20] See §5 in "Classical/quantum theory of 2-dimensional hydrogen" (notes for the Reed College Physics Seminar of 3 February 1999).

Here $k_1 + k_2 = 2k$ and $\epsilon_1 + \epsilon_2 = 0$. These equations can be written

$$\left.\begin{aligned}\left\{-\tfrac{\hbar^2}{2m}\left(\tfrac{d}{d\mu}\right)^2+\tfrac{1}{2}m\omega^2\mu^2\right\}M(\mu)&=(k_1+\epsilon_1)M(\mu)\\ \left\{-\tfrac{\hbar^2}{2m}\left(\tfrac{d}{d\nu}\right)^2+\tfrac{1}{2}m\omega^2\nu^2\right\}N(\nu)&=(k_2+\epsilon_2)N(\nu)\end{aligned}\right\}\tag{62}$$

$$\tfrac{1}{2}m\omega^2 \equiv -E : \text{ positive for bound states}\tag{63}$$

and—remarkably—place us in position to make formal use of the relatively simple *quantum theory of isotropic oscillators*. Note, however, that an element of *anharmonicity* is introduced into the latter theory at (63), and that the coordinates which enter into (62) are *non-Cartesian*.