A Mathematical Note

Occurances of the Lambert W Function

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September 2017

Introduction. The Lambert W function (known to *Mathematica* as the `ProductLog` function, of which `LambertW` is a silent alias) is a higher function that stands apart—does not support a population of identities relating it to other higher functions, and is not treated in any of the standard higher function handbooks[1]—but is remarkable for the extraordinary variety of the physical problems in which it has been found, during the last century or more, to occur.

The story began in 1758, when Johann Lambert (1728–1777) had occasion to study an equation

$$x^\alpha - x^\beta = (\alpha-\beta)y\,x^{\alpha+\beta}$$

that has come to be known as "Lambert's transcendental equation," and which came to Euler's attention in 1764. In the limit $\alpha \to \beta$ this assumes[2] the form

$$\log x = y\,x^\beta$$

which was studied by Euler in 1783. The notational adjustment $\beta \to k^{-1}$ produces

$$y = x^k \log x$$

which in the case $k = 0$ provides the fundamental statements

$$y = \log x \quad : \quad x = e^y$$

for which we are indebted to John Napier (1555–1617), and which in the case $k = 1$ becomes precisely the equation

$$y = x \log x \tag{1}$$

that sparked this entire discussion: the complex extension of the $x(y)$ produced

[1] See, however §4.13 in the Digital Library of Mathematical Functions, the successor to Abramowitz & Stegun.

[2] Use

$$\lim_{\alpha\to\beta}\frac{x^\alpha - x^\beta}{\alpha-\beta} = x^\beta \log x$$

by functional inversion of (1) is central to a 1899 paper by Arnold Sommerfeld (who was at the time unacquainted with the then-already-long history of (1) and its immediate relatives), of which Robert Warnock has provided an English translation.[3] A change of variable $x = e^w$ brings (1) to the form

$$we^w = y \tag{2}$$

which poses the inversion problem upon which the theory of Lambert's W function is founded: one has, for complex z,

$$W_k(z)e^{W_k(z)} = z \quad : \quad k = 0, \pm1, \pm2, \ldots$$

where k distinguishes the branches of the Lambert function. The more general inversion problem

$$z = w^k e^w \text{ generates the theory of } \begin{cases} \log z & \text{in the case } k = 0 \\ W(z) & \text{in the case } k = 1 \end{cases}$$

establishes the sense in which $\log(z)$ and $W(z)$ are siblings—each, in its way, as "strange"/isolated as the other.

In these pages I look to a few of the problems from which the Lambert W function emerges in a natural way.

Solution of a class of transcendental equations.[4] We look to the equation

$$p^x = ax + b = a(x + b/a) \quad : \quad a \neq 0 \tag{3}$$

Writing $u = x + b/a$ and using $p = e^{\log p}$, we have

$$e^{(u-b/a)\log p} = au \qquad \text{whence} \qquad \frac{e^{-(b/a)\log p}}{a} = ue^{-u\log p}$$

Set $w = -u\log p$ and obtain

$$we^w = y \quad : \quad y = -\log p \cdot \frac{e^{-(b/a)\log p}}{a} = -\frac{\log p}{a} \cdot p^{-b/a} \tag{3.1}$$

giving

$$w = W(y)$$

whence finally

$$x = -\frac{W(y)}{\log p} - b/a \tag{3.2}$$

Another class of transcendental equations.[5] We look to the equation

$$x^n = e^{-ax} \quad \text{which can be written} \quad x^n e^{ax} = 1$$

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[3] "On the propagation of electrodynamics waves along a wire" (unpublished).

[4] I borrow from the Wikipedia article "Lambert W function," which provides a list of 16 examples of the occurance of W functions, of which I discuss here **Example 1**.

[5] This is **Example 3** in the article just cited.

Extract the n^{th} root, get

$$x\,e^{ax/n} = 1$$

which can be written

$$w\,e^{w} = a/n \quad : \quad w = ax/n$$

giving

$$w = W(a/n) \quad \text{whence} \quad x = (n/a)W(a/n)$$

Solution of a simple delay differential equation.[6] We look to the DDE

$$\dot{x}(t) = a\,x(t-\tau) \tag{4}$$

of which $x(t) = Ae^{\lambda t}$ will be a solution if and only if

$$A\lambda e^{\lambda t} = Aae^{\lambda(t-\tau)}$$

which requires that λ be a solution of the characterstic equation $\lambda = ae^{-\lambda\tau}$, which written

$$\lambda\tau\,e^{\lambda\tau} = a\tau$$

gives $\lambda\tau = W_k(a\tau)$, whence

$$x(t) = A_k e^{W_k(a\tau)\,t/\tau}$$

where k identifies a branch of the W-function. If we require $x(t)$ to be real then we must take into account the facts that

- $W_0(a\tau)$ is real for all $a\tau > -\frac{1}{e}$, and positive or negative according as $a\tau \gtrless 0$;
- $W_{-1}(a\tau)$ is real (and negative) for $-\frac{1}{e} < a\tau < 0$, and otherwise complex;
- $W_k(a\tau)$ is in all other cases invariably complex.

Lucas Illing's stability problem. In the Appendix A of a recent paper[7] Lucas Illing discusses properties of the more general DDE

$$\dot{z}(t) = \alpha z(t) + \beta z(t-\tau) \tag{5}$$

for which the characteristic equation reads

$$\lambda = \alpha + \beta\,e^{-\lambda\tau}$$

and when written

$$e^{-\lambda\tau} = \beta^{-1}\lambda - \alpha\beta^{-1}$$

is seen to be of the form (3), with $x = -\lambda\tau$, $p = e$, $a = -1/\beta\tau$, $b = -\alpha/\beta$.

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[6] This is **Example 7** in the source just cited, and is treated also in the Wikipedia article "Delay differential equation" (see the § "The characteristic equation").

[7] "Amplitude death of identical oscillators in networks with direct coupling," Physical Review E **94**, 022215 1-10 (2016).

Making those substitutions into (3.1) and (3.2), we obtain solutions of the form

$$z_k(t) = \exp[\lambda_k t] \quad \text{with} \quad \lambda_k = \alpha + \tau^{-1} W_k(\beta\tau e^{-\alpha\tau})$$

If we set $\beta = 0$ (turn off the delay term) the DDE becomes $\dot{z}(t) = \alpha z(t)$, with a solution $z(t) = e^{\alpha t}$ that grows/dies exponentially according as $\alpha \gtrless 0$. From

$$W_k(0) = \begin{cases} 1 & : \quad k = 0 \\ -\infty & : \quad k \neq 0 \end{cases}$$

we then have

$$z_k(t) = \begin{cases} e^{\alpha t} & : \quad k = 0 \\ 0 & : \quad k \neq 0 \end{cases}$$

If, on the other hand, $\beta \neq 0$ we have

$$z_k(t) = \exp[\Re\{\alpha + \tau^{-1} W_k(\beta\tau e^{-\alpha\tau})\}t] \cdot \exp[i\Im\{W_k(\beta\tau e^{-\alpha\tau})\}t/\tau]$$

in which the second factor is oscillatory and the leading factor grows/dies exponentially. It is interesting that the delay term in (5) serves simultaneously to stimulate oscillations and—like the γ-term in $\ddot{x}+2\gamma\dot{x}+\omega^2 x = 0$—to modulate the exponential growth/damping factor. It is interesting also that Illing's DDE gives rise to expressions that are structurally similar to the $x = e^{W(y)}$ that arises from Sommerfeld's equation $y = x\log x$.

Critically damped oscillator with delayed self interaction. Look to the system

$$\ddot{z}(t) + 2\omega\dot{z}(t) + \omega^2 z(t) = az(t-\tau)$$

The characteristic equation $\lambda^2 + 2\omega\lambda + \omega^2 = ae^{-\lambda\tau}$ can be written

$$e^{-\lambda\tau} = a^{-1}(\lambda+\omega)^2$$

which with $x = \lambda + \omega$ becomes

$$x^2 e^{\tau x} = ae^{\omega}$$

and with $u = \tau x$ becomes

$$u^2 e^u = a\tau^2 e^{\omega} \equiv y$$

Arguing as at the top of page 3 we therefore have

$$u(y) = 2W(\pm\tfrac{1}{2}\sqrt{y}\,)$$

or

$$\lambda_k = \tfrac{2}{\tau} W_k(\pm\tfrac{1}{2}\tau\sqrt{a}e^{\frac{1}{2}\omega}\,) - \omega$$

giving

$$z_k(t) = \exp[\Re\{\lambda_k\}t] \cdot \exp[\Im\{\lambda_k\}t]$$

Notice that the argument hinges critically on the critical damping assumption,[8] and that all equations of the form $(\frac{d}{dt} + \omega)^p z(t) = az(t-\tau)$ yield to a similar argument.

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[8] Otherwise we would not have had $e^{-\lambda\tau} \sim (\text{term linear in } x)^{\text{power}}$.

Tetrations.[9] It was, I think, Euler[10] who first considered the evaluation of

$$h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = x^{h(x)} \tag{6}$$

The logarithm of (6) reads $\log h(x) = h(x)\log x$, which can be written

$$\frac{1}{h(x)}\log\frac{1}{h(x)} = -\log x$$

This is structurally identical to Sommerfeld's equation (1), so we have at once

$$\frac{1}{h(x)} = e^{W(-\log x)} \quad \text{or} \quad h(x) = e^{-W(-\log x)}$$

which when plotted is seen to range from 0 to e as x ranges from 0 to $e^{1/e}$, as had been established already by Euler himself. The function $h(x)$ is (when it exists) the limit of a recursive process that in *Mathematica* is achieved by the commands

```
q[x_,s_]:=x^s
h[x_,n_]:=Nest[q[x,#]&,x,n]
```

and the convergence of which is displayed by

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H[x_,x_]:=NestList[q[x,#]&,x,n]
```

The values produced by the process reside on the principal branch of the W function; we find, for example, that

$$\texttt{h[1.3,12]} = \exp[-W(-\log 1.3)] = 1.47099$$

and that convergence becomes slower as $x \to e^{1/e} = 1.44467$.

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[9] See **Example 4** in the Wikipedia article cited previously.[2] The problem is discussed also in the paper by Corless, Knuth *et al* to which that article provides a link, and from which I have taken my historical references.

[10] "De formulis exponentialibus replicatis," (1777). Interestingly, this date falls *between* the dates mentioned on page 1. Iterated exponentials are called "tetrations," and are the subject of an interesting Wikipedia article.