MEASUREMENT OF TRAPPED ATOM TEMPERATURE

Elementary theory of the TOF signal profile

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Introduction. Fundamental to the kinetic theory of ideal gases is the statement (Maxwell 1860) that in a thermalized sample of N molecules one can expect to find that a number

$$dN = N \cdot \underbrace{\left(\frac{1}{\alpha\sqrt{\pi}}\right)^3 e^{-(v/\alpha)^2} 4\pi v^2}_{f(v)}\, dv \tag{1}$$

have speeds that lie within the neighborhood dv of v. Here $\alpha \equiv \sqrt{2kT/m}$ and m refers to the mass of the individual molecules.[1] It follows that if such a "Maxwellian population" of runners were to race down a track[2] of length s the number $F(t)$ of runners who will have reached the finish line by time $t > 0$ can be described

$$F(t) = N \int_{s/t}^{\infty} f(v)\, dv$$

and the *rate at which runners cross* the finish line becomes

$$\begin{aligned} R(t) \equiv \tfrac{d}{dt}F(t) &= N(s/t^2)f(s/t) \\ &= N\frac{4s^3}{\sqrt{\pi}\alpha^3 t^4} e^{-s^2/\alpha^2 t^2} \end{aligned} \tag{2}$$

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[1] See THERMAL PHYSICS (2002), Chapter 6, page 252.

[2] Some readers may find jarring the fact that I use a result borrowed from 3-dimensional physics to lend structure to the participants in a 1-dimensional race, but will agree that I am permitted to do so, since my present intent is simply to illustrate a point of principle. In view of what follows it becomes interesting to note that *in the presence of a uniform gravitational field* the participants in a "3-dimensional Maxwellian race" would find themselves ultimately to be running all in the same direction.

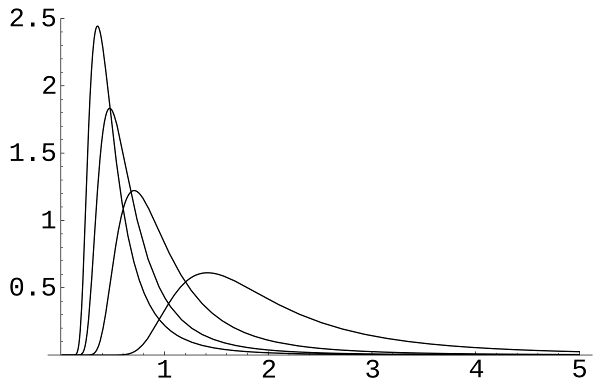

FIGURE 1: *Graphs of the activity $R(t)$ at the end of a Maxwellian race, based upon* (2). *The race was of length $s = 1$ and the parameter α was assigned the values* 0.5 (*broad curve*), 1.0, 1.5, 2.0. *Larger α refers to hotter/faster runners, who finish more quickly.*

which has the form shown in Figure 1. One readily verifies that

$$\int_0^\infty R(t)\,dt = N \quad : \quad \text{all runners eventually finish}$$

and finds that $R(t)$ peaks at

$$t_{\max} = \tfrac{1}{\sqrt{2}}(s/\alpha) \tag{3}$$

It was, in point of historical fact, from time-of-flight (TOF) data $R(t)$ that Maxwell's theoretical result (1) received its first experimental support. Today, with (1) secured, we are in position to turn the procedure around: from the observed structure of $R(t)$ we extract a measured value of $t_{\max}$, which we use in (3)—written

$$T = \frac{ms^2}{4k(t_{\max})^2}$$

—to obtain an estimate of the temperature of the Maxwellian population of runners/molecules. It is a variant of that procedure, now standard to what we might call the "atom trap industry," that concerns us:

The basic set-up is shown in Figure 2, and makes physical sense only if the blob is so cold that it remains reasonably compact for the duration of its descent. That requirement can be expressed

$$(\text{characteristic molecular speed } \alpha)\cdot(\text{flight time } \sqrt{2h/g}\,) \ll h$$

which gives $T \ll \frac{1}{4}mgh/k$. If—reasonably—we set $h = 5\,\text{cm}$ and assume the vapor to be composed of ${}^{87}\text{Rb}$ atoms ($m = 1.45 \times 10^{-25}$ kg) then we obtain $T \ll 10^{-3}\text{K}$.

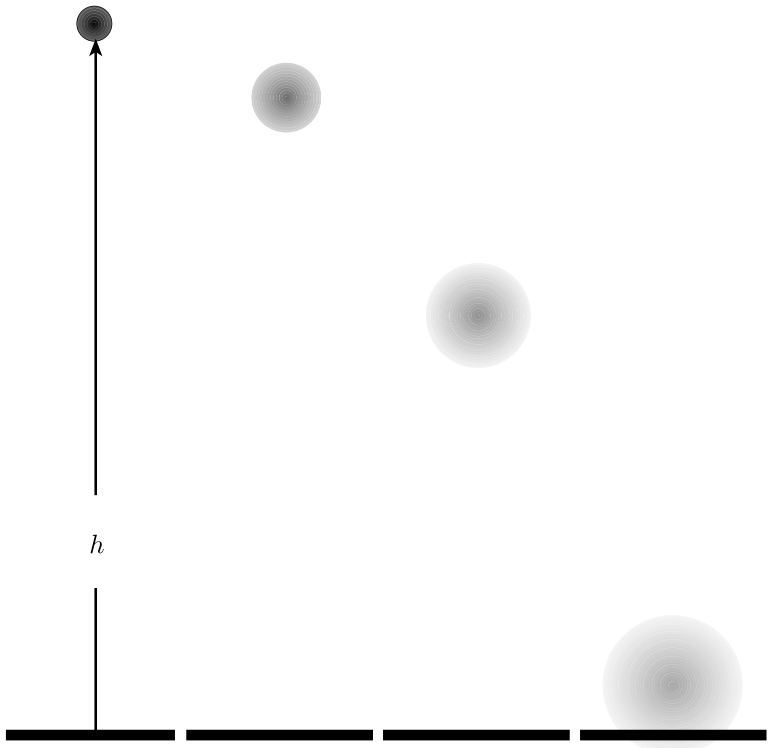

FIGURE 2: *Successive snapshots of a little blob of very cold vapor that has been dropped onto a sheet of laser light. Individual atoms flouresce and are counted as they pass through the sheet. Our assignment is deduce the theoretical profile of the signal $S(t)$ thus produced, and from the signal to extract an estimate of the temperature T of the blob.*

An early approach to the theoretical determination of $R(t)$—henceforth denoted $S(t)$ to emphasize that we are talking about a *signal*—is sketched in the appendix of a long paper by some atom trap pioneers,[3] but is susceptible to criticism on the ground that it advocates methods of a complexity that is grotesquely disproportionate to the intrinsic simplicity of the problem. A much more sophisticated line of argument—"sophisticated" because simpler, more illuminating, more elegantly apt—has been put forward more recently by I. Yavin *et al.*[4] Those authors proceed by elegantly executed ballistic analysis

[3] D. S. Weiss, E. Riis, Y. Shevy, P. J. Ungar & S. Chu, "Optical molasses and multilevel atoms: Experiment," J. Opt. Soc. Am. B **6**, 2072 (1989).

[4] I. Yavin, M. Weel, A. Andreyuk & A. Kumarakrishnan, "A calculation of the time-of-flight distribution of trapped atoms," AJP **70**, 149 (2001). Yavin and Weel were, by the way, undergraduates when this work was undertaken.

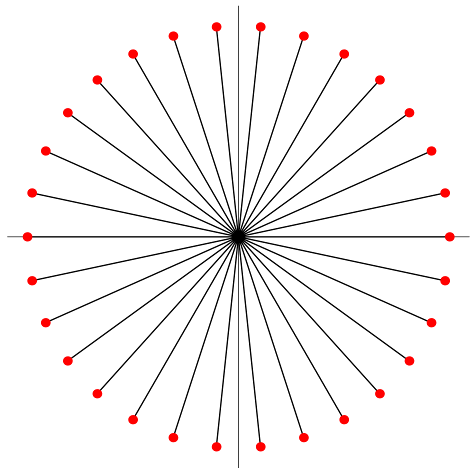

Figure 3: *Identical particles have fled isotropically with identical speeds* v.

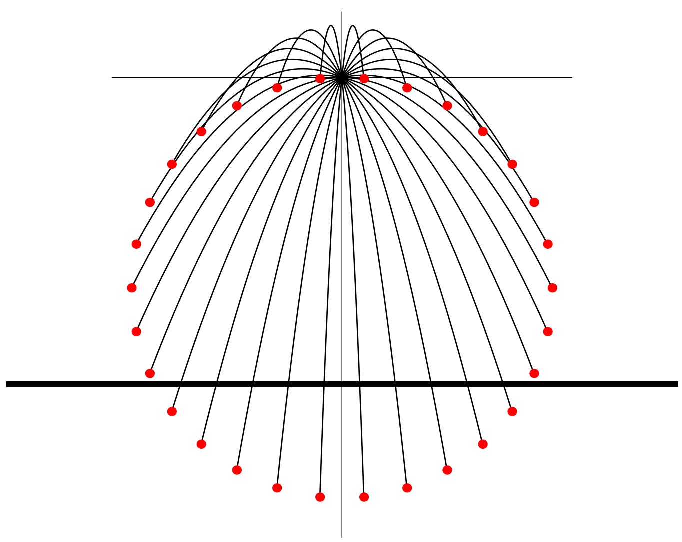

Figure 4: *"Fireworks display." Same physics as above, but in the presence of a gravitational field* (*or equivalently*: *as viewed by an upwardly accelerated observer*). *We have interest in the rate at which debris passes through the "detection plane"* (*represented in the figure by the heavy horizontal line*) *and realized physically by a slab-like laser beam.*

to the time-dependent *spatial distribution* of the atomic fallout . . . which is more than they need or want: they have, therefore, as a final step, to integrate out the space variables, which turns out not to be difficult, but lends unnecessary clutter to their work (and appears to have blinded them to certain fine points). I present here an argument that is elementary from start to finish, that is efficient in the sense that it does not labor to answer questions that were not asked, and that is physically illuminating.

1. Signal produced by a single-speed point source. The figures on the facing page are self-explanatory, and serve in themselves to capture the ballistic and geometrical essence of the argument. Suppose that, by action of some isotropic

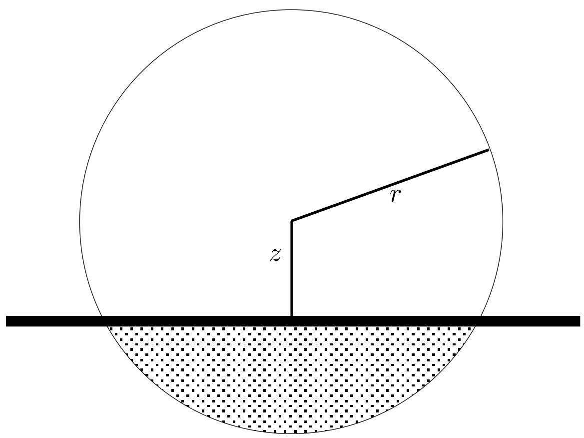

FIGURE 5: *A little elementary calculus serves to establish that the area of the spherical cap below the bold line (which represents the detection plane) can be described*

$$A(z,r) = 2\pi r^2\big[1-\tfrac{z}{r}\big] \quad \text{which becomes} \quad \begin{cases} 0 & \text{at} \quad z=+r \\ 2\pi r^2 & \text{at} \quad z=0 \\ 4\pi r^2 & \text{at} \quad z=-r \end{cases} \tag{4}$$

—the essential presumption being that $r \leqslant z \leqslant -r$; i.e., that $z^2 \leqslant r^2$. From the fact that A depends <u>*linearly*</u> *upon z it follows—somewhat counterintuitively—that* <u>*slices of equal thickness, wherever they may be taken from a sphere, all have the same surface area*</u>.

process, N points/atoms have been sprayed with statistical uniformity onto the surface of the sphere shown above. Assume, moreover, that—as suggested by Figures 2 & 4—z and r are time-dependent. The expected number of atoms on the sub-planar cap—the number of atoms that, riding on the sphere, have been transported past the detection plane and "completed their race"—can be described

$$F(z,r) = N\cdot\frac{\text{area of sub-planar cap}}{\text{area of entire sphere}}$$
$$= \begin{cases} N\cdot\dfrac{r-z}{2r} & \text{if } z^2\leqslant r^2\\ 0 & \text{otherwise}\end{cases}$$

and the *rate* at which atoms drift through the detection plane—physically: the "signal"—becomes

$$S(t) = \tfrac{d}{dt}F\big(z(t),r(t)\big) = \begin{cases} N\cdot\dfrac{z\dot r - r\dot z}{2r^2} & \text{if } z^2\leqslant r^2\\ 0 & \text{otherwise}\end{cases}$$
$$= N\Big\{\theta(z+r)-\theta(z-r)\Big\}\cdot\frac{z\dot r - r\dot z}{2r^2} \tag{5}$$

We have special interest in the case

$$\left.\begin{aligned} z(t) &= h + ut - \tfrac{1}{2}gt^2\\ r(t) &= vt\end{aligned}\right\} \tag{6}$$

with $v > 0$. We will say that the population was "*dropped* from height h" if $u = 0$, and in the contrary case that it was tossed or "lofted."[5] Returning with (6) to (5) we obtain

$$\begin{aligned} S(t) &= N\Big\{\theta(h+ut-\tfrac{1}{2}gt^2+vt)-\theta(h+ut-\tfrac{1}{2}gt^2-vt)\Big\}\\ &\qquad\cdot\frac{(h+ut-\tfrac{1}{2}gt^2)v - vt(u-gt)}{2(vt)^2}\\ &= N\Big\{\theta(h+ut-\tfrac{1}{2}gt^2+vt)-\theta(h+ut-\tfrac{1}{2}gt^2-vt)\Big\}\cdot\frac{h+\tfrac{1}{2}gt^2}{2vt^2} \end{aligned} \tag{7}$$

A simple argument establishes that the "switch factor" $\big\{etc.\big\}$ snaps on at the moment the expanding/falling sphere first strikes the detection plane

$$t_{\text{first}} = \frac{\sqrt{(u-v)^2+2gh}+(u-v)}{g} \tag{8.1}$$

and snaps off the instant

$$t_{\text{last}} = \frac{\sqrt{(u+v)^2+2gh}+(u+v)}{g} \tag{8.2}$$

[5] Experimentalists inform me of their suspicion that lofting may be a fact of life, an uncontrolled side-effect of the abrupt de-confinement of trapped atoms. We want to be in position to estimate the magnitude of the error thus introduced into their measurements.

the sphere sinks below the plane. Equation (7) can in this notation be rendered

$$S(t) = N\Big\{\theta(t - t_{\text{first}}) - \theta(t - t_{\text{last}})\Big\} \cdot \frac{h + \frac{1}{2}gt^2}{2vt^2} \tag{8.3}$$

and we are gratified to discover that *Mathematica*, working from (8), supplies

$$\int_{t_{\text{first}}}^{t_{\text{last}}} S(t)\,dt = N \quad : \quad \text{all parameter assignments}$$

2. Signal produced by a Maxwellian point source. Bringing (1) to (7) we obtain the signal

$$\begin{aligned} &S_{\text{thermalized point}}(t) \\ &\quad = N \cdot \int_0^\infty \Big(\frac{1}{\alpha\sqrt{\pi}}\Big)^3 e^{-(v/\alpha)^2} 4\pi v \Big\{\theta(vt+z) - \theta(-vt+z)\Big\} \cdot \frac{h + \frac{1}{2}gt^2}{2t^2}\, dv \end{aligned} \tag{9}$$

of a *Maxwellian superposition* of such populations. A little trickery permits the integration to be performed exactly. To that end, note first that (for $t > 0$)

$$\Big\{\theta(vt+z) - \theta(-vt+z)\Big\} = \Big\{\theta(v + z/t) - \theta(-v + z/t)\Big\}$$

But $\theta(x) = \frac{1}{2}\big\{1 + \text{sgn}(x)\big\}$ and for $\text{sgn}(x)$ we have[6] the integral representation

$$\text{sgn}(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin(xp)}{p}\, dp$$

so

$$\Big\{\theta(v + z/t) - \theta(-v + z/t)\Big\} = \frac{1}{\pi}\int_0^\infty \Big\{\frac{\sin[(v + z/t)p]}{p} + \frac{\sin[(v - z/t)p]}{p}\Big\} dp$$

Returning with this information to (9) we have

$$\begin{aligned} &S_{\text{thermalized point}}(t) \\ &\quad = N \cdot \frac{h + \frac{1}{2}gt^2}{2t^2} \int_0^\infty \Big(\frac{1}{\alpha\sqrt{\pi}}\Big)^3 e^{-(v/\alpha)^2} 4\pi v \\ &\qquad\qquad \Big\{\frac{1}{\pi}\int_0^\infty \Big\{\frac{\sin[(v + z/t)p]}{p} + \frac{\sin[(v - z/t)p]}{p}\Big\} dp\Big\} dv \end{aligned}$$

The idea now is to <u>reverse the order of integration</u>. *Mathematica* supplies

$$\begin{aligned} &= N \cdot \frac{h + \frac{1}{2}gt^2}{2t^2} \cdot \int_0^\infty \frac{2}{\pi}\cos[(z/t)p]e^{-\frac{1}{4}\alpha^2 p^2}\, dp \\ &= N \cdot \frac{h + \frac{1}{2}gt^2}{2t^2} \cdot \frac{2}{\alpha\sqrt{\pi}} \exp\Big\{-\frac{(z/t)^2}{\alpha^2}\Big\} \\ &= N \cdot \frac{h + \frac{1}{2}gt^2}{\alpha t^2 \sqrt{\pi}} \cdot \exp\Big\{-\frac{(h + ut - \frac{1}{2}gt^2)^2}{\alpha^2 t^2}\Big\} \qquad (10) \\ &\equiv N \cdot S(t; \alpha, g, h, u) \end{aligned}$$

which in the case $u = 0$ is precisely the result obtained (in quite another way)

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[6] See J. Spanier & K. B. Oldham, *An Atlas of Functions* (1987), page 65.

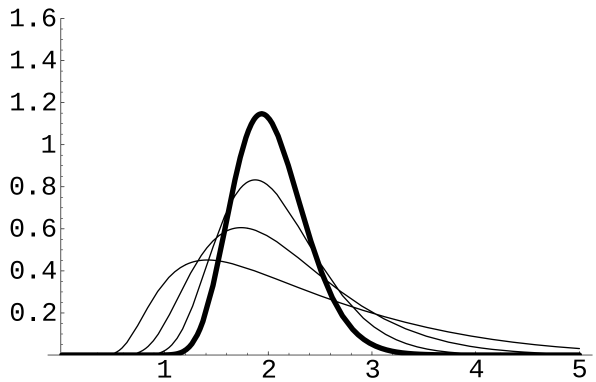

FIGURE 6: **Dropped Maxwellian point source.** *Curves derived from* (10) *in the case* $u = 0$. *In all cases* $g = 1$ *and* $h = 2$. *The heavy curve arose from* $\alpha = 0.5$; *progressively broader curves arose from setting* $\alpha = 0.7, 1.0, 1.5$; *i.e., from* *increasing the temperature*.

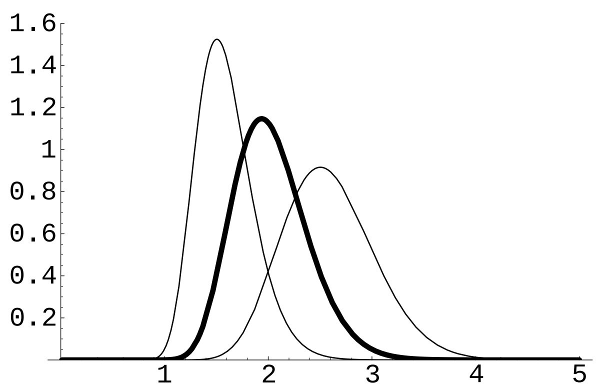

FIGURE 7: **Lofted Maxwellian point source.** *Lofting at speeds* u *much greater than about*

$$\frac{\text{characteristic diameter of workspace}}{\text{time } \sqrt{2h/g} \text{ required to fall a distance } h}$$

would very likely toss the sample right out of the apparatus. In all of the cases plotted $\alpha = 0.5$, $g = 1$, $h = 2$ (*and therefore* $\sqrt{2h/g} = 2$). *The heavy curve arose* (*as in the preceding figure*) *from setting* $u = 0$. *The broadened late-arriving curve arose from up-lofting the point source with speed* $u = 0.5$, *the narrowed early-arriving curve arose from down-lofting with that same speed.*

by Yavin *et al*, and that is fundamental to their paper. . . as it is also to most of what follows.

I describe now yet a third derivation of (10). We are informed by *Mathematica* that while

$$\int_a^\infty x^{\text{even}} e^{-x^2}\,dx = (\text{odd polynomial in } a)\cdot e^{-a^2} + (\text{numeric})\cdot[1-\text{erf}(a)]$$

—it was in a misguided attempt to avoid an anticipated error function that I devised the preceding argument (which I now find too pointlessly lovely to abandon)—the integral becomes elementary (which is to say, the error functions drop spontaneously away) in odd cases:[7]

$$\int_a^\infty x^{\text{odd}} e^{-x^2}\,dx = (\text{even polynomial in } a)\cdot e^{-a^2}$$

Returning now to (9), we note that

$$\theta(vt+z) - \theta(-vt+z) = \begin{cases} -1 & : \quad v < -z/t \quad (\text{unphysical}) \\ \;\;0 & : \quad -z/t < v < +z/t \\ +1 & : \quad v > +z/t \end{cases}$$

so we have

$$S_{\text{thermalized point}}(t) = N\cdot\Big(\frac{1}{\alpha\sqrt{\pi}}\Big)^3 4\pi\,\frac{h+\frac{1}{2}gt^2}{2t^2}\int_{z/t}^\infty v\,e^{-(v/\alpha)^2}\,dv$$

The integral is elementary, and immediately gives back (10). Figures 6 and 7 illustrate the signals produced by dropped/lofted Maxwellian point sources in some representative cases.

3. Signal produced by a Maxwellian blob. Let

$$R \equiv \frac{\text{characteristic ``diameter'' of the trapped population}}{2}$$

We expect to be able to treat the trapped population (or "source") as though it were point-like if and only if $R \ll h_{\text{mean}}$. The question now before us: What is the nature and extent of the effect that source structure has upon signal shape? The discussion turns on the observation that all aspects of source structure are irrelevant except those having to do with the fact that *the atoms in an extended source must fall assorted distances* to reach the detection plane. Given the design of a blob, we write

$$n(h)\,dh \equiv \text{number of atoms in the neighborhood } dh \text{ of height } h$$

Then $\int n(h)\,dh = N$ and we have

$$S_{\text{thermalized blob}}(t) = \int S(t;\alpha,g,h,u)\,n(h)\,dh \tag{11}$$

which we will examine in some illustrative special cases:

[7] I am indebted to David Griffiths for bringing this fact to my attention.

Spherically symmetric Gaussian blob We remark by way of preparation that if N points are normally distributed about the origin of a Cartesian coordinate system

$$\text{point density} = N\Big(\frac{1}{R\sqrt{2\pi}}\Big)^3 \exp\Big\{-\frac{x^2+y^2+z^2}{2R^2}\Big\}$$

then the expected number of points with coordinates in the neighborhood dx of z is given by

$$N\frac{1}{R\sqrt{2\pi}}\exp\Big\{-\frac{z^2}{2R^2}\Big\}dz$$

For a Gaussian blob of radius R, centered at height h_0, we therefore have

$$n(h) = N\frac{1}{R\sqrt{2\pi}}\exp\Big\{-\frac{(h-h_0)^2}{2R^2}\Big\}$$

Returning with this result to (11) we (for simplicity turn off u and) find

$$\begin{aligned}
&S_{\text{thermalized Gaussian blob}}(t)\\
&\quad = N\cdot\frac{h_0+\frac{1}{2}gt^2+2gR^2/\alpha^2}{\alpha t^2\sqrt{\pi}(1+2R^2/\alpha^2t^2)}\cdot\exp\Big\{-\frac{(h_0-\frac{1}{2}gt^2)^2}{\alpha^2t^2(1+2R^2/\alpha^2t^2)}\Big\} \qquad (12)\\
&\quad \downarrow\\
&\quad = S(t;\alpha,g,h_0,0)\quad \text{in the point-source limit } R\downarrow 0
\end{aligned}$$

The "extended source effect" can on this basis be argued to be relatively slight: see Figure 8.

Uniformly dense spherical blob A simple argument establishes that for a uniformly dense spherical blob of radius R, centered at height h_0,

$$n(h) = \begin{cases} N\dfrac{3}{4R^3}\big[R^2-(h-h_0)^2\big] & : \quad h_0-R\leqslant h\leqslant h_0+R\\ 0 & : \quad \text{otherwise}\end{cases}$$

Returning with this result to (11) we find that *Mathematica* is able and willing to do the integral, but produces a result that—even after simplification—is so uninformatively complicated as to be not worth reproducing on the page, but which when graphed (Figure 9) shows the extended source effect to be here even less significant than it was in the Gaussian case. It would appear that in realistic cases there is no reason to abandon the point-source assumption.

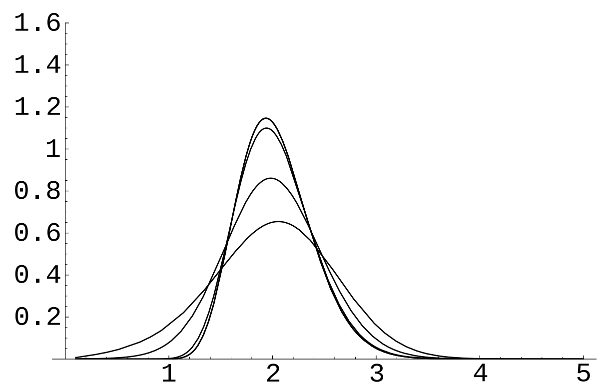

FIGURE 8: **Dropped Gaussian blob.** *Here superimposed are the signals that result from setting R to $0\%, 1\%, 10\%, 30\%$ and 50% of h_0. The parameters α, g and h_0 have been assigned the same values as in Figure 7. At the 1% setting the extended source effect is imperceptible.*

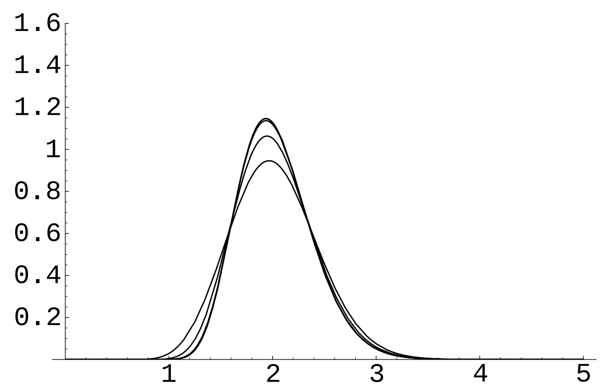

FIGURE 9: **Dropped uniformly dense spherical blob.** *The R/h_0 ratios and parameter setting are identical to those in the preceding figure. The extended source effect is even less pronounced.*

4. New method for extracting temperature from signal data. An atom thrust downward with speed $\alpha \equiv \sqrt{2kT/m}$ from height $h - R$ will intercept the detection plane at time

$$\begin{aligned} t_- &= \frac{\sqrt{\alpha^2 + 2g(h-R)} - \alpha}{g} \\ &= \frac{\sqrt{\alpha^2 + 2gh} - \alpha}{g} - \frac{1}{\sqrt{\alpha^2 + 2gh}} R - \frac{g}{2(\alpha^2 + 2gh)^{\frac{3}{2}}} R^2 - \cdots \end{aligned}$$

while an atom tossed upward with that same speed from height $h+R$ will be detected at time

$$\begin{aligned} t_+ &= \frac{\sqrt{\alpha^2+2g(h+R)}+\alpha}{g} \\ &= \frac{\sqrt{\alpha^2+2gh}+\alpha}{g} + \frac{1}{\sqrt{\alpha^2+2gh}}R - \frac{g}{2(\alpha^2+2gh)^{\frac{3}{2}}}R^2+\cdots \end{aligned}$$

Therefore

$$\begin{aligned} t_+ - t_- &= \frac{2\alpha}{g} + \frac{2}{\sqrt{2gh+\alpha^2}}R+\cdots \\ &= R\frac{1}{\sqrt{\frac{1}{2}gh}} + \frac{2\alpha}{g} + R\frac{1}{(2gh)^{\frac{3}{2}}}\alpha^2+\cdots \end{aligned}$$

and in the point-source approximation ($R=0$) we have the simple relation

$$\alpha = \tfrac{1}{2}g(t_+ - t_-) \tag{13}$$

It appears to be with the aid of (13) that experimentalists attempt to extract temperature estimates from signal data.[8] The problem is that even point-like trapped populations contain (at thermal equilibrium) atoms of *assorted* velocities, so the resulting signal cannot have a clearly defined "beginning" and "end." I turn now to the description of a practical alternative to (13).

Return to (10) and notice that $\frac{\partial}{\partial t}S(t;\alpha,g,h,0)=0$ if and only if t is a root of the polynomial

$$P(t;\alpha,g,h,0) \equiv 8h^3 + 4h(gh-2\alpha^2)t^2 - 2g^2ht^4 - g^3t^6$$

which, we observe, is cubic in t^2. At $\alpha=0$ (*i.e.*, at the zero of temperature)

$$P(t;0,g,h,0)=0 \quad\Longrightarrow\quad \begin{cases} t^2 = -2h/g \;:\; \text{double root, imaginary } t \\ t^2 = +2h/g \;:\; \text{physical root} \end{cases}$$

where the "physical root" $\sqrt{2h/g}$ is just the time it takes to drop a distance h. As the system warms up the time of signal maximality shifts downward (see

[8] See Hannah Noble, "Time of flight: measuring the temperature of trapped atoms in the Reed MOT," (Reed College thesis: May 2003), page 25.

again Figure 6), becoming[9]

$$t_{\max} = \sqrt{\frac{2h}{g}}\Big\{1 - \frac{1}{4gh}\alpha^2 - \frac{1}{32g^2h^2}\alpha^4 + \frac{1}{128g^3h^3}\alpha^6 + \cdots\Big\} \tag{14}$$

according to *Mathematica* (to whom I owe also all the results reported below). Now, writing $S'' \equiv \frac{\partial^2}{\partial t^2}S$, construct

$$\begin{aligned}\Omega(t;\alpha,g,h,u) &\equiv \frac{S''(t;\alpha,g,h,u)}{S(t;\alpha,g,h,u)}\\ &= \frac{A(t;\alpha,g,h) + u\cdot B(t;\alpha,g,h) + u^2\cdot C(t;\alpha,g,h)}{4t^6(2h+gt^2)\alpha^4}\end{aligned}$$

where

$$\begin{aligned}A &= 32h^5 + 16gh^4t^2 + g^3t^8(g^2t^2 - 2\alpha^2) - 16h^3t^2(g^2t^2 + 7\alpha^2)\\ &\qquad - 8h^2t^3(g^3t^3 + 3gt\alpha^2) + 2ht^4(g^4t^4 + 6g^2t^2\alpha^2 + 24\alpha^4)\\ B &= 64h^4t + 64gh^3t^3 - 4g^4t^9 - 96h^2t^3\alpha^2 - 2ht^4(8g^3t^3 + 24gt\alpha^2)\\ C &= 32h^3t^2 + 48gh^2t^4 + 24g^2ht^6 + 4g^3t^8\end{aligned}$$

Drawing upon (14) we find that in the absence of lofting

$$\Omega(t_{\max};\alpha,g,h,0) = -\frac{2g^2}{\alpha^2} - \frac{g}{2h} - \frac{\alpha^2}{2h^2} - \cdots$$

But

$$\Omega \equiv \frac{\text{curvature at the signal max}}{\text{maximal signal value}}$$

is a (negative) number that can, in principle, be extracted from data: assuming that to have been done, we have

$$\alpha^2 = 2kT/m = -\frac{2g^2}{\Omega} \tag{15}$$

as an alternative to (13). Equation (13) would appear to be most useful at very low temperatures, when the meaning of $t_+ - t_-$ is fairly clear. At higher

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[9] Isolate the real root of

$$8h^3 + 4h(gh - 2\alpha^2)\theta - 2g^2h\theta^2 - g^3\theta^3 = 0$$

(which is fairly complicated); develop that expression as a power series in α

$$\theta \equiv t^2 = \frac{2h}{g} - \frac{1}{g^2}\alpha^2 + \frac{0}{g^3h^1}\alpha^4 + \frac{1}{16g^4h^2}\alpha^6 + \frac{1}{32g^5h^3}\alpha^8 + \frac{1}{256g^6h^4}\alpha^{10} + \cdots$$

and then develop $t = \sqrt{\text{preceding expression}}$.

temperatures the meaning of $t_+ - t_-$ becomes vague, but the maximal signal remains clearly/unambiguously defined: it is, therefore, at higher temperatures that we expect (15) to be most useful. It may, however, prove difficult to extract useful estimates of Ω from noisy data.

5. Error introduced by weak lofting. Returning to the bottom of page 7, we find that $\frac{\partial}{\partial t}S(t;\alpha,g,h,u)=0$ if and only if t is a root of the polynomial

$$P(t;\alpha,g,h,u)\equiv P(t;\alpha,g,h,0)+u\cdot Q(t;g,h)$$

where $Q=8h^2t+8ght^3+2g^2t^5$ is (as it turns out) α-independent and terms of higher order in u are absent. Now, it is a general proposition that if x_0 is a root of the polynomial $p(x)$ then the requirement that $x_0+ux_1+u^2x_2+u^3x_3+\cdots$ be a root of $p(x)+uq(x)$ entails

$$p(x_0)+\big[q(x_0)+x_1p'(x_0)\big]u+\big[x_2p'(x_0)+x_1q'(x_0)+\tfrac{1}{2}x_1^2p''(x_0)\big]u^2+\cdots=0$$

and gives

$$\begin{aligned} x_1&=-\frac{q(x_0)}{p'(x_0)}\\ x_2&=-\frac{q(x_0)\big[q(x_0)p''(x_0)-2p'(x_0)q'(x_0)\big]}{3\big[p'(x_0)\big]^3}\\ &\ \ \vdots \end{aligned}$$

The object of immediate interest is $t_{\max}(u)=t_0+ut_1+u^2t_2+\cdots$, in which notation the object $t_{\max}$ described at (14) becomes $t_{\max}(0)=t_0$. The preceding remark supplies

$$\begin{aligned} t_1&=-\frac{Q(t_{\max})}{P'(t_{\max})}\\ &=-\frac{8h^2t+8ght^3+2g^2t^5}{8ht(gh-2\alpha^2)-8g^2ht^3-6g^3t^5}\bigg|_{t\to t_0}\\ &=\frac{1}{g}\Big\{1+\frac{0}{g^2h^1}\alpha^2-\frac{1}{8g^3h^2}\alpha^4-\frac{3}{32g^4h^3}\alpha^6-\cdots\Big\} \end{aligned}$$

giving

$$\begin{aligned} t_{\max}(u)=\ &\sqrt{\frac{2h}{g}}\Big\{1-\frac{1}{4gh}\alpha^2-\frac{1}{32g^2h^2}\alpha^4+\frac{1}{128g^3h^3}\alpha^6+\cdots\Big\}\\ &+u\cdot\frac{1}{g}\Big\{1\qquad\qquad-\frac{1}{8g^2h^2}\,\alpha^4-\frac{3}{32g^3h^3}\,\alpha^6-\cdots\Big\} \end{aligned}\tag{16}$$

Notice in this regard that at $\alpha=0$ thermal motion is extingusihed and the trapped population falls like a lofted rock $z(t)=h+ut-\frac{1}{2}gt^2$, landing at a time

$$t_{\text{lofted rock}}=\frac{\sqrt{2gh+u^2}+u}{g}=\sqrt{\frac{2h}{g}}+u\cdot\frac{1}{g}+\cdots$$

that is early or late according as u is less or greater than zero. Slight warming of the population serves, according to (16), to cause up-lofted populations to land a bit earlier than they otherwise would.

To discover the effect (in leading order) of lofting upon Ω we ask *Mathematica* to expand $\Omega(t_0+ut_1;\alpha,g,h,u)$ first in powers of α, then in powers of u . . . and obtain

$$\Omega_{\text{soft loft}} = \frac{1}{\alpha^2}\Big\{-2g^2 + u\sqrt{\frac{8g^3}{h}}\Big\} + \alpha^0\Big\{-\frac{g}{2h} + u\sqrt{\frac{8g}{h^3}}\Big\} + \alpha^2\Big\{-\frac{1}{2h^2} + u\sqrt{\frac{9}{2gh^5}}\Big\}$$

We are gratified to observe that all terms have the physical dimension of $(\text{time})^{-2}$, and that we recover the result reported on page 12 at $u = 0$. The upshot of the preceding discussion is that at low temperatures the effect of soft lofting can be described

$$\begin{aligned}\alpha^2 = 2kT/m &= -\frac{2g^2}{\Omega}\\ &\downarrow\\ &= -\frac{2g^2 - u\cdot 2g\sqrt{2g/h}}{\Omega}\\ &= -\frac{2g^2\big[1-u\cdot\sqrt{2/gh}\,\big]}{\Omega}\\ &= -\frac{2g^2}{\Omega}\cdot\big[1-2u/v_0\big]\end{aligned} \tag{17}$$

where $v_0 \equiv \sqrt{2gh}$ is the speed of a particle that has been dropped from height h. Evidently soft up-lofting—if not taken into account—will, by the Ω-method, lead to an artificially low temperature estimate, and down-lofting to an artificially high estimate.

Look by way of comparison to how $(t_+ - t_-)$ responds to soft lofting: we have

$$\begin{aligned}t_+ - t_- &= \frac{\sqrt{2gh+(\alpha+u)^2}+(\alpha+u)}{g} - \frac{\sqrt{2gh+(\alpha-u)^2}+(\alpha-u)}{g}\\ &= \alpha\frac{2}{g}\big[1+(u/v_0)\big] \quad : \quad \alpha \text{ and } u \text{ both small}\end{aligned}$$

giving

$$\alpha^2 = \tfrac{1}{4}g^2(t_+ - t_-)^2\cdot\big[1-2u/v_0\big] \tag{18}$$

Evidently the Ω-method and the $(t_+ - t_-)$-method are *equally/identically sensitive to soft-lofting*. Up-lofting tends, however, to increase the ambiguity that attaches in all cases to $(t_+ - t_-)$, and this is a little problem to which the Ω-method is immune.

6. Concluding remarks. Fundamental to the preceding discussion are certain physical assumptions: We have assumed that a trapped population of atoms can be treated as a blob of *classical ideal gas*, within which atoms move with isotropically distributed velocities and Maxwellian speeds. Yavin *et al*, in §5 of their paper,[3] entertain the possibility that the velocity distribution may be *an*isotropic; *i.e.*, that "the initial temperature of the atom cloud is not necessarily the same in all directions." An anisoptropic generalization of the analytical method sketched here is certainly imaginable, but threatens to be relatively tedious. Effort invested in the development of such refinements would seem to me to be effort misplaced if undertaken before one has assessed the effect of particle interactions . . . which at low temperatures one expects to be pronounced unless the gas has remained sufficiently dilute while cooled.

For gas samples so cold that

$$\text{de Broglie length } \lambda = \frac{\hbar}{\sqrt{2mkT}} \approx \text{sample diameter}$$

the system becomes so profoundly quantum mechanical that all appeals to naive classical imagery become highly suspect. Recent experiments[10] indicate that the explosions of abruptly de-confined Bose-Einstein condensates typically *are* anisotropic, and it seems to me unreasonable to expect classical theories of anisotropic explosions to shed useful light on the matter.

The short of it: We have obtained by novel means precisely the result first reported by I. Yavin, M. Weel, A. Andreyuk & A. Kumarakrishnan.[3] We have elaborated upon a few of the details latent in their work, and have in particular described a method for extracting temperature estimates from data that may in some instances offer advantages. But the physics of the matter remains precisely where Yavin *et al* left it.

I am an interloper in this field, and have benefited from conversations with John Essick and Hannah Nobel, who know things about its experimental realities. It is a special pleasure for me to acknowledge my enormous debt to David Griffiths for his close reading of an earlier draft and for his suggested improvements.

[10] E. A. Donley, N. R. Clausson, S. L. Cornish, J. L. Roberts, Eric A. Cornell & Carl E. Wieman, "Dynamics of collapsing and exploding Bose-Einstein condensates," Nature **412**, 295 (2001). For additional references go to http://jilawww.colorado.edu/bec/papers.html.