THE MAXIMAL LCM PROBLEM

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Introduction. Each of the $n!$ distinct permutations $\mathcal{P}$ of $\{1, 2, 3, \ldots, n\}$ acquires natural representation by a "permutation matrix" $\mathbb{P}$, the distinguishing feature of such matrices being that they have a 1 in every row/column, all other elements being 0. For example,

$$\mathcal{P} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$

acquires the representation

$$\mathbb{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

since

$$\mathbb{P}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \\ 1 \\ 5 \\ 4 \end{pmatrix}$$

All permutation matrices are inverted by transposition $\mathbb{P}^{-1} = \mathbb{P}^{\mathsf{T}}$, so are special instances of rotation matrices, proper or improper $\det \mathbb{P} = \pm 1$ according as the associated permutation is even or odd.

"Cyclic" permutations possess the structure

$$\mathcal{P}_{\text{cyclic}} = \begin{pmatrix} i_1 & i_2 & i_3 & \ldots & i_{\nu-1} & i_\nu \\ i_2 & i_3 & i_4 & \ldots & i_\nu & i_1 \end{pmatrix}$$

and are said to have “period” $\pi(\mathcal{P}_{\text{cyclic}}) = \nu$ because they give back the identity permutation $\mathcal{I}$ after ν repetitions. The associated permutation matrix $\mathbb{P}_{\text{cyclic}}$ is therefore periodic in the sense that $\mathbb{P}^{\nu}_{\text{cyclic}} = \mathbb{I}$ and therefore

$$\mathbb{P}^{k}_{\text{cyclic}} = \mathbb{P}^{k+m\nu}_{\text{cyclic}} \quad : \quad m = 0, \pm 1, \pm 2, \ldots$$

Every permutation $\mathcal{P}$ can be resolved into disjoint cycles

$$\mathcal{P} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p\}$$

Thus (returning to our previous example)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 1 & 5 & 4 \end{pmatrix} = \{\{1,6,4\}\{2,3\}\{5\}\}$$

which in matrix language amounts to the statement that $\mathbb{P} = \mathbb{C}_1\mathbb{C}_2\mathbb{C}_3$, with

$$\mathbb{C}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \bullet & 0 & 0 \end{pmatrix}, \quad \mathbb{C}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bullet & 0 & 0 & 0 \\ 0 & \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbb{C}_3 = \mathbb{I}$$

where to emphasize salient structural details I have written $\bullet = 1$. In this example $\mathbb{P}^6 = \mathbb{I}$, $\mathbb{C}_1^3 = \mathbb{I}$ and $\mathbb{C}_2^2 = \mathbb{I}$. More generally, if $\mathcal{P} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p\}$ then the integers $\{\pi(\mathcal{C}_1), \pi(\mathcal{C}_2), \ldots, \pi(\mathcal{C}_p)\}$ serve to *partition* n

$$\pi(\mathcal{C}_1) + \pi(\mathcal{C}_2) + \cdots + \pi(\mathcal{C}_p) = n$$

and

$$\pi(\mathcal{P}) = \text{LCM}\big(\pi(\mathcal{C}_1), \pi(\mathcal{C}_2), \ldots, \pi(\mathcal{C}_p)\big)$$

Our problem is to discover (or—for large values of n—at least to estimate) the *maximal* value $\pi_{\max}(\mathcal{P})$ assumed by $\pi(\mathcal{P})$ as $\mathcal{P}$ ranges over the set of all possible permutations of $\{1, 2, 3, \ldots, n\}$. This amounts to discovery (estimation) of the greatest possible value $\text{LCM}_{\max}(n)$ assumed by $\text{LCM}(\wp(n))$ as $\wp(n)$ ranges over the set of all possible partitions of n.

Preliminaries. To evaluate $\text{LCM}_{\max}(n)$ one has in principle only to list the partitions of n, compute the LCMs of the listed partitions and isolate the $\wp(n)$ that maximizes the LCM... all of which is easy work for *Mathematica*, which I used to generate the low-order data tabulated on the next page.

This naive procedure becomes, however, very time-consuming already by $n = 25$, for the simple reason that *Mathematica* has in that instance to examine a total of $p(25) = 1958$ partitions, most of which—for reasons to be discussed in a moment—can be dismissed out of hand as unreasonable LCM-maximization candidates. This “wasted effort problem” becomes rapidly more burdensome as n increases.

n	Maximizing Partition	Maximal LCM
1	1	1
2	2	2
3	3	3
4	4	4
5	3+2	6
6	6	6
	3+2+1	6
7	4+3	12
8	5+3	15
9	5+4	20
10	5+3+2	30
11	5+3+2+1	30
	6+5	30
12	5+4+3	60
13	5+4+3+1	60
14	7+4+3	84
15	7+5+3	105
16	7+5+4	140
17	7+5+3+2	210
18	7+5+3+2+1	210
19	7+5+4+3	420
20	7+5+4+3+1	420
21	7+5+4+3+1+1	420
22	7+5+4+3+1+1+1	420
23	8+7+5+3	840
24	8+7+5+3+1	840
25	9+7+5+4	1260

TABLE 1. *The case* $n = 6$ *is seen to be exceptional in that* two *distinct partitions of* 6 *are maximal. This curious detail traces to the circumstance that* 6 *is a "perfect" number* (*meaning equal to the sum of its divisors*). *The next perfect number—of which finitely many are known—is* $28 = 1 + 2 + 4 + 7 + 14$. *It is indeed the case that* $\text{LCM}(28) = \text{LCM}(1, 2, 4, 7, 14) = 28$, *but that number falls far short of* $\text{LCM}_{\text{max}}(28)$. *At* $n = 11$ *we encounter a more interesting instance of a case in which distinct partitions of a number have the same least common multiple. On the other hand, the table—though short—exposes many cases in which distinct numbers have the same* LCM, *all of which can be attributed to an obvious "*$n \to n + 1$ *mechanism."*

"Wasted computational effort" can be attributed principally to the occurance of repeated terms in a partitioning of n, since those—except for repeated ones, which we saw in TABLE 1 to be sometimes essential—contribute nothing toward elevation of the value of the LCM. On that same ground, we can dismiss partitions in which any pair of elements share a prime factor. In a sharpened version of the naive procedure described above we might restrict our attention to partitions in which all elements greater than one are *coprime* (or "relatively prime," meaning have GCD = 1). It is seen—by inspection, or by application of *Mathematica*'s **CoprimeQ** command—that each of the maximizing partitions listed in TABLE 1 possesses this property.[1]

The computational efficiency latent in the coprimality restriction becomes evident when one compares the number $q(n)$ of coprime partitions of n with the number $p(n)$ of unrestricted partitions, which is well known to grow rapidly, yet much (!!) less rapidly than the number $n!$ of permutations, as I illustrate:

n	$q(n)$	$p(n)$	$n!$
2	1	2	2
4	3	5	24
6	6	11	720
8	11	22	40320
10	17	42	3.6288×10^6
12	26	77	4.7900×10^8
14	37	135	8.7178×10^{10}
16	50	231	2.0923×10^{13}
18	69	385	6.4027×10^{15}
20	91	627	2.4329×10^{18}

Asymptotically, one has

$$p(n) \sim \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{2n/3}} \qquad \text{Hardy \& Ramanujan, 1918}$$

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad \text{de Moivre \& Stirling, 1730}$$

but I do not possess an asymptotic approximation to $q(n)$.[2] More seriously, I do not possess an algorithm for generating a *list* of the coprime partitions of n (it would be counterproductive to create such lists by filtering ever-longer lists of unrestricted partitions), and in the absence of such lists the "sharpened naive procedure" mentioned above will remain uselessly "latent."

Asymptotic estimation of LCM$_{\max}(n)$. The data reported in TABLE 1 suggests that as n increases the elements in the maximizing partition become (relative to n) progresssively smaller and more numerous. The elements of such partitions

[1] Strictly speaking, this is true only for $n \geq 5$ since the coprimality concept is inapplicable in the cases $n = 1, 2, 3, 4$.

[2] See, however, the ADDENDUM attached to the end of this note.

tend to cluster—as compactly as coprimality allows?—about a mean value given by

$$\text{mean maximizing element} \approx \frac{n}{\text{number of elements}(n)}$$

From coprimality if follows, moreover, that

$$\text{LCM(maximizing partition)} = \prod \text{(elements of maximizing partition)}$$

—an elementary proposition to which the data in TABLE I of course conforms.

We are led thus to expect the LCM-maximizing partition to display a k-member "coprime packet" with mean $\approx n/k$. If coprimality considerations are set temporarily aside, we expect therefore to have

$$\ell(n) = (n/k)^k \quad : \quad k\text{-value maximizes } \ell(n)$$

From

$$\frac{d}{dk}(n/k)^k = (n/k)^k\,[\log(n/k) - 1] = 0$$

we are led to set $k = k_{\max} = n/e$, giving finally

$$\ell(n) = e^{n/e} > \text{LCM}_{\max}(n)$$

where the inequality is an expression simply of my intuitive expectation that coprimality considerations will cause the actual $\text{LCM}_{\max}$ to fall short of our idealized estimate. When we use *Mathematica*'s **FindFit** command to discover the function of the form $a^{n/b}$ that best conforms to the low-order data reported in TABLE 1 we obtain

$$\mathcal{L}(n) = 1.27430^{\,n/0.848197}$$

Equivalently,

$$\log \mathcal{L}(n) = 0.285779\, n \quad : \quad \text{compare } \log \ell(n) = 0.367897\, n$$

which is to say

$$\log[\text{maximal period } \pi_{\max}(n)] \sim \alpha n \quad : \quad \alpha \approx 0.285779$$

Assuming this result—which, by the way, possesses precisely the structure anticipated by Richard Crandall[3]—to be correct in its structural essentials, we can expect expansion of the data set to lead simply to an adjustment of the value of α. One would like to possess a *theoretical* evaluation of α (description in terms of π, e, small integers, *etc.*) but the effort to construct such a result seems likely to require deep knowledge of the distribution of coprimes and God knows what else—material that lies far beyond my reach. I am reminded in this connection that[4]

$$\text{the probability that } k \text{ randomly chosen integers are coprime} = \frac{1}{\zeta(k)}$$

which is of no immediate relevance, but provides some indication of the anaytical riches that may lie close by.

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[3] Private communication, 19 April 2012.

[4] See http://en.wikipedia.org/wiki/Coprime

n	$\ell(n)$	$\mathcal{L}(n)$	$\text{LCM}_{\max}(n)$
1	1.46	1.33	1
2	2.09	1.77	2
3	3.02	2.36	3
4	4.36	3.14	4
5	6.29	4.17	6
6	9.09	5.56	6
7	13.13	7.39	12
8	18.97	9.84	15
9	27.41	13.09	20
10	39.60	17.42	30
11	57.21	23.19	30
12	82.65	30.86	60
13	119.39	41.06	60
14	172.50	54.65	84
15	249.18	72.73	105
16	359.99	96.78	140
17	520.06	128.80	210
18	751.32	171.40	210
19	1085.41	228.10	420
20	1568.05	303.56	420
21	2265.31	403.98	420
22	3272.63	537.61	420
23	4727.86	715.46	840
24	6830.18	952.13	840
25	9867.34	1267.09	1260

TABLE 2. *Comparison of the results predicted by $\ell(n)$ and $\mathcal{L}(n)$ with the facts of the matter. Of course, one cannot expect asymptotic formulae to be of much use when n is small. It is clear that $\ell(n)$ has been rendered worthless by the strong assumption that went into its construction. The evidence the $\mathcal{L}(n)$ remains valid asymptotically is entirely circumstantial, based upon very limited data; that it fits the data as well as it does is not surprising, since it was that data that was used to fix the value of α, its sole adjustable parameter. For a graphic display of the same information, see the accompanying Mathematica notebook.*

ADDENDUM. At the end of the notebook just mentioned I construct emperical evidence—without even the hint of a formal rationale—that the number of coprime partitions of n is given asymptotically by

$$q(n) \sim 4.81107\, e^{0.148647\, n}$$

Again, one would like to possess *theoretical* evaluations of the numerics.