A MATHEMATICAL NOTE

Frenet-Serret formulæ in higher dimension

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Introduction. In recent conversation with a colleague[1] I was asked whether I knew how to write analogs of the Frenet–Serret formulæ when the dimension of space is $N \geq 4$. I responded that I did not, but would give the matter thought, which I did while driving home that evening; my purpose here is to write out what I could not write in heavy traffic. What follows, therefore, is my own account of what turns out (not at all surprisingly) to be some fairly ancient mathematics.[2]

Let $\boldsymbol{x}(\lambda)$ provide (in reference to some Cartesian coordinate system) the arbitrarily parameterized description of some nice curve $\mathcal{C}$ in E^n. The length of the arc $\boldsymbol{x}(0) \xrightarrow[\mathcal{C}]{} \boldsymbol{x}(\lambda)$ is given by

$$s(\lambda) = \int_0^\lambda \sqrt{(dx^1/d\lambda)^2 + \cdots + (dx^n/d\lambda)^2}\, d\lambda$$

which entails $\frac{ds}{d\lambda} = \sqrt{(dx^1/d\lambda)^2 + \cdots + (dx^n/d\lambda)^2}$. Obvious simplifications result from the identification of λ with s; i.e., from the adoption of *arc-length parameterization*, which henceforth we agree to do, writing $\boldsymbol{X}(s)$. Then

$$\sqrt{(dx^1/ds)^2 + \cdots + (dx^n/ds)^2} = 1 \tag{1}$$

and (trivially), $s = \int_0^s ds$. We agree, moreover, to write $\mathring{\boldsymbol{X}}(s) \equiv \frac{d}{ds}\boldsymbol{X}(s)$, etc.; i.e., to use little circles to emphasize that we are differentiating with respect to a specialized parameter (and to reserve dots to indicate differention with respect to time, after the manner of Newton).

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[1] Oz Bonfim, 12 August 1998. Oz has at present a research interest in chaos, whence in Lyapunov exponents, and it was in pursuit of a method to compute the latter that he was led to pose his question.

[2] See, for example, §**111.D** in the *Encyclopedic Dictionary of Mathematics* (2nd edition, 1993).

1. Review of the 3-dimensional case. We observe first that

$$\boldsymbol{T}(s) \equiv \mathring{\boldsymbol{X}}(s) \quad \text{is tangent to } \mathcal{C} \text{ at } s \tag{2}$$

and is, by (1)—i.e., by $(ds)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$—a unit vector:

$$\boldsymbol{T}(s)\cdot\boldsymbol{T}(s) = 1 \tag{3}$$

Differentiation of the preceding equation teaches us that

$$\mathring{\boldsymbol{T}}(s) \equiv \kappa(s)\boldsymbol{U}(s) \quad \text{is} \perp \text{to } \boldsymbol{T}(s) \tag{4}$$

where $\boldsymbol{U}(s)$ is a unit vector:

$$\boldsymbol{U}(s)\cdot\boldsymbol{T}(s) = 0 \quad \text{and} \quad \boldsymbol{U}(s)\cdot\boldsymbol{U}(s) = 1 \tag{5}$$

$\boldsymbol{U}(s)$ describes the direction, and the $\kappa(s)$ the magnitude, of the local *curvature* of $\mathcal{C}$. Assume $\kappa(s) \neq 0$ and define $\boldsymbol{V}(s) \equiv \boldsymbol{T}(s) \times \boldsymbol{U}(s)$ which serves to complete the construction of an orthonormal triad at each (non-straight) point s of $\mathcal{C}$. Elementary arguments lead to the conclusions that

$$\mathring{\boldsymbol{U}}(s) = -\kappa(s)\,\boldsymbol{T}(s) - \tau(s)\boldsymbol{V}(s) \quad \text{and} \quad \mathring{\boldsymbol{V}}(s) = \tau(s)\boldsymbol{U}(s) \tag{6}$$

where $\tau(s)$ is the *torsion* of $\mathcal{C}$ at s. Briefly

$$\begin{pmatrix} \mathring{\boldsymbol{T}} \\ \mathring{\boldsymbol{U}} \\ \mathring{\boldsymbol{V}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{U} \\ \boldsymbol{V} \end{pmatrix} \tag{7}$$

which comprise the famous "Frenet-Serret formulae" (1847–1851).

That is (at least what I have always taken to be) the standard line of argument.[3] I give now a variant which, as will emerge, is better adapted to dimensional generalization. We have

$$\boldsymbol{T}\cdot\boldsymbol{T} = 1 \quad \Longrightarrow \quad \mathring{\boldsymbol{T}} \perp \boldsymbol{T} \tag{8}$$

and, on the assumption that $\kappa \equiv$ length of $\mathring{\boldsymbol{T}} \neq 0$, introduce the unit vector

$$\boldsymbol{U} \equiv \tfrac{1}{\kappa}\mathring{\boldsymbol{T}} \tag{9}$$

Now we have

$$\boldsymbol{U}\cdot\boldsymbol{U} = 1 \quad \Longrightarrow \quad \mathring{\boldsymbol{U}} \perp \boldsymbol{U} \tag{10}$$

[3] See §4 of *Geometrical Mechanics: Remarks Commemorative of Heinrich Hertz* (1994) for an earlier account of this same topic.

Noting that $\mathring{\boldsymbol{U}}$ is, however, (generally) *not* normal to $\boldsymbol{T}$, we appropriate the Gram-Schmidt orthogonalization techinque to write

$$\boldsymbol{V} \equiv -\frac{\mathring{\boldsymbol{U}} - (\mathring{\boldsymbol{U}}\cdot\boldsymbol{T})\boldsymbol{T}}{\text{length, call it } \tau} \tag{11}$$

where the overall sign has been introduced to achieve compliance with the chiarality convention implicit in $\boldsymbol{V} = \boldsymbol{T}\times\boldsymbol{U}$. By differentiation of $(\boldsymbol{U},\boldsymbol{T}) = 0$ we have $(\mathring{\boldsymbol{U}},\boldsymbol{T}) = -\kappa$, so we recover

$$\mathring{\boldsymbol{U}} = -\kappa\boldsymbol{T} - \tau\boldsymbol{V} \tag{12}$$

$\{\boldsymbol{T},\boldsymbol{U},\boldsymbol{V}\}_s$ is, by construction, an orthonormal triad, so we are assured that

$$\boldsymbol{V} = \pm(\boldsymbol{T}\times\boldsymbol{U})$$

and from this (together with results already in hand) it follows readily that $\mathring{\boldsymbol{V}} = \pm\tau\boldsymbol{U}$, but to resolve the sign ambiguity one must work a bit. I adopt, therefore, an alternative (and more readily generalizable) line of argument: I write

$$\mathring{\boldsymbol{V}} = (\mathring{\boldsymbol{V}}\cdot\boldsymbol{T})\boldsymbol{T} + (\mathring{\boldsymbol{V}}\cdot\boldsymbol{U})\boldsymbol{U} + (\mathring{\boldsymbol{V}}\cdot\boldsymbol{V})\boldsymbol{V}$$

which upon integration by parts[4] becomes

$$= -(\boldsymbol{V}\cdot\mathring{\boldsymbol{T}})\boldsymbol{T} - (\boldsymbol{V}\cdot\mathring{\boldsymbol{U}})\boldsymbol{U}$$

and with assistance from (9) and (12) obtain

$$= -\kappa(\boldsymbol{V}\cdot\boldsymbol{U})\boldsymbol{T} + \{\kappa(\boldsymbol{V}\cdot\boldsymbol{T}) + \tau(\boldsymbol{V}\cdot\boldsymbol{V})\}\boldsymbol{U}$$

Orthonormality can now be invoked to yield

$$= \tau\boldsymbol{U} \tag{13}$$

We have resolved the sign ambiguity (i.e., we have recovered $\boldsymbol{V} = \boldsymbol{T}\times\boldsymbol{U}$) and have reproduced the Frenet-Serret formulæ (7).

The orthonormal pair $\{\boldsymbol{U},\boldsymbol{V}\}_s$ provide a specialized basis—the "Frenet basis"—in the plane which stands (like the Dutch gentleman's collar in a Rembrandt) normal to $\mathcal{C}$ at s. The orthonormal triple $\{\boldsymbol{T},\boldsymbol{U},\boldsymbol{V}\}_s$ provides a similarly specialized decoration of $\mathcal{C}$; for an *arbitrary* orthonormal frame decoration one expects to have

$$\begin{pmatrix}\boldsymbol{T}\\\boldsymbol{U}\\\boldsymbol{V}\end{pmatrix}_{s+ds} = \mathbb{R}(s)\begin{pmatrix}\boldsymbol{T}\\\boldsymbol{U}\\\boldsymbol{V}\end{pmatrix}_s$$

$$\mathbb{R} = \mathbb{I} + \begin{pmatrix}0 & +\omega_3 & -\omega_2\\ -\omega_3 & 0 & +\omega_1\\ +\omega_2 & -\omega_1 & 0\end{pmatrix} \;:\; \text{infinitesimal rotation matrix}$$

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[4] No integrals are actually present, but I use $\boldsymbol{V}\cdot\boldsymbol{T} = 0 \Longrightarrow \mathring{\boldsymbol{V}}\cdot\boldsymbol{T} = -\boldsymbol{V}\cdot\mathring{\boldsymbol{T}}$, which is an instance of the identity that lies at the heart of the integration by parts technique.

giving

$$\begin{pmatrix} \overset{\circ}{\boldsymbol{T}} \\ \overset{\circ}{\boldsymbol{U}} \\ \overset{\circ}{\boldsymbol{V}} \end{pmatrix} = \begin{pmatrix} 0 & +\omega_3 & -\omega_2 \\ -\omega_3 & 0 & +\omega_1 \\ +\omega_2 & -\omega_1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{U} \\ \boldsymbol{V} \end{pmatrix}$$

Comparison with (7) shows that Frenet & Serret have, by their specialized procedure, arranged to kill ω_2 and to assign direct geometrical interpretations to ω_1 (negative of the "torsion") and ω_3 ("curvature").

2. 4-dimensional theory. The essential plan of attack is unchanged: from

$$\big(\tfrac{d}{ds}\big)^1\boldsymbol{X}, \quad \big(\tfrac{d}{ds}\big)^2\boldsymbol{X}, \quad \big(\tfrac{d}{ds}\big)^3\boldsymbol{X}, \quad \big(\tfrac{d}{ds}\big)^4\boldsymbol{X}$$

—taken in that order—we use Gram–Schmidt orthogonalization to erect at each point s of $\mathcal{C} \in E^4$ an orthonormal tetrad $\big\{\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}\big\}_s$, and then look to the structure forced upon the antisymmetric matrix $\mathbb{A}$ which enters into the "generalized Frenet–Serret equation"

$$\begin{pmatrix} \overset{\circ}{\boldsymbol{T}} \\ \overset{\circ}{\boldsymbol{U}} \\ \overset{\circ}{\boldsymbol{V}} \\ \overset{\circ}{\boldsymbol{W}} \end{pmatrix} = \mathbb{A} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{W} \end{pmatrix}$$

But the cross product—which we have learned to do without already in E^3—will not be available as a resource in E^4, and the algebra will motivate a couple of small adjustments.

We begin as before, introducing

$$\boldsymbol{T}(s) \equiv \overset{\circ}{\boldsymbol{X}}(s) \quad : \quad \text{unit vector tangent to } \mathcal{C} \text{ at } s$$

Then we introduce

$$\boldsymbol{U} \equiv \frac{\overset{\circ}{\boldsymbol{T}}}{\text{length, call it } \kappa_1} \quad \text{giving} \quad \overset{\circ}{\boldsymbol{T}} = \kappa_1 \boldsymbol{U}$$

and[5]

$$\boldsymbol{V} \equiv \frac{\overset{\circ}{\boldsymbol{U}} - (\overset{\circ}{\boldsymbol{U}}\cdot\boldsymbol{T})\,\boldsymbol{T}}{\text{length, call it } \kappa_2} \quad \text{giving} \quad \overset{\circ}{\boldsymbol{U}} = -\kappa_1\,\boldsymbol{T} + \kappa_2 \boldsymbol{V}$$

Only on this point do we encounter new details (though the pattern of the argument remains unchanged); we introduce

$$\boldsymbol{W} \equiv \frac{\overset{\circ}{\boldsymbol{V}} - (\overset{\circ}{\boldsymbol{V}}\cdot\boldsymbol{T})\,\boldsymbol{T} - (\overset{\circ}{\boldsymbol{V}}\cdot\boldsymbol{U})\,\boldsymbol{U}}{\text{length, call it } \kappa_3}$$

[5] Here we drop the leading minus sign from (11), since it has lost its former reason for being. And in place of τ we now write κ_2; "torsion" has become what differential geometers call the "2^{nd} curvature."

giving $\mathring{\boldsymbol{V}} = (\mathring{\boldsymbol{V}}\cdot\boldsymbol{T})\boldsymbol{T} + (\mathring{\boldsymbol{V}}\cdot\boldsymbol{U})\boldsymbol{U} + \kappa_3\boldsymbol{W}$. But

$$\begin{aligned}\mathring{\boldsymbol{V}}\cdot\boldsymbol{T} &= -\boldsymbol{V}\cdot\mathring{\boldsymbol{T}} = -\kappa_1(\boldsymbol{V}\cdot\boldsymbol{U}) = 0\\ \mathring{\boldsymbol{V}}\cdot\boldsymbol{U} &= -\boldsymbol{V}\cdot\mathring{\boldsymbol{U}} = +\kappa_1(\boldsymbol{V}\cdot\boldsymbol{T}) - \kappa_2(\boldsymbol{V}\cdot\boldsymbol{V}) = -\kappa_2\end{aligned}$$

so we have

$$\mathring{\boldsymbol{V}} = -\kappa_2\boldsymbol{U} + \kappa_3\boldsymbol{W}$$

Finally we write

$$\begin{aligned}\mathring{\boldsymbol{W}} &= \quad(\mathring{\boldsymbol{W}}\cdot\boldsymbol{T})\boldsymbol{T} + (\mathring{\boldsymbol{W}}\cdot\boldsymbol{U})\boldsymbol{U} + (\mathring{\boldsymbol{W}}\cdot\boldsymbol{V})\boldsymbol{V} + (\mathring{\boldsymbol{W}}\cdot\boldsymbol{W})\boldsymbol{W}\\ &= -(\boldsymbol{W}\cdot\mathring{\boldsymbol{T}})\boldsymbol{T} - (\boldsymbol{W}\cdot\mathring{\boldsymbol{U}})\boldsymbol{U} - (\boldsymbol{W}\cdot\mathring{\boldsymbol{V}})\boldsymbol{V}\\ &= -\kappa_1(\boldsymbol{W}\cdot\boldsymbol{U})\boldsymbol{T} + \big\{\kappa_1(\boldsymbol{W}\cdot\boldsymbol{T}) - \kappa_2(\boldsymbol{W}\cdot\boldsymbol{V})\ \big\}\boldsymbol{U}\\ &\qquad\qquad + \big\{\kappa_2(\boldsymbol{W}\cdot\boldsymbol{U}) - \kappa_3(\boldsymbol{W}\cdot\boldsymbol{W})\big\}\boldsymbol{V}\end{aligned}$$

to (by orthonormality) obtain

$$\mathring{\boldsymbol{W}} = -\kappa_3\boldsymbol{V}$$

Pulling these results together, we have

$$\begin{pmatrix}\mathring{\boldsymbol{T}}\\ \mathring{\boldsymbol{U}}\\ \mathring{\boldsymbol{V}}\\ \mathring{\boldsymbol{W}}\end{pmatrix} = \begin{pmatrix}0 & \kappa_1 & 0 & 0\\ -\kappa_1 & 0 & \kappa_2 & 0\\ 0 & -\kappa_2 & 0 & \kappa_3\\ 0 & 0 & -\kappa_3 & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{T}\\ \boldsymbol{U}\\ \boldsymbol{V}\\ \boldsymbol{W}\end{pmatrix} \tag{14}$$

When, in this enlarged setting, differential geometers speak of the “torsion” of $\mathcal{C}$ they allude to the “last curvature” (else to its negative): $\tau \equiv \kappa_3$.

3. The general case. The n-dimensional analog of (14) is obvious. The equations

$$\kappa_i = \kappa_i(s) \quad : \quad i = 1, 2, \ldots, n-1$$

are called the “natural equations” of $\mathcal{C} \in E^n$. The natural equations permit one to write—and in principle to solve—the Frenet–Serret equation (14), and by integration of $\boldsymbol{T}(s)$ to reconstruct the functions $\boldsymbol{X}(s)$ that describe the curve. Initial value data

$$\big\{\boldsymbol{X}(0), \boldsymbol{T}(0), \boldsymbol{U}(0), \boldsymbol{V}(0), \boldsymbol{W}(0), \ldots\big\}$$

remains specifiable: diverse curves support the same natural equations. The so-called “fundamental theorem of the theory of curves” states that any such curve can be brought into coincidence with any other by a Euclidean motion (combination of translation and rotation).

Frenet–Serret theory pertains only at points on $\mathcal{C}$ at which the vectors $\{\boldsymbol{X}(s),\boldsymbol{T}(s),\boldsymbol{U}(s),\boldsymbol{V}(s),\boldsymbol{W}(s),\ldots\}$ are linearly independent; i.e., at points where the Wronskian

$$\begin{vmatrix} T_1 & T_2 & T_3 & T_4 & \ldots \\ U_1 & U_2 & U_3 & U_4 & \ldots \\ V_1 & V_2 & V_3 & V_4 & \ldots \\ W_1 & W_2 & W_3 & W_4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} \neq 0$$

At points where this condition is violated the curve is contained within a subspace of E^n, and in retracing our steps we find ourselves forced to "divide by zero."

In decorating $\mathcal{C}$ with an s-parameterized family of orthonormal frames (of specialized design) we have in effect decorated $\mathcal{C}$ with a family $\mathbb{R}(s)$ of rotation matrices—and, latently, with all that entails. In E^3 every $\mathcal{C}$ supports in particular all the familiar spinor apparatus ($SU(2)$ representation of $O(3)$).

Direct physical embodiment of the Frenet–Serret triad $\{\boldsymbol{T}(s),\boldsymbol{U}(s),\boldsymbol{V}(s)\}$ is provided by the subject matter of what used to be called the "theory of elastica," which was created by Euler more than a century before Frenet and Serret took up their respective pens, and was probably known to them. Hold one end of a length of carpenter's tape[6] in one hand, the opposite end in the other; the centerline traces a curve (on which s is conveniently marked out), $\boldsymbol{T}(s)$ lies tangent to the curve, and the meanings of $\boldsymbol{U}(s)$ and $\boldsymbol{V}(s)$ are equally obvious. Kirchhoff noticed in 1858 that the static figure $\mathbb{R}(s)$ of such a tape can be read as a kind of "diagram" of the motion $\mathbb{R}(t)$ of a top, at least in cases where the top moves subject to the side condition

$$\omega_2(t) = 0 \quad \text{at all times}$$

Kirchhoff appears, by the way, to have had a special interest in the theory of tops, and did the work to which I have referred while the Frenet–Serret formulæ still had the status of "recent discoveries."[7]

Because $\boldsymbol{T}(s)$ lies everywhere tangent to $\mathcal{C}$ (which can be recovered by an integration procedure), it is perhaps well to take explicit note of the fact that the Frenet–Serret construction *draws only upon the differential geometry of the curve itself*, not upon the notions (borrowed from the geometry of the enveloping space) which give rise to the concept of "parallel transport." It becomes, in this light, natural to contemplate a "Frenet–Serret theory of curves (especially of geodesics) in curved space." Such a creation would be distinct

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[6] Such tape is designed to be flexible with respect to the transverse axis, but semi-rigid with respect to the longitudinal axis.

[7] See Chapter 22 in A. Gray, *Treatise on Gyrostatics & Rotational Motion* (1918); A. E. Love, *Treatise on the Mathematical Theory of Elasticity* (1927); or ANALYTICAL METHODS OF PHYSICS (1981), p. 474.

from the classic theory summarized here, but something very like it can be found at work in an obscure corner of geometrical optics. An inhomogeneous medium is, from an optical point of view, a "curved manifold," with metric structure supplied by the (scalar) index of refraction. "Rays" are, according to Fermat, geodesic with respect to that metric. In an *electromagnetic* theory of light rays the ray $\mathcal{C}$ acquires decoration: attached to each point on the ray is a directional/electric/magnetic triad of orthonormal vectors $\{\boldsymbol{T}(s), \boldsymbol{E}(s), \boldsymbol{B}(s)\}$, where s refers now not to Euclidean length but to "optical path length." One can—with labor—show[8] it to be an implication of Maxwell's equations that the triads which decorate any given ray are (with respect to the optical metric) parallel transports of one another.

4. Kinematics. To describe the motion of a point along a prescribed curve $\mathcal{C}$—or, less physically, to abandon arc-length parameterization in favor of *arbitrary* parameterization—one has only to write

$$s = s(t)$$

giving

$$\boldsymbol{x}(t) \equiv \boldsymbol{X}(s(t))$$

One then has (here I work from (7); i.e., in notation special to the 3-dimensional case)

$$\begin{aligned} \dot{\boldsymbol{x}} &= \dot{s}\,\mathring{\boldsymbol{X}} \\ &= \dot{s}\,\boldsymbol{T} \end{aligned} \tag{15.1}$$

$$\begin{aligned} \ddot{\boldsymbol{x}} &= \ddot{s}\,\boldsymbol{T} + \dot{s}^2\,\mathring{\boldsymbol{T}} \\ &= \ddot{s}\,\boldsymbol{T} + \dot{s}^2\kappa\,\boldsymbol{U} \end{aligned} \tag{15.2}$$

$$\dddot{\boldsymbol{x}} = (\,\dddot{s} - \dot{s}^3\kappa^2)\,\boldsymbol{T} + (3\dot{s}\ddot{s}\kappa + \dot{s}^3\,\mathring{\kappa})\,\boldsymbol{U} - \dot{s}^3\kappa\tau\,\boldsymbol{V} \tag{15.3}$$

$$\vdots$$

There exists a better way to organize this hierarchical series of calculations: Let the Frenet–Serret formula (14)—of whatever dimension—be notated

$$\mathring{\boldsymbol{F}}(s) = \mathbb{A}(s)\,\boldsymbol{F}(s)$$

Then

$$\dot{\boldsymbol{x}} \text{ is the top element of } \dot{s}\boldsymbol{F} \tag{16.1}$$

$$\ddot{\boldsymbol{x}} \text{ is the top element of } \ddot{s}\boldsymbol{F} + \dot{s}^2\mathbb{A}\boldsymbol{F} \tag{16.2}$$

$$\dddot{\boldsymbol{x}} \text{ is the top element of } \dddot{s}\boldsymbol{F} + 3\ddot{s}\dot{s}\mathbb{A}\boldsymbol{F} + \dot{s}^3\big\{\mathbb{A}^2 + \mathring{\mathbb{A}}\big\}\boldsymbol{F} \tag{16.3}$$

$$\begin{aligned} \ddddot{\boldsymbol{x}} \text{ is the top element of } &\ddddot{s}\boldsymbol{F} + 3(\dddot{s}\,\dot{s} + \ddot{s}^2)\mathbb{A}\boldsymbol{F} + 6\ddot{s}\dot{s}^2\big\{\mathbb{A}^2 + \mathring{\mathbb{A}}\big\}\boldsymbol{F} \\ &+ \dot{s}^4\big\{\mathbb{A}^3 + 2\mathring{\mathbb{A}}\mathbb{A} + \mathbb{A}\mathring{\mathbb{A}} + \mathring{\mathring{\mathbb{A}}}\,\big\}\boldsymbol{F} \end{aligned} \tag{16.4}$$

$$\vdots$$

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[8] For an account of the details, see my OPTICS (1982), pp. 71–79.

It is (especially with the assistance of *Mathematica*) not difficult to recover (15) from (16).

Some of the preceding details are actually familiar; look, for example, to the second term on the right side of (15.2). From

$$\text{curvature } \kappa \equiv \frac{1}{\text{radius of curvature } R}$$

we see that $\dot{s}^2\kappa$ is just the v^2/R familiar from the elementary theory of uniform circular motion. But the point I wish to make concerns not the details but the rapidly increasing qualitative *complexity* of the equations just obtained. That complexity derives from three distinct but intertwined circumstances: we have been computing successive derivatives of noncommutative[9] products of composite functions.

The lesson appears to be that the program initiated by Jean-Frédéric Frenet (1816–1900) and Joesph Alfred Serret (1819–1895) owes much of its swift efficiency to the adoption of arc-length parameterization.[10]

On a previous occasion[3] I have discussed the possibility and striking utility of using Frenet–Serret theory to cleave the fundamental problem of Newtonian dynamics

$$\ddot{\boldsymbol{x}} = -\tfrac{1}{m}\boldsymbol{\nabla}U(x)$$

into two distinct parts, the first of which addresses the *geometry of the trajectory* $\boldsymbol{X}(s)$ pursued with conserved energy E by the mass point m, and culminates in a mechanical analog

$$\tfrac{d}{ds}\big[\tfrac{1}{n}\tfrac{d}{ds}\boldsymbol{X}\big] - \boldsymbol{\nabla}\tfrac{1}{n} = \boldsymbol{0} \quad \text{with} \quad \tfrac{1}{n} \equiv \sqrt{\tfrac{2}{m}[E - U(\boldsymbol{x})]}$$

of the optical "ray equation." The second part of that program addresses *temporal progress* $s(t)$ *along that trajectory*.

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[9] I allude here to the fact that $\mathbb{A}$ and $\mathring{\mathbb{A}}$ fail to commute.

[10] But maybe I may misconstrue the situation; possibly if one brought Gramm–Schmidt orthogonalization to the sequence $\{\dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}, \dots\}$ one would obtain relatively simple relations which become complicated when translated back into arc-length parameterization.