Testing Effectiveness of a Method

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Introduction. In "Ray's Solution" (7 November 2013, henceforth denoted [A]) I provide an elaborated account of a method devised by Ray Mayer for constructing an estimate of the expected position of a random walker on $\mathbb{Z}$, given that the next-step protocol is of a certain type. In "Analytic theory of a Parrondo Game" (19 November 20123, henceforth denoted [B]) I attempt to adapt an "improved version" of Ray's Method to the more complicated next-step protocol devised by J. Parrondo, but on page 10 hit a snag. Here I use the methods of [B] to rehearse the argument of [A], in an effort to verify that those methods do indeed work in that simpler context. For the most part I adhere to the notational conventions of [B].

Setting the problem up. Picking up the argument at page 7 of [A], let

$$\mathbb{C} = \begin{pmatrix}
0 & y & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
Z & 0 & x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & Y & 0 & z & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & X & 0 & y & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & Z & 0 & x & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & Y & {\color{red}0} & z & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & X & 0 & y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & Z & 0 & x & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & Y & 0 & z & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & X & 0 & y \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & Z & 0
\end{pmatrix}$$

be the central section of an ∞-dimensional Markov matrix. Note the period-3 structure of $\mathbb{C}$, and that the stochasticity of the columns entails $X = 1-x$, etc.

Introduce basic period-3 vectors that are ∞-dimensional extensions of

$$\boldsymbol{F}_1 = \begin{pmatrix} 0\\1\\0\\0\\1\\0\\0\\1\\0 \end{pmatrix}, \quad \boldsymbol{F}_2 = \begin{pmatrix} 1\\0\\0\\1\\0\\0\\1\\0\\0 \end{pmatrix}, \quad \boldsymbol{F}_3 = \begin{pmatrix} 0\\0\\1\\0\\0\\1\\0\\0\\1 \end{pmatrix}, \quad \boldsymbol{F}_0 = \sum_{k=1}^{3} \boldsymbol{F}_k = \begin{pmatrix} 1\\1\\1\\1\\1\\1\\1\\1\\1 \end{pmatrix}$$

and define

$$\boldsymbol{e}_0 = \begin{pmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\0 \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} 4\\3\\2\\1\\0\\-1\\-2\\-3\\-4 \end{pmatrix}$$

Our objective is to develop the structure of $S_n(x,y,z) = (\boldsymbol{w}, \mathbb{C}^n\boldsymbol{e}_0)$, which described the expected position—after n steps—of a walker who departs from the origin. We have particular interest in the asymptotic structure of $S_n(x,y,z)$.

Ray's solution. Ray cleverly elects to work from

$$S_n(x,y,z) = (\boldsymbol{e}_0, \mathbb{D}^n\boldsymbol{w}) \quad \text{where} \quad \mathbb{D} = \mathbb{C}^{\mathsf{T}}$$

By computation

$$\mathbb{D}\boldsymbol{w} = \begin{pmatrix} 5y+3Y\\4x+2X\\3z+1Z\\2y+0Y\\1x-1X\\0z-2Z\\-1y-3Y\\-2x-4X\\-3z-5Z \end{pmatrix} = \begin{pmatrix} 3+2y\\2+2x\\1+2z\\0+2y\\-1+2x\\-2+2z\\-3+2y\\-4+2x\\-5+2z \end{pmatrix}$$

which can be written

$$\begin{aligned} \mathbb{D}\boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{G}_1 \qquad (1)\\ \boldsymbol{G}_1 &= (2x-1)\boldsymbol{F}_1 + (2y-1)\boldsymbol{F}_2 + (2z-1)\boldsymbol{F}_3\\ &= f(x)\boldsymbol{F}_1 + f(y)\boldsymbol{F}_2 + f(z)\boldsymbol{F}_3\\ &\equiv \alpha_1\boldsymbol{F}_1 + \beta_1\boldsymbol{F}_2 + \gamma_1\boldsymbol{F}_3 \end{aligned}$$

with $f(u) = 2u - 1 = u - (1-u) = u - U$. Iteration of (1) gives

$$\left.\begin{aligned} \mathbb{D}\ \boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{G}_1 \\ \mathbb{D}^2\boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{G}_1 + \boldsymbol{G}_2 \\ \mathbb{D}^3\boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{G}_1 + \boldsymbol{G}_2 + \boldsymbol{G}_3 \\ &\ \vdots \\ \mathbb{D}^n\boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{G}_1 + \boldsymbol{G}_2 + \cdots + \boldsymbol{G}_n \end{aligned}\right\} \tag{2}$$

where

$$\boldsymbol{G}_n = \mathbb{D}\,\boldsymbol{G}_{n-1} = \mathbb{D}^{n-1}\boldsymbol{G}_1$$

Looking now to the explicit evaluation of the ∞-dimensional $\boldsymbol{G}$-vectors, we by calculation have

$$\begin{aligned} \mathbb{D}\boldsymbol{F}_1 &= \qquad\quad Y\boldsymbol{F}_2 + z\boldsymbol{F}_3 & \mathbb{D}\boldsymbol{F}_1 &= g_1(x)\boldsymbol{F}_1 + g_2(y)\boldsymbol{F}_2 + g_3(z)\boldsymbol{F}_3 \\ \mathbb{D}\boldsymbol{F}_2 &= x\boldsymbol{F}_1 \qquad\quad + Z\boldsymbol{F}_3 & \mathbb{D}\boldsymbol{F}_2 &= g_3(x)\boldsymbol{F}_1 + g_1(y)\boldsymbol{F}_2 + g_2(z)\boldsymbol{F}_3 \\ \mathbb{D}\boldsymbol{F}_3 &= X\boldsymbol{F}_1 + y\boldsymbol{F}_2 & \mathbb{D}\boldsymbol{F}_3 &= g_2(x)\boldsymbol{F}_1 + g_3(y)\boldsymbol{F}_2 + g_1(z)\boldsymbol{F}_3 \end{aligned}$$

where we note that the functions

$$\begin{aligned} g_1(u) &= 0 & &: \quad \text{abbreviated } g_{1,u} \\ g_2(u) &= U \equiv 1 - u & &: \quad \text{abbreviated } g_{2,u} \\ g_3(u) &= u & &: \quad \text{abbreviated } g_{3,u} \end{aligned}$$

sum to unity. We now have

$$\begin{aligned} \boldsymbol{G}_2 = \alpha_2\boldsymbol{F}_1 + \beta_2\boldsymbol{F}_2 + \gamma_2\boldsymbol{F}_3 = \mathbb{D}\,\boldsymbol{G}_1 = &\quad\ \alpha_1 \cdot \{g_{1,x}\boldsymbol{F}_1 + g_{2,y}\boldsymbol{F}_2 + g_{3,z}\boldsymbol{F}_3\} \\ &+ \beta_1 \cdot \{g_{3,x}\boldsymbol{F}_1 + g_{1,y}\boldsymbol{F}_2 + g_{2,z}\boldsymbol{F}_3\} \\ &+ \gamma_1 \cdot \{g_{2,x}\boldsymbol{F}_1 + g_{3,y}\boldsymbol{F}_2 + g_{1,z}\boldsymbol{F}_3\} \end{aligned}$$

giving

$$\begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = \mathbb{G} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} \quad \text{with} \quad \mathbb{G} = \begin{pmatrix} g_{1,x} & g_{3,x} & g_{2,x} \\ g_{2,y} & g_{1,y} & g_{3,y} \\ g_{3,z} & g_{2,z} & g_{1,z} \end{pmatrix}$$

which is of the form

$$\boldsymbol{g}_2 = \mathbb{G}\,\boldsymbol{g}_1$$

and implies $\boldsymbol{g}_n = \mathbb{G}^{n-1}\boldsymbol{g}_1$. Here $\boldsymbol{g}_n$ is a 3-vector, assembled from the coordinates (with respect to the $\boldsymbol{F}$-basis) of the ∞-vector $\boldsymbol{G}_n$.

Ray recognized that, since $\boldsymbol{g}$-space is 3-dimensional, it must be possible to display every $\boldsymbol{g}_n$ as a linear combination of any linearly independent triplet, of which $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ is the most natural candidate. To accomplish that objective I draw upon some fairly elegant trickery. More than fifty years ago I devised a way to display the coefficients in the characteristic polynomial of any square matrix $\mathbb{M}$ as algebraic functions of the traces of powers of $\mathbb{M}$. In the 3-dimensional case we have[1]

$$\det(\mathbb{M}-\lambda\,\mathbb{I}) = \sum_{n=0}^{3}\tfrac{1}{n!}Q_n(-\lambda)^{3-n} = \tfrac{1}{6}Q_3 - \tfrac{1}{2}Q_2\lambda + Q_1\lambda^2 - Q_0\lambda^3$$

where

$$\begin{aligned} Q_0 &= 1\\ Q_1 &= T_1\\ Q_2 &= T_1^2 - T_2\\ Q_3 &= T_1^3 - 3T_1T_2 + 2T_3 = 6\det\mathbb{M} \end{aligned}$$

and $T_k = \mathrm{tr}\,\mathbb{M}^k$. It follows by the Cayley-Hamilton theorem that

$$\begin{aligned} \mathbb{M}^3 &= \tfrac{1}{6}Q_3\,\mathbb{I} - \tfrac{1}{2}Q_2\,\mathbb{M} + Q_1\,\mathbb{M}^2\\ &= \tfrac{1}{6}\big(T_1^3 - 3T_1T_2 + 2T_3\big)\,\mathbb{I} - \tfrac{1}{2}\big(T_1^2 - T_2\big)\,\mathbb{M} + T_1\,\mathbb{M}^2\\ &\equiv q_1\,\mathbb{I} + q_2\,\mathbb{M} + q_3\,\mathbb{M}^2 \end{aligned}$$

Returning with this result to the problem at hand (send $\mathbb{M}\to\mathbb{G}$ and multiply the result into $\boldsymbol{g}_1$), we find

$$\boldsymbol{g}_4 = q_1\,\boldsymbol{g}_1 + q_2\,\boldsymbol{g}_2 + q_3\,\boldsymbol{g}_3$$

In the present instance

$$\mathbb{G} = \begin{pmatrix} 0 & x & 1-x\\ 1-y & 0 & y\\ z & 1-z & 0 \end{pmatrix}$$

(note that $\mathbb{G}^\intercal$ is manifestly Markovian) and *Mathematica* supplies

$$\begin{aligned} q_1 &= 1-(x+y+z)+(xy+yz+zx) = \det\mathbb{G}\equiv\sigma\\ q_2 &= (x+y+z)-(xy+yz+zx) = 1-\det\mathbb{G}\\ q_3 &= 0 \end{aligned}$$

Importance will attach in a moment to the fact that

$$\begin{pmatrix} q_1\\ q_2\\ q_3 \end{pmatrix} = \begin{pmatrix} \sigma\\ 1-\sigma\\ 0 \end{pmatrix}\quad\text{is stochastic}$$

Returning with this information to (2), we have

[1] For a recent account of the old material to which I allude, see "Algorithm for the efficient evaluation of the trace of the inverse of a matrix" (1996), which was written to resolve a problem posed by Richard Crandall.

$$\left.\begin{aligned}
\mathbb{D}\ \boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{G}_1\\
\mathbb{D}^2 \boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{G}_1 + \boldsymbol{G}_2\\
\mathbb{D}^3 \boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{G}_1 + \boldsymbol{G}_2 + \boldsymbol{G}_3\\
\mathbb{D}^4 \boldsymbol{w} &= \boldsymbol{w} + (1+q_1)\boldsymbol{G}_1 + (1+q_2)\boldsymbol{G}_2 + (1+q_3)\boldsymbol{G}_3\\
&\equiv \boldsymbol{w} + a_4\boldsymbol{G}_1 + b_4\boldsymbol{G}_2 + c_4\boldsymbol{G}_3\\
&\ \vdots\\
\mathbb{D}^n \boldsymbol{w} &= \boldsymbol{w} + a_n\boldsymbol{G}_1 + b_n\boldsymbol{G}_2 + c_n\boldsymbol{G}_3\\
\mathbb{D}^{n+1} \boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{G}_1 + c_n q_1\boldsymbol{G}_1 + (a_n + c_n q_2)\boldsymbol{G}_2 + (b_n + c_n q_3)\boldsymbol{G}_3\\
&= \boldsymbol{w} + a_{n+1}\boldsymbol{G}_1 + b_{n+1}\boldsymbol{G}_2 + c_{n+1}\boldsymbol{G}_3
\end{aligned}\right\} \tag{3}$$

The coefficients $\{a, b, c\}$ are seen to increment by the inhomogeneous rule

$$\begin{pmatrix} a_{n+1}\\ b_{n+1}\\ c_{n+1}\end{pmatrix} = \begin{pmatrix} 0 & 0 & q_1\\ 1 & 0 & q_2\\ 0 & 1 & q_3\end{pmatrix}\begin{pmatrix} a_n\\ b_n\\ c_n\end{pmatrix} + \begin{pmatrix} 1\\ 0\\ 0\end{pmatrix} \tag{4}$$

To reproduce (3) we set

$$\begin{pmatrix} a_1\\ b_1\\ c_1\end{pmatrix} = \begin{pmatrix} 1\\ 0\\ 0\end{pmatrix}$$

and by (4) obtain

$$\begin{pmatrix} a_2\\ b_2\\ c_2\end{pmatrix} = \begin{pmatrix} 1\\ 1\\ 0\end{pmatrix},\quad \begin{pmatrix} a_3\\ b_3\\ c_3\end{pmatrix} = \begin{pmatrix} 1\\ 1\\ 1\end{pmatrix},\quad \begin{pmatrix} a_4\\ b_4\\ c_4\end{pmatrix} = \begin{pmatrix} 1+q_1\\ 1+q_2\\ 1+q_3\end{pmatrix},\quad \ldots$$

Equation (4) is of the form

$$\boldsymbol{g}_{n+1} = \mathbb{Z}\boldsymbol{g}_n + \boldsymbol{g}_1 \quad : \quad \mathbb{Z} = \begin{pmatrix} 0 & 0 & q_1\\ 1 & 0 & q_2\\ 0 & 1 & q_3\end{pmatrix},\quad \boldsymbol{g}_1 = \begin{pmatrix} 1\\ 0\\ 0\end{pmatrix}$$

which entails

$$\begin{aligned}
\boldsymbol{g}_2 &= \mathbb{Z}\boldsymbol{g}_1 + \boldsymbol{g}_1\\
\boldsymbol{g}_3 &= \mathbb{Z}(\mathbb{Z}\boldsymbol{g}_1 + \boldsymbol{g}_1) + \boldsymbol{g}_1\\
&= (\mathbb{Z}^2 + \mathbb{Z}^1 + \mathbb{Z}^0)\boldsymbol{g}_1\\
&\ \vdots\\
\boldsymbol{g}_{n+1} &= \sum_{k=0}^{n} \mathbb{Z}^k \boldsymbol{g}_1
\end{aligned} \tag{5}$$

By graphic analysis[2] we establish that the functions $q_k(x, y, z)$—which, as

[2] Use commands of the form

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Manipulate[Plot3D[f[x,y,z],{x,0,1},{y,0,1}],{z,0,1,0.1}]
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previously remarked, sum to unity—remain non-negative as the parameters $\{x, y, z\}$ range on $[0, 1]$, so $\mathbb{Z}$ is Markovian. The spectrum of $\mathbb{Z}$ has therefore the form

$$\left.\begin{aligned}\lambda_1 &= 1\\ \lambda_2(\sigma) &= \tfrac{1}{2}\big(-1+\sqrt{1-4\sigma}\,\big)\\ \lambda_3(\sigma) &= \tfrac{1}{2}\big(-1-\sqrt{1-4\sigma}\,\big)\end{aligned}\right\} \tag{6}$$

where λ_2 and λ_3—whether real or complex—have absolute values that are less than unity, as shown in the following figure:

Figure 1 goes here

FIGURE 1: *Graphs of the absolute values of* $\lambda_2(\sigma)$ (*red*) *and* $\lambda_3(\sigma)$ (*blue*) *as* σ *ranges on* $[0, 1]$.

With the assistance of *Mathematica* we compute column vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ that are right eigenvectors of $\mathbb{Z}$

$$\mathbb{Z}\,\boldsymbol{u}_k = \lambda_k \boldsymbol{u}_k$$

and row vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ that are left eigenvectors of $\mathbb{Z}$ (transposed right eigenvectors of $\mathbb{Z}^\mathsf{T}$)

$$\boldsymbol{v}_k\,\mathbb{Z} = \lambda_k \boldsymbol{v}_k$$

We use those to construct[3] matrices

$$\mathbb{P}_k = \frac{\boldsymbol{u}_k\,\boldsymbol{v}_k}{(\boldsymbol{v}_k\boldsymbol{u}_k)} \quad : \quad k = 1, 2, 3$$

which are demonstrably projective

$$\mathbb{P}_k^2 = \mathbb{P}_k \quad : \quad k = 1, 2, 3$$

[3] See "Generalized spectral resolution and some of its applications" (27 April 2009).

orthogonal

$$\mathbb{P}_j\mathbb{P}_k = \mathbb{O} \quad : \quad j \neq k$$

and complete

$$\mathbb{P}_1 + \mathbb{P}_2 + \mathbb{P}_3 = \mathbb{I}$$

and permit one to write

$$\begin{aligned} \mathbb{Z} &= \lambda_1\,\mathbb{P}_1 + \lambda_2\,\mathbb{P}_2 + \lambda_3\,\mathbb{P}_3 \\ &\Downarrow \\ \mathbb{Z}^n &= \lambda_1^n\,\mathbb{P}_1 + \lambda_2^n\,\mathbb{P}_2 + \lambda_3^n\,\mathbb{P}_3 \end{aligned}$$

Mathematica supplies explicit descriptions of the $\mathbb{P}$-matrices that can be written

$$\begin{aligned} \mathbb{P}_1 &= D_1{}^{-1}\begin{pmatrix} \sigma & \sigma & \sigma \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ \mathbb{P}_2 &= D_2{}^{-1}\begin{pmatrix} \xi+1-2\sigma & -(\xi+1)\sigma & 2\sigma^2 \\ 2\sigma & (\xi-1)\sigma & \frac{1}{2}(\xi-1)^2\sigma \\ -(\xi+1) & 2\sigma & (\xi-1)\sigma \end{pmatrix} \\ \mathbb{P}_3 &= D_3{}^{-1}\begin{pmatrix} \xi-1+2\sigma & -(\xi-1)\sigma & -2\sigma^2 \\ -2\sigma & (\xi+1)\sigma & -\frac{1}{2}(\xi+1)^2\sigma \\ -(\xi-1) & -2\sigma & (\xi+1)\sigma \end{pmatrix} \end{aligned}$$

with[4]

$$\xi \equiv \sqrt{1-4\sigma}$$

and

$$\left.\begin{aligned} D_1 &\equiv 2+\sigma \\ D_2 &\equiv \xi(1+2\sigma) - (4\sigma-1) \\ D_3 &\equiv \xi(1+2\sigma) + (4\sigma-1) \end{aligned}\right\} \tag{7}$$

Returning with this information to (5), we have

$$\boldsymbol{a}_{n+1} = \Big\{\sum_{k=0}^{n}\lambda_1^k\,\mathbb{P}_1 + \sum_{k=0}^{n}\lambda_2^k\,\mathbb{P}_2 + \sum_{k=0}^{n}\lambda_3^k\,\mathbb{P}_3\Big\}\boldsymbol{a}_1 \tag{8}$$

Typical low-order results

$$\boldsymbol{a}_5 = \begin{pmatrix} 1+\sigma \\ 2 \\ 2-\sigma \end{pmatrix} \qquad \boldsymbol{a}_6 = \begin{pmatrix} 1+2\sigma-\sigma^2 \\ 3-2\sigma+\sigma^2 \\ 2 \end{pmatrix}$$

$$\boldsymbol{a}_7 = \begin{pmatrix} 1+2\sigma \\ 3-\sigma^2 \\ 3-2\sigma+\sigma^2 \end{pmatrix} \qquad \boldsymbol{a}_8 = \begin{pmatrix} 1+3\sigma-2\sigma^2+\sigma^3 \\ 4-3\sigma+3\sigma^2-\sigma^3 \\ 3-\sigma^2 \end{pmatrix}$$

suggest that quite generally

$$\sum \text{elements of } \boldsymbol{a}_n = n$$

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[4] In this notation $\lambda_2(\sigma) = +\frac{1}{2}(\xi-1)$, $\lambda_3(\sigma) = -\frac{1}{2}(\xi+1)$.

For large n we have

$$\begin{aligned}\boldsymbol{a}_{n+1} &\sim \Big\{n\mathbb{P}_1 + \frac{1}{1-\lambda_2}\,\mathbb{P}_2 + \frac{1}{1-\lambda_3}\,\mathbb{P}_3\Big\}\boldsymbol{a}_1\\ &= \Big\{n\mathbb{P}_1 + \frac{2}{3-\xi}\,\mathbb{P}_2 + \frac{2}{3+\xi}\,\mathbb{P}_3\Big\}\boldsymbol{a}_1\\ &= \frac{n}{2+\sigma}\begin{pmatrix}\sigma\\1\\1\end{pmatrix} + \frac{1}{(2+\sigma)^2}\begin{pmatrix}4-\sigma\\ \sigma-1\\ -3\end{pmatrix}\end{aligned}$$

giving

$$\begin{pmatrix}a_{n+1}\\b_{n+1}\\c_{n+1}\end{pmatrix} \sim \mathcal{D}^{-1}\begin{pmatrix}(4-\sigma)+n(2+\sigma)\sigma\\(\sigma-1)+n(2+\sigma)\\(\quad-3)+n(2+\sigma)\end{pmatrix} \tag{9}$$

where again $\sigma = 1-(x+y+z)+(xy+yz+zx) = xyz + XYZ$ and where now

$$\mathcal{D} = (2+\sigma)^2 = \big[3-(x+y+z)+(xy+yz+zx)\big]^2 \tag{10}$$

Our objective—as posed by Ray and sharpened by (3)—is to evaluate

$$S_n(x,y,z) = \big(\boldsymbol{e}_0, \{\boldsymbol{w} + a_n\boldsymbol{G}_1 + b_n\boldsymbol{G}_2 + c_n\boldsymbol{G}_3\}\big) \tag{11}$$

With the coefficients $\{a_n, b_n, c_n\}$ now in hand, we look to the central elements of the basic $\boldsymbol{G}$-vectors $\{\boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{G}_3\}$. Working with *Mathematica*'s assistance from

$$\begin{aligned}\boldsymbol{G}_1 &= \mathbb{D}\,\boldsymbol{w} - \boldsymbol{w}\\ \boldsymbol{G}_2 &= \mathbb{D}\,\boldsymbol{G}_1\\ \boldsymbol{G}_3 &= \mathbb{D}\,\boldsymbol{G}_2\end{aligned}$$

we obtain

$$\boldsymbol{G}_1 = \begin{pmatrix}\vdots\\-1+2z\\-1+2y\\-1+2x\\-1+2z\\-1+2y\\\vdots\end{pmatrix}, \quad \boldsymbol{G}_2 = \begin{pmatrix}\vdots\\-1+2y+2z(x-y)\\-1+2x+2y(z-x)\\-1+2z+2x(y-z)\\-1+2y+2z(x-y)\\-1+2x+2y(z-x)\\\vdots\end{pmatrix}$$

$$\boldsymbol{G}_3 = \begin{pmatrix} \vdots \\ -1+2x-2(xy-yz+zx)+2z^2(1-x-y)+4xyz \\ -1+2z-2(zx-xy-yz)+2y^2(1-z-x)+4xyz \\ -1+2y-2(yz-zx+xy)+2x^2(1-y-z)+4xyz \\ -1+2x-2(xy-yz+zx)+2z^2(1-x-y)+4xyz \\ -1+2z-2(zx-xy-yz)+2y^2(1-z-x)+4xyz \\ \vdots \end{pmatrix}$$

all of which are seen to be manifectly 3-periodic and to possess the property that as one moves from element to next-higher element the variables $\{x, y, z\}$ advance in cyclic progression. These persistent patterns, inherited from the structure of $\mathbb{C}$, inspire confidence in the accuracy of our results, but because of the special structure of the initial state $\boldsymbol{e}_0$ it is only the central elements—shown in blue—that are relevant to the construction of $S_n(x, y, z)$. Returning with $(\boldsymbol{e}_0, \boldsymbol{w}) = 0$ and

$$\begin{aligned} (\boldsymbol{e}_0, \mathbb{G}_1) &\equiv G_{10} = -1+2x \\ (\boldsymbol{e}_0, \mathbb{G}_2) &\equiv G_{20} = -1+2z+2x(y-z) \\ (\boldsymbol{e}_0, \mathbb{G}_3) &\equiv G_{30} = -1+2y-2(yz-zx+xy)+2x^2(1-y-z)+4xyz \end{aligned} \tag{12}$$

to (11), we obtain finally

$$S_n(x, y, z) = (\boldsymbol{e}_0, \mathbb{D}^n \boldsymbol{w}) = a_n G_{10} + b_n G_{20} + c_n G_{30} \tag{13.1}$$

with

$$\boldsymbol{a}_n \equiv \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \left\{ n\mathbb{P}_1 + \sum_{k=0}^{n-1} \lambda_2^k \mathbb{P}_2 + \sum_{k=0}^{n-1} \lambda_3^k \mathbb{P}_3 \right\} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{13.2}$$

where the σ that enters into the construction (6) of $\lambda_2(\sigma)$ and $\lambda_3(\sigma)$ was defined

$$\sigma = 1 - (x+y+z) + (xy+yz+zx) = xyz + XYZ$$

and where the projection matrices $\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3\}$ were defined on page 7. For large n equations (13) give

$$S_n(x, y, z) \sim n\,\mathcal{P}(x, y, z) + \mathcal{Q}(x, y, z) \tag{14.1}$$

with

$$\begin{aligned} \mathcal{P}(x, y, z) &= \mathcal{D}^{-1}(2+\sigma)\big[\sigma G_{01} + G_{20} + G_{30}\big] \\ \mathcal{Q}(x, y, z) &= \mathcal{D}^{-1}\big[(4-\sigma)G_{10} + (\sigma-1)G_{20} - 3G_{30}\big] \end{aligned} \tag{14.2}$$

where

$$\mathcal{D} = (2+\sigma)^2 = \big[3 - (x+y+z) + (xy+yz+zx)\big]^2$$

Accuracy checks, and some instances of "polynomial similarity." We possess now two distinct ways to approach the evaluation of $S_n(x, y, z)$. The naive approach (which I employed in some earlier *Mathematica*-based work) proceeds from

$$S_n(x, y, z) = (\boldsymbol{w}, \mathbb{C}^n \boldsymbol{e}_0)$$

so involves raising large matrices[5] to high powers. This works well enough for small n, but at some point raising large matrices to high powers becomes unfeasible. The method supplies

$$\begin{aligned} S_n(x,y,z) &= \text{homogeneous polynomial of degree } n \text{ in } \{x,X,y,Y,z,Z\} \\ &= \text{inhomogeneous polynomial of degree } n \text{ in } \{x,y,z\} \end{aligned}$$

but provides no insight into the structure of those complicated polynomials. Ray's method, on the other hand, proceeds from

$$S_n(x,y,z) = (\boldsymbol{e}_0, \mathbb{D}^n \boldsymbol{w}) \quad : \quad \mathbb{D} = \mathbb{C}^\mathsf{T}$$

to formulae (13) that involve no matrix multiplication at all, that supply precise results in every order and that yield quite a simple result (14) in asymptotic approximation. *Mathematica* reports that the two methods produce identical results

$$\begin{aligned} S_4(x,y,z) = {} & -4+4x+2y-2xy+2x^2y+2xy^2-2x^2y^2+2z-2x^2z+4xyz-2xy^2z \\ & +2z^2-4xz^2+2x^2z^2-2yz^2+2xyz^2 \\ S_5(x,y,z) = {} & -5+4x+2x^3+2y+2xy-4x^3y+2y^2-4xy^2+2x^3y^2+4z-4xz+4x^2z \\ & -4x^3z+2xyz-2x^2yz+4x^3yz-4y^2z+8xy^2z-2x^2y^2z+2xz^2-4x^2z^2 \\ & +2x^3z^2-2yz^2+4xyz^2-2x^2yz^2+2y^2z^2-4xy^2z^2 \\ S_6(x,y,z) = {} & -6+4x+4x^2-2x^3+4y-8x^2y+6x^3y-2xy^2+8x^2y^2-6x^3y^2+2xy^3-4x^2y^3 \\ & +2x^3y^3+4z-6x^2z+2x^3z-4yz+8xyz+8x^2yz-4x^3yz+2y^2z+2xy^2z \\ & -10x^2y^2z+2x^3y^2z-4xy^3z+4x^2y^3z+2xz^2-4x^2z^2+2x^3z^2+4yz^2 \\ & -10xyz^2+8x^2yz^2-2x^3yz^2-4y^2z^2+2xy^2z^2+2xy^3z^2+2z^3-6xz^3 \\ & +6x^2z^3-2x^3z^3-4yz^3+8xyz^3-4x^2yz^3+2y^2z^3-2xy^2z^3 \\ S_7(x,y,z) = {} & -7+6x+2x^2+4y-4xy+6x^2y-6x^4y+8xy^2-14x^2y^2+6x^4y^2+2y^3-8xy^3 \\ & +8x^2y^3-2x^4y^3+4z-6x^2z+8x^3z-6x^4z+8xyz+2x^2yz-16x^3yz \\ & +12x^4yz+2y^2z-20xy^2z+20x^2y^2z+8x^3y^2z-6x^4y^2z-6y^3z+20xy^3z \\ & -14x^2y^3z+4z^2-12xz^2+18x^2z^2-16x^3z^2+6x^4z^2-6yz^2+12xyz^2 \\ & -16x^2yz^2+16x^3yz^2-6x^4yz^2-4y^2z^2+20xy^2z^2-18x^2y^2z^2+6y^3z^2 \\ & -16xy^3z^2+6x^2y^3z^2-2z^3+8xz^3-12x^2z^3+8x^3z^3-2x^4z^3+2yz^3 \\ & -4xyz^3+2x^2yz^3+2y^2z^3-8xy^2z^3+6x^2y^2z^3-2y^3z^3+4xy^3z^3 \end{aligned}$$

through order $n=7$. We expect to have

[5] To avoid "boundary errors" in the naive evaluation of $S_n(x,y,z)$ the $\nu\times\nu$ matrix $\mathbb{C}$ must have dimension not less than $2n+1$. Working with $\nu=15$ I could by that method ascend only to order 7, by which point my typographic patience had already been pressed to its limit.

$$
\begin{aligned}
S_n(0,\,0,\,0,\,0) &= -n\\
S_n(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) &= 0\\
S_n(1,\,1,\,1,\,1) &= +n
\end{aligned}
\tag{15}
$$

and are informed by *Mathematica* that each of the results reported above conforms to that expectation. That the two methods yield results that—increasing complicated though they rapidly become—are in precise agreement through order 7, and that in those cases they conform to (15), is fairly convincing evidence that our results are accurate.

In the asymptotic limit we by (14) have

$$S_n(x,y,z) \sim S_{n,\infty}(x,y,z) = n\,\mathcal{P}(x,y,z) + \mathcal{Q}(x,y,z) \sim n\,\mathcal{P}(x,y,z)$$

where

$$
\begin{aligned}
\mathcal{P}(x,y,z) &= \mathcal{D}^{-1}\mathfrak{p} \quad \text{where} \quad \mathfrak{p}(x,y,z) = (2+\sigma)\big[\sigma G_{10} + G_{20} + G_{30}\big]\\
\mathcal{Q}(x,y,z) &= \mathcal{D}^{-1}\mathfrak{q} \quad \text{where} \quad \mathfrak{q}(x,y,z) = (4-\sigma)G_{10} + (\sigma-1)G_{20} - 3G_{30}\\
\mathcal{D}(x,y,z) &= (2+\sigma)^2
\end{aligned}
$$

when spelled out in explicit detail read

$$
\begin{aligned}
\mathfrak{p}(x,y,z) = {} & -9+12x-3x^2+12y-18xy+6x^2y-3y^2+6xy^2-3x^2y^2+12z-18xz+6x^2z\\
& -18yz+36xyz-12x^2yz+6y^2z-12xy^2z+6x^2y^2z-3z^2+6xz^2-3x^2z^2\\
& +6yz^2-12xyz^2+6x^2yz^2-3y^2z^2+6xy^2z^2\\
\mathfrak{q}(x,y,z) = {} & 6x-4x^2-6y+8xy+2x^2y-2xy^2+2x^2y^2-6xz+6x^2z+4yz-12xyz\\
& +2xy^2z-2z^2+4xz^2-2x^2z^2+2yz^2-2xyz^2\\
\mathcal{D}(x,y,z) = {} & 9-6x+x^2-6y+8xy-2x^2y+y^2-2xy^2+x^2y^2-6z+8xz-2x^2z+8yz\\
& -6xyz+2x^2yz-2y^2z+2xy^2z+z^2-2xz^2+x^2z^2-2yz^2+2xyz^2+y^2z^2
\end{aligned}
$$

giving

$$S_{n,\infty}(x,y,z) = \frac{n(5^{\text{th}}\text{ order}) + (4^{\text{th}}\text{ order})}{4^{\text{th}}\text{ order}} \tag{16}$$

It is insructive to look to the special case $x = y = z$ (*i.e.*, to the simplest unbalanced walk). In that case $G_{10} = G_{20} = G_{30} = 2x-1$, $\sigma = 1-3x+3x^2$ and

$$
\begin{aligned}
\mathfrak{p}(x,x,x) &= -9+36x-63x^2+72x^3-45x^4+18x^5\\
\mathfrak{q}(x,x,x) &= 0\\
D(x,x,x) &= 9-18x+27x^2-18x^3+9x^4
\end{aligned}
$$

from which it follows that

$$S_{n,\infty}(x,x,x) = n(2x-1) \tag{17}$$

—exactly as one might have anticipated. For consider an ensemble of walkers, each of whom advances one step with probability x, retreats one step with probability $X = 1-x$. The mean single-step advance is

$$S_1(x) = x - X = 2x - 1$$

so the mean n-step advance is

$$S_n(x) = nS_1(x) = n(2n-1)$$

—in precise agreement with (17). The asymptotic formula has in this instance been found to be exact for all n.

Walkers who advance by the simple site-independent rule just considered can expect to "break even" (make no net n-step progress) if $S_n(x) = 0$, which entails $x = \frac{1}{2}$. The n-step beak-even conditions for walkers who advance by the site-dependent rule $\mathbb{C}_{x,y,z}$ read

$$S_n(x, y, z) = 0 \tag{18}$$

—each of which inscribes a "null surface" within the unit cube in $\{x, y, z\}$-space. When plotted,[6] those surfaces are found to resemble one another ever more closely as n ascends, and for n greater than about 10 to become virtually indistinguishable from the surface defined

$$S_{n,\infty}(x, y, z) \sim n\,\mathcal{P}(x, y, z) = 0$$

Which is a little perplexing, since the multinomials $S_n(x, y, z)$—of ascending high order—do not at all resemble one another, and $\mathcal{P}(x, y, z)$ is a *ratio* of low order multinomials. The mystery would disappear if it were the case that

$$S_{n+1}(x, y, z) = S_n(x, y, z) + \text{higher order terms}$$

but that is manifestly *not* the case. I will return later to discussion of that "polynomial similarity problem."

Parrondo's paradoxical game. Juan Parrondo's discovery derives from his interest in "Brownian ratchets," Smoluchowski's realization—popularized by Feynman—of Maxwell's Demon, but as a matter of expository convenience adopted game-theoretic language when he first reported his paradoxical result.[7] I continue in that tradition.

Player A, who by flip of a loaded coin places a penny on the table with probability x, removes a penny with probability $X = 1-x$. Player A, as recently remarked, can expect to break even if $x = \frac{1}{2}$. Parrondo's player B uses one or the other of two coins, depending upon whether or not the money on the table is $2 \bmod 3$. When that is the case player B deposits a penny with probability z (withdraws one with probability $Z = 1 - z$), but in all other cases he deposits with probability y, withdraws with probability $Y = 1 - y$. To discover his long term prospects, we return to (14) and set $x = y$; we look, in other words, to

---

[6] In *Mathematica* v7 use the command

$$\texttt{ContourPlot3D}[S_n(x, y, z) == 0, \{x, 0, 1\}, \{y, 0, 1\}, \{z, 0, 1\}]$$

Such figures are supplied in a companion notebook.

[7] The seminal document is a slide entitled "How to cheat a bad mathematician" that he used to illustrate a lecture entitled "Efficiency of Brownian motors" which he presented at a Complexity and Chaos Workshop that took place in Torino, Italy in July, 1996. See his homepage at

http://seneca.fis.ucm.es/parr/

$$\mathcal{P}(y, y, z) = 0 \tag{19}$$

where

$$\begin{aligned}\mathcal{P}(y, y, z) = \mathcal{D}^{-1}\big\{ &-9 + 24y - 24y^2 + 12y^3 - 3y^4 + 12z - 36yz + 48y^2 z \\ &- 24y^3 z + 6y^4 z - 3z^2 + 12yz^2 - 18y^2 z^2 + 12y^3 z^2 \big\} \\ \mathcal{D} = 9 - 12y &+ 10y^2 - 4y^3 + y^4 - 6z + 16yz - 10y^2 z \\ &- 10y^2 z + 4y^3 z + z^2 - 4yz^2 + 4y^2 z^2\end{aligned}$$

The asymptotic break-even condition (18) inscribes "null curve" within the unit square in $\{y, z\}$-space: see the following figure:

Figure 2 goes here

FIGURE 2: *The null curve derived from setting* $\mathcal{P}(y, y, z) = 0$. *The B player wins only if* $\{y, z\}$ *falls above the curve. In the figure,* $y \in [0, 1]$ *runs* $\rightarrow$, $z \in [0, 1]$ *runs* $\uparrow$.

Suppose now that players A and B move (deposit or withdraw pennies) alternately. Player A's move is generated by

$$\mathbb{A} = \mathbb{C}_{a,a,a}$$

while player B's move is generated by

$$\mathbb{B} = \mathbb{C}_{y,y,z}$$

The composite result of such a pair of moves is generated by $\mathbb{S} = \mathbb{A}\mathbb{B}$ (which is Markovian since all products of Markov matrices are Marcovian). Coincidences (such as $x = y$) tend to obscure patterns, so we look to the more general

4-parameter case that results from setting $\mathbb{B} = \mathbb{C}_{x,y,z}$ and will set $x = y$ only at the end of the argument.

Looking only to the illustrative central secti of a 15-dimensional $\mathbb{S}$-matrix, we have

$$\mathbb{S} = \begin{pmatrix}
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & az & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & ay & 0 & 0 & 0 & \cdots \\
\cdots & Az + aZ & 0 & ax & 0 & 0 & \cdots \\
\cdots & 0 & Ay + aY & 0 & az & 0 & \cdots \\
\cdots & AZ & 0 & Ax + aX & 0 & ay & \cdots \\
\cdots & 0 & AY & 0 & Az + aZ & 0 & \cdots \\
\cdots & 0 & 0 & AX & 0 & Ay + aY & \cdots \\
\cdots & 0 & 0 & 0 & AZ & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & AY & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots
\end{pmatrix}$$

We verify that the column elements sum to unity, and note the 3-periodicity of $\mathbb{S}$. The matrices $\mathbb{A}$ and $\mathbb{B}$ refer to nearest-neighbor walks with stand-in-place excluded, so have 0 on their diagonals. But with a second step such a walker can *return* to place, which accounts for the non-zero elements on the diagonal of $\mathbb{S}$. By computation

$$\begin{aligned}
\mathbb{T}\,\boldsymbol{w} = \boldsymbol{w} &+ 2(a + x - 1)\boldsymbol{F}_1 \\
&+ 2(a + y - 1)\boldsymbol{F}_2 \quad : \quad \mathbb{T} = \mathbb{S}^{\intercal} \\
&+ 2(a + z - 1)\boldsymbol{F}_3
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{T}\boldsymbol{F}_1 &= (Ax + aX)\boldsymbol{F}_1 + ay\,\boldsymbol{F}_2 + AZ\,\boldsymbol{F}_3 \\
\mathbb{T}\boldsymbol{F}_2 &= AX\,\boldsymbol{F}_1 + (Ay + aY)\boldsymbol{F}_2 + az\,\boldsymbol{F}_3 \\
\mathbb{T}\boldsymbol{F}_3 &= ax\,\boldsymbol{F}_1 + AY\,\boldsymbol{F}_2 + (Az + aZ)\boldsymbol{F}_3
\end{aligned}$$

which in notation that mimics that of pages 2 and 3 become

$$\begin{aligned}
\mathbb{T}\,\boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{G}_1 \\
\boldsymbol{G}_1 &= f(x)\boldsymbol{F}_1 + f(y)\boldsymbol{F}_2 + f(z)\boldsymbol{F}_3 \\
&= \quad \alpha_1\boldsymbol{F}_1 + \quad \beta_1\boldsymbol{F}_2 + \quad \gamma_1\boldsymbol{F}_3
\end{aligned}$$

with $f(u) = 2(a + u - 1) = (a - A) + (u - U)$ and

$$\begin{aligned}\mathbb{T}\boldsymbol{F}_1 &= g_1(x)\boldsymbol{F}_1 + g_2(y)\boldsymbol{F}_2 + g_3(z)\boldsymbol{F}_3\\ \mathbb{T}\boldsymbol{F}_2 &= g_3(x)\boldsymbol{F}_1 + g_1(y)\boldsymbol{F}_2 + g_2(z)\boldsymbol{F}_3\\ \mathbb{T}\boldsymbol{F}_3 &= g_2(x)\boldsymbol{F}_1 + g_3(y)\boldsymbol{F}_2 + g_1(z)\boldsymbol{F}_3\end{aligned}$$

where

$$\begin{aligned} g_1(u) &= a+u-2au &&: \quad \text{abbreviated } g_{1,u}\\ g_2(u) &= au &&: \quad \text{abbreviated } g_{2,u}\\ g_3(u) &= 1-a-u+au &&: \quad \text{abbreviated } g_{3,u}\end{aligned}$$

are seen to sum to unity.

We are led now as we were on pages 3–5 (except that our symbols bear now different meanings) to write

$$\begin{aligned}\boldsymbol{G}_2 = \alpha_2\boldsymbol{F}_1 + \beta_2\boldsymbol{F}_2 + \gamma_2\boldsymbol{F}_3 = \mathbb{T}\,\boldsymbol{G}_1 = \quad &\alpha_1\cdot\{g_{1,x}\boldsymbol{F}_1 + g_{2,y}\boldsymbol{F}_2 + g_{3,z}\boldsymbol{F}_3\}\\ &+\beta_1\cdot\{g_{3,x}\boldsymbol{F}_1 + g_{1,y}\boldsymbol{F}_2 + g_{2,z}\boldsymbol{F}_3\}\\ &+\gamma_1\cdot\{g_{2,x}\boldsymbol{F}_1 + g_{3,y}\boldsymbol{F}_2 + g_{1,z}\boldsymbol{F}_3\}\end{aligned}$$

giving

$$\begin{pmatrix}\alpha_2\\ \beta_2\\ \gamma_2\end{pmatrix} = \mathbb{G}\begin{pmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{pmatrix} \quad\text{with}\quad \mathbb{G} = \begin{pmatrix} g_{1,x} & g_{3,x} & g_{2,x}\\ g_{2,y} & g_{1,y} & g_{3,y}\\ g_{3,z} & g_{2,z} & g_{1,z}\end{pmatrix}$$

so again the $\mathbb{F}$-coordinates of the ∞-dimensional $\mathbb{G}$-vectors increment by the rule

$$\boldsymbol{g}_2 = \mathbb{G}\,\boldsymbol{g}_1 \quad\Longrightarrow\quad \boldsymbol{g}_n = \mathbb{G}^{\,n-1}\boldsymbol{g}_1$$

The matrix $\mathbb{G}$ is 3×3, so again we have

$$\begin{aligned}\mathbb{G}^{\,3} &= \tfrac{1}{6}\big(T_1^3 - 3T_1T_2 + 2T_3\big)\,\mathbb{I} - \tfrac{1}{2}\big(T_1^2 - T_2\big)\,\mathbb{G} + T_1\,\mathbb{G}^2\\ &\equiv q_1\,\mathbb{I} + q_2\,\mathbb{G} + q_3\,\mathbb{G}^{\,2}\end{aligned}$$

where by computation the coefficients are given now by

$$\begin{aligned} q_1 &= \big[1-3a+3a^2\big]\big[1-(x+y+z)+(xy+yz+zx)\big]\\ q_2 &= 1-q_1-q_3\\ q_3 &= 3a-(2a-1)(x+y+z)\end{aligned}$$

which clearly sum to unity. Finally, we have

$$\boldsymbol{g}_{n+1} = \sum_{k=0}^{n}\mathbb{Z}^{\,k}\boldsymbol{g}_1 \quad\text{with}\quad \mathbb{Z} = \begin{pmatrix}0 & 0 & q_1\\ 1 & 0 & q_2\\ 0 & 1 & q_3\end{pmatrix},\ \ \boldsymbol{g}_1\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

by precisely the argument which gave (5), but with this important difference: the matrix $\mathbb{Z}$ is *not Markovian* because its third column is not stochastic. Its elements sum to unity, but do not all fall within the unit interval. I have been unable to develop a clean analytical demonstration of the latter point, but can supply persuasive statistical evidence. Assigning random unit interval values to $\{a, x, y, z\}$ I found after 100 trials that in every instance $q_1 \in [0,1]$,[8] $q_2 < 0$

[8] In this instance the analytical demonstration is elementary.

and $q_3 > 1$. Looking to the statistics of 10,000 such trials, I found

$$\begin{aligned} \text{mean } q_1 &= 0.125\,, \quad \Delta q_1 = 0.097 \\ \text{mean } q_2 &= -0.624\,, \quad \Delta q_2 = 0.305 \\ \text{mean } q_3 &= \underline{1.498}\,, \quad \Delta q_1 = 0.288 \\ &0.999 \end{aligned}$$

So $\mathbb{Z}$ is non-Markovian. But—surprisingly/fortunately—the spectral properties of $\mathbb{Z}$ do mimic those of a Markov matrix: examination of 100 randomized trials showed that (*i*) in every case the leading eigenvalue was unity; (*ii*) in every case $|\lambda_2| < 1$ and $|\lambda_3| < 1$; (*iii*) in about 40% of cases λ_2 and λ_3 were both real (and in all other cases complex conjugates of one another).

If we sought exact description of $S_n(a, x, y, z)$ we would have to construct an exact evaluation of $\mathbb{Z}^{n-1}$, so would at this point undertake to produce the spectral decomposition of $\mathbb{Z}$. But we have interest only in the form assumed by $S_n(a, x, y, z)$ when n is sufficiently large we can spare ourselves that labor, exploiting what we know about the spectrum of $\mathbb{Z}$ to write

$$\mathbb{Z}^{n-1} \sim n\mathbb{P}_1$$

where projects onto the leading eigenvector of $\mathbb{Z}$:

$$\boldsymbol{h} = \frac{1}{2 + q_1 - q_3}\begin{pmatrix} q_1 \\ 1 - q_3 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \quad : \quad h_1 + h_2 + h_3 = 1$$

We then have[9]

$$\lim_{n\to\infty} \mathbb{Z}^n \boldsymbol{h}_0 = \boldsymbol{h} \quad : \quad \text{all stochastic } \boldsymbol{h}_0$$

from which it follows in particular that in asymptotic approximation

$$\boldsymbol{g}_n = n\boldsymbol{h}$$

[9] These results are stranger than they appear, as I demonstrate: one random parameter assignment produced

$$\mathbb{Z} = \begin{pmatrix} 0 & 0 & 0.52701 \\ 1 & 0 & -1.98017 \\ 0 & 1 & 2.45309 \end{pmatrix}$$

which gives

$$\mathbb{Z}^{50}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 7.1248 \\ -19.6424 \\ 13.5176 \end{pmatrix} \equiv \boldsymbol{h}_{50} \;\approx\; \boldsymbol{h} = \begin{pmatrix} 7.1248 \\ -19.6425 \\ 13.5177 \end{pmatrix}$$

The vector $\boldsymbol{h}_{50}$ is clearly not stochastic, though its elements do sum to 1.0000.

We conclude that in asymptotic approximation

$$\begin{aligned} S_n(a,x,y,z) &= n\,\mathcal{P}(a,x,y,z) \\ \mathcal{P}(a,x,y,z) &= h_1G_{10}+h_2G_{20}+h_3G_{30} \end{aligned}$$

where by calculation (see again pages 8–9)

$$\begin{aligned} G_{10} &= -2+2a+2x \\ G_{20} &= -6+6a+4x+2ax+2x^2-4ax^2+2y-2ay-2xy+2axy+2axz \\ G_{30} &= -10+10a+4x+4ax+2a^2x+4x^2-2ax^2-10a^2x^2+2x^3-8ax^3+8a^2x^3+4y-4a^2y \\ &\quad -2xy-4axy+6a^2xy-2x^2y+4ax^2y-2a^2x^2y+2y^2-6ay^2+4a^2y^2-2xy^2+6axy^2 \\ &\quad -4a^2xy^2+2z-4az+2a^2z-2xz+8axz+2a^2xz-2a^2x^2z-2yz+4ayz-2a^2yz \\ &\quad +2xyz-4axyz+4a^2xyz+2axz^2-4a^2xz^2 \end{aligned}$$

Writing

$$\begin{aligned} h_1 &= \mathcal{D}^{-1}k_1 \quad \text{with} \quad k_1 = q_1 \\ h_2 &= \mathcal{D}^{-1}k_2 \quad \text{with} \quad k_2 = 1-q_3 \\ h_3 &= \mathcal{D}^{-1}k_3 \quad \text{with} \quad k_3 = 1 \end{aligned}$$

we have finally

$$\begin{aligned} \mathcal{P}(a,x,y,z) &= \mathcal{D}^{-1}\big[k_1G_{10}+k_2G_{20}+k_3G_{30}\big] \equiv \mathcal{D}^{-1}\,\mathcal{R}(a,x,y,z) \\ \mathcal{D} &= 2+q_1-q_3 \end{aligned}$$

The equation $\mathcal{R}(a,x,y,z)=0$ inscribes a null hypersurface within the 4-cube, of which we can only plot 3-dimensional sections at (say) selected values of a. Parrondo, however, restricts his interest to the SPECIAL CASE that results from setting $x=y$, and the surface $\mathcal{R}(a,y,y,z)=0$ *does* admit of graphic display. In that case the relevant expressions are, in fact, fairly easy to write out; we find

$$\begin{aligned} \mathcal{R}(a,y,y,z) &= -18+42a-30a^2+6a^3+32y-72ay+52a^2y-12a^3y-14y^2+30ay^2 \\ &\quad -18a^2y^2+6a^3y^2+10z-30az+26a^2z-6a^3z-16yz+48ayz-36a^2yz \\ &\quad +12a^3yz+6y^2z-18ay^2z+18a^2y^2z \\ \mathcal{D}(a,y,y,z) &= 2-6a+3a^2-4y+10ay-6a^2y+y^2-3ay^2+3a^2y^2 \\ &\quad -2z+5az-3a^2z+2yz-6ayz+6a^2yz \end{aligned}$$

which are again multinomials of orders 5 and 3, respectively.

The naive construction of $S_n(a,z,y,z)$ requires—if boundary errors are to be avoided—that $\mathbb{S}\equiv\mathbb{AB}$ be $\nu\times\nu$ with $\nu\geqslant 4n+1$. The $\mathbb{S}$ of page 14 supplies

$$(\boldsymbol{w},\mathbb{S}^n\boldsymbol{e}_0) = \begin{cases} \text{homogeneous multinomial of degree } 2n \\ \text{in } \{a,x,y,z,A,X,Y,Z\} \end{cases}$$

Replacements of the form $U \to 1 - u$ produce

$$S_n(a,x,y,z) = \begin{cases} \text{inhomogeneous multinomial of degree} \leqslant 2n \\ \text{in } \{a,x,y,z\} \end{cases}$$

To ascend to order 7 I set $\nu = (4\cdot 7+1) = 29$, whereupon *Mathematica* supplied

$$\begin{aligned} S_7(a,x,y,z) &= \text{inhomogeneous sum of 741 terms} \\ S_7(a,y,y,z) &= \text{inhomogeneous sum of 230 terms} \end{aligned}$$

and also

$$\begin{aligned} S_7(0,\,0,\,0,\,0) &= -14 \\ S_7(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) &= \quad 0 \\ S_7(1,\,1,\,1,\,1) &= +14 \end{aligned}$$

. . . which (compare (15)) make intuitive good sense: a walker who with certainty advances/retreats one step with every step of a composite 2-step move can expect to advance/retreat 14 steps in seven such moves, and to make no progress at all if the probabilities of advancing/retreating are at every step equal.