A Paradox Involving Sommerfeld's Function

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Introduction. In 1899 Arnold Sommerfeld (1868–1951) wrote a paper of which Robert Warnock (Reed '52, my friend of 65 years) has recently prepared an English translation: "On the propagation of electrodynamic waves along a wire." The paper provides what is claimed to be the "first rigorous solution" of a basic physical problem which to this day still provokes dispute, but the physics need not concern us here.

Central to Sommerfeld's discussion is the (complex extension of) the function $y(x)$ defined by functional inversion of

$$x(y) = y \log y$$

Looking to the graph of $x(y)$ [Figure 1] we see that $x(0) = 0$, the curve descends to a minimum $x(1/e) = -1/e$ and then ascends monotonically, passing through $x(1) = 0$. From $\frac{d}{dx}[y(x) \log y(x) - x = 0]$ we obtain

$$y'(x)[1 + \log y(x)] - 1 = 0$$

for which *Mathematica* provides solutions

$$y(x) = \frac{x + c}{\text{LambertW}(x + c)}$$

but with this WARNING: "Inverse functions are being used by `Solve`, so some solutions may not be found." To achieve $y(0) = 1$ we set the constant of integration $c = 0$, so are led to

$$y(x) = \frac{x}{\text{LambertW}(x)} \tag{1.1}$$

If we write $y(x) = e^{w(x)}$ we are led by a similar argument to

$$y(x) = e^{\text{Lambert}(x)} \tag{1.2}$$

Properties of the LambertW function—which henceforth I will denote $w(x)$ —are developed in none of the usual handbooks,[1] but Google leads one to a

[1] Abramowitz & Stegun, Erdélyi *et al*, Magnus & Oberhettinger, Gradshteyn & Ryzhik, Spanier & Oldham.

great many web sources; I have found the Wolfram MathWorld article, the Wikipedia article "Lambert W Function" and especially a paper by Corless *et al*[2] to be helpful. The function originates in work of Lambert (1758), came to Euler's attention in 1764 and some of its properties were developed by Euler in 1783. Since that time, because it arises in scientific applications of such remarkable variety, it has been reinvented every decade or so: Sommerfeld's reinvention is but one among the many.

Lagrange inversion. The consistency of equations (1) hinges on the Lambert identity $e^{w(x)} = x/w(x)$; *i.e.*, upon the fact that $w(x)$ is the functional inverse of

$$w\,e^w = x(w)$$

—$x(w) = w\,e^w$ is plotted inFigure 2—just as $u(x) = \log x$ is the functional inverse of

$$e^u = x(u)$$

The functions $w(x)$ and $\log x$ are evidently close relatives, and it is no surprise that the analytic properties of the former mimic those of the latter.

Because the leading (constant) coefficient in

$$e^u = 1 + u + \tfrac{1}{2}u^2 + \tfrac{1}{6}u^3 + \tfrac{1}{24}u^4 + \cdots = x$$

is non-zero we cannot use Lagrange's inversion formula to develop $u(x)$ as a power series in x, though from

$$e^u - 1 = 0 + u + \tfrac{1}{2}u^2 + \tfrac{1}{6}u^3 + \tfrac{1}{24}u^4 + \cdots = x$$

we by Lagrange inversion[3] obtain

$$\log(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots$$

On the other hand, the leading term in

$$w\,e^w = w + w^2 + \tfrac{1}{2}w^3 + \tfrac{1}{6}w^4 + \tfrac{1}{24}w^5 + \cdots$$

does vanish, so Lagrange inversion is available, and produces

$$w(x) = x - x^2 + \tfrac{3}{2}x^3 - \tfrac{8}{3}x^4 + \tfrac{125}{24}x^5 + \cdots \tag{2}$$

which is precisely the result produced by `Series[LambertW[x],{x,0,5}]`.

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[2] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffery & D. E. Knuth, "On the Lambert W function," Advances in Computational Mathematics **5**, 329–359 (1996). Corless, Hare and Jeffery are Canadian, so prefer Maple over *Mathematica*. The first four of those authors wrote "Lambert's W function in Maple," The Maple Technical Newsletter **9**, 12–22 (no date given).

[3] *Mathematica* permits one to avoid the complications that attend Lagrange inversion: define $g(u) = e^u - 1$ and command

```
InverseSeries[Series[g[x],{x,0,4}]]
```

From (2) Sommerfeld might have obtained

$$y(x) = e^{w(x)} = \frac{x}{w(x)} = 1 + x - \tfrac{1}{2}x^2 + \tfrac{2}{3}x^3 - \tfrac{9}{8}x^4 + \tfrac{32}{15}x^5 - \cdots \tag{3}$$

which—since he worked long before computers were available to provide high-precision evaluations of $w(x)$, but had interest only in (complex) x-values in the that lay close to the origin—would seeming have served his purposes well;[4] carrying the series (3) to tenth order, we find $y(0.1) = 1.09557$ which agrees precisely with the reported evaluation of `Exp[LamberrtW[0.1]]`.

First hint of a mystery. The expansion (3) supplies $y(0) = 1$ but provides no indication of the fact—evident from the graph of $x(y) = y \log y$—that also $y(0) = 0$, or that $y(x)$ is in fact double-valued on the interval $-1/e \leqslant x \leqslant 0$. We have

$$\begin{aligned} x(1) &= 0 \\ x(0) &= \text{indeterminate, though } \lim_{y\to 0} x(y) = 0 \end{aligned}$$

Expansion of $x(y)$ about $y = 1$ gives

$$x(y) = -1 + \tfrac{1}{2}(x-1)^2 - \tfrac{1}{6}(x-1)^3 + \tfrac{1}{12}(x-1)^4 - \tfrac{1}{20}(x-1)^5 + \cdots$$

in which the leading coefficient is non-zero, so it is rather mysterious (?) that the command

```
InverseSeries[Series[x Log[x],{x,1,5}]]
```

gives back (3).

Iterative evaluation of Lambert's function. Today one can evaluate $w(x)$—or even $w(z)$, with z complex—by simple command `LambertW[z]`.[5] But until about 1970 people doing numerical calculations involving $\log x$, $\sin x$, etc. had to consult tables. To evaluate $w(z)$ Lambert, Euler, ..., Sommerfeld, ... had, in the absence of tables, to devise iterative algorithms, of which several are described in the sources cited above (see especially Corless *et al*). One, based on Newton's method, proceeds

$$\begin{aligned} w_0 &= \text{seed}(x) \\ w_{n+1} &= w_n - \frac{w_n e^{w_n} - x}{e^{w_n}(w_n + 1)} \quad : \quad n = 0, 1, 2, \ldots \end{aligned} \tag{4}$$

Another, based on Halley's refinement of Newton's method, proceeds

$$\begin{aligned} w_0 &= \text{seed}(x) \\ w_{n+1} &= w_n - \frac{w_n e^{w_n} - x}{e^{w_n}(w_n + 1) - \dfrac{(w_n + 2)(w_n e^{w_n} - z)}{2(w_n + 1)}} \end{aligned} \tag{5}$$

What value should be assigned to the seed? I follow suggestions of Corless *et al*:

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[4] That it would in fact not have is the "paradox"—or at least the surprise—at issue.

[5] To obtain the value assumed at z by $w(z)$ on the n^{th} sheet command `LambertW[n,z]`.

At undistinguished typical points take the seed to be given by the first two terms in the asymptotic expansion of $w(z)$:

$$\text{seed}(z) = \log z - \log\log z$$

In the neighborhood of the branch point[6] use

$$\text{seed}(z) = -1 + \sqrt{2(ez+1)}$$

In the neighborhood of the origin use

$$\begin{aligned}\text{seed}(z) &= \texttt{PadeApproximant[LambertW[z],\{z,0,\{3,2\}\}]}\\ &= \frac{z + \frac{19}{10}z^2 + \frac{17}{60}z^3}{1 + \frac{29}{10}z + \frac{101}{60}z^2}\\ &= z - z^2 + \tfrac{3}{2}z^3 - \tfrac{8}{3}z^4 + \cdots\end{aligned}$$

or more simply

$$\begin{aligned}\text{seed}(z) &= \texttt{PadeApproximant[LambertW[z],\{z,0,\{2,1\}\}]}\\ &= \frac{z + \frac{1}{2}z^2}{1 + \frac{3}{2}z}\\ &= z - z^2 + \tfrac{3}{2}z^3 - \tfrac{9}{4}z^4 + \cdots\end{aligned}$$

Numerical experiments (use $z = z_{\text{typical}} + re^{i\varphi}$, where modest real values are assigned to r and random real values to φ) show that the iteration process (4) converges rapidly (typically in fewer than half a dozen steps) to an estimate (accurate to the 9$^{\text{th}}$ decimal) of the value of $w(z)$, and that (5) converges even more rapidly.

In the near neighborhood of the origin one could alternatively use the series (2). But continuation of the series

$$\begin{aligned}w(z) = x - x^2 + \tfrac{3}{2}x^3 - \tfrac{8}{3}x^4 + \tfrac{125}{24}x^5 - \tfrac{54}{5}x^6 + \tfrac{16807}{720}x^7 - \tfrac{16384}{315}x^8\\ + \tfrac{531441}{4480}x^9 - \cdots - \tfrac{32000000000000000}{14849255421}x^{20} + \cdots\end{aligned}$$

shows that the coefficients alternate in sign, and grow very rapidly; their growth is in fact log-linear, with $c_n \approx 10^{0.317}$. The series is useful for $|z| < 0.2$. But for larger values of $|z|$ the partial sums converge very slowly, and not at all for $|z|$ greater than about 4.6.

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[6] Figures produced by the commands

```
Plot3D[Abs[LambertW[x+iy]],{x,-2,2},{y,-2,2}]

Plot3D[Arg[LambertW[x+iy]],{x,-2,2},{y,-2,2}]
```

show that the principal sheet of $w(z)$ possesses a branch cut that runs along the real axis from $-\infty$ to a branch point at $z = -1/e + 0i$. The branch points of higher-order sheets stand at the complex origin. The command `Abs` produces the modulus (`Abs[1+i]`$=\sqrt{2}$) and `Arg` produces the phase (`Arg[1+i]`$=\frac{1}{4}\pi$).

Iterative evaluation of Sommerfeld's function. From the iteratively generated sequence $\{w_0, w_1, w_2, \ldots\}$ one obtains by computations $y_n = e^{w_n}$ a sequence $\{y_0, y_1, y_2, \ldots\}$ that approaches a solution of Sommerfeld's equation $y \log y = z$. But that y-sequence is inferred, not directly iterative.[7] Sommerfeld, however, has proposed an algorithm that proceeds without reference Lambert's function. He writes

$$\left.\begin{aligned} y_0 &= z \\ y_1 &= z/\log y_0 \\ y_2 &= z/\log y_1 \\ &\vdots \\ y_{n+1} &= z/\log y_n \end{aligned}\right\} \tag{6}$$

which he abbreviates

$$y(z) = \cfrac{z}{\log \cfrac{z}{\log \cfrac{z}{\log \cfrac{z}{\cdots}}}}$$

and calls a "non-terminating continued fraction," though it is in point of fact not a continued fraction but a non-terminating nested function.

The *Mathematica* command `NestList[f[#]&,x,n]//TableForm` produces a list of the successive nestings (through order n) of $f(x)$—thus[8]

$$\begin{aligned} &x \\ &\cos x \\ &\cos(\cos x) \\ &\cos(\cos(\cos x)) \end{aligned}$$

—while `NestList[f[#]&,x,k]` produces the k^{th} entry in that list.

From Figure 1 we see that $0 < y < 1/e$ entails $-1/e < x < 0$. To implement (6) we define

$$f(x, s) = \frac{x}{\log s}$$

and at $x = -0.01$ obtain $y_{20} = 0.00154493 + 3.67862 \times 10^{-19} i$, which does (with the indicated precision) satisfy

$$y_{20} \log y_{20} = -0.01$$

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[7] I have been unable to cast that sequence as a directly iterative sequence $\{y_0 \to y_1 \to y_2 \to \cdots\}$.

[8] One is reminded that when, in the early 1970s, the HP-45 appeared on the market it did not take long for people to notice that when x was assigned any value between 0 and $\frac{1}{2}\pi$ and the COS key pressed repeatedly (the HP-45 used RPN), the number 0.739085 invariably appeared, for the reason illustrated in Figure 3.

and does so even if the imaginary part of y_{20} is abandoned. But

$$y_{\text{analytic}} = \exp[w(-0.01)] = 0.989949$$

is *also* a solution of $y \log y = -0.01$. **The Sommerfeld algorithm has in this instance led to a solution distinct from the analytic solution.**

This situation persists as x proceeds downward toward $-1/e = -0.367879$, but one must carry Sommerfeld's iteration to progressively higher order, as illustrated below:

$$\begin{aligned} x &= -0.1 \\ y_{20} &= 0.0279552 + 7.44244 \times 10^{-13} i \\ y_{\text{analytic}} &= 0.894194 \end{aligned}$$

$$\begin{aligned} x &= -0.2 \\ y_{30} &= 0.0786584 + 1.13041 \times 10^{-13} i \\ y_{\text{analytic}} &= 0.771691 \end{aligned}$$

$$\begin{aligned} x &= -0.3 \\ y_{40} &= 0.168413 + 1.38626 \times 10^{-11} i \\ y_{\text{analytic}} &= 0.612993 \end{aligned}$$

$$\begin{aligned} x &= -1/e \\ y_{1000} &= 0.367143 + 7.95736 \times 10^{-7} i \\ y_{\text{analytic}} &= 0.367879 = 1/e \end{aligned}$$

where calculation shows that in all cases the imaginary parts of y_n can be abandoned. Also that the real parts of y_n stabilize long before the imaginary parts, which appear to meander in a leisurely way toward zero.

For positive real values of x the Sommerfeld algorithm produces sequences in which the real parts stabilize promptly, as do the absolute values of the imaginary parts (which no longer diminish), but the latter alternate in sign: the sequence therefore does not converge, so is useless.

Moving way from the real axis, I have set

$$z = -x(1 + \tfrac{1}{2}e^{i\varphi}) \quad : \quad x = 0.1, 0.2, 0.3 \text{ and } \varphi \text{ random}$$

and found that again the Sommerfeld algorithm converges to a solution of $y \log y = z$, but a solution distinct from y_{analytic}. In particular, Sommerfeld had physical reason (see page 28 of Warnock's translation) to look to the case

$$\begin{aligned} z &= -1.44(1 - i) \times 10^{-7} \\ y_{20} &= (7.49012 - 8.20184 i) \times 10^{-9} \\ y_{\text{analytic}} &= 1.00 + 1.44 \times 10^{-7} \end{aligned}$$

Again, we have distinct solutions of $y \log y = z$, and have to wonder why Sommerfeld assigned physical significance to one but not the other (of which, however, he was ignorant).

The problem in a nutshell. The Sommerfeld algorithm converges—when it converges—to a solution distinct from the solution $y(z) = e^{\text{LambertW}(z)}$ of Sommerfeld's equation $y \log y = z$. The problem is to construct an analytic description of $y_{\text{Sommerfeld}}$. *Mathematica* warned us already at the outset that we might have missed such a function. How to recover it? The problem looks to be difficult because the desired function can exist only within the Sommerfeld iteration algorithm's domain of convergence, and that—as will emerge from the last few of the following figures—appears to be difficult to describe.

Figure captions. The 3D figures in the following collection are best viewed in the *Mathematica* notebooks from which they were taken, for there they can be viewed from all angles.

Figure 1. Graph of $x(y) = y \log y$, a real valued function of a real argument.

Figure 2. Graph of $x(w) = w\, e^w$, a real valued function of a real argument, the result of setting $y = e^w$.

Figure 3. Representation of the inevitable result of nesting the function $\cos x$, for any $x : 0 < x < \frac{1}{2}\pi$.

ANALYTICAL PROPERTIES OF LAMBET'S FUNCTION

Figure 4. Modulus of the principle value of $w(x + iy) = \text{LambertW}(x + iy)$, where x ranges $(-2 \rightarrow +1)$, y ranges $(-1 \rightarrow +1)$, $|w(z)|$ ranges $(0 \rightarrow 1.7)$. The blue line stands at the origin. The red lines indicate the branch cut (which runs along the negative real axis) and the location of the branch point (which stands at $x = -1/e$).

Figure 5. Moduli of the first three sheets of $w(z)$. The branch points of the second, third and all higher sheets stand at the origin (red line). x and y range as before, while $|w(z)|$ now ranges $0 \rightarrow 15$.

Figure 6. Phase of the principal value of $w(z)$. The red line indicates the branch cut, the red/blue dashed line indicates the location of the banch point, which stands now at the origin. x ranges $(-2 \rightarrow +2)$ and y range as before, while phase ranges $(-\pi \rightarrow +\pi)$.

Figure 7. Phase of the second sheet of $w(z)$. x and y range as before, while the phase ranges $(1.6 \rightarrow 2.3)$. The tent is peculiar to the second sheet, and the discontinuity at the cut is much reduced.

Figure 8. Ascending sections of the tent shown in the preceding figure.

Figure 9. Phase of the first three sheets of $w(z)$. Successive sheets become progressively flatter and more compact.

Figure 10. Central detail of the preceding figure, showing the tent on the second sheet. All branch points stand at the origin.

ANALYTICAL PROPERTIES OF SOMMERFELD'S FUNCTION

Figure 11. Modulus of the principle value of Sommerfeld's function

$$S(z) = e^{w(z)} = \frac{z}{w(z)}$$

The branch point stands at $x = -1/e$ and the branch cut runs along the negative real axis. x ranges $(-1.5 \to 0.5)$, y ranges $(-0.5 \to 0.5)$, the vertical scale ranges $(0 \to 1.7)$.

Figure 12. Moduli of the first three sheets of $S(z)$. The sheets descend with ascending order. The branch points of all but the principal sheet stand at the origin.

Figure 13. Phase of the principal value of $S(z)$.

Figure 14. Phase of the second sheet of $S(z)$. The discontinuity at the branch cut that runs along the negative x-axis (and terminates now not at $x = -1/e$ but at $x = 0$) is now much reduced, and a second curvilinear branch cut has appeared.

Figure 15. Phase of the first three sheets of $S(w)$.

SURFACES GENERATED BY SOMMERFELD ITERATION

Figure 16. I define $F(z, s) = z/\log s$ and use `G[z_]:=Nest[F[z,#]&,z,n]` to produce the result of n-fold Sommerfeld iteration at z. The command

$$\texttt{AbsData=Table[Abs[G[}\frac{0.01 + j + ki}{10 * 50}\texttt{]],\{k,-50,50\},\{j,-50,50\}]}$$

tabulates the values assumed by the moduli of each of the 100×100 points of a lattice with vertices at $\frac{1}{10}(\pm 1 \pm i)$, which defines what I call the "large domain." Experiment shows that 7-place stability is achieved on that domain with $n = 15$. The "0.01" shifts the lattice a bit so as to avoid the fact that the function $z/\log z$ is formally indeterminate at the origin. I use `ListPlot3D` to display the data. In the figure the red line marks the x-axis, the blue line marks the y-axis, the black line locates the origin. The vertical range is $(0 \to 0.04)$. Ripples just shy of the 45° lines on both sides of the positive x-axis mark—as will emerge—the boundaries of the domain where Sommerfeld iteration fails to converge (because the sign of the phase is an alternating function of n).

Figure 17. That same procedure was used here to produce iterative moduli on the "small domain" with vertices at $\frac{1}{10000}(\pm 1 \pm i)$. The vertical range is now $(0 \to 13 \times 10^{-6})$.

Figure 18. Iterative phase on the large domain, with $n = 15$ (odd). The vertical range is $(-4.5 \to +4.5)$. Produced as before with `Abs` replaced by `Arg`, and displayed by `ListPlot3D[Data, InterpolationOrder→3]`; *i.e.*, with interpolation accomplished by means of a cubic polynomial.

Figure 19. The same, but with $n = 16$ (even). Note the phase reversal in the triangular non-convergent domain.

Figure 20. Iterative phase on the small domain, with $n = 15$ (odd). The vertical range is $(-3 \to +3)$. The non-convergent domain is now narrower, indicating that it curves away from the x-axis as x increases.

Figure 21. The same, but with $n = 16$ (even).

Figure 22. Here, by way of contrast, is a display of the modulus of the principal value of Sommerfeld's function $S(z)$ on the small domain. The vertical range is $(0.99990 \to 1.00010)$, which is to say: the modulus ranges in the neighborhood of one, while (Figure 17) the iterative modulus ranges in the neighborhood of zero.

Figure 23. Display of the phase of the principal value of $S(z)$ on the small domain. Compare Figures 20 & 21.

Concluding remark. Figures 18–21 indicate why one can expect the production of an analytic function that reproduces the convergent points of the Sommerfeld iteration procedure to be difficult, particularly since in the neighborhood of the origin the convergence domain is known to be awkwardly shaped, and its boundary in regions remote from the origin remain uncharted.

ADDENDUM: THE PROBLEM SOLVED

No sooner had the preceding material been rendered as a pdf draft than it occurred (actually re-occured) to me that my problem might trace to the circumstance—made evident by the graph of $x(y) = y \log y$ [Figure 1] and remarked already on page 3—that while $y(x)$ is double-valued on the interval $-1/e \leqslant x \leqslant 0$, the function $y(x) = \exp[\text{LambertW}(x)]$ isn't (compare Figures 24 & 25; the latter fails to reproduce the red portion of the former).

But the Lambert W-function is, of course, *infinitely* multivalued; the construction

$$y_n(x) = \texttt{Exp[LambertW[n,x]]} \quad : \quad n = 0, 1, 2, \ldots$$

produces an infinitude of functions. But of those,

$$f_n(z) \equiv y_n(z) \log y_n(z) - z$$

vanishes for (all z: see Figure 26) only in the case $n = 0$; *i.e.*, on the principal sheet.[9] In higher order $(n = 1, 2, \ldots)$ those functions fail to vanish *aywhere*, which is why I have dismissed them from consideration.

There exists, however, yet another branch of the Lambert W-function. It arises from setting $n = -1$, and is real on (and only on) the real interval $-1/e \leqslant x \leqslant 0$ [Figure 27]. From Figure 28 we see that *it does precisely the job that is required*. Analytic properties of $y_{-1}(z)$ are illustrated in Figures 29–32.

[9] I call $f_n(z)$ the "analytical fault function of order n" because when non-zero it quantifies the extent to which $y_n(z)$ fails to satisfy Sommerfeld's equation.

At $z = -1.44(1-i)\times 10^{-7}$ we obtain $y_{-1}(z) = (7.49012-8.20184i)\times 10^{-9}$, in precise agreement with Sommerfeld's interative y_{20}, as reported on page 6: the function $y_{-1}(z)$ has here reproduced a result obtained by convergent Sommerfeld iteration. One might infer from this isolated result that (i) the analytic fault function $f_{-1}(z)$ vanishes for all z, and (ii) that it reproduces the result of Sommerfeld iteration when that process converges. Both inferences are, however, profoundly in error, as I will demonstrate.

Figure 33 (compare Figure 26) indicates that $f_{-1}(z)$ vanishes only on a domain that is largely confined to the upper left quadrant of the complex plane; elsewhere it assumes complex non-zero values (Figures 34 & 35).

The "iterative fault function" $F(z)$ arises from

$$f(x,s) = \frac{x}{\log s}$$

by

```
Y[z_]:= Nest[f[z,#]&, z, 25]
F[z_]:= Y[z]Log[Y[z]] - z
```

Figures 36 & 37, which display (respectively) the real and imaginary parts of $F(z)$, indicate that Sommerfeld iteration produces solutions of the Sommerfeld equation except on the curvilinear wedge where—as shown in Figures 18–21—it fails to converge (because phase has become an oscillatory function of iterative order).

Comparison of Figures 33–35 with Figures 36–37 indicates that while the null zones of $f_{-1}(z)$ and F(z) intersect (mainly on the upper left quadrant), they are not coextensive: there are points z at which both vanish, points at which one but not the other vanish, points at which neither vanish, as I demonstrate:

$$\text{at } z = -0.1+0.1i \; \begin{cases} f_{-1}(z) = 0 \\ F(z) = 0 \end{cases}$$

$$\text{at } z = -0.4+0.1i \; \begin{cases} f_{-1}(z) = 0 \\ F(z) = 0.00002 + 2.48989\times 10^{-7}\, i \end{cases}$$

$$\text{at } z = +0.1-0.1i \; \begin{cases} f_{-1}(z) = -0.14383 + 0.01782\, i \\ F(z) = 0 \end{cases}$$

$$\text{at } z = +0.4-0.1i \; \begin{cases} f_{-1}(z) = -0.48910 - 0.13555\, i \\ F(z) = -0.52998 - 0.23586\, i \end{cases}$$

Figures 36–37 are, however, deceptive, mainly because they refer to a complex domain (vertices at $0.1(\pm 1 \pm i)$) too small to include the point $x = -1/e$, but also because of the limited resolution. Sommerfeld, in his 1899 paper, remarks[10] that "convergence questions [relating to his iteration procedure] are settled completely (at least for the case of real z) are settled completely in a special

[10] Page 26 in Warnock's translation.

note.[11] Simple convergence is, of course, quite another thing than *convergence to a solution* of the Sommerfeld equation. Figures 38 & 39 show, respectively, the real and imaginary parts of the values assumed by $F(z)$ on the real line. The oscillations bunch ever closer to $x = -1/e$ as the order of the iteration is increased, but the sign of the slope of the ramp (x positive) in Figure 39 depends on the parity of the order. The asymptotic implication appears to be that $F(x)$ vanishes for $x \neq -1/e < 0$.

But all hell breaks loose when one moves off the real line: see Figure 40.

Epilogue. These pages are simply the record of some experimental mathematics. They provide analytical discussion or proof of absolutely nothing, which is to say: they contain no mathematics. I feel obliged in the light of this circumstance to mention that the Wikipedia article cited on page 2 provides a list of sixteen simple-seeming problems that lead by proper mathematics to occurances of the Lambert W-function. I look to a single example (Example 4, where the topic is treated by other means than those that follow), selected because of its close relation to Sommerfeld's problem.

It was, I think, Euler[12] who first considered the evaluation of

$$h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = x^{h(x)} \tag{7.1}$$

The logarithm of (7) reads $\log h(x) = h(x) \log x$, which can be written

$$\frac{1}{h(x)} \log \frac{1}{h(x)} = -\log x$$

But this is an instance of Sommerfeld's equation, so we have at once

$$\frac{1}{h(x)} = \frac{-\log x}{w(-\log x)} \quad \text{or} \quad h(x) = \frac{w(-\log x)}{-\log x} \tag{7.2}$$

Corless *et al*, who treat the problem briefly on their pages 4–5 (and are the source of all my historical references), report that this is an ancient result, due to G. Eisenstein,[13] and that Euler himself had established that the exponential iteration converges for $e^{-e} < x < e^{1/e}$; *i.e.*, for $0.065988 < x < 1.44467$. But when one plots [Figure 41] the expression on the right side of (7.2) one finds convergence for $0 < x < e^{1/e}$, with $h(0) \approx 0.102$ (seems implausible) and $h(e^{1/e}) = e$. Convergence for complex values of x has been discussed by

[11] Whereupon he cites "Uber die numerische Anflösnug transcendenter Gleichungen durch successive Approximationen," Gött. Nachr., December 1898, of which Warnock has promised an English translation.

[12] "De formulis exponentialibus replicatis," (1777). Iterated exponentials are called "tetrations," and are the subject of an interesting Wikipedia article.

[13] "Entwicklung von $a^{a^{\cdot^{\cdot^{\cdot}}}}$," J. reine angewandte Math **28**, 49–52 (1844).

I. N. Baker & P. J. Rippon.[14]

I look finally to a closely related story in which again Euler is a principal player. Chapter 5 of Julian Havil's challenging Martin Gardneresque little book *Impossible: Surprising Solutions to Counterintuitive Conundrums* (2008) is entitled "The power of complex numbers," and proceeds from an account to the history of the problem of evaluating i^i. Johann Bernoulli had argued that $\log i = 0$, but Leibniz demonstrated that such a result leads to a contradiction. It was Euler who established that $\log i = i(\frac{1}{2}\pi)$, and that more generally

$$\log i = i(\tfrac{1}{2}\pi + 2\pi n)$$

giving
$$i^i = [e^{i(\frac{1}{2}\pi+2\pi n)}]^i = e^{-(\frac{1}{2}\pi+2\pi n)} \tag{8}$$

It is a familiar fact that $\log z$ is multi-valued, but—as Euler himself pointed out—quite remarkable that i^i assumes an infinitude of *real* values:

$$i^i = \begin{cases} 59609.7 & : \quad n = -2 \\ 111.318 & : \quad n = -1 \\ 0.2078795763507 & : \quad \text{Euler's hand-calculated result} \\ 0.00039 & : \quad n = +1 \\ 7.24947 \times 10^{-7} & : \quad n = +2 \end{cases}$$

If we define $p(x, a) = x^a$ then the command `NestList[p[x,#]&,x,3]` produces $\{x, x^x, x^{x^x}, x^{x^{x^x}}\}$. As remarked above, Euler has described the real x-values for which that exponential iteration converges. Looking to `NestList[p[i,#]&,i,n]` we find that the result is imaginary (namely i) in the case $n = 0$, real in the case $n = 1$, and complex in all higher order cases. And that the sequence shows clear evidence of convergence. In particular, we find

$$\texttt{Nest[p[i,\#]\&,i,100]} = 0.438283 + 0.360599\,i$$

while
$$\texttt{Nest[p[i,\#]\&,i,}\infty\texttt{]} \equiv h(i) = 0.438283 + 0.360592\,i$$

Captions for addendum figures

Figure 24. Graph of $x(y) = y \log y$ (compare Figure 1), showing in red the sector that gives rise to the double-valuedness of $y(x)$.

Figure 25. Graph of $y_0(x) = \exp[\text{LambertW}(0, x)]$.

Figure 26. Graph of the analytic fault function $f_0(z) = y_0(z) \log y_0(z) - z$, which is seen to vanish everywhere except on the cut. By fixed convention, the x and y are red/blue. The red vertical marks the location of $x = -1/e$.

Figure 27. Graph of $y_{-1}(x) = \exp[\text{LambertW}(-1, x)]$.

Figure 28. Superposition of Figures 25 & 27. Shows that both W sectors are required to reproduce the double-valuedness of Figure 24.

[14] "A note on complex iteration," Amer. Math. Monthly **92**, 501–504 (1985).

Figure 29. Modulus of $y_{-1}(z)$. The x (red) ranges $(-1.5 \to 1.0)$, y (blue) ranges $(-1.0 \to 1.0)$, the vertical scale ranges $(0 \to 1)$. The cut runs all the way to the origin (black axis). The dashed red line marks the location of $x = -1/e$.

Figure 30. Phase of $y_{-1}(z)$. Same conventions as above, except that the vertical scale ranges now $(-\pi \to \pi)$. In addition to the discontinuity that runs along the negative x axis there is now a curved phase discontinuity. A green line segment marks points where the phase vanishes ($y_{-1}(z)$ becomes real).

Figure 31. A magnified view (at higher resolution) of the central region of the preceding figure: x ranges $(-1/e - 0.1 \to 0.1)$, y ranges $(-0.1 \to 0.1)$.

Figure 32. Lines of phase discontinuity for $y_{-1}(z)$.

Figure 33. Graph of the analytic fault function $f_{-1}(z) = y_{-1}(z) \log y_{-1}(z) - z$, of which the null zone is seen to lie mainly in the upper left quadrant of the complex plane.

Figure 34. Real part of $f_{-1}(z)$. x ranges $(-4 \to 2)$, y ranges $(-3 \to 3)$ and the red vertical marks $x = -1/e$.

Figure 35. Imaginary part of $f_{-1}(z)$. The null zone is again clearly evident.

Figure 36. Real part of the iterative fault function $F(z)$, with indication of the curvilinear wedge where iteration fails to converge.

Figure 37. Imaginary part $F(z)$. The part within the wedge flips sign when the iterative order advances from even to odd.

Figure 38. Values assumed by the real part of $F(x)$ on the negatively extended real line. The oscillations in the region just below $x = -1/e$ bunch ever more tightly as the iterative order is increased, but the spike at that critical point appears to remain.

Figure 39. Values assumed by the imaginary part of $F(x)$ on the negatively extended real line. The slope of the ramp to the right of the origin—where x has entered into the interior of the curvilinear wedge—flips sign when the iterative order advances from even to odd, indicating a failure of convergence.

Figure 40. Real part of the iterative fault function $F(z)$, the same as Figure 36 except that x has been extended past $-1/e$ to -0.4, revealing the onset of great complexities, of which Figure 38 exposed a symptom.

Figure 41. Graph of Euler's iterated exponential $h(x)$.