RAY MAYER'S SOLUTION OF STEIN'S PROBLEM

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Introduction. On 7 March 2015, Peter Renz relayed to me a problem posed by his friend, Sherman Stein, a mathematician retired from the UC/Davis faculty. Let

$$F(x) = \tfrac{1}{e}\cdot(1+x)^{1/x}$$

Stein asks for the expansion of $F(x)$ about $x=0$. *Mathematica* supplies

$$F(x) = 1 - \tfrac{1}{2}x + \tfrac{11}{24}x^2 - \tfrac{7}{16}x^3 + \tfrac{2447}{5760}x^4 - \tfrac{959}{2304}x^5 + \tfrac{238043}{580608}x^6 - \tfrac{67223}{165888}x^7 + \cdots$$

where the leading term provides a statement of the familiar result

$$\lim_{z\to\infty}\big(1+\tfrac{1}{z}\big)^z = e$$

Writing

$$F(x) = a_0 - a_1x + a_2x^2 - a_3x^3 + \cdots = \sum_{n=0}^{\infty} a_n(-x)^n$$

Stein asks for constructions of the coefficients a_n. I brought Stein's problem to the attention of Ray Mayer, from whom I now quote.

Mayer's solution. Write

$$\begin{aligned} F(x) = e^{f(x)} \quad \text{where} \quad f(x) &= \log F(x) \\ &= \tfrac{1}{x}\log(1+x) - 1 \\ &= -\tfrac{1}{2}x + \tfrac{1}{3}x^2 - \tfrac{1}{4}x^3 + \tfrac{1}{5}x^4 - \tfrac{1}{6}x^5 + \cdots \\ &\equiv \sum_{n=0}^{\infty} c_n(-x)^n \end{aligned}$$

with $c_0 = 0$, $c_n = \frac{1}{n+1}$ $(n = 1,2,3,\ldots)$. Application of $x\frac{d}{dx}$ to $F = e^f$ gives

$$xF' = F\cdot xf'$$

or

$$\begin{aligned} \sum_{n=0}^{\infty}(-)^n a_n n x^n = \sum_{j=0}^{\infty}(-)^j a_j x^j \cdot \sum_{k=0}^{\infty}(-)^k c_k k x^k &= \sum_{j,k=0}^{\infty}(-)^{j+k} a_j c_k k x^{j+k} \\ &= \sum_{n=0}^{\infty}(-)^n \sum_{j=0}^{n-1}(n-j)a_j c_{n-j} x^n \end{aligned}$$

where the reduced upper limit on the second summation arises from $c_0 = 0$. Equating the coefficients of x^n on left and right gives the recursion relation

$$
\begin{aligned}
a_n &= \tfrac{1}{n}\sum_{j=0}^{n-1}(n-j)c_{n-j}a_j \\
&= \tfrac{1}{n}\sum_{j=0}^{n-1}\tfrac{n-j}{n-j+1}a_j \\
&= \tfrac{1}{n}\big\{\tfrac{n}{n+1}a_0 + \tfrac{n-1}{n}a_1 + \tfrac{n-2}{n-1}a_2 + \cdots + \tfrac{2}{3}a_{n-2} + \tfrac{1}{2}a_{n-1}\big\}
\end{aligned}
\tag{1}
$$

Mayer's construction (1) agrees precisely with the recursion relation that was promptly obtained (almost certainly by the same argument) by Don Chakerian (Stein's colleague, also retired from the mathematics faculty at UC/Davis).

Some ramifications. In (1) a_n is seen to depend linearly on $a_0, a_1, \ldots, a_{n-1}$, which places at our disposal the resources of linear algebra. Let

$$
\mathbb{A} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\frac{1}{1}\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\frac{1}{2}\frac{2}{3} & \frac{1}{2}\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\frac{1}{3}\frac{3}{4} & \frac{1}{3}\frac{2}{3} & \frac{1}{3}\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\frac{1}{4}\frac{4}{5} & \frac{1}{4}\frac{3}{4} & \frac{1}{4}\frac{2}{3} & \frac{1}{4}\frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\
\frac{1}{5}\frac{5}{6} & \frac{1}{5}\frac{4}{5} & \frac{1}{5}\frac{3}{4} & \frac{1}{5}\frac{2}{3} & \frac{1}{5}\frac{1}{2} & 0 & 0 & 0 & \cdots \\
\frac{1}{6}\frac{6}{7} & \frac{1}{6}\frac{5}{6} & \frac{1}{6}\frac{4}{5} & \frac{1}{6}\frac{3}{4} & \frac{1}{6}\frac{2}{3} & \frac{1}{6}\frac{1}{2} & 0 & 0 & \cdots \\
\frac{1}{7}\frac{7}{8} & \frac{1}{7}\frac{6}{7} & \frac{1}{7}\frac{5}{6} & \frac{1}{7}\frac{4}{5} & \frac{1}{7}\frac{3}{4} & \frac{1}{7}\frac{2}{3} & \frac{1}{7}\frac{1}{2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{pmatrix}
$$

and

$$
\boldsymbol{a}_0 = \begin{pmatrix} 1 \\ * \\ * \\ * \\ \vdots \end{pmatrix}
$$

Then

$$
\mathbb{A}\boldsymbol{a}_0 = \begin{pmatrix} 1 \\ a_1 \\ * \\ * \\ \vdots \end{pmatrix} \equiv \boldsymbol{a}_1,\ \ \mathbb{A}\boldsymbol{a}_1 = \mathbb{A}^2\boldsymbol{a}_0 = \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ * \\ * \\ \vdots \end{pmatrix} \equiv \boldsymbol{a}_2,\ \ \ldots\ ,\ \mathbb{A}^n\boldsymbol{a}_0 = \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_n \\ * \\ * \\ \vdots \end{pmatrix} \equiv \boldsymbol{a}_n
$$

where the values of the $*$-terms are irrelevant. To see more clearly how this comes about, we look to the case $\mathbb{A}^5$ where (according to *Mathematica*) we have

$$\mathbb{A}^5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{11}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{7}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{2447}{5760} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{959}{2304} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{148769}{362880} & \frac{1}{23040} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{588073}{1451520} & \frac{1}{6048} & \frac{1}{80640} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

and notice that the coefficients $\{a_0, a_1, a_2, a_3, a_4, a_5\}$ stand in sequence at the top of the leading column, with the consequence that manifestly $\boldsymbol{a}_5 = \mathbb{A}^5\boldsymbol{a}_0$.

To gain additional insight, we look to this 8×8 truncated version of $\mathbb{A}$

$$\tilde{\mathbb{A}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{1}\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\frac{2}{3} & \frac{1}{2}\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}\frac{3}{4} & \frac{1}{3}\frac{2}{3} & \frac{1}{3}\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4}\frac{4}{5} & \frac{1}{4}\frac{3}{4} & \frac{1}{4}\frac{2}{3} & \frac{1}{4}\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{5}\frac{5}{6} & \frac{1}{5}\frac{4}{5} & \frac{1}{5}\frac{3}{4} & \frac{1}{5}\frac{2}{3} & \frac{1}{5}\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{6}\frac{6}{7} & \frac{1}{6}\frac{5}{6} & \frac{1}{6}\frac{4}{5} & \frac{1}{6}\frac{3}{4} & \frac{1}{6}\frac{2}{3} & \frac{1}{6}\frac{1}{2} & 0 & 0 \\ \frac{1}{7}\frac{7}{8} & \frac{1}{7}\frac{6}{7} & \frac{1}{7}\frac{5}{6} & \frac{1}{7}\frac{4}{5} & \frac{1}{7}\frac{3}{4} & \frac{1}{7}\frac{2}{3} & \frac{1}{7}\frac{1}{2} & 0 \end{pmatrix}$$

of which the eigenvalues are obviously $\{1, 0, 0, 0, 0, 0, 0, 0\}$. From the values (see page 1) of $\{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ we construct a column vector $\tilde{\boldsymbol{a}}$ and are informed by *Mathematica* that vectors proportional to $\tilde{\boldsymbol{a}}$ are eigenvectors associated with the eigenvalue $\lambda = 1$:

$$\tilde{\mathbb{A}}\tilde{\boldsymbol{a}} = \tilde{\boldsymbol{a}}$$

(NOTE: *Mathematica*, when asked for the leading eigenvector, produces $\tilde{\boldsymbol{a}}/a_7$.)

Asymptotics. We have $a_0 = 1$ and are satisfied by low-order numerical evidence that the a_n decrease monotonically, as also does their rate of descent:

$$a_n > a_{n+1} \quad \text{and} \quad a_{n-1} - a_n > a_n - a_{n+1}$$

Noting that one has $a_n < 0.4$ for $n \geqslant 9$, that the rate of descent is by that point already pretty slow ($a_9 - a_{10} = 0.003045$), and that one appears to have

$$a_n > 1/e = 0.367879 < 0.4 \quad : \quad \text{all } n$$

Chakerian conjectured that

$$\lim_{n\to\infty} a_n = 1/e$$

Equivalently,

$$\lim_{n\to\infty} \log a_n = -1$$

which is to say: we expect $\log a_n$ to descend monotonically from 0 to -1, very like functions of the form

$$S(x\,;\alpha,p) \equiv e^{-\alpha\,x^p} - 1$$

To achieve coincidence at $\log a_{100} = -0.99054$ we set

$$\alpha = \alpha(p) = -\frac{\log(1+\log a_{100})}{100^{\,p}} = \frac{4.66068}{100^{\,p}}$$

to obtain a function

$$S(x;p) = \exp\Big[-4.66068\Big(\frac{x}{100}\Big)^p\Big] - 1$$

that gives $S(100;p) = \log a_{100}$ for all p. To achieve coincidence also at $\log a_{200} = -0.995155$ we set

$$p = \frac{\log\big[-\log(1+\log a_{200})\big] - \log 4.66068}{\log 2} = 0.193543$$

We on this basis expect the numbers $\log a_n$ to be well approximated by the function

$$S(x) = \exp\Big[-4.66068\Big(\frac{x}{100}\Big)^{0.193543}\Big] - 1 \quad : \quad x = 0,1,2,3,\dots$$

When with *Mathematica*'s assistance we (i) `ListPlot` the numbers

$$\log a_0, \log a_1, \log a_2, \dots, \log a_{200}$$

(ii) `Plot` the function $S(x) : 0 \leqslant x \leqslant 200$, and (iii) superimpose the two graphs, we find that initially $S(n)$ underestimates $\log a_n$ by an amount that falls to less than 1% at $n = 20$ and has fallen to 0.001% at $n = 99$. At $n = 101$ the error switches signs and rises to 0.006% at $n = 150$, then falls again to 0.0001% at $n = 199$. It switches signs again at $n = 201$. The available evidence suggests that a formula of the type $S(n) \sim \log a_n$ becomes ever more precise as n becomes larger; *i.e.*, that Chakerian's conjecture is correct. If one had at hand an analytical (rather than a recursive) description of $\log a_n$ one would expect to be able to assemble improved analogs of 4.66068 and 0.193543 from mathematical constants. But the analytical description of $\log a_n$ appears to require the importation of some new ideas.