Kronecker Product with *Mathematica*

Nicholas Wheeler
Reed College Physics Department
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Introduction

In several recent essays I have drawn extensively on properties of the *Kronecker product*—a concept not treated in most standard introductions to matrix theory. It is to open the door to experimentation in the area, and to describe the tools I used in some of my own exploratory work, that I offer the following material.

0. Matrices with doubly-indexed elements

The objects of interest to us are ordinary symbolic matrices. *Mathematica* has no objection to entries of the design

```
a = ( a11  a12  a13
      a21  a22  a23 )
```

```
{{a11, a12, a13}, {a21, a22, a23}}
```

```
% // MatrixForm
```

```
( a11  a12  a13
  a21  a22  a23 )
```

```
Clear[a]
```

But when we try to enter the subscripted design

$$\mathbf{a} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

```
— $RecursionLimit::reclim : Recursion depth of 256 exceeded.

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— General::stop : Further output of
     $RecursionLimit::reclim will be suppressed during this calculation.

$Aborted
```

Mathematica complains. The curious fact—for which I cannot at present account—is that

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{pmatrix}$$
$$\mathbf{Q} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

$\{\{p_{11}, p_{12}, p_{13}\}, \{p_{21}, p_{22}, p_{23}\}\}$

$\{\{q_{11}, q_{12}\}, \{q_{21}, q_{22}\}\}$

work perfectly well: *Mathematica* appears to resolve the letters of the alphabet into two classes: some letters are "subscriptable," some aren't!

1. *Mathematica*'s "Outer" command

The following command

```
K = Outer[Times, P, Q]
```

$$\{\{\{\{p_{11}\,q_{11}, p_{11}\,q_{12}\}, \{p_{11}\,q_{21}, p_{11}\,q_{22}\}\},$$
$$\{\{p_{12}\,q_{11}, p_{12}\,q_{12}\}, \{p_{12}\,q_{21}, p_{12}\,q_{22}\}\},$$
$$\{\{p_{13}\,q_{11}, p_{13}\,q_{12}\}, \{p_{13}\,q_{21}, p_{13}\,q_{22}\}\}\},$$
$$\{\{\{p_{21}\,q_{11}, p_{21}\,q_{12}\}, \{p_{21}\,q_{21}, p_{21}\,q_{22}\}\},$$
$$\{\{p_{22}\,q_{11}, p_{22}\,q_{12}\}, \{p_{22}\,q_{21}, p_{22}\,q_{22}\}\},$$
$$\{\{p_{23}\,q_{11}, p_{23}\,q_{12}\}, \{p_{23}\,q_{21}, p_{23}\,q_{22}\}\}\}\}$$

```
MatrixForm[K]
```

$$\begin{pmatrix} \begin{pmatrix} p_{11}\,q_{11} & p_{11}\,q_{12} \\ p_{11}\,q_{21} & p_{11}\,q_{22} \end{pmatrix} & \begin{pmatrix} p_{12}\,q_{11} & p_{12}\,q_{12} \\ p_{12}\,q_{21} & p_{12}\,q_{22} \end{pmatrix} & \begin{pmatrix} p_{13}\,q_{11} & p_{13}\,q_{12} \\ p_{13}\,q_{21} & p_{13}\,q_{22} \end{pmatrix} \\ \begin{pmatrix} p_{21}\,q_{11} & p_{21}\,q_{12} \\ p_{21}\,q_{21} & p_{21}\,q_{22} \end{pmatrix} & \begin{pmatrix} p_{22}\,q_{11} & p_{22}\,q_{12} \\ p_{22}\,q_{21} & p_{22}\,q_{22} \end{pmatrix} & \begin{pmatrix} p_{23}\,q_{11} & p_{23}\,q_{12} \\ p_{23}\,q_{21} & p_{23}\,q_{22} \end{pmatrix} \end{pmatrix}$$

captures well the essential meaning of the Kronecker product. But it yields a result which is *not a matrix* because not a list of lists (it is instead a list of lists of lists). And—since not a matrix—it cannot be manipulated like a matrix to establish properties of the Kronecker product, or to evaluate such Kronecker products as arise in particular calculations. We confront this problem: **How to remove the internal parentheses?**

Resolution of the problem requires familiarity with various **list manipulation resources**. Look at the following:

```
Flatten[{Part[K, 1, 1, 1], Part[K, 1, 2, 1], Part[K, 1, 3, 1]}]
```

$$\{p_{11}\,q_{11},\ p_{11}\,q_{12},\ p_{12}\,q_{11},\ p_{12}\,q_{12},\ p_{13}\,q_{11},\ p_{13}\,q_{12}\}$$

```
{Flatten[{Part[K, 1, 1, 1], Part[K, 1, 2, 1], Part[K, 1, 3, 1]}],
 Flatten[{Part[K, 1, 1, 2], Part[K, 1, 2, 2], Part[K, 1, 3, 2]}],
 Flatten[{Part[K, 2, 1, 1], Part[K, 2, 2, 1], Part[K, 2, 3, 1]}],
 Flatten[{Part[K, 2, 1, 2], Part[K, 2, 2, 2], Part[K, 2, 3, 2]}]}
```

$$\{\{p_{11}\,q_{11},\ p_{11}\,q_{12},\ p_{12}\,q_{11},\ p_{12}\,q_{12},\ p_{13}\,q_{11},\ p_{13}\,q_{12}\},$$
$$\{p_{11}\,q_{21},\ p_{11}\,q_{22},\ p_{12}\,q_{21},\ p_{12}\,q_{22},\ p_{13}\,q_{21},\ p_{13}\,q_{22}\},$$
$$\{p_{21}\,q_{11},\ p_{21}\,q_{12},\ p_{22}\,q_{11},\ p_{22}\,q_{12},\ p_{23}\,q_{11},\ p_{23}\,q_{12}\},$$
$$\{p_{21}\,q_{21},\ p_{21}\,q_{22},\ p_{22}\,q_{21},\ p_{22}\,q_{22},\ p_{23}\,q_{21},\ p_{23}\,q_{22}\}\}$$

```
% // MatrixForm
```

$$\begin{pmatrix} p_{11}\,q_{11} & p_{11}\,q_{12} & p_{12}\,q_{11} & p_{12}\,q_{12} & p_{13}\,q_{11} & p_{13}\,q_{12} \\ p_{11}\,q_{21} & p_{11}\,q_{22} & p_{12}\,q_{21} & p_{12}\,q_{22} & p_{13}\,q_{21} & p_{13}\,q_{22} \\ p_{21}\,q_{11} & p_{21}\,q_{12} & p_{22}\,q_{11} & p_{22}\,q_{12} & p_{23}\,q_{11} & p_{23}\,q_{12} \\ p_{21}\,q_{21} & p_{21}\,q_{22} & p_{22}\,q_{21} & p_{22}\,q_{22} & p_{23}\,q_{21} & p_{23}\,q_{22} \end{pmatrix}$$

We have here removed the interior parentheses "by hand." The procedure works, but it would be tedious to have to execute all the steps each time we encounter a fresh Kronecker product. What we need are commands that serve to **automate** the procedure.

2. Automated construction of the Kronecker product

Recall, by way of preparation, that dimensions of an arbitrary matrix can be obtained by the commands illustrated below:

```
Dimensions[P]⟦1⟧ (* number of rows *)
```

```
2
```

```
Dimensions[P]⟦2⟧ (* number of columns *)
```

```
3
```

The double backets are entered ESC[[ESC and ESC]]ESC. Note also that "circled times" is produced by the keyboard strokes ESCc*ESC.

Much experimentation and many false starts have led me to the following composite command

```
Flatten[
  Table[Flatten[Table[Part[K, i, j, k], {j, Dimensions[P]⟦2⟧}]],
   {i, Dimensions[P]⟦1⟧}, {k, Dimensions[Q]⟦1⟧}], 1] // MatrixForm
```

$$\begin{pmatrix} p_{11}\,q_{11} & p_{11}\,q_{12} & p_{12}\,q_{11} & p_{12}\,q_{12} & p_{13}\,q_{11} & p_{13}\,q_{12} \\ p_{11}\,q_{21} & p_{11}\,q_{22} & p_{12}\,q_{21} & p_{12}\,q_{22} & p_{13}\,q_{21} & p_{13}\,q_{22} \\ p_{21}\,q_{11} & p_{21}\,q_{12} & p_{22}\,q_{11} & p_{22}\,q_{12} & p_{23}\,q_{11} & p_{23}\,q_{12} \\ p_{21}\,q_{21} & p_{21}\,q_{22} & p_{22}\,q_{21} & p_{22}\,q_{22} & p_{23}\,q_{21} & p_{23}\,q_{22} \end{pmatrix}$$

and thus to the following **definition**:

```
P_⊗Q_ :=
 Flatten[Table[Flatten[Table[Part[Outer[Times, P, Q], i, j, k],
     {j, Dimensions[P]⟦2⟧}]],
   {i, Dimensions[P]⟦1⟧}, {k, Dimensions[Q]⟦1⟧}], 1]
```

We check it out in the generic case

```
P⊗Q // MatrixForm
```

$$\begin{pmatrix} p_{11}\,q_{11} & p_{11}\,q_{12} & p_{12}\,q_{11} & p_{12}\,q_{12} & p_{13}\,q_{11} & p_{13}\,q_{12} \\ p_{11}\,q_{21} & p_{11}\,q_{22} & p_{12}\,q_{21} & p_{12}\,q_{22} & p_{13}\,q_{21} & p_{13}\,q_{22} \\ p_{21}\,q_{11} & p_{21}\,q_{12} & p_{22}\,q_{11} & p_{22}\,q_{12} & p_{23}\,q_{11} & p_{23}\,q_{12} \\ p_{21}\,q_{21} & p_{21}\,q_{22} & p_{22}\,q_{21} & p_{22}\,q_{22} & p_{23}\,q_{21} & p_{23}\,q_{22} \end{pmatrix}$$

and in a couple of cases the definition "has not seen before:"

$$\mathbf{a} = \begin{pmatrix} \mathbf{a1} \\ \mathbf{a2} \\ \mathbf{a3} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} \mathbf{b1} \\ \mathbf{b2} \\ \mathbf{b3} \\ \mathbf{b4} \end{pmatrix}$$

```
{{a1}, {a2}, {a3}}

{{b1}, {b2}, {b3}, {b4}}
```

a⊗b // MatrixForm

$$\begin{pmatrix} a1\,b1 \\ a1\,b2 \\ a1\,b3 \\ a1\,b4 \\ a2\,b1 \\ a2\,b2 \\ a2\,b3 \\ a2\,b4 \\ a3\,b1 \\ a3\,b2 \\ a3\,b3 \\ a3\,b4 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} \mathbf{r11} & \mathbf{r12} \\ \mathbf{r21} & \mathbf{r22} \end{pmatrix}$$

$$\mathbf{s} = \begin{pmatrix} \mathbf{s11} & \mathbf{s12} \\ \mathbf{s21} & \mathbf{s22} \end{pmatrix}$$

```
{{r11, r12}, {r21, r22}}

{{s11, s12}, {s21, s22}}
```

r⊗s // MatrixForm

$$\begin{pmatrix} r11\,s11 & r11\,s12 & r12\,s11 & r12\,s12 \\ r11\,s21 & r11\,s22 & r12\,s21 & r12\,s22 \\ r21\,s11 & r21\,s12 & r22\,s11 & r22\,s12 \\ r21\,s21 & r21\,s22 & r22\,s21 & r22\,s22 \end{pmatrix}$$

Seems to work!

Some Kronecker product identities

At (3) in "Toy Quantum Field Theory: Populations of Indistinguishable Finite-State Systems" (Notes for a Reed College Physics Seminar, 1 Novembeer 2000) I list basic properties of the Kronecker product. Earlier versions of the list can be found on pages 32–33 of *Classical Theory of Fields* (1999) and at (63) in Chapter 1 of *Advanced Quanatum Topics* (2000), where I cite my ultimate sources (E. P. Wigner's *Group Theory & its Application to the Quantum Theory of Atomic Spectra*, P. Lancaster's *Theory of Matrices* and Richard Bellman's *Introduction to Matrix Analysis*). Here I demonstrate those properties in illustrative concrete cases.

First we introduce some matrices to work with:

```
Clear[P, Q]
```

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{pmatrix}$$

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \end{pmatrix}$$

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

$$W = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{ww} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}$$

$\{\{p_{11}, p_{12}, p_{13}\}, \{p_{21}, p_{22}, p_{23}\}\}$

$\{\{q_{11}, q_{12}, q_{13}\}, \{q_{21}, q_{22}, q_{23}\}\}$

$\{\{u_{11}, u_{12}\}, \{u_{21}, u_{22}\}\}$

$\{\{v_{11}, v_{12}\}, \{v_{21}, v_{22}\}\}$

$\{\{w_{11}, w_{12}, w_{13}\}, \{w_{21}, w_{ww}, w_{23}\}, \{w_{31}, w_{32}, w_{33}\}\}$

- **Scalar associativity**:

```
6 (P⊗Q) == (6 P)⊗Q
```

```
True
```

```
6 (P⊗Q) == P⊗(6 Q)
```

```
True
```

- **Distributivity:**

```
Simplify[(P + Q)⊗U == P⊗U + Q⊗U]
```

```
True
```

- **Kronecker associativity:**

```
Simplify[(P⊗U)⊗W == P⊗(U⊗W)]
```

```
True
```

- **Transposition rule:**

```
Simplify[Transpose[P⊗U] == Transpose[P]⊗Transpose[U]]
```

```
True
```

- **Trace rule:**

```
Simplify[Tr[P⊗U] == Tr[P] Tr[U]]
```

```
True
```

NOTE that *Mathematica* assigns a special, non-matrix-theoretic meaning to the word "trace."

- **Determinantal rule (both matrices square, but not necessarily co-dimensional):**

```
Dimensions[U]
Dimensions[W]
```

```
{2, 2}
```

```
{3, 3}
```

```
Simplify[Det[U⊗W] == Det[U]^3 Det[W]^2]
```

```
True
```

NOTE that each factor on the right wears the *other*'s dimension as an exponent.

- **Inversion rule:**

```
Simplify[Inverse[U⊗W] == Inverse[U]⊗Inverse[W]]
```

```
True
```

NOTE: That is a fairly amazing property, and took *Mathematica* several seconds to verify.

- **The amazing criss-cross rule:**

Define a pair of additional matrices:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{pmatrix}$$

$$Z = \begin{pmatrix} z_1 & z_2 \end{pmatrix}$$

$\{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\}$

$\{\{y_{11}, y_{12}\}, \{y_{21}, y_{22}\}, \{y_{31}, y_{32}\}\}$

$\{\{z_1, z_2\}\}$

```
Dimensions[P]
Dimensions[X]
Dimensions[Y]
Dimensions[Z]
```

```
{2, 3}

{5, 1}

{3, 2}

{1, 2}
```

The point is that we have now in hand a quartet of matrices for which all of the ordinary matrix products encountered in the following identity are *meaningful*.

```
Simplify[(P⊗X).(Y⊗Z) == (P.Y)⊗(X.Z)]
```

```
True
```

The matrix in question is a 10 x 4 mess, which I do not write down only because it overruns the right margin.

NOTE: Lancaster discusses this identity only in the case in which P and Y are co-dimensionally square (m x m, let us say), and so are X and Z, though the latter may be of some *different* square dimension (n x n). He shows the identity to have a number of interesting corollaries. But that it holds under much weaker conditions (any conditions sufficient to insure multiplicative conformation) is my own little discovery, and central to the work cited in my introduction.