```
2 - 2
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```

Random Walks on a Hexagonal Lattice

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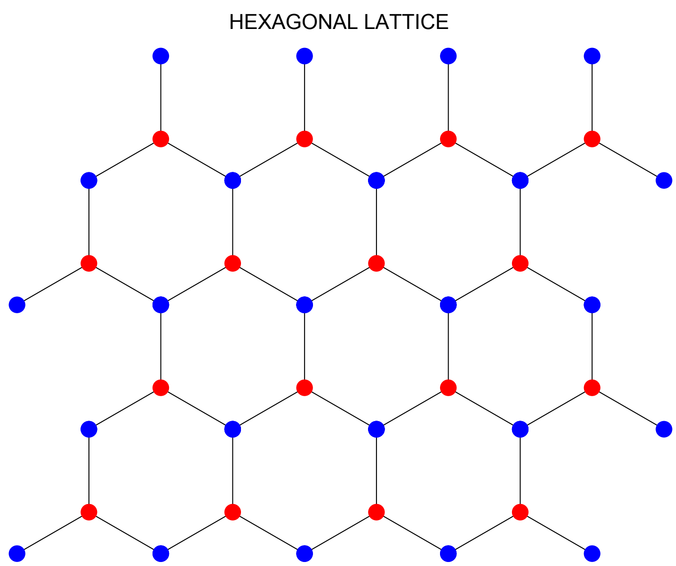
Introduction

This work is motivated by my 2nd conversation with Sina Zeytinoglu (whose previous visit motivated me to write "Iteration Techniques" (23 November 2011)). Here is a hexagonal lattice:

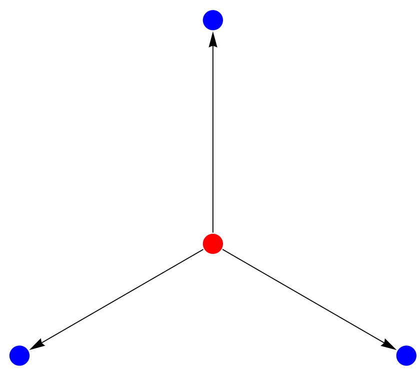


Notice that this is what I have been motivated elsewhere to call a "blinking" design: the node on which the walker stands changes color with each step.

The following vectors describe the steps (of unit length) available to a walker who stands on ●

```
R1 = {Cos[π/2], Sin[π/2]};
R2 = {Cos[π/2 - 4 (2π)/12], Sin[π/2 - 4 (2π)/12]};
R3 = {Cos[π/2 - 8 (2π)/12], Sin[π/2 - 8 (2π)/12]};
```

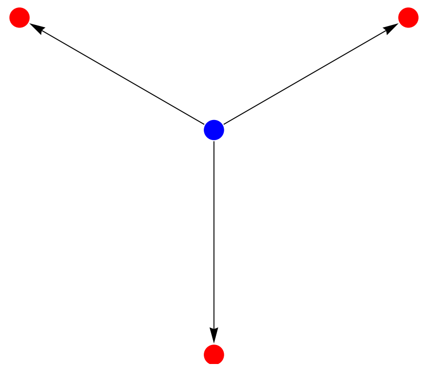
```
RedOptions = Graphics[{{Red, PointSize[0.05], Point[{0, 0}]},
   {Blue, PointSize[0.05], Point[R1]},
   {Blue, PointSize[0.05], Point[R2]},
   {Blue, PointSize[0.05], Point[R3]},
   {Arrow[{{0, 0}, R1}, .05]},
   {Arrow[{{0, 0}, R2}, .05]},
   {Arrow[{{0, 0}, R3}, .05]}}]
```



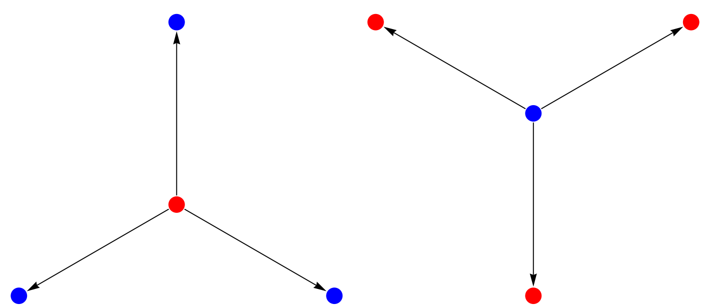
while these vectors—the negatives of the preceding vectors—describe the unit steps available to a walker who stands momentarily on ●:

```
B1 = -R1;
B2 = -R2;
B3 = -R3;
```

```
BlueOptions = Graphics[{{Blue, PointSize[0.05], Point[{0, 0}]},
   {Red, PointSize[0.05], Point[B1]},
   {Red, PointSize[0.05], Point[B2]},
   {Red, PointSize[0.05], Point[B3]},
   {Arrow[{{0, 0}, B1}, .05]},
   {Arrow[{{0, 0}, B2}, .05]},
   {Arrow[{{0, 0}, B3}, .05]}}]
```



```
GraphicsArray[{RedOptions, BlueOptions}]
```

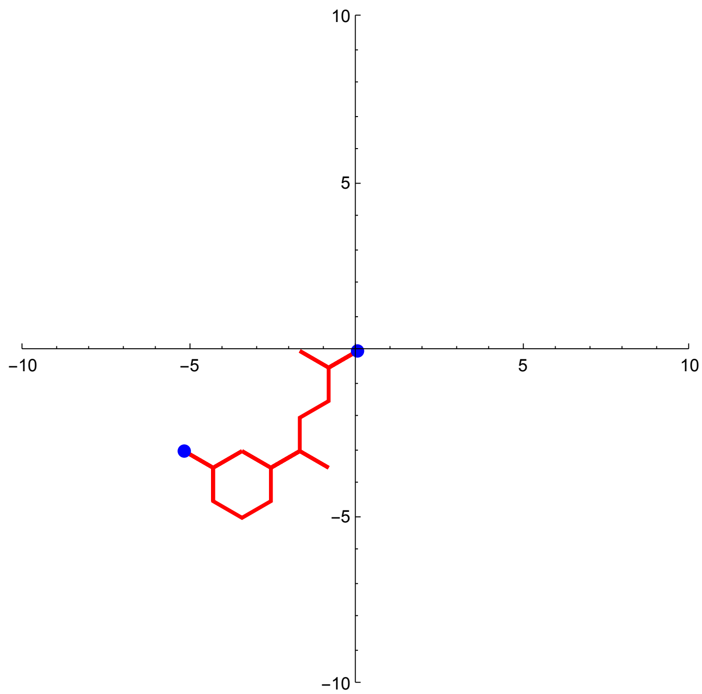


"By hand" construction of a 30-step hexagonal walk

The following multiple command serves to construct 30-step hexagonal walks "by hand":

```
S1 = RandomChoice[{R1, R2, R3}];
S2 = RandomChoice[{B1, B2, B3}];
S3 = RandomChoice[{R1, R2, R3}];
S4 = RandomChoice[{B1, B2, B3}];
S5 = RandomChoice[{R1, R2, R3}];
S6 = RandomChoice[{B1, B2, B3}];
S7 = RandomChoice[{R1, R2, R3}];
S8 = RandomChoice[{B1, B2, B3}];
S9 = RandomChoice[{R1, R2, R3}];
S10 = RandomChoice[{B1, B2, B3}];
S11 = RandomChoice[{R1, R2, R3}];
S12 = RandomChoice[{B1, B2, B3}];
S13 = RandomChoice[{R1, R2, R3}];
S14 = RandomChoice[{B1, B2, B3}];
S15 = RandomChoice[{R1, R2, R3}];
S16 = RandomChoice[{B1, B2, B3}];
S17 = RandomChoice[{R1, R2, R3}];
S18 = RandomChoice[{B1, B2, B3}];
S19 = RandomChoice[{R1, R2, R3}];
S20 = RandomChoice[{B1, B2, B3}];
S21 = RandomChoice[{R1, R2, R3}];
S22 = RandomChoice[{B1, B2, B3}];
S23 = RandomChoice[{R1, R2, R3}];
S24 = RandomChoice[{B1, B2, B3}];
S25 = RandomChoice[{R1, R2, R3}];
S26 = RandomChoice[{B1, B2, B3}];
S27 = RandomChoice[{R1, R2, R3}];
S28 = RandomChoice[{B1, B2, B3}];
S29 = RandomChoice[{R1, R2, R3}];
S30 = RandomChoice[{B1, B2, B3}];
SuccessivePositions = Table[Pk = Sum[Sj, {j, 1, k}], {k, 1, 30}];

HexagonalWalk = Prepend[SuccessivePositions, {0, 0}];

Walk30 = ListLinePlot[HexagonalWalk, AspectRatio → Automatic, AxesOrigin → {0, 0},
   PlotRange → {{-10, 10}, {-10, 10}}, PlotStyle → {Thick, Red}];

EndPoints30 = Graphics[{{Blue, PointSize[0.02], Point[{0, 0}]},
    {Blue, PointSize[0.02], Point[Last[HexagonalWalk]]}}];

Show[{Walk30, EndPoints30}]
```

In the notebook, all subscripts are printed as subscripts (S_1, R_1, B_1, P_k, S_j), and the sum is printed as $\sum_{j=1}^{k} S_j$.



Many repetitions later, I am surprised by the frequency with which the walker doubles back, retraces previous steps.

Automating the construction of such walks

I can't get the **Nest** command to construct *alternating* RBRBRBRB sequences, so adopt a relatively clumsy procedure of which I provide first a sketch:

```
A = Table[Transpose[
   {{RandomChoice[{R1, R2, R3}], RandomChoice[{B1, B2, B3}]}}], {k, 1, 10}]

B = Table[MatrixPower[(0 1
                       1 0), k].A[[k]], {k, 1, 10}]

Steps = Table[Flatten[B[[k]][[1]]][[1]], {k, 1, 10}]
```

```
{{{R3}, {B3}}, {{R3}, {B2}}, {{R2}, {B2}}, {{R1}, {B3}}, {{R1}, {B3}},
 {{R3}, {B2}}, {{R3}, {B1}}, {{R3}, {B1}}, {{R2}, {B1}}, {{R2}, {B2}}}

{{{B3}, {R3}}, {{R3}, {B2}}, {{B2}, {R2}}, {{R1}, {B3}}, {{B3}, {R1}},
 {{R3}, {B2}}, {{B1}, {R3}}, {{R3}, {B1}}, {{B1}, {R2}}, {{R2}, {B2}}}

{B3, R3, B2, R1, B3, R3, B1, R3, B1, R2}
```

In the context at hand the objects R1, R2, ..., B3 are 2-vectors. I find that *Mathematica* objects to subscripted notation in this context, so I renotate my vectors:

$$R1 = \left\{\text{Cos}\left[\frac{\pi}{2}\right], \text{Sin}\left[\frac{\pi}{2}\right]\right\};$$

$$R2 = \left\{\text{Cos}\left[\frac{\pi}{2} - 4\,\frac{2\,\pi}{12}\right], \text{Sin}\left[\frac{\pi}{2} - 4\,\frac{2\,\pi}{12}\right]\right\};$$

$$R3 = \left\{\text{Cos}\left[\frac{\pi}{2} - 8\,\frac{2\,\pi}{12}\right], \text{Sin}\left[\frac{\pi}{2} - 8\,\frac{2\,\pi}{12}\right]\right\};$$

```
B1 = -R1;
B2 = -R2;
B3 = -R3;
```

Step 1 of the procedure sketched above produces

```
A = Table[
  Transpose[{{RandomChoice[{R1, R2, R3}], RandomChoice[{B1, B2, B3}]}}], {k, 1, 10}]
```

$$\left\{\left\{\left\{\left\{-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}, \left\{\left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}\right\}, \{\{\{0, 1\}\}, \{\{0, -1\}\}\}, \left\{\{\{0, 1\}\}, \left\{\left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}\right\}, \left\{\left\{\left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}, \left\{\left\{-\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}\right\}, \left\{\{\{0, 1\}\}, \left\{\left\{-\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}\right\}, \left\{\left\{\left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}, \left\{\left\{-\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}\right\}, \left\{\left\{\left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}, \{\{0, -1\}\}\right\}, \{\{\{0, 1\}\}, \{\{0, -1\}\}\}, \left\{\left\{\left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}, \left\{\left\{-\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}\right\}, \left\{\left\{\left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}, \{\{0, -1\}\}\right\}\right\}$$

```
A[[1]]
```

$$\left\{\left\{\left\{-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}, \left\{\left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}\right\}$$

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ serves still to reverse those vectors

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\%$$

$$\left\{\left\{\left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}, \left\{\left\{-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}\right\}$$

so we construct

```
B = Table[MatrixPower[(0 1
                        1 0), k].A[[k]], {k, 1, 10}]
```

$$\left\{\left\{\left\{\left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}, \left\{\left\{-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}\right\}, \{\{\{0, 1\}\}, \{\{0, -1\}\}\},\right.$$
$$\left\{\left\{\left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}, \{\{0, 1\}\}\right\}, \left\{\left\{\left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}, \left\{\left\{-\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}\right\}, \left\{\left\{\left\{-\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}, \{\{0, 1\}\}\right\},$$
$$\left\{\left\{\left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}, \left\{\left\{-\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}\right\}, \left\{\{\{0, -1\}\}, \left\{\left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}\right\},$$
$$\left.\{\{\{0, 1\}\}, \{\{0, -1\}\}\}, \left\{\left\{\left\{-\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}\right\}, \left\{\left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}\right\}, \left\{\left\{\left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}, \{\{0, -1\}\}\right\}\right\}$$

Commands such as these serve to extract the leading member of each vector pair:

```
B[[1]][[1]][[1]]
B[[2]][[1]][[1]]
B[[3]][[1]][[1]]
```

$$\left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}$$

$$\{0, 1\}$$

$$\left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}$$

We do so, and prepend the zero vector `{0,0}` so that our walks begin at the origin. We arrive thus as this set of random alternating BRBRBRB steps

```
Steps = Prepend[Table[B[[k]][[1]][[1]], {k, 1, 10}], {0, 0}]
```

$$\left\{\{0, 0\}, \left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}, \{0, 1\}, \left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}, \left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\},\right.$$
$$\left.\left\{-\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}, \left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}, \{0, -1\}, \{0, 1\}, \left\{-\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}, \left\{\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\}\right\}$$

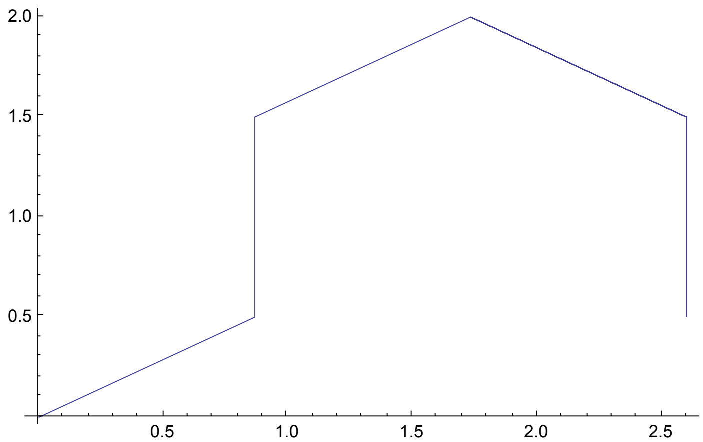
which we link together to build up a 10-step walk:

```
Walk = Table[Sum[Steps[[j]], {j, 1, k}], {k, 1, 11}]
```

$$\left\{\{0, 0\}, \left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}, \left\{\frac{\sqrt{3}}{2}, \frac{3}{2}\right\}, \left\{\sqrt{3}, 2\right\}, \left\{\frac{3\sqrt{3}}{2}, \frac{3}{2}\right\}, \left\{\sqrt{3}, 2\right\},\right.$$
$$\left.\left\{\frac{3\sqrt{3}}{2}, \frac{3}{2}\right\}, \left\{\frac{3\sqrt{3}}{2}, \frac{1}{2}\right\}, \left\{\frac{3\sqrt{3}}{2}, \frac{3}{2}\right\}, \left\{\sqrt{3}, 2\right\}, \left\{\frac{3\sqrt{3}}{2}, \frac{3}{2}\right\}\right\}$$

The walk thus generated looks like this:

```
ListLinePlot[Walk]
```



We now consolidate those commands to produce at random a 200-step hexagonal walk with a single composite command:

```
A200 = Table[Transpose[
    {{RandomChoice[{R1, R2, R3}], RandomChoice[{B1, B2, B3}]}}], {k, 1, 200}];

B200 = Table[MatrixPower[(0 1
                          1 0), k].A200[[k]], {k, 1, 200}];

Steps200 = Prepend[Table[B200[[k]][[1]][[1]], {k, 1, 200}], {0, 0}];

Walk200 = Table[Sum[Steps200[[j]], {j, 1, k}], {k, 1, 201}];

HexagonalWalk200 =
  ListLinePlot[Walk200, AspectRatio → Automatic, AxesOrigin → {0, 0},
   PlotRange → {{-20, 20}, {-20, 20}}, PlotStyle → {Thick, Red}];

EndPoints200 = Graphics[{{Blue, PointSize[0.02], Point[{0, 0}]},
    {Blue, PointSize[0.02], Point[Last[Walk200]]}}];

Show[{HexagonalWalk200, EndPoints200}, PlotLabel → "200-Step Hexagonal Walk"]
```