Some Remarks Concerning the

Sine-Gordon Equation

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Introduction. Interest in the non-linear partial differential equation

$$\omega_{uv} = \sin\omega \tag{1.1}$$

—an equation that entered the literature of physics in work by Frenkel and Kontorova (1939) on the theory of crystal dislocations, and in the second half of the 20^{th} became central to a subject remarkable for the variety of its physical applications, the theory of solitons—sprang in the first instance from differential geometry, specifically from the discovery (1862) by Edmond Bour (1832–1866) that if "asymptotic coordinate curves" u and v are inscribed on the surface of a pseudosphere[1] and if $\omega(u,v)$ is the angle subtended by the u-curve and v-curve at their point of intersection, then $\omega(u,v)$ satisfies (1.1).[2]

A change of variables $\{u,v\} \longrightarrow \{x,t\}$ by

$$\begin{aligned} u &= \tfrac{1}{2}(x+t) \\ v &= \tfrac{1}{2}(x-t) \end{aligned} \quad \text{inversely} \quad \begin{aligned} x &= (u+v) \\ t &= (u-v) \end{aligned}$$

gives

$$\begin{aligned} \partial_u &= \partial_x + \partial_t \\ \partial_v &= \partial_x - \partial_t \end{aligned}$$

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[1] A pseudosphere is a tractrix of revolution, a surface notable for the fact that it has constant negative curvature. Such surfaces, which resemble trumpets placed bell to bell, were given their name by Eugenio Beltrami (1835–1900), who—pursuing ideas introduced in 1840 by Ferdinand Minding (1806–1885)—used them to construct the first explicit model of a non-Euclidean (hyperbolic) geometry (1868), but formulae for the surface area ($4\pi r^2$) and enclosed volume ($\frac{2}{3}\pi r^3$) had been given already in 1693 by Christiaan Huygens.

[2] Details will be developed in a companion essay.

whence $\partial_u\partial_v = \partial_x^2 - \partial_t^2$ and causes (1.1) to assume a (non-linear) form

$$\omega_{xx} - \omega_{tt} = \sin\omega \tag{1.2}$$

that is structurally reminiscent of the (linear relativistic) Klein-Gordon equation

$$\psi_{xx} - (1/c^2)\psi_{tt} = \mu^2\,\psi$$

It is on the strength therefore of a weak but inevitable pun that (1) has come to be called the "sine-Gordon equation," though neither Osker Klein nor Walter Gordon nor any of the many other claimants to invention (1926) of the K-G equation had anything to do with it.

Factoring the sine-Gordon equation. Assume $\omega(u,v)$ to be a solution of the following pair of non-linear partial differential equations:

$$\omega_u = 2\sin(\tfrac{1}{2}\omega) \tag{2.1}$$

$$\omega_v = 2\sin(\tfrac{1}{2}\omega) \tag{2.2}$$

Immediately

$$\begin{aligned}\omega_{uv} = \cos(\tfrac{1}{2}\omega)\cdot\omega_v &= 2\cos(\tfrac{1}{2}\omega)\sin(\tfrac{1}{2}\omega)\\ &= \sin\omega\end{aligned}$$

Solution of the 1^{st}-order equations (2) will yield therefore a solution of the 2^{nd}-order sine-Gordon equation. But equations (2) can be solved by quadrature: looking first to (2.1), we have

$$\int du = \int \frac{1}{2\sin(\frac{1}{2}\omega)}d\omega$$

whence

$$u - k = \log(\tan(\tfrac{1}{4}\omega))$$

which gives

$$\omega = 4\arctan(e^{u-k}) \tag{3.1}$$

where k is a constant of integration (an arbitrary function of v). Returning, by way of verification, with this result to (2.1), we find

$$\omega_u = \frac{4\,e^{u-k}}{1+e^{2(u-k)}} = 2\,\text{sech}(u-k)$$

To evaluate $2\sin(\frac{1}{2}\omega)$ we use $\arctan\alpha = \frac{1}{2}i\log\frac{1-i\,\alpha}{1+i\,\alpha}$ (with which *Mathematica* responds to the command `TrigToExp[ArcTan`α`]`) to obtain

$$\omega = 2\,i\log\frac{1-i\,e^{u-k}}{1+i\,e^{u-k}} \tag{3.2}$$

and find[3]

$$2\sin(\tfrac{1}{2}\omega) = \frac{4\,e^{u-k}}{1+e^{2(u-k)}} = 2\,\text{sech}(u-k)$$

[3] I have entrusted the labor to *Mathematica*'s `Simplify` and `FullSimplify` commands.

which completes the demonstration. Equation (2.2) yields, of course, to the same argument (u and v exchange roles), and the results combine to produce[4]

$$\begin{aligned} \omega(u,v) &= 4\arctan(e^{u+v}) && (4.1)\\ &= 2i\log\frac{1-i\,e^{u+v}}{1+i\,e^{u+v}} && (4.2)\end{aligned}$$

Mathematica confirms that (4.2) does indeed satisfy the non-linear equation

$$\omega_{uv} = \sin\omega \tag{5.1}$$

and supplies

$$= -2\,\mathrm{sech}(u+v)\tanh(u+v) \tag{5.2}$$

Associated linear equation. We were led at (4) to a sine-Gordon function $\omega(u,v)$ into which u and v enter only in additive combination, so

$$\omega_{uu} = \omega_{uv} = \omega_{vv}$$

and more generally $\partial_u^m\omega = \partial_u^{m-n}\partial_v^n\omega = \partial_v^m\omega$ ($m = 0,1,\ldots;\ n = 0,1,\ldots,m$). Selecting from an unlimited set of similar examples, we find that $\omega(u,v)$ satisfies also the *linear* equation

$$\omega_{uu} - \omega_{vv} = (\partial_u - \partial_v)(\partial_u + \partial_v)\omega = 0$$

of which the general solution has the form $f(u+v) + g(u-v)$. At (4) we see a particular solution of type $f(\bullet)$.

By the aforementioned change of variables $\{u,v\} \longrightarrow \{x,t\}$ we have

$$\partial_u^2 - \partial_v^2 = (\partial_x + \partial_t)^2 - (\partial_x - \partial_t)^2 = 4\partial_x\partial_t$$

so

$$\left.\begin{aligned} \omega_{uv} &= \sin\omega\\ \omega_{uu} - \omega_{vv} &= 0\end{aligned}\right\}\quad \text{become} \quad \left\{\begin{aligned} \omega_{xx} - \omega_{tt} &= \sin\omega\\ \omega_{xt} &= 0\end{aligned}\right.$$

In $\{x,t\}$-variables the function (4.1) becomes the t-independent function

$$\omega(x) = 4\arctan e^x \tag{6}$$

We note in this connection that (2), in t-independent cases, supplies redundant copies of

$$\omega_x = 2\sin(\tfrac{1}{2}\omega) \quad\Rightarrow\quad \omega_{xx} = \sin\omega$$

which by quadrature give back (6), from which follows (compare (5.2))

$$\omega_{xx} = \sin\omega = -2\sinh x\tanh x$$

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[4] Here and henceforth I have set the translational constant $k = 0$ to reduce distracting notational clutter.

A small—but consequential—generalization. The substitutions

$$u \to a\,u, \quad v \to a^{-1}v$$

send

$$\partial_u \to a^{-1}\partial u, \quad \partial_v \to a\,\partial v$$

but preserve $\partial_u\partial_v$. We are led therefore—in place of (2)—to write

$$\omega_u = 2a \quad \sin(\tfrac{1}{2}\omega) \tag{7.1}$$
$$\omega_v = 2a^{-1} \sin(\tfrac{1}{2}\omega) \tag{7.2}$$

which, by the argument rehearsed in the preceding section, lead to a somewhat enlarged a-parameterized class of sine-Gordon functions[4]

$$\omega(u,v) = 4\arctan(e^{au+\frac{1}{a}v}) \tag{8.1}$$
$$= 2\,i\log\frac{1-i\,e^{au+\frac{1}{a}v}}{1+i\,e^{au+\frac{1}{a}v}} \tag{8.2}$$

which give back (4) when $a=1$. *Mathematica*, which finds it easiest to work with (8.2), confirms/supplies

$$\omega_{uv} = \sin\omega \tag{9.1}$$
$$= -2\sinh(au+a^{-1}v)\tanh(au+a^{-1}v) \tag{9.2}$$

Because u and v enter into $\omega(u,v)$ only as the weighted sum $au+a^{-1}v$ we have

$$a^{-2}\omega_{uu} = \omega_{uv} = a^2\omega_{vv}$$

from which follows in particular the linear equation

$$\omega_{uu} - a^4\omega_{vv} = (\partial_u - a^2\partial_v)(\partial_u + a^2\partial_v) = 0$$

of which the general solution is of the form $f(u+a^{-2}v)+g(u-a^{-2}v)$. At (8) we see a particular solution of type $f(\bullet)$.

Which brings me to the point that initially puzzled me, and which one of my objectives has been to clarify. When we pass as before from $\{u,v\}$ to $\{x,t\}$-variables and set

$$a = \sqrt{\frac{1-\beta}{1+\beta}}$$

the function (9.1) becomes

$$\omega(x,t) = 4\arctan\Big[\exp\Big(\frac{x-\beta t}{\sqrt{1-\beta^2}}\Big)\Big] \tag{10}$$

It becomes clear at this point why $\{x,t\}$ are often called "spacetime variables," even though x and t do not possess the physical dimensions of length and time (which in the relativistic Klein-Gordon theory they do). And clear also why the asymptotic coordinates are called "lightcone variables," since in 2-dimensional spacetime (set $c = 1$) $u = 0$ and $v = 0$ define the respective branches of the lightcone.[5] The function defined at (10) is of the form $f(x - \beta t)$, therefore describes a right-running rigid solution of the (linear) wave equation

$$\omega_{xx} - \beta^{-2}\omega_{tt} = 0 \tag{11.1}$$

How, therefore, can it be a solution also of the (*non*-linear) sine-Gordon equation

$$\omega_{xx} - \omega_{tt} = \sin\omega \tag{11.2}$$

The answer—anticipated in preceding remarks—is elementary. Let the wave equation (11.1) be written

$$\omega_{xx} - \omega_{tt} = (\beta^{-2} - 1)\,\omega_{tt}$$

Consistency with (11.2) requires that

$$(\beta^{-2} - 1)\,\omega_{tt} = \sin\omega \tag{12}$$

We are informed by *Mathematica* that on the one hand

$$\begin{aligned}(\beta^{-2} - 1)\,\partial_t^2\, 4\arctan\Big[\exp\Big(\frac{x - \beta t}{\sqrt{1-\beta^2}}\Big)\Big]&\\ &= -2\,\mathrm{sech}\Big(\frac{x - \beta t}{\sqrt{1-\beta^2}}\Big)\tanh\Big(\frac{x - \beta t}{\sqrt{1-\beta^2}}\Big)\\ &= -2\,\mathrm{sech}\,\xi\,\tanh\xi \qquad \text{with} \qquad \xi = \frac{x - \beta t}{\sqrt{1-\beta^2}}\end{aligned}$$

while on the other hand

$$\sin\Big[2i\log\Big(\frac{1 - i\,e^{\xi}}{1 + i\,e^{\xi}}\Big)\Big] = -2\,\mathrm{sech}\,\xi\,\tanh\xi$$

which together serve to establish (12).

The function (11), when plotted as a function of x with t given and fixed, is in good approximation 0 for $x \ll \beta t$, rises rapidly from 0 to 2π in the vicinity of $x = \beta t$, and is in good approximation 2π for $x \gg \beta t$. The animated graph therefore resembles what Lamb[6] calls a right-sliding "shelf."

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[5] $\sqrt{\frac{1-\beta}{1+\beta}}$ and $\sqrt{\frac{1+\beta}{1-\beta}}$ are central objects in Hermann Bondi's "k-calculus formulation" (1980) of special relativity, and play prominent roles also in my own "How Einstein might have been led to relativity already in 1895" (text of a lecture presented on Einstein's 100$^{\text{th}}$ birthday [Wednesday, 14 March 1979] and rendered as a pdf file in 1999), where it is pointed out that they are the eigenvalues of the Lorentz matrix

$$\gamma\begin{pmatrix}1 & \beta\\ \beta & 1\end{pmatrix} \quad : \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

[6] G. L. Lamb, *Elements of Soliton Theory* (1980), page 145.

The function

$$\omega_x = \frac{2}{\sqrt{1-\beta^2}}\text{sech}\Big(\frac{x-\beta t}{\sqrt{1-\beta^2}}\Big)$$

has therefore the form of a rigidly right-sliding pulse, centered at $x = \beta t$. That function is *not* a solution of the sine-Gordon equation, since $\partial_x \sin\omega \neq \sin\omega_x$, so it is misleading to refer to it as a "solitonic solution of the sine-Gordon equation" (as, according to Lamb, is commonly done); it is in fact a solution of the wave equation (11.1) and of the non-linear equation

$$\omega_{xx} - \omega_{tt} = \omega_x \cos\omega$$

Of course, *every* $f(x-\beta t)$ satisfies

$$f_{xx} - \beta^{-2} f_{tt} = 0$$

which can *always* be written

$$f_{xx} - f_{tt} = (\beta^{-2} - 1) f_{tt}$$

And one can always break linearity by stipulating that

$$(\beta^{-2} - 1) f_{tt} = \mathcal{F}(f)$$

In $\{u,v\}$-variables one then has

$$f_{uv} = \mathcal{F}(f)$$

which can be factored by writing

$$\begin{aligned} f_u &= \mathcal{U}(f) \\ f_v &= \mathcal{V}(f) \end{aligned}$$

and imposing upon the functions $\mathcal{U}$ and $\mathcal{V}$ the conditions

$$\mathcal{U}'\mathcal{V} = \mathcal{U}\mathcal{V}' = \mathcal{F}$$

The first equality, written $\mathcal{U}'/\mathcal{U} = \mathcal{V}'/\mathcal{V}$, entails $\log\mathcal{U} = \log\mathcal{V} + \log k$ or $\mathcal{V} = k\mathcal{U}$. Without loss of generality we set the constant $k = 1$ and by the second equality have

$$\mathcal{U}'\mathcal{U} = \tfrac{1}{2}(\mathcal{U}^2)' = \mathcal{F} \quad\Longrightarrow\quad \mathcal{U} = \sqrt{2\int\mathcal{F}}$$

In the case $\mathcal{F}(f) = \sin f$ we have

$$f_u = f_v = \mathcal{U}(f)$$

with $\mathcal{U} = \sqrt{-2\cos f}$ giving $f_{uv} = \mathcal{U}'\mathcal{U} = \sin f$. Interestingly, this variant of the factorization (2) proceeds to completion without the assistance of a fortunate trigonometric identity.

Bäcklund's trick. Impose upon $\omega(u,v)$ and $\Omega(u,v)$ the conditions

$$\begin{aligned}\omega_u - \Omega_u &= 2a\ \ \sin(\tfrac{1}{2}\omega + \tfrac{1}{2}\Omega) &(13.1)\\ \omega_v + \Omega_v &= 2a^{-1}\sin(\tfrac{1}{2}\omega - \tfrac{1}{2}\Omega) &(13.2)\end{aligned}$$

which give back (7) in the special case $\Omega = 0$. Then

$$\begin{aligned}\omega_{uv} - \Omega_{uv} &= a\cos(\tfrac{1}{2}\omega + \tfrac{1}{2}\Omega)\cdot(\omega_v + \Omega_v)\\ &= 2\cos(\tfrac{1}{2}\omega + \tfrac{1}{2}\Omega)\sin(\tfrac{1}{2}\omega - \tfrac{1}{2}\Omega)\end{aligned}$$

which—again by a fortunate trigonometric identity—can be written

$$= \sin\omega - \sin\Omega$$

giving

$$\begin{aligned}\omega_{uv} - \sin\omega &= \Omega_{uv} - \sin\Omega\\ &= 0\quad \underline{\text{if } \Omega \text{ is a known solution of the sine-Gordon equation}}\end{aligned}$$

By integration of

$$\begin{aligned}\omega_u &= \ \ \ \Omega_u + 2a\ \ \sin(\tfrac{1}{2}\omega + \tfrac{1}{2}\Omega) &(14.1)\\ \omega_v &= -\Omega_v + 2a^{-1}\sin(\tfrac{1}{2}\omega - \tfrac{1}{2}\Omega) &(14.2)\end{aligned}$$

one has—by "Bäcklund transformation," symbolized $\Omega \to \omega = \mathbb{B}_a\Omega$—then produced from $\Omega(u,v)$ a *new* solution of the sine-Gordon equation. The process can, in principle, be iterated. It was, in effect, by Bäcklund transformation of the trivial sine-Gordon function $\Omega(u,v) = 0$ that we constructed (8).

Enter: The Riccati equation. Integration of (14) in the case $\Omega = 0$ was found at (8) to be elementary, accomplished by simple quadrature. But in general ($\Omega \neq 0$) it poses a challenge. We proceed from the observation that equations (14) can be written

$$\left.\begin{aligned}\omega_u &= \ \ \ \Omega_u + 2a\ \ \cos\tfrac{1}{2}\Omega\cdot\sin\tfrac{1}{2}\omega + 2a\ \ \sin\tfrac{1}{2}\Omega\cdot\cos\tfrac{1}{2}\omega\\ \omega_v &= -\Omega_v + 2a^{-1}\cos\tfrac{1}{2}\Omega\cdot\sin\tfrac{1}{2}\omega - 2a^{-1}\sin\tfrac{1}{2}\Omega\cdot\cos\tfrac{1}{2}\omega\end{aligned}\right\} \quad (15)$$

which, in terms of $\varphi = \frac{1}{2}\omega$, become

$$\left.\begin{aligned}\varphi_u &= \ \ \ \tfrac{1}{2}\Omega_u + a\ \ \cos\tfrac{1}{2}\Omega\cdot\sin\varphi + a\ \ \sin\tfrac{1}{2}\Omega\cdot\cos\varphi\\ \varphi_v &= -\tfrac{1}{2}\Omega_v + a^{-1}\cos\tfrac{1}{2}\Omega\cdot\sin\varphi - a^{-1}\sin\tfrac{1}{2}\Omega\cdot\cos\varphi\end{aligned}\right\} \quad (16)$$

of which the former (to which I will restrict my explicit attention) is of the form

$$\varphi_u = f(u) + g(u)\sin\varphi + h(u)\cos\varphi \tag{17}$$

and the latter of similar form, but with altered $\{f, g, h\}$ coefficients. Those coefficients acquire $\{u, v\}$-dependence from the known function $\Omega(u, v)$, but in (17) the v variable is a mere spectator, so reference to it has been omitted.

Change dependent variables once again, writing

$$\varphi = 2\arctan x$$

Then (17) assumes the form

$$\begin{aligned} x_u &= \tfrac{1}{2}f(u)[1+x^2] + g(u)\,x + \tfrac{1}{2}h(u)[1-x^2] \\ &= \frac{f(u)+h(u)}{2} + g(u)\,x + \frac{f(u)-h(u)}{2}\,x^2 \\ &\equiv q_0(u) + q_1(u)\,x + q_2(u)\,x^2 \end{aligned} \tag{18}$$

of the celebrated (nonlinear 1st-order ordinary) Riccati equation. Assume $q_2(u) \neq 0$ and write

$$y = q_2(u)\,x$$

Then

$$\begin{aligned} y_u &= x\,q_{2,u} + q_2\cdot(q_0 + q_1\,x + q_2\,x^2) \\ &= (y/q_2)\,q_{2,u} + q_0q_2 + q_1q_2(y/q_2) + q_2q_2(y/q_2)^2 \\ &= y^2 + R(u)\,y + S(u) \end{aligned} \tag{19.1}$$

which is the Riccati equation in "standard form." Here

$$\left.\begin{aligned} R &= q_1 + q_{2,u}/q_2 \\ S &= q_0q_2 \end{aligned}\right. \tag{19.2}$$

Contact with the theory of *linear* homogeneous differential equations *of* 2nd *order* is established by setting

$$y = -\,z_u/z$$

Then

$$y_u = z_u^2/z^2 - z_{uu}/z = y^2 - z_{uu}/z$$

which by (19.1) entails

$$-z_{uu}/z = -R\,z_u/z + S$$

or finally

$$z_{uu} - R(u)\,z_u + S(u)\,z = 0 \tag{20}$$

Conversely, if $z(u)$ is a solution of (20) then $y(u) = -(\log z(u))_u$ is a solution of the Riccati equation (19.1).[7]

[7] I have here only scratched the surface of a subject that is explored in elaborate detail in Keisuke Hasegawa, "The Riccati Equation and its Applications in Physics" (Reed College thesis, 2001).

In the case $\Omega = 0$ we have $f = h = 0$, $g = a$ whence $q_0 = q_2 = 0$, $q_1 = a$. Equation (17) assumes the form $\varphi_u = a \sin \varphi$, which by quadrature gives back the familiar result

$$\varphi(u) = \tfrac{1}{2}\omega(u) = 2 \arctan e^{au+k}$$

Equation (18), on the other hand, becomes simply $x_u = ax$, giving

$$x(u) = \tan \tfrac{1}{2}\varphi(u) = \tan \tfrac{1}{4}\omega(u) = e^{au+k}$$

which amounts to the same thing. Because $q_2 = 0$ the Ricatti equation (18) is in this simple instance linear, and cannot be brought to standard form.

All of which is quite elementary. But when we attempt (with, for simplicity, the same a: see below) to carry the Bäcklund transformation one step further—returning to (14) with[8]

$$\Omega(u, v) = 4 \arctan e^{au+bv} \tag{21}$$

—things become at once more complicated, for the coefficients in (17) have become[9]

$$\begin{aligned} f(u) &= a\,\mathrm{sech}(au + bv) \\ g(u) &= a \cos\big(2 \arctan e^{au+bv}\big) \\ h(u) &= a\, \sin\big(2 \arctan e^{au+bv}\big) \end{aligned}$$

or more simply[10]

$$\begin{aligned} f(u) &= a\,\frac{2\,e^{au+bv}}{1 + e^{2(au+bv)}} \\ g(u) &= a\,\frac{1 - e^{2(au+bv)}}{1 + e^{2(au+bv)}} \\ h(u) &= a\,\frac{2\,e^{au+bv}}{1 + e^{2(au+bv)}} = f(u) \end{aligned}$$

and the coefficients in (18) have therefore become

$$\begin{aligned} q_0(u) &= a\,\frac{2\,e^{au+bv}}{1 + e^{2(au+bv)}} \\ q_1(u) &= a\,\frac{1 - e^{2(au+bv)}}{1 + e^{2(au+bv)}} \\ q_2(u) &= 0 \end{aligned}$$

By lucky happenstance, (18) has assumed the linear form $x_u = q_0 + q_1 x$, of which the general solution can be written

$$x(u) = e^{-p(u)}\bigg\{\int^u q_0(w)e^{p(w)}dw + k\bigg\} \quad \text{with} \quad p(u) = -\int^u q_1(w)dw$$

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[8] We adopt temporarily the abbreviation $a^{-1} = b$.

[9] The v-terms are spectators so far as (17) and related u-equations are concerned, but acquire significance when we turn to the associated v-equations.

[10] Use `TrigToExp`.

which gives

$$x(u) = \quad au\frac{2e^{au+bv}}{1+e^{2(au+bv)}} + k(v)\frac{e^{au}}{1+e^{2(au+bv)}}$$

where the function $k(v)$—by nature a constant of integration—is arbitrary. The associated v-equation leads by a similar argument to

$$x(v) = -bv\frac{2e^{au+bv}}{1+e^{2(au+bv)}} + K(u)\frac{e^{bv}}{1+e^{2(au+bv)}}$$

where $K(u)$ is arbitrary. The only way to have it both ways—to merge those functions—is to set $k(v) = -bv\,e^{bv}$ and $K(u) = 2au\,e^{au}$. We verify that the resulting function

$$\begin{aligned} x(u,v) &= (au - bv)\frac{2e^{au+bv}}{1+e^{2(au+bv)}} \\ &= (au - bv)\,\mathrm{sech}(au+bv) \end{aligned}$$

is indeed a simultaneous solution of both (18) and its v-mate.

We are led thus by Bäcklund transformation from the $\Omega(u,v)$ of (21)—which is to say: by the process

$$\Omega_0 \to \Omega = \mathbb{B}_a\Omega_0 \to \omega_a = \mathbb{B}_a\Omega = \mathbb{B}_a\mathbb{B}_a\Omega_0 \quad \text{with} \quad \Omega_0 = 0$$

—to the a-parameterized family of functions

$$\omega_a(u,v) = 4\arctan\big[(au - bv)\,\mathrm{sech}(au+bv)\big]\Big|_{b=a^{-1}} \tag{21}$$

each of which, as *Mathemtica* confirms, is indeed a sine-Gordon function:

$$\omega_{a,uv} = \sin\omega_a$$

REMARK: We have greater interest in 2-step Bäcklund processes of the form

$$\Omega_0 \to \omega_{12} = \mathbb{B}_{a_2}\mathbb{B}_{a_1}\Omega_0$$

But we then have[11]

$$\begin{aligned} q_0(u) &= \alpha\,\frac{2\,e^{au+bv}}{1+e^{2(au+bv)}} \\ q_1(u) &= \alpha\,\frac{1-e^{2(au+bv)}}{1+e^{2(au+bv)}} \\ q_2(u) &= 0 \end{aligned}$$

with the consequence that whereas before we had

$$e^p = e^{-au}\big(1+e^{2(au+bv)}\big)$$

we now have
$$e^p = e^{-\alpha u}\big(1+e^{2(au+bv)}\big)^{\alpha/a}$$

[11] To reduce notational clutter I write a for a_1 and α for a_2, while retaining the abbreviations $a^{-1} = b$, $\alpha^{-1} = \beta$.

Before we had

$$e^{-p(u)}\int^u q_0(w)e^p(w)dw = au\frac{2e^{au+bv}}{1+e^{2(au+bv)}}$$

but now (according to *Mathematica*) have

$$\begin{aligned} e^{-p(u)}\int^u q_0(w)e^p(w)dw &= \frac{\alpha}{a-\alpha}\frac{2\,e^{au+bv}}{\big(1+e^{2(au+bv)}\big)^{\alpha/a}} \\ &\quad\times \text{Hypergeometric2F1}\Big(1-\frac{\alpha}{a},\frac{a-\alpha}{2a},\frac{3a-\alpha}{2a},-e^{2(au+bv)}\Big) \end{aligned}$$

which cannot be correct: in the limit $\alpha \to a$ the hypergeometric factor $\to 1$ but the factor $\alpha/(a{-}\alpha)$, instead of approaching au, becomes singular. I am therefore presently unable to carry discussion of this process—fundamental though it is—to completion. END OF REMARK

Analytic iteration of Bäcklund transformations: Bianchi's permutability theorem . We cannot expect to be always so fortunate as we were in the case just discussed, where $f = h$ gave $q_2 = 0$, which caused the non-linear Riccati equation to be replaced by a soluable linear differential equation. Remarkably, Luigi Bianchi (1856–1928) devised in 1879 a procedure by which, given a single sine-Gordon ω_0 and two of its Bäcklund transforms (ω_1 and ω_2), it becomes possible to construct a fourth such function—indeed, an infinite network of such functions—*by purely algebraic means, without recourse to any integration procedures at all*.

To reproduce Bianchi's result we look comparatively to two two-step Bäcklund transformations that proceed from the same seed:

$$\begin{aligned} \omega_0 \xrightarrow[a_1]{} \omega_1 \xrightarrow[a_2]{} \omega_{12} &= \mathbb{B}_{a_2}\mathbb{B}_{a_1}\omega_0 \\ \omega_0 \xrightarrow[a_2]{} \omega_2 \xrightarrow[a_1]{} \omega_{21} &= \mathbb{B}_{a_1}\mathbb{B}_{a_2}\omega_0 \end{aligned}$$

Looking for the moment only to the u-component of the Bäcklund equations (14), we have

$$\begin{aligned} \omega_{1,u} - \omega_{0,u} &= 2a_1 \sin\Big(\frac{\omega_1+\omega_0}{2}\Big) \\ \omega_{12,u} - \omega_{1,u} &= 2a_2 \sin\Big(\frac{\omega_{12}+\omega_1}{2}\Big) \\ \omega_{2,u} - \omega_{0,u} &= 2a_2 \sin\Big(\frac{\omega_2+\omega_0}{2}\Big) \\ \omega_{21,u} - \omega_{2,u} &= 2a_1 \sin\Big(\frac{\omega_{21}+\omega_2}{2}\Big) \end{aligned}$$

which on the assumption that the transformations commute ($\omega_{12} = \omega_{21} \equiv \Omega$) become

$$\begin{aligned}
\omega_{1,u} - \omega_{0,u} &= 2a_1 \sin\Big(\frac{\omega_1 + \omega_0}{2}\Big) \\
\Omega_u - \omega_{1,u} &= 2a_2 \sin\Big(\frac{\Omega + \omega_1}{2}\Big) \\
\omega_{2,u} - \omega_{0,u} &= 2a_2 \sin\Big(\frac{\omega_2 + \omega_0}{2}\Big) \\
\Omega_u - \omega_{2,u} &= 2a_1 \sin\Big(\frac{\Omega + \omega_2}{2}\Big)
\end{aligned}$$

Summing first the first pair, then the second pair, we have

$$\begin{aligned}
\Omega_u - \omega_u &= 2a_2 \sin\Big(\frac{\Omega + \omega_1}{2}\Big) + 2a_1 \sin\Big(\frac{\omega_1 + \omega_0}{2}\Big) \\
\Omega_u - \omega_u &= 2a_1 \sin\Big(\frac{\Omega + \omega_2}{2}\Big) + 2a_2 \sin\Big(\frac{\omega_2 + \omega_0}{2}\Big)
\end{aligned}$$

whence[12]

$$a_1\Big[\sin\Big(\frac{\omega_1 + \omega_0}{2}\Big) - \sin\Big(\frac{\Omega + \omega_2}{2}\Big)\Big] + a_2\Big[\sin\Big(\frac{\Omega + \omega_1}{2}\Big) - \sin\Big(\frac{\omega_2 + \omega_0}{2}\Big)\Big] = 0$$

The expression on the left can be written

$$\begin{aligned}
&\frac{a_1 + a_2}{2}\Big\{\Big[\sin\Big(\frac{\omega_1 + \omega_0}{2}\Big) - \sin\Big(\frac{\Omega + \omega_2}{2}\Big)\Big] + \Big[\sin\Big(\frac{\Omega + \omega_1}{2}\Big) - \sin\Big(\frac{\omega_2 + \omega_0}{2}\Big)\Big]\Big\} \\
+&\frac{a_1 - a_2}{2}\Big\{\Big[\sin\Big(\frac{\omega_1 + \omega_0}{2}\Big) - \sin\Big(\frac{\Omega + \omega_2}{2}\Big)\Big] - \Big[\sin\Big(\frac{\Omega + \omega_1}{2}\Big) - \sin\Big(\frac{\omega_2 + \omega_0}{2}\Big)\Big]\Big\}
\end{aligned}$$

which when—Bianchi's inspiration!—multiplied by

$$\frac{1}{4} \sec\Big(\frac{\Omega - \omega_0}{4}\Big) \sec\Big(\frac{\omega_2 - \omega_1}{4}\Big) \sec\Big(\frac{\Omega + \omega_0 + \omega_1 + \omega_2}{4}\Big)$$

becomes (according to *Mathematica*)

$$\frac{a_1 + a_2}{2} \tan\Big(\frac{\omega_1 - \omega_2}{4}\Big) + \frac{a_1 - a_2}{2} \tan\Big(\frac{\omega_0 - \Omega}{4}\Big)$$

giving

$$\tan\Big(\frac{\Omega - \omega}{4}\Big) = \frac{a_2 + a_1}{a_2 - a_1} \tan\Big(\frac{\omega_2 - \omega_1}{4}\Big) \tag{22.1}$$

or

$$\Omega = \omega_0 + 4 \arctan\Big[\frac{a_2 + a_1}{a_2 - a_1} \tan\Big(\frac{\omega_2 - \omega_1}{4}\Big)\Big] \tag{22.2}$$

[12] C. Rogers & W. Schief (*Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory* (2002), page 29) dismiss the whole of the argument that leads from here to (22) with an unhelpful "whence." One of my main objectives in this essay has been to fill some of the gaps in their beautiful monograph.

Argument from the v-component of (14) leads to the same result. Direct *proof* of commutivity ($\omega_{12} = \omega_{21}$)—which has been assumed—is precluded by my previously remarked inability to construct explicit descriptions of

$$\omega_{12} = \mathbb{B}_{a_1}\mathbb{B}_{a_2}\omega_0 \quad \text{and} \quad \omega_{21} = \mathbb{B}_{a_2}\mathbb{B}_{a_1}\omega_0$$

even in the simple case $\omega_0 = 0$, much less that when ω_0 it taken to be an arbitrary sine-Gordon function. Such proof amounts to a demonstration that if ω_0 is a sine-Gordon function (which assures that ω_1 and ω_2 also are) then so also is the Ω of (22.2). I can report only that *Mathematica* has, with much coaxing, managed to confirm the claim in the simple case mentioned; *i.e.*, when

$$\Omega = 4\arctan\Big[\frac{a_2 + a_1}{a_2 - a_1}\tan\Big(\arctan e^{a_2 u + b_2 v} - \arctan e^{a_1 u + b_1 v}\Big)\Big]$$

Having used (22.2) to proceed

$$\omega_0 \rightarrow \begin{pmatrix} \mathbb{B}_{a_1}\omega_0 = \omega_1 \\ \\ \mathbb{B}_{a_2}\omega_0 = \omega_2 \end{pmatrix} \rightarrow \begin{pmatrix} \mathbb{B}_{a_2}\omega_1 \\ \\ \mathbb{B}_{a_1}\omega_2 \end{pmatrix} = \omega_{12}$$

and similarly

$$\omega_0 \rightarrow \begin{pmatrix} \mathbb{B}_{a_2}\omega_0 = \omega_2 \\ \\ \mathbb{B}_{a_3}\omega_0 = \omega_3 \end{pmatrix} \rightarrow \begin{pmatrix} \mathbb{B}_{a_3}\omega_2 \\ \\ \mathbb{B}_{a_2}\omega_3 \end{pmatrix} = \omega_{23}$$

we find ourselves in position, taking now ω_2 as our seed, to proceed

$$\omega_2 \rightarrow \begin{pmatrix} \mathbb{B}_{a_1}\omega_2 = \omega_{12} \\ \\ \mathbb{B}_{a_3}\omega_2 = \omega_{23} \end{pmatrix} \rightarrow \begin{pmatrix} \mathbb{B}_{a_3}\omega_{12} \\ \\ \mathbb{B}_{a_1}\omega_{23} \end{pmatrix} = \omega_{123}$$

By replication of this process (introducing a fresh a-parameter with each replication) one can create, by purely algebraic means, an unlimited number—a "Bianchi lattice"—of sine-Gordon functions, all descended from the same ancestral ω_0.

Bianchi's "permutability theorm" (21.1), upon which the preceding construction is based, is frequently referred to (with, it seems to me, some license) as a "non-linear superposition principle."[13]

The pseudospheric soliton. As was remarked at the outset, it was to describe a property of asymptotically inscribed pseudospheres that the sine-Gordon equation (1.1) first entered the mathematical literature. I return here to the birthplace of our subject. I draw as needed upon details developed in the companion essay cited previously.[2]

The asymptotically parameterized pseudosphere is described

$$\boldsymbol{r} = \begin{pmatrix} \operatorname{sech}(\frac{u+v}{2})\cos(\frac{v-u}{2}) \\ \operatorname{sech}(\frac{u+v}{2})\sin(\frac{v-u}{2}) \\ (\frac{u+v}{2}) - \tanh(\frac{u+v}{2}) \end{pmatrix} \tag{23}$$

[13] For a recent contribution to this subjct, see Raphael Boll, "On Bianchi permutability of Bäcklund transformations for asymmetric quad-equations," arXiv:1211.4374v2 [nlin.SI] 29 May 2013.

which was used to produce Figure 1. Application of ∂_u and ∂_v produces vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ which when normalized become $\boldsymbol{R}_u = 2\,\boldsymbol{r}_u$ and $\boldsymbol{R}_u = 2\,\boldsymbol{r}_u$. One then has

$$\cos\omega(u,v) = \boldsymbol{R}_u\boldsymbol{\cdot}\boldsymbol{R}_v = 1 - \frac{4}{1+\cosh(u+v)}$$

whence

$$\omega(u,v) = \arccos\Big(1 - \frac{4}{1+\cosh(u+v)}\Big)$$

For the reason remarked already on page 3, $\omega_{uu} = \omega_{uv} = \omega_{vv}$ so (trivially)

$$\omega_{uu} - \omega_{vv} = 0$$

and we are informed by *Mathematica* that

$$4\omega_{uv} = \sin\omega$$

The function

$$\begin{aligned}\Omega(u,v) = \omega(2u,2v) &= \arccos\Big(1 - \frac{4}{1+\cosh(2u+2v)}\Big)\\ &= \arccos\big(1 - 2\,\mathrm{sech}^2(u+v)\big)\end{aligned} \tag{24}$$

therefore satisfies

$$\begin{aligned}\Omega_{uv} &= \sin\Omega\\ &= 2\sqrt{\mathrm{sech}^2(u+v)\tanh^2(u+v)}\end{aligned}$$

where—curiously—we have encountered a function very closely related to a function that was encountered already, in quite a different context, at (5.2) (case $k=0$). I digress now to explain how this comes about, adopting as a matter of notational convenience the abbreviation $u+v=\xi$. The functions $2\sqrt{\mathrm{sech}^2\xi\tanh^2\xi}$ and $2\,\mathrm{sech}\,\xi\tanh\xi$ are even/odd respectively, and stand in this relationship

$$2\sqrt{\mathrm{sech}^2\xi\tanh^2\xi} = \begin{cases} -2\,\mathrm{sech}\,\xi\tanh\xi & : \quad \xi < 0\\ +2\,\mathrm{sech}\,\xi\tanh\xi & : \quad \xi > 0\end{cases}$$

Moreover, the functions

$$\begin{aligned}\omega(\xi) &= 4\arctan e^{\xi}\\ \Omega(\xi) &= \arccos\big(1 - 2\,\mathrm{sech}^2\xi\big)\end{aligned}$$

—of which the first is shelf-like, the second tent-like—are seen graphically to stand in the relationship

$$\Omega(\xi) = \vartheta(-\xi)\,\omega(\xi) + \vartheta(\xi)\big[2\pi - \omega(\xi)\big] \tag{25}$$

where $\vartheta(\xi) = \begin{cases} 0 \;:\; \xi < 0\\ 1 \;:\; \xi > 0\end{cases}$ is the Heaviside step function. So one has

$$\sin\Omega(\xi) = \begin{cases} +\sin\omega(\xi) & : \quad \xi < 0 \\ -\sin\omega(\xi) & : \quad \xi > 0 \end{cases}$$

Equation (14) provides indication of how distinct solutions $\omega_1, \omega_2, \ldots$ of the sine-Gordon equation can be snipped and spliced to produce new solutions.[14]

Note that Ω does *not* conform to the factorization scheme (2):

$$\Omega_u = \Omega_v = -2\coth(u+v)\sqrt{\operatorname{sech}^2(u+v)\tanh^2(u+v)}$$

while $$2\sin\tfrac{1}{2}\Omega = 2\sin\left(\tfrac{1}{2}\arccos(1-2\,\mathrm{sech(u+v)}\right)$$

and those two expressions are inequivalent. Graphic experimentation serves, however, to establish—remarkably—that

$$|\Omega_u| = 2\sin\tfrac{1}{2}\Omega$$

which is to say:

$$\big\{\vartheta(-\xi) - \vartheta(\xi)\big\}\Omega_u(\xi) = 2\sin\tfrac{1}{2}\Omega(\xi)$$

It appears that the factorization scheme may admit of some gneralization. This is a topic to which I may return on some future occasion.

FIGURE 1: `ParametricPlot3D` *based upon* (23) *shows the upper half of a pseudosphere, inscribed with asymptotic curves. The curves of constant* u/*constant* v *twist* $\circlearrowleft$ / $\circlearrowright$, *respectively. At their points of intersection they subtend an angle* $\omega(u,v)$ *which is* π *at* $z=0$, *approaches* 0 *as* $z\to\infty$ *and satisfies* $4\omega_{uv} = \sin\omega$.

[14] In which connection it is useful to notice that if ω is a sine-Gordon function then so also is $-\omega$.

Out[2]=