ALGEBRAIC THEORY OF SPHERICAL HARMONICS [‡]

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Introduction. To think of "the partial differential equations of physics" is to think of equations such as the following:

$$\begin{aligned}
\text{HEAT EQUATION}: &\quad \nabla^2\varphi = \ \ D\tfrac{\partial}{\partial t}\varphi\\
\text{SCHRÖDINGER EQUATION}: &\quad \nabla^2\varphi = i\,D\tfrac{\partial}{\partial t}\varphi + W\varphi\\
\text{WAVE EQUATION}: &\quad \nabla^2\varphi = \ \ D\tfrac{\partial^2}{\partial t^2}\varphi\\
\text{POISSON EQUATION}: &\quad \nabla^2\varphi = W\\
&\qquad\quad \vdots\\
\text{LAPLACE EQUATION}: &\quad \nabla^2\varphi = \ 0
\end{aligned}\tag{1}$$

The solution of such equations (subject to specified side conditions) is a task which—as their names already suggest—has engaged the attention of leading mathematicians for centuries. By separation of variables (when it can be effected) such partial differential equations give rise to ordinary differential equations of 2^{nd} order—soil from which sprang classical material associated with the names of Gauß, Legendre, Leguerre, Hermite, Bessel and many others, the stuff of "higher analysis," which found synthesis in Sturm-Liouville theory and the theory of orthogonal polynomials. The diverse ramifications of the mathematical problems thus posed have proven to be virtually inexhaustible. And central to the whole story has been the differential operator ∇^2, the theory latent in (1).

Nor is it difficult to gain an intuitive sense of the reason ∇^2 is encountered so ubiquitously. Let $\varphi(x,y)$ be defined on a neighborhood containing the point (x,y) on the Euclidian plane. At points on the boundary of a disk centered at (x,y) the value of φ is given by

$$\begin{aligned}
\varphi(x+r\cos\theta, y+r\sin\theta) &= e^{r\cos\theta\frac{\partial}{\partial x}+r\cos\theta\frac{\partial}{\partial x}}\varphi(x,y)\\
&= \varphi + r\big(\varphi_x\cos\theta+\varphi_y\cos\theta\big)\\
&\qquad + \tfrac{1}{2}r^2\big(\varphi_{xx}\cos^2\theta+2\varphi_{xy}\cos\theta\sin\theta+\varphi_{yy}\sin^2\theta\big)+\cdots
\end{aligned}\tag{2}$$

[‡] Notes for a Reed College Math Seminar presented 7 March 1996.

The *average* of the values assumed by φ on the boundary of the disk is given therefore by

$$\begin{aligned}\langle\varphi\rangle &= \frac{1}{2\pi r}\int_0^{2\pi}\big\{\text{right side of (2)}\big\}r\,d\theta\\ &= \varphi + 0 + \tfrac{1}{4}r^2\big\{\varphi_{xx}+\varphi_{yy}\big\}+\cdots\end{aligned}$$

So we have[1]

$$\langle\varphi\rangle - \varphi = \tfrac{1}{4}r^2\cdot\nabla^2\varphi \tag{3}$$

in leading approximation. Laplace's equation asserts simply that the φ-function is "relaxed":

$$\nabla^2\varphi = 0 \quad\iff\quad \langle\varphi\rangle = \varphi \quad \text{everywhere} \tag{4}$$

If the φ-function descriptive of a physical field is, on the other hand, *not* relaxed, it is natural to set

$$\begin{aligned}\text{restoring force} &= k\big\{\langle\varphi\rangle-\varphi\big\}\\ &= \text{mass element}\,\cdot\,\text{acceleration}\\ k\tfrac{1}{4}r^2\cdot\nabla^2\varphi &= 2\pi r^2\rho\,\cdot\,\tfrac{\partial^2}{\partial t^2}\varphi\end{aligned}$$

and thus to recover the wave equation, with $D = 8\pi\rho/k$.

The ∇^2 operator is actually quite a rubust construct. Within the exterior calculus one has[2]

$$\nabla^2 = \mathbf{d}*\mathbf{d}* + (-)^{p(n-p+1)}*\mathbf{d}*\mathbf{d}$$

while on Riemannian manifolds one has the "Laplace-Beltrami operator"

$$g^{ij}\varphi_{;i;j} \equiv \nabla^2(0)\,\varphi = \left\{\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\sqrt{g}g^{ij}\frac{\partial}{\partial x^i}\right\}\varphi \tag{5}$$

provided φ transforms as a scalar density of weight $W = 0$. In the more general case (as, for example, in quantum mechanics, where in order to preserve $\int\psi^*\psi\,d^nx = 1$ the wave function must transform as a density of weight $W=\frac{1}{2}$) the explicit description of $\nabla^2(W)$ is very messy, but practical work is much simplified by the following little-known identity:

$$\nabla^2(W)\cdot g^{+\frac{1}{2}W} = g^{+\frac{1}{2}W}\cdot\nabla^2(0) \tag{6}$$

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[1] We have achieved here—by averaging over the surface of a small enveloping 2-sphere (which is to say: a small disk centered at the field-point in question)—a result which is more commonly achieved by averaging over nearest-neighboring lattice points. In n-dimensions the lattice argument gives

$$\langle\varphi\rangle - \varphi = \tfrac{1}{2n}r^2\cdot\nabla^2\varphi$$

which is precisely the result which, as I have (with labor!) shown elsewhere, is obtained when one averages over the surface of an enveloping n-sphere.

[2] See p. 17 of "Electrodynamical applications of the exterior calculus" (1996) for the origin and meaning of the sign factor.

Several analogs of the "shift rule" (6) will be encountered in subsequent pages.

So much by way of general orientation. To study $\nabla^2\varphi = 0$ is in Euclidean n-space to study

$$\left\{\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 + \cdots + \left(\frac{\partial}{\partial x_n}\right)^2\right\}\varphi(x_1, x_2, \cdots, x_n) = 0$$

The case $n = 2$ is in many ways exceptional, for this familiar reason: if the complex-valued function $f(z) = u(x, y) + iv(x, y)$ is analytic, then by virtue of the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

the real-valued functions $u(x, y)$ and $v(x, y)$ are "conjugate harmonic"

$$\nabla^2 u = \nabla^2 v = 0$$

It is therefore very easy to exhibit solutions of the Laplace equation in two dimensions (though not so easy to satisfy imposed boundary conditons). When $n \geq 3$ one has access, unfortunately, to no such magic carpet, and the situation is, in some respects, more "crystaline." For example, L. P. Eisenhart established in 1934 that there exist precisely eleven coordinate systems (of a certain type) in which the 3-dimensional Laplace equation

$$\left\{\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2\right\}\varphi(x, y, z) = 0$$

separates; they are the

Rectangular
Circular-cylindrical
Elliptic-cylindrical
Parabolic-cylindrical
Spherical
Prolate spheroidal
Oblate spheroidal
Parabolic
Conical
Ellipsoidal
Paraboloidal

coordinate systems, the detailed descriptions of which are spelled out in the handbooks; see, for example, *Field Theory Handbook* by P. Moon & D. Spencer (Springer, 1961).

Central to the quantum mechanics of a particle moving in a prescribed force field is the time-independent Schrödinger equation, which has the form

$$\nabla^2\psi(x,y,z) = \big[W(x,y,z)+\lambda\big]\psi(x,y,z) \tag{7.1}$$

The presence of the W-factor serves to destroy separability except in favorable special cases. For example, if the force field is rotationally invariant

$$W(x,y,z) = U(r) \quad \text{with} \quad r = \sqrt{x^2+y^2+z^2} \tag{7.2}$$

then (7) does separate in *spherical* coordinates. Moreover

$U(r) = kr^{+2}$ permits separation also in *rectangular* coordinates
$U(r) = kr^{-1}$ permits separation also in *parabolic* coordinates

The former is familiar as the "isotropic spring potential," and the latter as the "Kepler potential." These are, as it happens (according to Bertrand's theorem[3]), the only central potentials in which all (bounded) orbits close upon themselves; the connection between double separability and orbital closure is a deep one, but it is a story for another day.

1. Standard analytical construction of spherical harmonics. My main objective today is to describe a novel approach[4] to the spherical separation of (7)—a novel approach to the *theory of spherical harmonics*—and it is to underscore the novelty (and the merit!) of the method that I pause now to outline the *standard* approach to the spherical separation problem. One writes

$$\left.\begin{aligned} x &= r\,\sin\theta\cos\phi \\ y &= r\,\sin\theta\,\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \tag{8}$$

giving

$$(ds)^2 = (dx)^2+(dy)^2+(dz)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2$$

and from (5) obtains

$$\nabla^2 = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}r^2\sin\theta\frac{\partial}{\partial r} + \frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}\csc\theta\frac{\partial}{\partial\phi}\right\}$$

One writes $\psi(x,y,z) = \Psi(r,\theta,\phi)$ and assumes

$$\Psi(r,\theta,\phi) = R(r)\cdot Y(\theta,\phi) \tag{9}$$

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[3] For an excellent discussion see Appendix A in the 2nd edition of Goldstein's *Classical Mechanics* (1980).

[4] Can a method first described nearly seventy years ago fairly be said to be "novel"? "Little-known and non-standard" is perhaps a better description.

to obtain

$$\left\{\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \big[U(r)+\lambda\big] - \frac{\alpha}{r^2}\right\}R(r) = 0 \tag{10.1}$$

$$\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}Y(\theta,\phi) = -\alpha Y(\theta,\phi) \tag{10.2}$$

where α is a "separation constant." It is notable that the particulars of the problem, as written onto $U(r)$, enter into the structure of the "radial equation" (10.1), but are completely absent from (10.2), which looks only to what we might call the "sphericity" of the problem. To complete the separation, one writes

$$Y(\theta,\phi) = \Theta(\theta)\cdot\Phi(\phi) \tag{11}$$

and obtains

$$\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + \alpha - \frac{\beta}{\sin^2\theta}\right\}\Theta(\theta) = 0 \tag{12.1}$$

$$\left\{\frac{d^2}{d\phi^2} + \beta\right\}\Phi(\phi) = 0 \tag{12.2}$$

where β is again a separation constant. From (12.2) and the requirement that solutions be *regular on the whole sphere* one is led easily to the orthonormal functions

$$\Phi_m(\phi) \equiv \frac{1}{\sqrt{2\pi}}e^{im\phi} \quad \text{where} \quad m = 0, \pm1, \pm2, \cdots \tag{13}$$

and to the conclusion that $\beta = m^2$. Returning with the latter information to (12.1) one confronts a more intricate problem. A change of variables

$$\theta \longrightarrow \omega \equiv \cos\theta \tag{14}$$

produces

$$\mathcal{D}(m^2)P(\omega) = 0 \tag{15}$$

where $P(\cos\theta) = \Theta(\theta)$ and the operator $\mathcal{D}(m^2)$ is defined

$$\begin{aligned}\mathcal{D}(m^2) &\equiv \left\{\frac{d}{d\omega}(1-\omega^2)\frac{d}{d\omega} + \alpha - \frac{m^2}{1-\omega^2}\right\}\\ &= \left\{(1-\omega^2)\frac{d^2}{d\omega^2} - 2\omega\frac{d}{d\omega} + \alpha - \frac{m^2}{1-\omega^2}\right\}\end{aligned} \tag{16}$$

Remarkably (compare the "shift rule" (6)),

$$\mathcal{D}(m^2)\cdot\big(1-\omega^2\big)^{\frac{1}{2}m}\left(\frac{d}{d\omega}\right)^m = \big(1-\omega^2\big)^{\frac{1}{2}m}\left(\frac{d}{d\omega}\right)^m\cdot\mathcal{D}(0) \tag{17}$$

with this implication: if $P(\omega)$ is a solution of $\mathcal{D}(0)P(\omega) = 0$ then

$$P^m(\omega) \equiv \big(1-\omega^2\big)^{\frac{1}{2}m}\left(\frac{d}{d\omega}\right)^m P(\omega) \quad \text{is a solution of} \quad \mathcal{D}(m^2)P^m(\omega) = 0 \tag{18}$$

So one studies

$$\left\{\frac{d}{d\omega}(1-\omega^2)\frac{d}{d\omega}+\alpha\right\}P(\omega)=0 \tag{19}$$

Highly non-trivial analysis leads to the conclusion that solutions regular on the sphere exist if and only if

$$\alpha=\ell(\ell+1) \quad \text{with} \quad \ell=0,1,2,\cdots \tag{20}$$

in which case the solutions are in fact the famous Legendre polynomials, which can be described

$$P_\ell(\omega)=\frac{1}{2^\ell\,\ell!}\left(\frac{d}{d\omega}\right)^\ell\left(\omega^2-1\right)^\ell \tag{21}$$

Thus, when all the dust has settled, is one led to the functions

$$Y_\ell^m(\theta,\phi)\equiv\sqrt{\frac{2\,\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\cdot P_\ell^m(\cos\theta)e^{im\phi} \tag{22}$$

where $\ell=0,1,2,\cdots$ and $m=0,\pm1,\pm2,\cdots,\pm\ell$. These "spherical harmonics" are orthonormal on the sphere

$$\int_0^{2\pi}\int_0^{\pi}\Big(Y_{\ell'}^{m'}(\theta,\phi)\Big)^*\Big(Y_{\ell''}^{m''}(\theta,\phi)\Big)\sin\theta\,d\theta\,d\phi=\delta^{m'm''}\delta_{\ell'\ell''}$$

and put one in position to do "Fourier analysis on the sphere," just as the functions (13) permit one to do Fourier analysis on the circle. This is a wonderful accomplishment, of high practical importance in a great variety of applications. But the argument which led us to the construction of the functions $Y_\ell^m(\theta,\phi)$ is notable for its opaque intricacy, and has left us deeply indebted to Legendre, who was clearly no slouch!

2. Harmonic polynomials: Kramers' construction. It was by straightforward application of precisely such classical analysis (*and* its relatively less well known parabolic counterpart) that Schrödinger, in his very first quantum mechanical publication (1926), constructed the quantum theory of the hydrogen atom. Almost immediately thereafter the Dutch physicist H. A. Kramers, drawing inspiration from Schrödinger's accomplishment, sketched an alternative approach to the theory of spherical harmonics which has, in my view, much to recommend it, but which remains relatively little known. It was in the (misplaced) hope of rectifying the latter circumstance that H. C. Brinkman (formerly a student of Kramers') published in 1956 the slim monograph (*Applications of Spinor Invariants in Atomic Physics*, North-Holland) which has been my principal source.

The germinal idea resides in two assumptions. First we assume $\psi(x,y,z)$ to have—compare (9)—the factored structure

$$\psi(x,y,z)=\underbrace{F_\ell(r)\cdot(\boldsymbol{a}\cdot\boldsymbol{r})^\ell}_{\text{manifestly rotation-invariant}} \tag{23}$$

Introduction of (23) into (7) leads straightforwardly to the equation

$$(\boldsymbol{a}\cdot\boldsymbol{r})^\ell\cdot\left\{\frac{d^2}{dr^2}+\frac{2(\ell+1)}{r}\frac{d}{dr}-\big[W(r)+\lambda\big]\right\}F_\ell(r)+F_\ell(r)\cdot\nabla^2(\boldsymbol{a}\cdot\boldsymbol{r})^\ell=0$$

The statement

$$\nabla^2(\boldsymbol{a}\cdot\boldsymbol{r})^\ell=0 \tag{24.1}$$

is now not forced by the usual separation argument, so will simply be *assumed*; the implicit companion of that assumption is the modified radial equation

$$\left\{\frac{d^2}{dr^2}+\frac{2(\ell+1)}{r}\frac{d}{dr}-\big[W(r)+\lambda\big]\right\}F_\ell(r)=0 \tag{24.2}$$

It is to (24.1) that we henceforth confine our attention. Immediately

$$\nabla^2(\boldsymbol{a}\cdot\boldsymbol{r})^\ell=\ell(\ell-1)(\boldsymbol{a}\cdot\boldsymbol{r})^{\ell-2}(\boldsymbol{a}\cdot\boldsymbol{a}) \tag{25}$$

so to achieve (24.1) we must have $\boldsymbol{a}\cdot\boldsymbol{a}=0$. The impliction is that the null 3-vector $\boldsymbol{a}$ must be complex:

$$\boldsymbol{a}=\boldsymbol{b}+i\boldsymbol{c}\quad\text{with}\quad b^2=c^2\quad\text{and}\quad\boldsymbol{b}\cdot\boldsymbol{c}=0 \tag{26}$$

Since $a_1^2+a_2^2+a_3^2=0$ entails $a_3=i\sqrt{(a_1+ia_2)(a_1-ia_2)}$ it becomes fairly natural to introduce complex variables

$$\begin{aligned}u&\equiv\sqrt{a_1+ia_2}\\ v&\equiv\sqrt{a_1-ia_2}\end{aligned}$$

Then

$$\left.\begin{aligned}a_1&=\tfrac{1}{2}\big(u^2+v^2\big)\\ a_2&=-i\,\tfrac{1}{2}\big(u^2-v^2\big)\\ a_3&=i\,uv\end{aligned}\right\} \tag{27}$$

The $\boldsymbol{a}(u,v)$ thus defined has the property that

$$\mathbf{a}(u,v)=\mathbf{a}(-u,-v) \tag{28}$$

We conclude that as (u,v) ranges over complex 2-space $\mathbf{a}(u,v)$ ranges *twice* over the set of null 3-vectors.

With the aid of (27) we obtain

$$\begin{aligned}(\boldsymbol{a}\cdot\boldsymbol{r})^\ell&=\big[\tfrac{1}{2}\big(u^2+v^2\big)x-i\,\tfrac{1}{2}\big(u^2-v^2\big)y+i\,uvz\big]^\ell\\ &=\big[\tfrac{1}{2}u^2\big(x-iy\big)+i\,uvz+\tfrac{1}{2}v^2\big(x+iy\big)\big]^\ell\\ &=\Big(\frac{r}{2}\Big)^\ell\big[u^2\sin\theta e^{-i\phi}+2iuv\cos\theta+v^2\sin\theta e^{+i\phi}\big]^\ell\end{aligned}$$

which, provided ℓ is an integer, we can notate

$$\equiv \left(\frac{r}{2}\right)^{\ell} \sum_{m=-\ell}^{m=+\ell} u^{\ell-m}v^{\ell+m}Q_\ell^m(\theta,\phi) \tag{29}$$

Since (29) is, by construction, harmonic for all u and v we have

$$\nabla^2\big\{r^\ell Q_\ell^m(\theta,\phi)\big\} = 0$$

which by (10) entails

$$\left\{\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{\alpha}{r^2}\right\} r^\ell = 0 \quad \text{giving back again} \quad \alpha = \ell(\ell+1)$$

and

$$\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \ell(\ell+1)\right\} Q_\ell^m(\theta,\phi) = 0$$

Our assignment now is to construct explicit descriptions of the functions $Q_\ell^m(\theta,\phi)$; our expectation, of course, is that we will find

$$Q_\ell^m(\theta,\phi) \sim Y_\ell^m(\theta,\phi) \tag{30}$$

Proceeding in Kramer's clever footsteps, we write

$$\begin{aligned}
\sum_{m=-\ell}^{m=+\ell} u^{\ell-m}v^{\ell+m}Q_\ell^m(\theta,\phi) &= \big[u^2\sin\theta\, e^{-i\phi} + 2iuv\cos\theta + v^2\sin\theta\, e^{+i\phi}\big]^\ell \\
&= \big(u^2e^{-i\phi}\big)^\ell\,\big[\sin\theta + 2i\,(v/u)\,e^{i\phi}\cos\theta + (v^2/u^2)\,e^{2i\phi}\sin\theta\big]^\ell \\
&= \big(u^2e^{-i\phi}\big)^\ell\,\big[\big(1-Z^2\big)\sin\theta + 2Z\cos\theta\big]^\ell \\
&\qquad\qquad Z \equiv i\,(v/u)\,e^{i\phi} \\
&= \left(\frac{u^2e^{-i\phi}}{\sin\theta}\right)^\ell \big[\big(1-Z^2\big)\sin^2\theta + 2Z\sin\theta\cos\theta\big]^\ell \\
&= \left(\frac{u^2e^{-i\phi}}{\sin\theta}\right)^\ell \Big[1 - \big(\cos\theta - Z\sin\theta\big)^2\Big]^\ell \\
&= \left(\frac{u^2e^{-i\phi}}{\sin\theta}\right)^\ell \sum_{k=0}^{2\ell}\frac{1}{k!}Z^k\left\{\left(\frac{d}{dZ}\right)^k\Big[1 - \big(\cos\theta - Z\sin\theta\big)^2\Big]^\ell\right\}_{Z=0}
\end{aligned} \tag{31}$$

A change of variable

$$Z \longrightarrow \Omega(Z) \equiv \cos\theta - Z\sin\theta$$

gives $\frac{d}{dZ} = (-\sin\theta)\frac{d}{d\Omega}$ whence $\big(\frac{d}{dZ}\big)^k = \big(-\sin\theta\big)^k\big(\frac{d}{d\Omega}\big)^k$ so

$$\begin{aligned}\left\{\left(\frac{d}{dZ}\right)^k\Big[1-\big(\cos\theta - Z\sin\theta\big)^2\Big]^\ell\right\}_{Z=0}\\ &= \big(-\sin\theta\big)^k\left\{\left(\frac{d}{d\Omega}\right)^k\big[1-\Omega^2\big]^\ell\right\}_{\Omega=\Omega(0)}\\ &= \big(-\sin\theta\big)^k\left(\frac{d}{d\omega}\right)^k\big[1-\omega^2\big]^\ell\end{aligned}$$

where (consistently with prior usage) $\Omega(0) = \cos\theta \equiv \omega$. Returning with this information to (31) we obtain

$$\begin{aligned}&\sum_{m=-\ell}^{\ell} u^{\ell-m}v^{\ell+m}Q_\ell^m(\theta,\phi) = \left(\frac{u^2e^{-i\phi}}{\sin\theta}\right)^\ell\sum_{k=0}^{2\ell}\frac{1}{k!}\big(-Z\sin\theta\big)^k\left(\frac{d}{d\omega}\right)^k\big[1-\omega^2\big]^\ell\\ &= \left(\frac{u^2e^{-i\phi}}{\sin\theta}\right)^\ell\sum_{m=-\ell}^{\ell}\frac{1}{(\ell+m)!}\underbrace{\big(-Z\sin\theta\big)^{\ell+m}}_{=(-i)^{\ell+m}v^{\ell+m}u^{-\ell-m}e^{i(\ell+m)\phi}(\sin\theta)^{\ell+m}}\left(\frac{d}{d\omega}\right)^{\ell+m}\big[1-\omega^2\big]^\ell\\ &= \sum_{m=-\ell}^{\ell}u^{\ell-m}v^{\ell+m}\left\{\frac{(-i)^{\ell+m}}{(\ell+m)!}e^{im\phi}\underbrace{\big(\sin\theta\big)^m\left(\frac{d}{d\omega}\right)^{\ell+m}\big[1-\omega^2\big]^\ell}_{\substack{=(-)^\ell\big(1-\omega^2\big)^{\frac{1}{2}m}\left(\frac{d}{d\omega}\right)^{\ell+m}\big(\omega^2-1\big)^\ell \\ =(-)^\ell 2^\ell\ell!P_\ell^m(\omega)}}\right\}\end{aligned}$$

giving

$$Q_\ell^m(\theta,\phi) = (-)^\ell(-i)^{\ell+m}\frac{2^\ell\ell!}{(\ell+m)!}\cdot e^{im\phi}P_\ell^m(\cos\theta) \tag{32}$$

—precisely as anticipated at (30). Remarkably, we have achieved (32) without having had to solve any 2$^{\text{nd}}$-order differential equations, without imposing any regularity conditions (these were latent in our initial assumption), without acquiring indebtedness to Legendre.

It is instructive to compare the preceeding line of argument with its 2-dimensional counterpart. Since $a_1^2+a_2^2=0$ entails $a_2 = ia_1$ it becomes natural to write

$$\left.\begin{aligned}a_1 &= u\\ a_2 &= iu\end{aligned}\right\} \tag{33}$$

Then

$$\begin{aligned}(\boldsymbol{a}\cdot\boldsymbol{r})^m &= [ux+iuy]^m\\ &= [ur\cos\phi + iur\sin\phi]^m\\ &= r^m[u^me^{im\phi}]\end{aligned} \tag{34}$$

where I have forced myself to proceed in pedantic imitation of the argument that led to (29). The $\sum$ consists now of but a single term. Were I to continue in my pedantry, I would write

$$Q_m(\phi) = e^{im\phi} \tag{35}$$

and observe that

$$\nabla^2\big\{r^m Q_m(\phi)\big\} = 0$$

though this is hardly a surprise; $r^1 Q_1(\phi) = x + iy \equiv z$ so $r^m Q_m(\phi) = z^m$, which is an analytic function, and therefore is assuredly harmonic. Look now to the manifestly rotation-invariant expressions

$$C_{mn} \equiv \int_0^{2\pi} (\boldsymbol{A}\cdot\boldsymbol{r})^{*\,m}\,(\boldsymbol{a}\cdot\boldsymbol{r})^n\,d\phi \tag{36.1}$$

$$= r^{m+n}\cdot U^{*\,m}u^n \cdot \int_0^{2\pi} Q^*{}_m(\phi)\,Q_n(\phi)\,d\phi \tag{36.2}$$

In the u-representation rotation entails

$$u \longrightarrow e^{i\vartheta}u \tag{37}$$

under which $U^{*\,m}u^n$ is invariant if and only if $m = n$. The implication is that

$$\int_0^{2\pi} Q^*{}_m(\phi)\,Q_n(\phi)\,d\phi = 0 \quad \text{unless } m = n$$

The functions $Q_m(\phi)$ are, in other words, *orthogonal*. It follows moreover from the ϕ-independence of $(\boldsymbol{A}\cdot\boldsymbol{r})^*(\boldsymbol{a}\cdot\boldsymbol{r}) = (U^*x - iU^*y)(ux+iuy) = r^2\cdot U^*u$ that

$$C_{mm} = \big[r^2\cdot U^*u\big]^m \cdot \int_0^{2\pi} d\phi = \big[r^2\cdot U^*u\big]^m\cdot 2\pi$$

so in fact we have

$$\int_0^{2\pi} Q^*{}_m(\phi)\,Q_n(\phi)\,d\phi = \begin{cases} 2\pi & \text{if } m = n \\ 0 & \text{otherwise}\end{cases} \tag{38}$$

The integral relations just established are, of course, trivial implications of the definitions (35) of the functions $Q_m(\phi)$. Note, however, that in arriving at (38) we did not have actually to integrate anything; the mode of argument was entirely algebraic, rooted in the *transformational* aspects of the formalism at hand. Moreover, the line of argument sketched above has (as will emerge) the property that it admits of natural generalization.

3. Rotational ramifications. At (27) we set up an association between complex null 3-vectors $\boldsymbol{a} = \boldsymbol{b} + i\boldsymbol{c}$ and the points of a complex 2-space:

$$\begin{pmatrix} u \\ v \end{pmatrix} \xrightarrow[\text{2 to 1}]{\hspace{3cm}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

By computation

$$\begin{aligned}\boldsymbol{a}^*\cdot\boldsymbol{a} &= a_1^*a_1 + a_2^*a_2 + a_3^*a_3\\ &= \tfrac{1}{2}(u^*u+v^*v)^2 = \tfrac{1}{2}\left\{\begin{pmatrix}u\\v\end{pmatrix}^{\mathrm{t}}\begin{pmatrix}u\\v\end{pmatrix}\right\}^2\end{aligned}\tag{39}$$

The right side of (39) will be invariant under

$$\begin{pmatrix}u\\v\end{pmatrix}\longrightarrow\begin{pmatrix}\overline{u}\\\overline{v}\end{pmatrix}=\mathbb{U}\begin{pmatrix}u\\v\end{pmatrix}$$

if and only if $\mathbb{U}$ is unitary: $\mathbb{U}^{\mathrm{t}}\mathbb{U}=\mathbb{I}$. Without loss of generality one can write $\mathbb{U}=e^{i\vartheta}\mathbb{S}$ with $\mathbb{S}$ unimodular: $\mathbb{S}^{\mathrm{t}}\mathbb{S}=\mathbb{I}$ and $\det\mathbb{S}=1$. It follows from (27) that

$$\begin{pmatrix}u\\v\end{pmatrix}\longrightarrow\begin{pmatrix}\overline{u}\\\overline{v}\end{pmatrix}=e^{i\theta}\begin{pmatrix}u\\v\end{pmatrix}\quad\text{induces}\quad\boldsymbol{a}\longrightarrow\overline{\boldsymbol{a}}=e^{2i\vartheta}\boldsymbol{a}\tag{40}$$

To study the action $\boldsymbol{a}\longrightarrow\overline{\boldsymbol{a}}$ similarly induced by $\mathbb{S}$ we find it most efficient to proceed infinitesimally; we write

$$\mathbb{S}=\mathbb{I}+\tfrac{1}{2}\delta\varphi\cdot\mathbb{L}\tag{41}$$

and observe that the unimodularity of $\mathbb{S}$ entails the traceless anti-hermiticity of $\mathbb{L}$:

$$\mathbb{L}=\begin{pmatrix}i\lambda_3 & \lambda_1+i\lambda_2\\ -\lambda_1+i\lambda_2 & -i\lambda_3\end{pmatrix}\tag{42}$$

where without loss of generality we assume $\det\mathbb{L}=\lambda_1^2+\lambda_2^2+\lambda_3^2=1$. Introducing (42) into (41) and (41) into

$$\begin{pmatrix}u\\v\end{pmatrix}\longrightarrow\begin{pmatrix}\overline{u}\\\overline{v}\end{pmatrix}=\mathbb{S}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}u\\v\end{pmatrix}+\delta\begin{pmatrix}u\\v\end{pmatrix}$$

we obtain

$$\delta\begin{pmatrix}u\\v\end{pmatrix}=\tfrac{1}{2}\delta\varphi\cdot\mathbb{L}\begin{pmatrix}u\\v\end{pmatrix}=\tfrac{1}{2}\delta\varphi\cdot\begin{pmatrix}i\lambda_3u\ +\ (\lambda_1+i\lambda_2)v\\(-\lambda_1+i\lambda_2)u-i\lambda_3v\end{pmatrix}$$

which by (27) induces

$$\boldsymbol{a}\longrightarrow\overline{\boldsymbol{a}}=\boldsymbol{a}+\delta\boldsymbol{a}$$

with

$$\begin{aligned}
\delta\boldsymbol{a} &= \frac{\partial\boldsymbol{a}}{\partial u}\delta u + \frac{\partial\boldsymbol{a}}{\partial v}\delta v \\
&= \begin{pmatrix} u \\ -iu \\ iv \end{pmatrix}\delta u + \begin{pmatrix} v \\ +iv \\ iu \end{pmatrix}\delta v \\
&= \tfrac{1}{2}\delta\varphi\cdot\begin{pmatrix} 2i\lambda_2 uv \;+\; i\lambda_3(u^2-v^2) \\ \lambda_3(u^2+v^2) \;-\; 2i\lambda_1 uv \\ -i\lambda_1(u^2-v^2)-\lambda_2(u^2+v^2) \end{pmatrix} \quad \text{after simplifications} \\
&= \tfrac{1}{2}\delta\varphi\cdot\begin{pmatrix} 2\lambda_2 a_3 - 2\lambda_3 a_2 \\ 2\lambda_3 a_1 - 2\lambda_1 a_3 \\ 2\lambda_1 a_2 - 2\lambda_2 a_1 \end{pmatrix} \qquad \text{by appeal once again to (27)}
\end{aligned}$$

Thus do we obtain

$$\delta\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \delta\varphi\cdot\underbrace{\begin{pmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{pmatrix}}_{\text{generates rotations about the } \boldsymbol{\lambda}\text{-axis}}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

We have now in hand an association of the form

$$\mathbb{S}(\varphi,\boldsymbol{\lambda}) \longleftrightarrow \mathbb{R}(\boldsymbol{\varphi},\boldsymbol{\lambda})$$

between the elements

$$\mathbb{S}(\varphi,\boldsymbol{\lambda}) = \exp\left\{\tfrac{1}{2}\boldsymbol{\varphi}\begin{pmatrix} i\lambda_3 & \lambda_1 + i\lambda_2 \\ -\lambda_1 + i\lambda_2 & -i\lambda_3 \end{pmatrix}\right\} \tag{43}$$

of $SU(2)$ and the elements

$$\mathbb{R}(\varphi,\boldsymbol{\lambda}) = \exp\left\{\boldsymbol{\varphi}\begin{pmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{pmatrix}\right\} \tag{44}$$

of $O(3)$. In

$$\mathbb{R}(\varphi+2\pi,\boldsymbol{\lambda}) = \mathbb{R}(\varphi,\boldsymbol{\lambda}) \quad \text{but} \quad \mathbb{S}(\varphi+2\pi,\boldsymbol{\lambda}) = -\,\mathbb{S}(\varphi,\boldsymbol{\lambda})$$

(I omit the easy proof) we see the source of the biuniqueness of the association.

Look now again to (29), which we may notate

$$(\boldsymbol{a}\cdot\boldsymbol{r})^\ell = \left(\frac{r}{2}\right)^\ell \cdot \sum_{m=-\ell}^{m=+\ell} \xi^m(\ell)Q_m(\ell) \tag{45}$$

with

$$\begin{aligned} Q_m(\ell) &\equiv Q_\ell^m(\theta,\phi)\\ \xi^m(\ell) &\equiv u^{\ell-m}v^{\ell+m}\end{aligned}$$

Explicitly

$$\xi(0) = (\,1\,),\, \xi(\tfrac{1}{2}) = \begin{pmatrix} u\\ v\end{pmatrix}, \xi(1) = \begin{pmatrix} u^2\\ uv\\ v^2\end{pmatrix}, \xi(\tfrac{3}{2}) = \begin{pmatrix} u^3\\ u^2v\\ uv^2\\ v^3\end{pmatrix}, \xi(2) = \begin{pmatrix} u^4\\ u^3v\\ u^2v^2\\ uv^3\\ v^4\end{pmatrix}, \cdots$$

The object $\xi \equiv \xi(\frac{1}{2})$, with coordinates

$$\xi^\mu \equiv \xi^\mu(\tfrac{1}{2}) = \begin{cases} u & \text{if } \mu = -\frac{1}{2}\\ \\ v & \text{if } \mu = +\frac{1}{2}\end{cases}$$

lives in a 2-dimensional complex vector space $\mathfrak{S}$ called "spin space." The complex numbers

$$\xi^{\mu_1\mu_2\cdots\mu_r} \equiv \xi^{\mu_1}\xi^{\mu_2}\cdots\xi^{\mu_r}$$

are the components of a "2-dimensional spinor of rank r," which lives in the space $\mathfrak{S}\times\mathfrak{S}\times\cdots\times\mathfrak{S}$. The object $\xi(\ell)$ provides a columnar display of the components of the objects that live in the *symmetrized* product space $\mathfrak{S}^r\equiv\ \mathfrak{S}\vee\mathfrak{S}\vee\cdots\vee\mathfrak{S}$; $r = 2\ell$ and $\dim\mathfrak{S}^r = r+1 = 2\ell+1$.

"Spinor algebra" and "spinor analysis," generally conceived, can be understood to be the straightforward complex generalizations of tensor algebra and tensor analysis. One studies the transformations in product spaces which are induced by transformations in the base space, and pays special attention to objects which are transformationally invariant. The base space can, in general, be n-dimensional, but in the literature is frequently understood to be (as for us it presently is) 2-dimensional. Such, then, is the general context within which we ask this relatively narrow question: What can we say concerning the transformations within $\mathfrak{S}^r$ which are induced by unimodular transformations within $\mathfrak{S}$? By way of preparation for an attack on the problem, we note that (42) entails $\mathbb{L}^2 = -\mathbb{I}$, so from (43) it follows that

$$\mathbb{S}(\varphi,\boldsymbol{\lambda}) = \cos\tfrac{1}{2}\varphi\cdot\mathbb{I} + \sin\tfrac{1}{2}\varphi\cdot\mathbb{L} = \begin{pmatrix} \cos\frac{1}{2}\varphi + i\lambda_3\sin\frac{1}{2}\varphi & (\lambda_1+i\lambda_2)\sin\frac{1}{2}\varphi\\ -(\lambda_1-i\lambda_2)\sin\frac{1}{2}\varphi & \cos\frac{1}{2}\varphi - i\lambda_3\sin\frac{1}{2}\varphi\end{pmatrix}$$

can be described

$$\mathbb{S} = \begin{pmatrix} \alpha & \beta\\ -\beta^* & \alpha^*\end{pmatrix} \quad \text{with} \quad \alpha^*\alpha + \beta^*\beta = 1 \tag{46.1}$$

where

$$\begin{aligned}\alpha &= \alpha(\varphi,\boldsymbol{\lambda}) \equiv \cos\tfrac{1}{2}\varphi + i\lambda_3\sin\tfrac{1}{2}\varphi \\ \beta &= \beta(\varphi,\boldsymbol{\lambda}) \equiv (\lambda_1 + i\lambda_2)\sin\tfrac{1}{2}\varphi\end{aligned} \tag{46.2}$$

are the so-called "Cayley-Klein parameters." Consider now the transformation

$$\xi^\mu \longrightarrow \overline{\xi^\mu} = S^\mu{}_\nu(\alpha,\beta)\,\xi^\nu \quad \text{in } \mathfrak{S} \tag{47}$$

Explicitly

$$\begin{aligned}\overline{u} &= \alpha\,u + \ \beta\,v \\ \overline{v} &= -\beta^* u + \alpha^* v\end{aligned} \tag{48}$$

which in $\mathfrak{S}^{2\ell}$ entails

$$\begin{aligned}\xi^m(\ell) \longrightarrow \overline{\xi^m(\ell)} &= (\alpha u + \beta v)^{\ell-m}(-\beta^* u + \alpha^* v)^{\ell+m} \\ &= \text{polynomial of degree } 2\ell \text{ in the variables } \{u,v\}, \\ &\qquad\qquad\qquad \text{expressible therefore as follows:} \\ &= \sum_{n=-\ell}^{n=+\ell} S^m{}_n(\alpha,\beta;\ell)\,\xi^n(\ell) \\ &\equiv S^m{}_n(\ell)\,\xi^n(\ell)\end{aligned} \tag{49}$$

and gives back (47) at $\ell = \frac{1}{2}$. Explicit description of the $(2\ell+1)\times(2\ell+1)$ matrix $S^m{}_n(\ell)$ is straightforward in principle, if tedious in practice. In the case $\ell = 1$ one obtains, for example,

$$\mathbb{S}(1) = ||S^m{}_n(1)|| = \begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ -\alpha\beta^* & (\alpha^*\alpha - \beta^*\beta) & \beta\alpha^* \\ \beta^{*2} & -2\alpha^*\beta^* & \alpha^{*2}\end{pmatrix} \tag{50}$$

Actually, the case $\ell = 1$ acquires special interest from the following curious circumstance: (27) can be notated

$$\boldsymbol{a} = \mathbb{C}\,\boldsymbol{\xi}(1) \quad \text{with} \quad \mathbb{C} = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ -i & 0 & +i \\ 0 & 2i & 0\end{pmatrix}$$

so

$$\boldsymbol{\xi}(1) \longrightarrow \overline{\boldsymbol{\xi}(1)} = \mathbb{S}(1)\,\boldsymbol{\xi}(1) \qquad \Longleftrightarrow \qquad \boldsymbol{a} \longrightarrow \overline{\boldsymbol{a}} = \underbrace{\mathbb{C}\,\mathbb{S}(1)\,\mathbb{C}^{-1}}_{\mathbb{R}\text{ matrix of (44)}}\cdot\boldsymbol{a} \tag{51}$$

Noting that $\mathbb{C}$ admits of the decomposition

$$\mathbb{C} = \mathbb{D}\cdot\mathbb{U}$$

where

$$\mathbb{D} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{is diagonal and real, while}$$

$$\mathbb{U} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -i\frac{1}{\sqrt{2}} & 0 & +i\frac{1}{\sqrt{2}} \\ 0 & i & 0 \end{pmatrix} \quad \text{is unitary}$$

we have $\mathbb{R} = \mathbb{D}\,\mathbb{U}\,\mathbb{S}\,\mathbb{U}^{\mathsf{t}}\,\mathbb{D}^{-1}$ whence $\mathbb{R}^{\mathsf{T}} = \mathbb{R}^{\mathsf{t}} = \mathbb{D}^{-1}\,\mathbb{U}\,\mathbb{S}^{\mathsf{t}}\,\mathbb{U}^{\mathsf{t}}\,\mathbb{D}$. From $\mathbb{R}^{\mathsf{t}}\,\mathbb{R} = \mathbb{I}$ it therefore follows that $\mathbb{S}^{\mathsf{t}}\cdot\mathbb{U}^{\mathsf{t}}\,\mathbb{D}\,\mathbb{D}\,\mathbb{U}\cdot\mathbb{S} = \mathbb{U}^{\mathsf{t}}\,\mathbb{D}\,\mathbb{D}\,\mathbb{U}$; i.e., that $\mathbb{S} \equiv \mathbb{S}(1)$ is "unitary with respect to an induced metric":

$$\mathbb{S}^{\mathsf{t}}\,\mathbb{G}\,\mathbb{S} = \mathbb{G} \quad \text{where} \quad \mathbb{G} \equiv \mathbb{U}^{\mathsf{t}}\,\mathbb{D}\,\mathbb{D}\,\mathbb{U} = \mathbb{C}^{\mathsf{t}}\,\mathbb{C} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{52}$$

$$= \text{ induced metric matrix in } \mathfrak{S}^2$$

and that $\mathbb{T} \equiv \mathbb{C}\,\mathbb{S}\,\mathbb{C}^{-1} = \mathbb{D}\,\mathbb{U}\,\mathbb{S}\,\mathbb{U}^{\mathsf{t}}\,\mathbb{D}^{-1}$ is literally unitary: $\mathbb{T}^{\mathsf{t}}\,\mathbb{T} = \mathbb{I}$. The situation just encountered is (I assert without proof) entirely general:

$$\mathbb{S}^{\mathsf{t}}(\ell)\,\mathbb{G}(\ell)\,\mathbb{S}(\ell) = \mathbb{G}(\ell) \tag{53}$$

where the $2\ell+1$-dimensional "induced metric" is real, diagonal, and symmetric about the anti-diagonal:

$$\mathbb{G}(\ell) = \begin{pmatrix} G_\ell &&&&&& \\ & G_{\ell-1} &&&&& \\ && \ddots &&&& \\ &&& G_0 &&& \\ &&&& \ddots && \\ &&&&& G_{\ell-1} & \\ &&&&&& G_\ell \end{pmatrix} \tag{54}$$

We observe also that the matrix elements of $\mathbb{S}(\ell)$ are polynomials of degree 2ℓ in $\{\alpha, \alpha^*, \beta, \beta^*\}$, so

$$\{\alpha,\beta\} \longrightarrow \{-\alpha,-\beta\} \quad \text{induces} \quad \begin{cases} \mathbb{S}(\ell) \longrightarrow +\mathbb{S}(\ell) : \ell = 0,\ 1,\ 2,\ 3, \ldots \\ \\ \mathbb{S}(\ell) \longrightarrow -\mathbb{S}(\ell) : \ell = \ \frac{1}{2}, \frac{3}{2}, \frac{5}{2},\ \ldots \end{cases} \tag{55}$$

I am in position now to sketch the argument by which one might establish the orthonormality of the spherical harmonics $Q_\ell^m(\theta\phi)$. We look—compare (36) —to the expressions

$$C_{\ell'\ell''} \equiv \int_0^{2\pi}\int_0^{\pi} (\boldsymbol{a}\cdot\boldsymbol{r})^{*\ell'}(\boldsymbol{b}\cdot\boldsymbol{r})^{\ell''}\cdot\,\sin\theta\,d\theta\,d\phi \tag{56.1}$$

and note that these are on the one hand manifestly rotation-invariant, but (according to (45)) can on the other hand be described

$$C_{\ell'\ell''} = \left(\frac{r}{2}\right)^{\ell'+\ell''} \cdot \sum_{m'=-\ell'}^{m'=+\ell'} \sum_{m''=-\ell''}^{m''=+\ell''} \xi^{*m'}(\ell')\,\eta^{m''}(\ell'') \cdot \underbrace{\int_0^{2\pi}\int_0^{\pi} Q_{\ell'}^{*m'}(\theta,\phi) Q_{\ell''}^{m''}(\theta,\phi)\cdot \sin\theta\, d\theta\, d\phi} \tag{56.2}$$

We argue that of necessity $\ell' = \ell''$ (so we write ℓ in place of both) and

$$= \text{constant}\cdot G_{m'm''}(\ell)$$

Orthogonality then follows from the diagonality of $\mathbb{G}(\ell)$. To get a handle on the value of the multiplicative constant, we set $\boldsymbol{a} = \boldsymbol{b}$ and obtain

$$C_{\ell\ell} = \int_0^{2\pi}\int_0^{\pi} (\boldsymbol{a}\cdot\boldsymbol{r})^{*\ell}(\boldsymbol{a}\cdot\boldsymbol{r})^{\ell}\cdot \sin\theta\, d\theta\, d\phi$$

Without loss of generality we set

$$\boldsymbol{a} = a\begin{pmatrix}1\\ i\\ 0\end{pmatrix}$$

and obtain $(\boldsymbol{a}\cdot\boldsymbol{r})^*(\boldsymbol{a}\cdot\boldsymbol{r}) = a^*a\cdot(x-iy)(x+iy) = a^*a\cdot(r^2-z^2) = a^*a\cdot r^2\cos^2\theta$, giving

$$C_{\ell\ell} = r^{2\ell}\cdot(a^*a)^{\ell}\cdot 2\pi\cdot \underbrace{\int_0^{2\pi}\cos^{2\ell}\theta\cdot\sin\theta\, d\theta}_{=\frac{2}{2\ell+1}}$$

Evidently

$$2^{-\ell}\cdot\boldsymbol{\xi}^{\mathrm{t}}\mathbb{G}\boldsymbol{\xi}\cdot\text{constant} = (a^*a)^{\ell}\cdot 2\pi\cdot\frac{2}{2\ell+1} \tag{57}$$

I am, however, in position to carry the argument to completion only in the case $\ell = 1$, where we have

$$\begin{aligned}\boldsymbol{\xi}^{\mathrm{t}}\mathbb{G}\boldsymbol{\xi} = \begin{pmatrix}u^2\\ uv\\ v^2\end{pmatrix}^{\mathrm{t}}\begin{pmatrix}\frac{1}{2} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}u^2\\ uv\\ v^2\end{pmatrix} &= \tfrac{1}{2}\big(u^*u+v^*v\big)^2\\ &= a^*a \qquad \text{by (39)}\end{aligned}$$

giving

$$\text{constant} = \frac{8\pi}{3} \text{ in the case } \ell = 1$$

Finally we note it to be an implication of results now in hand that the spherical harmonics of given order ℓ transform among themselves in such a way as to establish a $2\ell+1$-dimensional representation of the rotation group $O(3)$. In this sense: let the functions

$$Q_m(\ell) \equiv Q_\ell^m(\theta,\phi): \quad m = -\ell, -(\ell-1), \ldots, -1, 0, +1, \ldots, +(\ell-1), +\ell$$

relate in the familiar way to a cartesian frame in 3-space, and let

$$\tilde{Q}_m(\ell) \equiv Q_\ell^m(\tilde{\theta},\tilde{\phi}): \quad m = -\ell, -(\ell-1), \ldots, -1, 0, +1, \ldots, +(\ell-1), +\ell$$

relate in that same way to a rotated frame. Looking to (45), we conclude from the rotational invariance of the expression on the left that in association with the rotationally-induced contravariant transformation of the spinor components $\xi^m(\ell)$ is a covariant transformation of the functions $Q_m(\ell)$:

$$\begin{aligned} \xi^m(\ell) &\xrightarrow[\mathbb{S}(\ell)]{} \tilde{\xi}^m(\ell) \\ Q_m(\ell) &\xrightarrow[\mathbb{S}^{-1}(\ell)]{} \tilde{Q}_m(\ell) \end{aligned} \tag{58}$$

4. Analytic theory of hyperspherical harmonics. Interesting problems emerge when one looks—as is from several points of view quite natural—to the *N-dimensional generalization* of the preceeding material. To write

$$\begin{aligned} y &= r\,\sin\phi \\ x &= r\cos\phi \end{aligned}$$

$$\begin{aligned} y &= r\,\sin\theta\,\sin\phi \\ x &= r\,\sin\theta\cos\phi \\ z &= r\cos\theta \end{aligned}$$

$$\begin{aligned} y &= r\,\sin\theta_2\,\sin\theta_1\,\sin\phi \\ x &= r\,\sin\theta_2\,\sin\theta_1\cos\phi \\ z_1 &= r\,\sin\theta_2\cos\theta_1 \\ z_2 &= r\cos\theta_2 \end{aligned}$$

is to see quite clearly the pattern of the nested construction by means of which spherical coordinates $\{r, \phi, \theta_1, \theta_2, \ldots, \theta_{N-2}\}$ are introduced in the general case. At equatorial points $(\theta_2 = \frac{1}{2}\pi)$ on the 4-sphere we recover the spherical coordinatization of 3-space, while at $\theta_2 = \theta_1 = \frac{1}{2}\pi$ we recover the polar coordinatization of the plane. Familiarly

$$\begin{aligned} (ds)^2_{\text{2-dimensional}} &= [d(r\cos\phi)]^2 + [d(r\sin\phi)]^2 \\ &= (dr)^2 + r^2(d\phi)^2 \end{aligned} \tag{59.1}$$

from which it follows that

$$
\begin{aligned}
(ds)^2_{\text{3-dimensional}} &= \underbrace{[d(r\cos\theta)]^2 + \big\{[d(r\sin\theta)]^2}_{=\,(dr)^2 + r^2(d\theta)^2} + (r\sin\theta)^2(d\phi)^2\big\}\\
&= (dr)^2 + (r\sin\theta)^2(d\phi)^2 + r^2(d\theta)^2 \qquad (59.2)
\end{aligned}
$$

$$
\begin{aligned}
&(ds)^2_{\text{4-dimensional}}\\
&\quad= \underbrace{[d(r\cos\theta_2)]^2 + \big\{[d(r\sin\theta_2)]^2}_{=\,(dr)^2 + r^2(d\theta_2)^2} + (r\sin\theta_2\sin\theta_1)^2(d\phi)^2 + (r\sin\theta_2)^2(d\theta_1)^2\big\}\\
&\quad= (dr)^2 + (r\sin\theta_2\sin\theta_1)^2(d\phi)^2 + (r\sin\theta_2)^2(d\theta_1)^2 + r^2(d\theta_2)^2 \qquad (59.3)
\end{aligned}
$$

$$
= \begin{pmatrix} dr \\ d\phi \\ d\theta_1 \\ d\theta_2 \end{pmatrix}^{\mathsf{T}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (r\sin\theta_2\sin\theta_1)^2 & 0 & 0 \\ 0 & 0 & (r\sin\theta_2)^2 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix}}_{=\,||g_{ij}(4)||} \begin{pmatrix} dr \\ d\phi \\ d\theta_1 \\ d\theta_2 \end{pmatrix} \qquad (60)
$$

where $||g_{ij}(4)||$ is the Euclidian metric in hyperspherical coordinates. Similarly (proceeding by what might be called the "method of dimensional ascent") we have

$$
||g_{ij}(5)|| = \begin{pmatrix} 1 & & & & \\ & (g_0)^2 & & & \\ & & (g_1)^2 & & \\ & & & (g_2)^2 & \\ & & & & (g_3)^2 \end{pmatrix}
$$

where

$$
\begin{aligned}
g_0 &\equiv r\sin\theta_3\sin\theta_2\sin\theta_1\\
g_1 &\equiv r\sin\theta_3\sin\theta_2\\
g_2 &\equiv r\sin\theta_3\\
g_3 &\equiv r
\end{aligned}
$$

entails $\sqrt{g(5)} = g_0g_1g_2g_3 = r^4\sin^3\theta_3\sin^2\theta_2\sin\theta_1$. Working from (5) we find that the Laplacian in (for example) 5-space can be described

$$
\begin{aligned}
\Delta_5 = \frac{1}{r^4}\frac{\partial}{\partial r}r^4\frac{\partial}{\partial r} &+ \frac{1}{r^2\sin^2\theta_3\sin^2\theta_2\sin^2\theta_1}\frac{\partial^2}{\partial\phi^2}\\
&+ \frac{1}{r^2\sin^2\theta_3\sin^2\theta_2}\cdot\frac{1}{\sin\ \theta_1}\frac{\partial}{\partial\theta_1}\sin\ \theta_1\frac{\partial}{\partial\theta_1}\\
&+ \frac{1}{r^2\sin^2\theta_3}\cdot\frac{1}{\sin^2\theta_2}\frac{\partial}{\partial\theta_2}\sin^2\theta_2\frac{\partial}{\partial\theta_2}\\
&+ \frac{1}{r^2}\cdot\frac{1}{\sin^3\theta_3}\frac{\partial}{\partial\theta_3}\sin^3\theta_3\frac{\partial}{\partial\theta_3}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{r^4}\frac{\partial}{\partial r}r^4\frac{\partial}{\partial r} + \frac{1}{r^2}\left\{\frac{1}{\sin^3\theta_3}\frac{\partial}{\partial\theta_3}\sin^3\theta_3\frac{\partial}{\partial\theta_3}\right. \\
&\qquad + \frac{1}{\sin^2\theta_3}\left[\frac{1}{\sin^2\theta_2}\frac{\partial}{\partial\theta_2}\sin^2\theta_2\frac{\partial}{\partial\theta_2}\right. \\
&\qquad + \left.\left.\left.\frac{1}{\sin^2\theta_2}\left[\frac{1}{\sin\ \theta_1}\frac{\partial}{\partial\theta_1}\sin\ \theta_1\frac{\partial}{\partial\theta_1} + \frac{1}{\sin^2\theta_1}\left[\frac{\partial^2}{\partial\phi^2}\right]\right]\right]\right\}
\end{aligned}
\tag{61}
$$

A standard line of argument leads from (61) to the conclusion that the fully-separated function

$$F = R(r)\cdot\varPhi(\phi)\cdot Z_1(\theta_1)\cdot Z_2(\theta_2)\cdot Z_3(\theta_3)$$

will be harmonic ($\Delta_5 F = 0$) if an only if

$$
\begin{aligned}
\left\{\frac{1}{r^2}\frac{d}{dr}r^4\frac{d}{dr} - \alpha_4\right\}R(r) &= 0\\
\left\{\frac{1}{\sin\theta_3}\frac{d}{d\theta_3}\sin^3\theta_3\frac{d}{d\theta_3} + \alpha_4\sin^2\theta_3 - \alpha_3\right\}Z_3(\theta_3) &= 0\\
\left\{\frac{d}{d\theta_2}\sin^2\theta_2\frac{d}{d\theta_2} + \alpha_3\sin^2\theta_2 - \alpha_2\right\}Z_2(\theta_2) &= 0\\
\left\{\sin\theta_1\frac{d}{d\theta_1}\sin\ \theta_1\frac{d}{d\theta_1} + \alpha_2\sin^2\theta_1 - \alpha_1\right\}Z_1(\theta_1) &= 0\\
\left\{\frac{d^2}{d\phi^2} + \alpha_1\right\}\varPhi(\phi) &= 0
\end{aligned}
\tag{62}
$$

where α_1, α_2, α_3 and α_4 are separation constants. Equivalently we have, in reversed order (i.e., in the order in which the equations are serially to be solved),

$$\left\{\frac{d^2}{d\phi^2} + \alpha_1\right\}\varPhi(\phi) = 0 \tag{63.1}$$

$$\left\{(1-\omega_1^2)\frac{d^2}{d\omega_1^2} - 2\omega_1\frac{d}{d\omega_1} + \alpha_2 - \frac{\alpha_1}{1-\omega_1^2}\right\}P_1(\omega_1) = 0 \tag{63.21}$$

$$\left\{(1-\omega_2^2)\frac{d^2}{d\omega_2^2} - 3\omega_2\frac{d}{d\omega_2} + \alpha_3 - \frac{\alpha_2}{1-\omega_2^2}\right\}P_2(\omega_2) = 0 \tag{63.22}$$

$$\left\{(1-\omega_3^2)\frac{d^2}{d\omega_3^2} - 4\omega_3\frac{d}{d\omega_3} + \alpha_4 - \frac{\alpha_3}{1-\omega_3^2}\right\}P_3(\omega_3) = 0 \tag{63.23}$$

$$\left\{\frac{1}{r^2}\frac{d}{dr}r^4\frac{d}{dr} - \alpha_4\right\}R(r) = 0 \tag{63.3}$$

where $\omega_k \equiv \cos\theta_k$. Solutions of (63.1) are of the form $\varPhi(\phi) \sim e^{im\phi}$ and a regularity condition entails (compare (13))

$$\alpha_1 = m^2 \quad \text{with} \quad m = 0, \pm1, \pm2, \cdots$$

Equation (63.21)—equivalently (62)/ $\sin^2\theta_1$—is precisely the equation which (in the discussion subsequent to (15)) was seen already to entail

$$\alpha_2 = \ell(\ell+1) \quad \text{with} \quad \ell = m, m+1, m+2, \cdots$$

and to give rise to the Legendre functions

$$P_\ell^m(\omega) \sim \big(1-\omega^2\big)^{\frac{1}{2}m}\left(\frac{d}{d\omega}\right)^{\ell+m}\big(\omega^2-1\big)^\ell$$

Equation (63.22) assumes therefore the structure

$$\left\{\big(1-\omega^2\big)\frac{d^2}{d\omega^2} - 3\omega\frac{d}{d\omega} + \alpha - \frac{\ell(\ell+1)}{1-\omega^2}\right\}P_2(\omega) = 0 \tag{64}$$

In preparation now for a chain of argument (63.21)⟶(63.22)⟶(63.23)⟶ $\cdots$ we observe that the following statement (which provides yet another instance of a "shift rule")

$$\begin{aligned}&\big(1-\omega^2\big)^{\frac{1}{2}m}D^m\cdot\Big\{\big(1-\omega^2\big)D^2 - \big(2+k\big)\omega D + \alpha\Big\}\\ &\qquad = \left\{\big(1-\omega^2\big)D^2 - \big(2+k\big)\omega D + \alpha - \frac{m(m+k)}{1-\omega^2}\right\}\cdot\big(1-\omega^2\big)^{\frac{1}{2}m}D^m\end{aligned} \tag{65}$$

(here $D \equiv \frac{d}{d\omega}$) holds as an operator identity (and gives back (17) in the case $k=0$), and has this implication: if $G(\omega)$ is a solution of

$$\Big\{\big(1-\omega^2\big)D^2 - \big(2+k\big)\omega D + \alpha\Big\}G(\omega) = 0 \tag{66.1}$$

then $F(\omega) \equiv \big(1-\omega^2\big)^{\frac{1}{2}m}D^m G(\omega)$ is a solution of

$$\left\{\big(1-\omega^2\big)D^2 - \big(2+k\big)\omega D + \alpha - \frac{m(m+k)}{1-\omega^2}\right\}F(\omega) = 0 \tag{66.2}$$

It becomes evident on this basis that

$$F^\ell(\omega) \equiv \big(1-\omega^2\big)^{\frac{1}{2}\ell}D^\ell G(\omega) \tag{67.1}$$

will be a solution of (64) if $G(\omega)$ is a solution of (66.1) with $k=1$. But equations of the type (66.1) were studied in the 1890's by Gegenbauer, who found that regular solutions exist if an only if

$$\alpha = \ell(\ell+1+k) \quad \text{with} \quad \ell = 0, 1, 2, \cdots \tag{67.2}$$

and are in such cases given by the Gegenbauer polynomials $G_\ell(\omega;k)$, which are generated as follows

$$\left[\frac{1}{\sqrt{1-2x\omega+x^2}}\right]^{1+k} = \sum_{n=0}^\infty G_\ell(\omega;k)x^\ell \qquad (67.3$$

$$G_\ell(\omega;k) \sim \underbrace{\big(1-\omega^2\big)^{-\frac{1}{2}k}\left(\frac{d}{d\omega}\right)^\ell\big(\omega^2-1\big)^{\ell+\frac{1}{2}k}}_{\text{polynomial of degree } \ell}$$

and reduce to the Legendre polynomials at $k=0$. What I call the "associated Gegenbauer functions" are defined

$$G_\ell^m(\omega;k)\equiv\big(1-\omega^2\big)^{\frac{1}{2}m}D^mG_\ell(\omega;k)\quad:\quad m=0,1,2,\ldots,\ell$$

which give back the associated Legendre functions at $k=0$. We are in position now to proceed directly to the following distillation of the implications of (63):

$$\left.\begin{aligned}\Big\{\frac{d^2}{d\phi^2}+m^2\Big\}e^{\pm im\phi}&=0\\ \Big\{\big(1-\omega_1^2\big)\frac{d^2}{d\omega_1^2}-2\omega_1\frac{d}{d\omega_1}+\ell_1(\ell_1+1)-\frac{m^2}{1-\omega_1^2}\Big\}G_{\ell_1}^m(\omega_1;0)&=0\\ \Big\{\big(1-\omega_2^2\big)\frac{d^2}{d\omega_2^2}-3\omega_2\frac{d}{d\omega_2}+\ell_2(\ell_2+2)-\frac{\ell_1(\ell_1+1)}{1-\omega_2^2}\Big\}G_{\ell_2}^{\ell_1}(\omega_2;1)&=0\\ \Big\{\big(1-\omega_3^2\big)\frac{d^2}{d\omega_3^2}-4\omega_3\frac{d}{d\omega_3}+\ell_3(\ell_3+3)-\frac{\ell_2(\ell_2+2)}{1-\omega_3^2}\Big\}G_{\ell_3}^{\ell_2}(\omega_3;2)&=0\\ \Big\{\frac{1}{r^2}\frac{d}{dr}r^4\frac{d}{dr}-\ell_3(\ell_3+3)\Big\}R(r)&=0\end{aligned}\right\}\tag{68}$$

The general solution of the last equation is seen easily to have the form

$$R_{\ell_3}(r)=Ar^{\ell_3}+B\frac{1}{r^{\ell_3+3}}\tag{69}$$

The orthogonality (and normalization) of the functions

$$\begin{aligned}&Y_{\ell_p\cdots\ell_2\ell_1}^m(\theta_p,\cdots,\theta_2,\theta_1,\phi)\\&\quad=\text{normalization factor}\cdot G_{\ell_p}^{\ell_{p-1}}(\theta_p;p-1)\cdots G_{\ell_2}^{\ell_1}(\theta_2;1)\cdot G_{\ell_1}^m(\theta_1;0)e^{im\phi}\\&\qquad\qquad\qquad\qquad\ell_p\geq\ell_{p-1}\geq\cdots\geq\ell_2\geq\ell_1\geq|m|\geq0\\&\qquad\qquad\qquad\qquad p=N-2\end{aligned}$$

can be extracted from these two facts: differential surface area on the unit sphere in N-space is given by

$$d\Omega=\sin^p\theta_p\cdots\sin^2\theta_2\sin\theta_1\,d\theta_1\,d\theta_2\cdots d\theta_p$$

and the Gegenbauer polynomials have this orthogonality property:

$$\int_0^\pi G_m(\cos\theta;k)G_n(\cos\theta;k)\sin^{k+1}\theta\,d\theta=\begin{cases}0 & \text{if } m\neq n\\ \text{complicated factor} & \text{if } m=n\end{cases}$$

I must, however, refer readers to the standard handbooks for the details.

The hyperspherical harmonics $Y_{\ell_p\cdots\ell_2\ell_1}^m(\theta_p,\cdots,\theta_2,\theta_1,\phi)$ permit one to do "Fourier analysis" on the surface of an $N=p+2$-dimensional hypersphere. In a curious sense we have labored harder and harder to do less and less, for the surface area of an N-sphere of unit radius is given by $S_N=2\pi^{\frac{N}{2}}/\Gamma(\frac{N}{2})$, which is maximal at $N=7$ vanishes in the limit $N\to\infty$.

5. Concluding remarks & open questions. Such then, in bald outline, is the inexhaustibly rich (and intricate!) analytic theory of hyperspherical harmonics, which I have reviewed here in order to be in position to pose this question: Can Kramers' technique be generalized in such a way as to yield an *algebraic* theory of hyperspherical harmonics? In (for example) 5-space $(\boldsymbol{a}\cdot\boldsymbol{r})^{\ell_3}$ is clearly rotationally invariant, and harmonic if $\boldsymbol{a}$ is null. Can one, in imitation of (27), complex-parameterize the set of null 5-vectors and thus, from an analog of (29), recover the functional data displayed in (68)? On the evidence only of my many failures, I have come to the very tentative conclusion that *Kramers' technique does not generalize; constructions imitative of (29) are impossible for* $N > 3$. What I would like to see is either (i) a constructive solution of the problem, or (ii) a clear indication of *why* such a construction is impossible.

Returning now to the 3-dimensional context of our initial discussion, we found that the transformational theory, as it emerged from the algebraic line of argument, was unexpectedly richer than the classical theory of spherical harmonics. Looking specifically to (55), we note that while

$$(\mathbf{a}\cdot\mathbf{r})^\ell \quad \text{is a polynomial for } \ell = 0, 1, 2, \cdots$$

$(\mathbf{a}\cdot\mathbf{r})^\ell$ is *not* a polynomial if $\ell = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$. In quantum mechanics the half-integral representations (the "spin representations") of $O(3)$ do play an important role, and announce their presence by the characteristic appearance of multi-component wave functions, with

$$\text{number of components} = 2\ell + 1 = 2, 4, 6, \cdots$$

There is, however, an alternative which (though precluded by the quantum mechanical requirement that the wave function be single-valued) is available in principle to some applications. For the functions

$$Y_\ell^m(\theta,\phi) \sim e^{im\phi}\big(1-\omega^2\big)^{\frac{1}{2}m}\left(\frac{d}{d\omega}\right)^{\ell+m}\big(\omega^2-1\big)^\ell$$

remain meaningful even when ℓ and m are (both) half-integral. Such functions —the construction of which (since $\ell+m$ is integral-valued) does not even entail the concept of "fractional differentiation"[5]—would appear to permit one to perform Fourier analysis on the *double*-sphere. In quantum mechanics one has sometimes to distintinguish 720° rotations (which are equivalent to the identity) from 360° rotations (which aren't), and it was to illustrate this fact that Dirac invented his famous "spinor spanner."

There have been two principal actors in my story, as I have told it: *algebra* and *analysis*. A third actor—*group representation theory*—would have heavy

[5] We touch here, interestingly, on yet another subject of which Laplace was a founding father; see Chapter I of K. S. Miller & B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (1993).

contributions to make in any more complete account, and the potential for cross-talk seems inexhaustible. Concerning the group-theoretic aspect of my topic I must on the present occasion be content to record only a few incidental remarks.

The sense in which the hyperspherical harmonics $Y^m_{\ell_p\cdots\ell_2\ell_1}(\theta_p,\cdots,\theta_2,\theta_1,\phi)$ can be expected to "fold among themselves in such a manner as to provide representations of $O(N)$" is most transparently evident in the case $N=2$, where the "spherical harmonics" are (we disregard all normalization factors)

$$Y^0(\phi)=1 \quad\text{and}\quad Y^{\pm m}(\phi)=e^{\pm im\phi} \quad : \quad m=1,2,\ldots$$

which organize naturally into an array of this design:

$$\begin{array}{ccccccc} & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots \\ \bullet & & & & & & \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots \end{array}$$

Action of the elements of $O(2)$ can be described $Y(\phi)\longrightarrow Y(\tilde\phi)=Y(\phi+\alpha)$ which in real terms entails

$$\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}\longrightarrow\begin{pmatrix}\cos m\tilde\phi\\ \sin m\tilde\phi\end{pmatrix}=\underbrace{\begin{pmatrix}\cos m\alpha & -\sin m\alpha\\ \sin m\alpha & \cos m\alpha\end{pmatrix}}_{\mathbb{R}(\alpha;m)}\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}$$

Evidently (certain real linear combinations of) the "2-dimensional spherical harmonics" of leading non-trivial order $m=1$ transform in direct imitation of the elements of $O(2)$

$$\begin{pmatrix}x\\ y\end{pmatrix}\longrightarrow\begin{pmatrix}\tilde x\\ \tilde y\end{pmatrix}=\mathbb{R}(\alpha)\begin{pmatrix}x\\ y\end{pmatrix}\quad\text{with}\quad\mathbb{R}(\alpha)\equiv\mathbb{R}(\alpha;1)$$

while those of higher order $m>1$ provide a population of alternative 2×2 matrix representations $\mathbb{R}(\alpha;m)=\mathbb{R}^m(\alpha)$ of $O(2)$. Turning from $O(2)$ to $O(3)$, the spherical harmonics organize into an array of the design

$$\begin{array}{cccccccl}
 & & & & & & \bullet & \\
 & & & & & \circ & & \\
 & & & & \bullet & & \bullet & \\
 & & & \circ & & \circ & & \\
 & & \bullet & & \bullet & & \bullet & \\
 & \circ & & \circ & & \circ & & \\
\bullet & & \bullet & & \bullet & & \bullet & \quad\text{ascending } \ell \longrightarrow \\
 & \circ & & \circ & & \circ & & \\
 & & \bullet & & \bullet & & \bullet & \\
 & & & \circ & & \circ & & \\
 & & & & \bullet & & \bullet & \\
 & & & & & \circ & & \\
 & & & & & & \bullet &
\end{array}$$

Here again, (certain real linear combinations of) the 3-dimensional spherical harmonics of leading non-trivial order $\ell = 1$ transform in direct imitation

$$\begin{pmatrix} \bullet \\ \bullet \\ \bullet \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{\bullet} \\ \tilde{\bullet} \\ \tilde{\bullet} \end{pmatrix} = \mathbb{R} \begin{pmatrix} \bullet \\ \bullet \\ \bullet \end{pmatrix}$$

of the geometrical action of the elements of $O(3)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \mathbb{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

while the spherical harmonics of higher order $\ell > 1$ provide $O(3)$ representations of ascending odd dimension. Interleaved among those are the even-dimensional spinor representations of ascending dimension

$$\begin{pmatrix} \circ \\ \circ \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{\circ} \\ \tilde{\circ} \end{pmatrix} = \mathbb{S} \begin{pmatrix} \circ \\ \circ \end{pmatrix}$$

One expects the pattern of these remarks to be repeated in the N-dimensional case, but I look here only to the "numerological" aspects of the situation. Enlarging upon prior usage, let us agree to call $\ell \equiv \ell_p$ the "order" of the hyperspherical function $Y^m_{\ell_p \cdots \ell_2 \ell_1}(\theta_p, \cdots, \theta_2, \theta_1, \phi)$, and let

$$\#(\ell; N) \equiv \text{number of } Y\text{-functions of order } \ell \text{ in the } N\text{-dimensioinal case}$$

Familiarly

$$\#(\ell; 2) = \begin{cases} 1 & \text{for} \quad \ell = 0 \\ 2 & \text{for} \quad \ell = 1, 2, 3, \ldots \end{cases}$$

and

$$\#(\ell; 3) = 2\ell + 1 \quad \text{for} \quad \ell = 0, 1, 2, 3, \ldots$$

And from $\ell \geq \ell_{p-1} \geq \ell_{p-2} \geq \cdots \geq \ell_1$ we obtain

$$\#(\ell; N) = \sum_{k=0}^{k=\ell} \#(k; N-1)$$

which gives rise to the following self-explanatory table:

$$\begin{array}{ccccccccccccc}
1 & & 2 & & 2 & & 2 & & 2 & & 2 & & \ldots \\
1 & \circ & 3 & \circ & 5 & \circ & 7 & \circ & 9 & \circ & 11 & & \ldots \\
1 & & 4 & & 9 & & 16 & & 25 & & 36 & & \ldots \\
1 & & 5 & & 14 & & 30 & & 55 & & 91 & & \ldots \\
1 & & 6 & & 20 & & 50 & & 105 & & 196 & & \ldots \\
1 & & 7 & & 27 & & 77 & & 182 & & 378 & & \ldots \\
1 & & 8 & & 35 & & 112 & & 294 & & 672 & & \ldots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Dimension N marches down the second column; it appears therefore plausible that a statement of the form "(certain real linear combinations of) the N-dimensional spherical harmonics of leading non-trivial order $\ell = 1$ transform in direction imitation of the geometrical action of the elements of $O(N)$" will hold *generally*, not just in the cases $N = 2$ and $N = 3$. I have marked $\circ$ the interstices which in the case $N = 3$ are occupied by the spin representations of $O(3)$. The absence of such simply-patterned interstices for $N > 3$ lends seeming weight to my conclusion that "Kramers' construction does not generalize." We notice, however, that the preceding display is strongly reminiscent of what might be called the "table of hypertrangular numbers"

$$
\begin{array}{lcccccccc}
N=1 & : & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
N=2 & : & 1 & 3 & 6 & 10 & 15 & 21 & \ldots \\
N=3 & : & 1 & 4 & 10 & 20 & 35 & 56 & \ldots \\
N=4 & : & 1 & 5 & 15 & 35 & 70 & 126 & \ldots \\
N=5 & : & 1 & 6 & 21 & 56 & 126 & 252 & \ldots \\
N=6 & : & 1 & 7 & 28 & 84 & 210 & 462 & \ldots \\
N=7 & : & 1 & 8 & 36 & 120 & 330 & 792 & \ldots \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

and that if one (i) doubles every entry and (ii) makes the replacement $N \to N{+}2$ one does obtain a plausible "table of spinorial interstitials." While I attach no great weight to results achieved by such "mere numerology," I do note with sharpened interest that such constructions do in fact occur in accounts of the "irreducible representations of $O(N)$."[6]

Two final observations: Kramers' method brings to mind some aspects of a line of argument due to Maxwell.[7] And it is vividly evocative of an operator-algebraic quantum theory of angular momentum which was devised (but never properly published) by J. Schwinger.[8] It would be amusing to work out the detailed interconnections, and to discover more particularly what Maxwell/Schwinger might have to say if the world were N-dimensional.

Notes & references. Laplace's equation $\nabla^2 V(x, y, z) = 0$ appears for the first time in a paper of 1789 concerned with the stability of the rings of Saturn—a problem which later was to engage the attention also of Maxwell.

Hendrik Anthony Kramers (1894-1952) was a student of Paul Ehrenfest, and in 1934 succeeded Ehrenfest at the University of Leiden. Kramers is remembered by physicists today mainly as the "K" in the "WKB method,"

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[6] See, for example, Chapter V, §7 of H. Weyl, *The Classical Groups: Their Invariants and Representations* (Second edition, 1946).

[7] See, for example, Volume I, Chapter VII, §5 of R. Courant & D. Hilbert, *Methods of Mathematical Physics* (1953) or my own ELECTRODYNAMICS (1972), pp. 397–402.

[8] "On angular momentum," publication NYO–3071 (26 January 1952) of the U. S. Atomic Energy Commission.

but during his short life he made deep contributions—all notable for their mathematical sophistication—to a wide assortment of topical areas. The work reported here was apparently based on Weyl's treatment of the rotation group and on work then current on the theory of invariants. The theory of spinors was roughly contemporaneous with the work of Kramers; van der Waerden's "Spinoranalyse" appeared in 1929, and in 1935 R. Brauer & H. Weyl published an account of É. Cartan's "Spinors in n Dimensions" (American Journal of Mathematics, **57**, 425). A good modern source is Cartan's *The Theory of Spinors* (Dover, 1981, translated from the French edition of 1937).

The Gegenbauer polynomials (also called "ultraspherical polynomials") are relatively late additions to the population of "special functions of mathematical physics." They generalize the theory of Legendre polynomials, and have very close associations with the Laguerre, Hermite and Tschebyscheff polynomials. All are special cases of Gauß' hypergeometric function

$$F(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots$$

which, interestingly, predates most of the classic theory of special functions (Gauß published in 1813) and derive their name by allusion not to "geometry in hyperspace" but (via Euler to a usage introduced by Wallis in 1655) to "generalized geometric series." My Gegenbauer notation is eccentric (intended to simplify expression of the results of most immediate interest to me). I have found A. Erdélyi's *Higher Transcendental Functions* (Bateman Manuscript Project, McGraw-Hill, 1953), §3.15; the Appendix to the Fourth Chapter of W. Magnus & F. Oberhettinger's *Formulas & Theorems for the Functions of Mathematical Physics* (Chelsea, 1954); and B. C. Carlson's *Special Functions of Applied Mathematics* (Academic Press, 1977) to be particularly helpful. An elaborate account of the theory of "Spherical & Hyperspherical Harmonic Polynomials" can be found in Erdélyi's Chapter XI (which was reportedly based on unpublished course notes by G. Herglotz); for an alternative account (without attribution) of Kramers' method, see §11.5.1.

A wonderfully engaging account—in verse yet!—of the theory of Dirac's spinor spanner can be found on pp. 93–98 of L. H. Kauffman's *On Knots* (Annals of Mathematics Studies Number 115 (Princeton 1987)). Kaufman, writing under the title "Quaternions and the Belt Trick," makes explicit contact with the "Pauli matrices" which come to light when (42) is written

$$\mathbb{L} = i\big\{\lambda_1\boldsymbol{\sigma}_1 + \lambda_2\boldsymbol{\sigma}_2 + \lambda_3\boldsymbol{\sigma}_3\big\}$$

Also of interest in this connection is A. Jursišić, "The Mercedes knot problem," (Amer. Math. Monthly **103**, 756 (1996)).

Kramer's algebraic method is seldom encountered in textbooks. A fairly detailed account of the method, and useful references, can, however, be found in §7–2 of J. L. Powell & B. Crassmann, *Quantum Mechanics* (1961). When I had occasion (1998) with Crassmann, he responded "Oh, that was Powell's contribution. He had learned of the method when a student of E. P. Wigner, and was always fond of it."