Alternative Formulations of the Consistency Argument
that leads

From Pseudosphere to sine-Gordon Equation

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Introduction. From the parameterized description $\boldsymbol{r}(u,v)$ of a surface Σ in 3-space we construct matrices

$$\mathbb{G}(u,v)=\begin{pmatrix}\boldsymbol{r}_u\cdot\boldsymbol{r}_u & \boldsymbol{r}_u\cdot\boldsymbol{r}_v\\ \boldsymbol{r}_u\cdot\boldsymbol{r}_v & \boldsymbol{r}_v\cdot\boldsymbol{r}_v\end{pmatrix}\equiv\begin{pmatrix}g_{11} & g_{12}\\ g_{12} & g_{22}\end{pmatrix}\equiv\begin{pmatrix}E & F\\ F & G\end{pmatrix}\tag{1.1}$$

$$\mathbb{H}(u,v)=\begin{pmatrix}\boldsymbol{r}_{uu}\cdot\boldsymbol{N} & \boldsymbol{r}_{uv}\cdot\boldsymbol{N}\\ \boldsymbol{r}_{uv}\cdot\boldsymbol{N} & \boldsymbol{r}_{vv}\cdot\boldsymbol{N}\end{pmatrix}\equiv\begin{pmatrix}h_{11} & h_{12}\\ h_{12} & h_{22}\end{pmatrix}\equiv\begin{pmatrix}e & f\\ f & g\end{pmatrix}\tag{1.2}$$

that embody the content of the 1^{st} and 2^{nd} fundamental forms. Here

$$\boldsymbol{N}(u,v)=\frac{\boldsymbol{n}}{\sqrt{\boldsymbol{n}\cdot\boldsymbol{n}}}\quad\text{with}\quad\boldsymbol{n}=\boldsymbol{r}_u\times\boldsymbol{r}_v\tag{2}$$

is the unit normal at the point $P\in\Sigma$ with coordinates $\{u,v\}$. $\mathbb{G}(u,v)$ describes the local metric structure of Σ, while $\mathbb{H}(u,v)$ alludes[1] to the local curvature of Σ. The Gauss and Weingarten equations[2] read

$$\left.\begin{aligned}\boldsymbol{r}_{uu}&=\Gamma^1{}_{11}\boldsymbol{r}_u+\Gamma^2{}_{11}\boldsymbol{r}_v+h_{11}\boldsymbol{N}\\ \boldsymbol{r}_{uv}&=\Gamma^1{}_{12}\boldsymbol{r}_u+\Gamma^2{}_{12}\boldsymbol{r}_v+h_{12}\boldsymbol{N}\\ \boldsymbol{r}_{vv}&=\Gamma^1{}_{22}\boldsymbol{r}_u+\Gamma^2{}_{22}\boldsymbol{r}_v+h_{22}\boldsymbol{N}\end{aligned}\right\}\tag{3.1}$$

$$\left.\begin{aligned}\boldsymbol{N}_u&=\frac{fF-eG}{\mathfrak{D}}\boldsymbol{r}_u+\frac{eF-fE}{\mathfrak{D}}\boldsymbol{r}_v\equiv\gamma_{11}\boldsymbol{r}_u+\gamma_{12}\boldsymbol{r}_v\\ \boldsymbol{N}_v&=\frac{gF-fG}{\mathfrak{D}}\boldsymbol{r}_u+\frac{fF-gE}{\mathfrak{D}}\boldsymbol{r}_v\equiv\gamma_{21}\boldsymbol{r}_u+\gamma_{22}\boldsymbol{r}_v\end{aligned}\right\}\tag{3.2}$$

respectively, where $\mathfrak{D}\equiv\det\mathbb{G}=EG-F^2$. That Gauss-Weingarten information

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[1] Note that $(\boldsymbol{r}_x\cdot\boldsymbol{N})_y=0_y=\boldsymbol{r}_x\cdot\boldsymbol{N}_y+\boldsymbol{r}_{xy}\cdot\boldsymbol{N}$ gives $h_{xy}=-\boldsymbol{r}_x\cdot\boldsymbol{N}_y$, which vanishes when—as on the flat plane—$\boldsymbol{N}_y=\boldsymbol{0}$.

[2] See "Surfaces in 3-Space" (December, 2015), pages 4–5 for the derivations. Here, as in the preceding descriptions of $\mathbb{G}$ and $\mathbb{H}$, I find it sometimes convenient to employ numerical instead of alphabetic indices; $\Gamma^1{}_{12}$ means $\Gamma^u{}_{uv}$, *etc.*

can be deployed

$$\left.\begin{aligned}\begin{pmatrix}\boldsymbol{r}_u\\ \boldsymbol{r}_v\\ \boldsymbol{N}\end{pmatrix}_u &= \begin{pmatrix}\Gamma^1{}_{11} & \Gamma^2{}_{11} & h_{11}\\ \Gamma^1{}_{12} & \Gamma^2{}_{12} & h_{12}\\ \gamma_{11} & \gamma_{12} & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{r}_u\\ \boldsymbol{r}_v\\ \boldsymbol{N}\end{pmatrix} \equiv \mathbb{U}\begin{pmatrix}\boldsymbol{r}_u\\ \boldsymbol{r}_v\\ \boldsymbol{N}\end{pmatrix}\\ \begin{pmatrix}\boldsymbol{r}_u\\ \boldsymbol{r}_v\\ \boldsymbol{N}\end{pmatrix}_v &= \begin{pmatrix}\Gamma^1{}_{12} & \Gamma^2{}_{12} & h_{12}\\ \Gamma^1{}_{22} & \Gamma^2{}_{22} & h_{22}\\ \gamma_{21} & \gamma_{22} & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{r}_u\\ \boldsymbol{r}_v\\ \boldsymbol{N}\end{pmatrix} \equiv \mathbb{V}\begin{pmatrix}\boldsymbol{r}_u\\ \boldsymbol{r}_v\\ \boldsymbol{N}\end{pmatrix}\end{aligned}\right\} \tag{4}$$

where the Christoffel symbols—which arise from the metric $\mathbb{G}(u,v)$, and on 2-dimensional manifolds are six in number—are given by[3]

$$\left.\begin{aligned}\Gamma^1{}_{11} &= \mathfrak{D}^{-1}\big\{\ \tfrac{1}{2}g_{22}g_{11,u} - g_{12}g_{12,u} + \tfrac{1}{2}g_{12}g_{22,v}\big\}\\ \Gamma^2{}_{11} &= \mathfrak{D}^{-1}\big\{-\tfrac{1}{2}g_{12}g_{11,u} + g_{11}g_{12,u} - \tfrac{1}{2}g_{11}g_{22,v}\big\}\\ \Gamma^1{}_{12} = \Gamma^1{}_{21} &= \mathfrak{D}^{-1}\big\{\ \tfrac{1}{2}g_{22}g_{11,v} - \tfrac{1}{2}g_{12}g_{22,u}\big\}\\ \Gamma^2{}_{12} = \Gamma^2{}_{21} &= \mathfrak{D}^{-1}\big\{-\tfrac{1}{2}g_{12}g_{11,v} + \tfrac{1}{2}g_{11}g_{22,u}\big\}\\ \Gamma^1{}_{22} &= \mathfrak{D}^{-1}\big\{-\tfrac{1}{2}g_{12}g_{22,v} + g_{22}g_{12,v} - \tfrac{1}{2}g_{22}g_{22,u}\big\}\\ \Gamma^2{}_{22} &= \mathfrak{D}^{-1}\big\{\ \tfrac{1}{2}g_{11}g_{22,v} - g_{12}g_{12,v} + \tfrac{1}{2}g_{12}g_{22,u}\big\}\end{aligned}\right\} \tag{5}$$

The Mainardi-Codazzi consistency condition[2] can now be formulated

$$\begin{pmatrix}\boldsymbol{r}_u\\ \boldsymbol{r}_v\\ \boldsymbol{N}\end{pmatrix}_{uv} = \begin{pmatrix}\boldsymbol{r}_u\\ \boldsymbol{r}_v\\ \boldsymbol{N}\end{pmatrix}_{vu}$$

which by (4) entails $\mathbb{U}_v + \mathbb{U}\mathbb{V} = \mathbb{V}_u + \mathbb{V}\mathbb{U}$ or

$$\mathbb{W} \equiv \mathbb{U}_v - \mathbb{V}_u + \mathbb{U}\mathbb{V} - \mathbb{V}\mathbb{U} = \mathbb{O} \tag{6}$$

All of which is quite general, in the sense that it pertains to arbitrarily parameterized arbitrary surfaces Σ in 3-space. It was from a specific instance of (6)—namely: from the asymptotically parameterized pseudosphere—that Edmond Bour was led (1862) to the nonlinear partial differential equation that physicists—many decades later, and for their own reasons—found it natural to call the "sine-Gordon equation." C. Rogers & W. K. Schief, in their recent review of developments that stem from Bour's work,[4] have drawn attention[5] to the remarkable *variety* of the forms in which Bour's argument can be cast. It is my objective here to provide a comparative review of those, with indication of the extent to which they may be generalizable, not specific to the pseudosphere.

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[3] See equations (17.2) on page 12 in the source just cited.

[4] *Bäcklund and Darboux Transformations: Geometry & Modern Applications in Soliton Theory* (2002).

[5] See Chapter 2: "The Motion of Curves and Surfaces. Soliton Connections" in the work just cited.

The pseudosphere as tractrix of revolution. The pseudosphere, thus conceived, acquires the natural parameterization

$$\boldsymbol{r}(u,v) = \begin{pmatrix} \operatorname{sech} u \, \cos v \\ \operatorname{sech} u \, \sin v \\ u - \tanh u \end{pmatrix} \tag{7}$$

By quick calculation[6]

$$\left.\begin{aligned} \mathbb{G} &= \begin{pmatrix} \tanh^2 u & 0 \\ 0 & \operatorname{sech}^2 u \end{pmatrix} \\ \mathbb{H} &= \begin{pmatrix} -\operatorname{sech} u \tanh u & 0 \\ 0 & \operatorname{sech} u \tanh u \end{pmatrix} \end{aligned}\right\} \tag{8}$$

—from which it is immediately evident that the Gaussian curvature

$$K = \det \mathbb{H} / \det \mathbb{G} = -1$$

—and, by calculation that would be less quick only if done by hand,

$$\mathbb{U} = \begin{pmatrix} \operatorname{csch} u \operatorname{sech} u & 0 & -\operatorname{sech} u \tanh u \\ 0 & -\tanh u & 0 \\ \operatorname{csch} u & 0 & 0 \end{pmatrix}$$

$$\mathbb{V} = \begin{pmatrix} 0 & -\tanh u & 0 \\ \operatorname{csch} u \operatorname{sech} u & 0 & -\operatorname{sech} u \tanh u \\ 0 & -\sinh u & 0 \end{pmatrix}$$

whence (here the assistance of *Mathematica* is indispensable)

$$\mathbb{W} = \mathbb{U}_v - \mathbb{V}_u + \mathbb{U}\mathbb{V} - \mathbb{V}\mathbb{U} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Which is to say: The naturally parameterized pseudosphere leads to Gauss-Weingarten equations that are manifestly consistent.

The asymptotically parameterized pseudosphere. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be tangent vectors at $P \in \Sigma$. They are said to be "conjugate" if $(\boldsymbol{x}, \mathbb{H}\boldsymbol{y}) = 0$ and $\boldsymbol{x}$ is said to be "self-conjugate" or "asymptotic" if $(\boldsymbol{x}, \mathbb{H}\boldsymbol{x}) = 0$. If the curvature of Σ is negative at P then, by an easy argument, there exists at P a conjugate *pair* of asymptotic vectors, and if the curvature of Σ is *everywhere* negative then through every point P pass a pair of "asymptotic curves" (curves with asymptotic tangents), which serve collectively to provide the "asymptotic parameterization" of Σ. So simple is the (traceless diagonal) structure (1.2) of $\mathbb{H}(u,v)$ that we are led

[6] I have prepared a package of *Mathematica* commands to produce all the computational implications of equations (1–6) that I will have occasion to report.

almost immediately to this asymptotic parameterization of the pseudosphere:

$$\boldsymbol{r}(x,y) = \begin{pmatrix} \operatorname{sech}(x+y)\,\cos(x-y) \\ \operatorname{sech}(x+y)\,\sin(x-y) \\ (x+y) - \tanh(x+y) \end{pmatrix} \tag{9}$$

Direct calculation[7] supplies

$$\begin{aligned} \mathbb{G}(x,y) &= \begin{pmatrix} 1 & \tanh^2(x+y) - \operatorname{sech}^2(x+y) \\ \tanh^2(x+y) - \operatorname{sech}^2(x+y) & 1 \end{pmatrix} \\ \mathbb{H}(x,y) &= \begin{pmatrix} 0 & -2\operatorname{sech}(x+y)\tanh(x+y) \\ -2\operatorname{sech}(x+y)\tanh(x+y) & 0 \end{pmatrix} \end{aligned}$$

From the diagonal of $\mathbb{G}(x,y)$ we learn that the asymptotic tangent vectors $\boldsymbol{r}_x(x,y)$ and $\boldsymbol{r}_y(x,y)$ are already normalized, and therefore that the off-diagonal term describes the cosine of the angle

$$\begin{aligned} \omega(x,y) &= \arccos\bigl[\tanh^2(x+y) - \operatorname{sech}^2(x+y)\bigr] \\ &= \arccos\bigl[1 - 2\operatorname{sech}^2(x+y)\bigr] \end{aligned} \tag{10}$$

with which they intersect. In this notation

$$\mathbb{G}(x,y) = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} 1 & \cos\omega(x,y) \\ \cos\omega(x,y) & 1 \end{pmatrix} \tag{11.1}$$

which gives $\det\mathbb{G}(x,y) = \sin^2\omega(x,y)$, from which (by $K=-1$) we infer that $\mathbb{H}(x,y)$ can be written

$$\mathbb{H}(x,y) = \begin{pmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{pmatrix} = \begin{pmatrix} 0 & \sin\omega(x,y) \\ \sin\omega(x,y) & 0 \end{pmatrix} \tag{11.2}$$

Working from (11), *Mathematica* supplies

$$\begin{aligned} \mathbb{U}(x,y) &= \begin{pmatrix} \omega_x\cot\omega & -\omega_x\csc\omega & 0 \\ 0 & 0 & \sin\omega \\ \cot\omega & -\csc\omega & 0 \end{pmatrix} \\ \mathbb{V}(x,y) &= \begin{pmatrix} 0 & 0 & \sin\omega \\ -\omega_y\csc\omega & \omega_y\cot\omega & 0 \\ -\csc\omega & \cot\omega & 0 \end{pmatrix} \end{aligned}$$

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[7] Alternatively, look to the tensor transformations

$$\mathbb{G}(u,v) \longrightarrow \mathbb{G}(x,y), \quad \mathbb{H}(u,v) \longrightarrow \mathbb{H}(x,y)$$

that result from $u(x,y) = x+y,\ v(x,y) = x-y$; see "Transformed fundamental forms" (April 2016), page 4.

whence

$$\mathbb{W}(x,y)=\begin{pmatrix}-\cos\omega\cdot[1-\omega_{xy}\csc\omega] & [1-\omega_{xy}\csc\omega] & 0\\ -[1-\omega_{xy}\csc\omega] & \cos\omega\cdot[1-\omega_{xy}\csc\omega] & 0\\ 0 & 0 & 0\end{pmatrix}$$

From the Gauss-Weingarten consistency condition $\mathbb{W}(x,y)=\mathbb{O}$ we therefore acquire the statement $[1-\omega_{xy}\csc\omega]=0$ or

$$\omega_{xy}=\sin\omega \tag{12}$$

Note that the SINE-GORDON EQUATION (12) has emerged here not as a side-condition—a constraint imposed to achieve Gauss-Weingarten consistency—but as a *corollary*, peculiar to the asymptotic parameterization of the pseudosphere, of that consistency, which was preordained. And that natural parameterization led to no such corollary.

That the function (10) is indeed a solution of (12)—as by the preceding argument it must be—can be verified by direct calculation, which supplies

$$\omega_{xy}=\sin\omega=2\sqrt{\operatorname{sech}^2(x+y)\tanh^2(x+y)}$$

2×2 formalism. The asymptotic tangent vectors $\boldsymbol{r}_x(x,y)$ and $\boldsymbol{r}_y(x,y)$ are, as previously noted, unit vectors, and so (by construction) is the vector normal to them

$$\boldsymbol{N}(x,y)=\frac{\boldsymbol{r}_x\times\boldsymbol{r}_y}{\sin\omega}$$

We erect at $\{x,y\}$ an orthonormal triad $\{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\}$ by writing

$$\begin{aligned}\boldsymbol{A}=\boldsymbol{r}_x,\quad \boldsymbol{C}=\boldsymbol{N},\quad \boldsymbol{B}&=\boldsymbol{N}\times\boldsymbol{r}_x\\ &=\frac{1}{\sin\omega}(\boldsymbol{r}_x\times\boldsymbol{r}_y)\times\boldsymbol{r}_x\\ &=\frac{1}{\sin\omega}[(\boldsymbol{r}_x\cdot\boldsymbol{r}_x)\boldsymbol{r}_y-(\boldsymbol{r}_x\cdot\boldsymbol{r}_y)\boldsymbol{r}_x]\\ &=\csc\omega\,\boldsymbol{r}_y-\cot\omega\,\boldsymbol{r}_x\end{aligned}$$

Then $\boldsymbol{r}_y=\cos\omega\,\boldsymbol{A}+\sin\omega\,\boldsymbol{B}$ gives

$$\begin{pmatrix}\boldsymbol{r}_x\\ \boldsymbol{r}_y\\ \boldsymbol{N}\end{pmatrix}=\begin{pmatrix}1&0&0\\ \cos\omega&\sin\omega&0\\ 0&0&1\end{pmatrix}\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{B}\\ \boldsymbol{C}\end{pmatrix}\quad\text{abbreviated}\quad \boldsymbol{\rho}=\mathbb{Z}\boldsymbol{\alpha}$$

and the merged Gauss-Weingarten equations

$$\boldsymbol{\rho}_x=\mathbb{U}\boldsymbol{\rho},\quad \boldsymbol{\rho}_y=\mathbb{V}\boldsymbol{\rho}$$

become $$\boldsymbol{\alpha}_x=\mathbb{Z}^{-1}(\mathbb{U}\mathbb{Z}-\mathbb{Z}_x)\boldsymbol{\alpha},\quad \boldsymbol{\alpha}_y=\mathbb{Z}^{-1}(\mathbb{V}\mathbb{Z}-\mathbb{Z}_y)\boldsymbol{\alpha}$$

or

$$\boldsymbol{\alpha}_x = \tilde{\mathbb{U}}\boldsymbol{\alpha}, \quad \boldsymbol{\alpha}_y = \tilde{\mathbb{V}}\boldsymbol{\alpha} \tag{13.1}$$

where

$$\tilde{\mathbb{U}} = \begin{pmatrix} 0 & -\omega_x & 0 \\ \omega_x & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \tilde{\mathbb{V}} = \begin{pmatrix} 0 & 0 & \sin\omega \\ 0 & 0 & -\cos\omega \\ -\sin\omega & \cos\omega & 0 \end{pmatrix} \tag{13.2}$$

are antisymmetric matrices *from which I will henceforth drop the tildes*. From the orthonormality of the vectxors $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ it follows that the matrix

$$\mathbb{R} = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

is a rotation matrix $\mathbb{R}^{\intercal}\mathbb{R} = \mathbb{I}$, in terms of which (13.1) can be written

$$\mathbb{R}_x = \mathbb{U}\mathbb{R}, \quad \mathbb{R}_y = \mathbb{V}\mathbb{R} \tag{14}$$

Whether we work from (13) of from (14), we have again the compatability condition (6)

$$\mathbb{W} \equiv \mathbb{U}_y - \mathbb{V}_x + \mathbb{U}\mathbb{V} - \mathbb{V}\mathbb{U} = \mathbb{O}$$

in which the old symbols have acquired now new $\tilde{}$ meanings. By calculation we recover precisely the $\mathbb{W}(x, y)$ that appears on the preceding page. Since the orthogonalization procedure $\{\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{N}\} \to \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ has served simply to reorganize the argument, we are not surprised to have been led back again to the sine-Gordon equation. The reorganization has, however, served to open some formal doors.

It follows quite generally from the fact that $\mathbb{R}$ is a rotation matrix (element of $O(3)$) that $\mathbb{R}_z = \mathbb{A}\mathbb{R}$, where $\mathbb{A}$ is 3×3 antisymmetric. Equations (14) conform to this fact since, as previously remarked, the matrices (13.2) are antisymmetric. Generally, we have $\mathbb{A} = a_1\mathbb{L}_1 + a_2\mathbb{L}_2 + a_3\mathbb{L}_3$ where

$$\mathbb{L}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbb{L}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathbb{L}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

have commutation properties ($[\mathbb{L}_1, \mathbb{L}_2] = \mathbb{L}_3$, *etc.*) characteristic of $O(3)$ and familiar from the quantum theory of angular momentum. In the present instance we have

$$\mathbb{U} = \omega_x\mathbb{L}_3 - \mathbb{L}_1$$
$$\mathbb{V} = \cos\omega\,\mathbb{L}_1 + \sin\omega\,\mathbb{L}_2$$

The Pauli matrices

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy $[\boldsymbol{\varpi}_1, \boldsymbol{\varpi}_2] = 2i\boldsymbol{\varpi}_3$, *etc.* so the matrices

$$\mathbb{l}_1 = \tfrac{1}{2i}\boldsymbol{\varpi}_1, \quad \mathbb{l}_2 = \tfrac{1}{2i}\boldsymbol{\varpi}_2, \quad \mathbb{l}_3 = \tfrac{1}{2i}\boldsymbol{\varpi}_3,$$

mimic precisely the commutation properties of the $\mathbb{L}$-matrices. So if we had occasion to require of a 2-comnponent spinor $\psi(x, y)$ that it satisfy equations

$$\psi_x = \mathbb{P}\psi, \quad \psi_y = \mathbb{Q}\psi \tag{15.1}$$

with

$$\left.\begin{aligned} \mathbb{P}(x,y) &= \omega_x \mathbb{l}_3 - \mathbb{l}_1 = \tfrac{i}{2}\begin{pmatrix} -\omega_x & 1 \\ 1 & \omega_x \end{pmatrix} \\ \mathbb{Q}(x,y) &= \cos\omega\,\mathbb{l}_1 + \sin\omega\,\mathbb{l}_2 = -\tfrac{i}{2}\begin{pmatrix} 0 & e^{-i\omega} \\ e^{i\omega} & 0 \end{pmatrix} \end{aligned}\right\} \tag{15.2}$$

and that it conform to the condition $\psi_{xy} = \psi_{yx}$ then we would have once again an equation of the familiar form

$$\mathbb{P}_y - \mathbb{Q}_x + \mathbb{P}\mathbb{Q} - \mathbb{Q}\mathbb{P} = \mathbb{O}$$

Working out the expression on the left, we obtain

$$\tfrac{i}{2}\begin{pmatrix} [\sin\omega - \omega_{xy}] & 0 \\ 0 & -[\sin\omega - \omega_{xy}] \end{pmatrix}$$

from which the sine-Gordon equation follows very neatly.

To recapitulate: The asymptotically parameterized pseudosphere supplied (9), our point of departure. The orthogonalization step $\{\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{N}\} \to \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ was fairly natural (though its inversion $\{\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{N}\} \leftarrow \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ would appear not to be: what would motivate one to introduce $\boldsymbol{r}_y = \cos\omega\,\boldsymbol{A} + \sin\omega\,\boldsymbol{B}$?) and led us to associate a rotation matrix $\mathbb{R}(x, y)$ with each of the points $\{x, y\}$ of the pseudosphere. With the appearance of elements of $O(3)$ it became—here as always—natural to look to the $SU(2)$ representation of the theory, from which the sine-Gordan equation again emerged, but in which no explicit description of the function $\omega(x, y)$ survived, and from which the pseudospheric surface Σ had pretty much disappeared from view. One might attempt to recover Σ by running the argument in reverse, but such an effort would be impeded by the $\{\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{N}\} \leftarrow \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ problem noted above.

And what would *motivate* one—especially one with pseudospheres in mind —to write equations (15)? Rogers & Scheif apparently wrote out the $SU(2)$ formalism to establish contact with AKNS theory[8]. The pseudosphere may lurk somewhere within the "AKNS hierarchy," and provide the answer to that question.

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[8] M. J. Ablowitz, D. J. Kaup, A. C. Newell & Harvey Segur, "The inverse scattering transform-Fourier analysis for nonlinear problems," Studies in Appl. Math. **53**, 249–315 (1974).

Asymptotic curves as space curves. If we assign a constant value to x then (9) describes a y-parameterized space curve $\mathcal{C}_x$, inscribed on the pseudosphere Σ. Alternatively, assign a constant value to y and obtain an x-parameterized space curve $\mathcal{C}_y$. From the diagonal elements of $\mathbb{G}(x,y)$ we see that y and x serve actually to achieve *arc-length parameterization* of $\mathcal{C}_x$ and $\mathcal{C}_y$, respectively. We bring to those asymptotic pseudospheric curves the apparatus standard to the theory of arc-length parameterized space curves $\mathcal{C}$, as devised by Jean Frédéric Frenet (1816–1900) and Joseph Alfred Serret (1819–1885). I begin with brief review of the apparatus in question.

Let the 3-vector $\boldsymbol{X}(s)$ describe, relative to a Cartesian frame, such a space curve $\mathcal{C}$. Then

$$\boldsymbol{T}(s) = \tfrac{d}{ds}\boldsymbol{X}(s) \tag{16.1}$$

is the unit vector tangent to $\mathcal{C}$ at s. Differentiation of $\boldsymbol{T}(s)\cdot\boldsymbol{T}(s) = 1$ establishes that $\frac{d}{ds}\boldsymbol{T}(s) \perp \boldsymbol{T}(s)$ and leads one to write

$$\tfrac{d}{ds}\boldsymbol{T}(s) \equiv \kappa(s)\boldsymbol{U}(s) \tag{16.2}$$

where the unit vector $\boldsymbol{U}(s)$, directed to the center of curvature, describes the direction, and $\kappa(s)$ the magnitude, of the local *curvature* of $\mathcal{C}$. Assume $\kappa(s) \neq 0$ and define the "binormal" unit vector

$$\boldsymbol{V}(s) \equiv \boldsymbol{T}(s) \times \boldsymbol{U}(s) \tag{16.3}$$

which serves to complete the construction of an orthonormal triad at each regular point s of $\mathcal{C}$. Elementary arguments[9] lead to the conclusion that

$$\tfrac{d}{ds}\boldsymbol{U}(s) = -\kappa(s)\boldsymbol{T}(s) + \tau(s)\boldsymbol{V}(s) \quad \text{and} \quad \tfrac{d}{ds}\boldsymbol{V}(s) = -\tau(s)\boldsymbol{U}(s) \tag{16.4}$$

where $\tau(s)$ is the *torsion* of $\mathcal{C}$ at s. Briefly,

$$\boldsymbol{\xi}_s(s) \equiv \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{U} \\ \boldsymbol{V} \end{pmatrix}_s = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{U} \\ \boldsymbol{V} \end{pmatrix} \equiv \mathbb{S}(s)\,\boldsymbol{\xi}(s) \tag{17}$$

which are the famous "Frenet-Serret formulæ" (1847–1851). The antisymmetric matrix $\mathbb{S}(s)$ is the generator of the infinitesimal rotation that produces the orthonormal frame at $s + ds$ from the frame at s. It is the upshot of the "Fundamental Theorem of Space Curves" that—given initial data—one can by integrating (17) reproduce the entire curve $\mathcal{C}$ from the prescribed local data written into $\kappa(s)$ and $\tau(s)$. Ideas developed in the preceding section are, of course, immediately applicable; they could be used to reformulate (17) in a way that makes explicit the relevance of $O(3)$ and $SU(2)$ to the differential geometry of space curves.

Look in this light to the curve $\mathcal{C}_y$ inscribed on the pseudosphere Σ,

[9] See, for example, "A Mathematical Note: Frenet-Serret formulæ in higher dimension" (August 1998), page 2. Or any relevant textbook.

which by notational adjustment of (9) we describe[10]

$$\boldsymbol{Y}(s,y) = \begin{pmatrix} \operatorname{sech}(s+y)\,\cos(s-y) \\ \operatorname{sech}(s+y)\,\sin(s-y) \\ (s+y)-\tanh(s+y) \end{pmatrix} \tag{18}$$

Bringing (18) to (16), we calculate

$$\kappa(y,s) = 2\operatorname{sech}(y+s) \quad : \quad \text{curvature of } \mathcal{C}_y \text{ at } s \tag{19.1}$$

and

$$\tau(y,s) = -1 \quad : \quad \text{torsion of } \mathcal{C}_y \text{ at } s \tag{19.2}$$

so the curves $\mathcal{C}_y$ are curves of variable curvature but constant torsion, and (17) becomes

$$\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{U} \\ \boldsymbol{V} \end{pmatrix}_s = \begin{pmatrix} 0 & 2\operatorname{sech}(y+s) & 0 \\ -2\operatorname{sech}(y+s) & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{U} \\ \boldsymbol{V} \end{pmatrix}$$

The same can be said of the curves $\mathcal{C}_x$, except that for such curves the sign of the torsion is reversed.

From temporally twisted curves to the sine-Gordon equation. For any curve $\mathcal{C}$ we have (17), which describes the *differential geometry* of the curve. Rogers & Schief (their §2.2) enrich the subject by looking to what they find it convenient to think of as the *motion* of curves—not of curves in general but of the curves $\mathcal{C}_y$. The simplest such motion is the rigid motion that arises from writing $y = y_0 + ct$; the curve $\mathcal{C}_{y_0}$ then translates temporally through the entire population of asymptotic $\mathcal{C}_y$-curves, tracing out the pseudospheric surface Σ. But this temporal process supplies no information about the $\mathcal{C}_x$-curves, therefore none about the angle $\omega(x,y)$ of their intersection with $\mathcal{C}_y$-curves, therefore cannot lead by the familiar route to the sine-Gordon equation. Rogers & Schief look instead to the class of motions that result when the orthonormal Frenet-Serret triads $\{\boldsymbol{T},\boldsymbol{U},\boldsymbol{V}\}_y$ associated with $\mathcal{C}_y$ are subjected

[10] Similarly, to describe $\mathcal{C}_x$ we would write

$$\boldsymbol{X}(x,s) = \begin{pmatrix} \operatorname{sech}(x+s)\,\cos(x-s) \\ \operatorname{sech}(x+s)\,\sin(x-s) \\ (x+s)-\tanh(x+s) \end{pmatrix}$$

I digress to note that there are now associated with every point P on $\mathcal{C}_y$ *two* orthonormal triads, one with elements $\{\boldsymbol{T},\boldsymbol{U},\boldsymbol{V}\}_y$ that refer to the local geometry of $\mathcal{C}_y$ and another with elements $\{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\}_P$ that refer to the local geometry of Σ. The vectors $\boldsymbol{T}$ and $\boldsymbol{A}$ both refer to the unit tangent originally denoted $\boldsymbol{r}_x(x,y)$. The vectors $\boldsymbol{U},\boldsymbol{V}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ all lie in the plane normal to the tangent, but $\boldsymbol{U}$ (normal to $\mathcal{C}_y$) and $\boldsymbol{C}$ (normal to Σ) are generally distinct, and so also therefore are $\boldsymbol{V}$ and $\boldsymbol{B}$. Yet a third orthonormal frame $\{\boldsymbol{T},\boldsymbol{U},\boldsymbol{V}\}_x$ refers to the local geometry of the $\mathcal{C}_x$ that passes through P.

to local orthonormality-preserving *rotations*. I will describe sequentially

- The "local twisting transformation" as it relates to curves in general.
- The natural emergence of the sine-Gordon equation in cases where local twisting has been designed to preserve a prescribed constancy of the torsion.
- Circumstance which give rise to *solutions* of the sine-Gordon equation.

Let $\mathcal{C}$ be any one of the (congruent) curves that arise from the prescribed curvature and torsion functions, $\kappa(s)$ and $\tau(s)$. We have then the Frenet-Serret formulae (17), which we write

$$\boldsymbol{\xi}_s(s,t) = \mathbb{S}(s,t)\boldsymbol{\xi}(s,t) \quad \text{with} \quad \mathbb{S}(s,t) = \begin{pmatrix} 0 & \kappa(s,t) & 0 \\ -\kappa(s,t) & 0 & \tau(s,t) \\ 0 & -\tau(s,t) & 0 \end{pmatrix}$$

in anticipation of our intention to (in a manner of speaking) "launch $\mathcal{C}$ into twisting motion." Which we do by writing

$$\boldsymbol{\xi}_t(s,t) = \mathbb{T}(s,t)\,\boldsymbol{\xi}(s,t) \quad \text{with} \quad \mathbb{T}(s,t) = \begin{pmatrix} 0 & c(s,t) & b(s,t) \\ -c(s,t) & 0 & a(s,t) \\ -b(s,t) & -a(s,t) & 0 \end{pmatrix}$$

where the antisymmetric matrix $\mathbb{T}(s,t)$ generates at s an infinitesimal rotational "twist" $\boldsymbol{\xi}(s,t) \to \boldsymbol{\xi}(s,t+dt)$ about the axis[11]

$$\boldsymbol{\lambda}(s,t) = \begin{pmatrix} a \\ -b \\ c \end{pmatrix}$$

We impose now the stipulation that operations of s-translation and t-translation *commute*[12]

$$\boldsymbol{\xi}_{st} = \boldsymbol{\xi}_{ts} \tag{20.1}$$

which can be formulated

$$\mathbb{C} = \mathbb{S}_t - \mathbb{T}_s + \mathbb{S}\mathbb{T} - \mathbb{T}\mathbb{S} = \mathbb{O} \tag{20.2}$$

The matrix $\mathbb{C}(s,t)$ is antisymmetric, so $\mathbb{C} = \mathbb{O}$ produces three equations that by calculation are found to read

$$\left.\begin{aligned} a_s &= -\kappa b + \tau_t \\ b_s &= \kappa a - \tau c \\ c_s &= \tau b + \kappa_t \end{aligned}\right\} \tag{21}$$

and to entail

$$(a^2 + b^2 + c^2)_s = 2(c\kappa_t + a\tau_t) \tag{22}$$

[11] Here the minus sign is a contrivance introduced to make things work out most simply.

[12] This Rogers & Schief—for reasons that I do not quite understand—refer to as imposition of an *inextensibility* requirement.

So much for the general theory of temporally twisted curves. We look now to the special circumstances that lead to the sine-Gordon equation. Stipulate

$$\tau_t = c = 0$$

and (22) becomes[13]

$$(a^2 + b^2)_s = 0$$

This we achieve (another contrivance) by setting

$$\begin{aligned} a(s,t) &= f(t)\cos\sigma(s,t) \\ b(s,t) &= f(t)\sin\sigma(s,t) \end{aligned}$$

The commutivity conditions (21) have become

$$\begin{aligned} -f(t)\,\sigma_s \sin\sigma &= -\kappa f(t)\sin\sigma \\ f(t)\,\sigma_s \cos\sigma &= \quad \kappa f(t)\cos\sigma \\ 0 &= \tau f(t)\sin\sigma(s,t) + \kappa_t \end{aligned}$$

of which the first pair give

$$\kappa = \sigma_s \tag{23}$$

which by the last of those conditions (set $f(t) = 1$ and assign to τ its $\mathcal{C}_y$ pseudospheric value $\tau = -1$) gives the sine-Gordon equation

$$\sigma_{st} = \sin\sigma \tag{24}$$

We have been led to set

$$\mathbb{S}(s,t) = \begin{pmatrix} 0 & \sigma_s & 0 \\ -\sigma_s & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbb{T}(s,t) = \begin{pmatrix} 0 & 0 & \sin\sigma \\ 0 & 0 & \cos\sigma \\ -\sin\sigma & -\cos\sigma & 0 \end{pmatrix}$$

in terms of which the compatability condition (20.2) reads

$$\mathbb{C} = \begin{pmatrix} 0 & [\sigma_{st} - \sin\sigma] & 0 \\ -[\sigma_{st} - \sin\sigma] & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{O} \tag{25}$$

We have here an efficient summary of the "twisted curve with constant torsion" argument that led to the sine-Gordon equation (24). If we assign to the torsion its $\mathcal{C}_x$ pseudospheric value $\tau = +1$ and at the same time reverse the sense of the twist (*i.e.*, replace $\mathbb{T}$ by its transpose) we again recover (25).

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[13] One might alternatively set $\kappa_t = a = 0$ and obtain $(b^2 + c^2)_s = 0$. It is—see again (19)—a property of the pseudospheric asymptotic curves $\mathcal{C}_x$ and $\mathcal{C}_y$ that has led me to dismiss that option, which is considered in Rogers & Schief's §2.1.2.

But production of the sine-Gordon equation as a "natural object" is quite a different thing from production of a *solution* of the sine-Gordon equation. In this connection the pseudosphere can be brought again into play. At (19) we have a discription of the $\mathcal{C}_y$ that gives $\tau = -1$ and $\kappa(s,t) = 2\,\mathrm{sech}(s+t)$. Integration of $\sigma_s = \kappa$ gives

$$\sigma(s,t) = 4\arctan\Big[\tanh\Big(\frac{s+t}{2}\Big)\Big] + \pi$$

where the additive constant of integration has been fixed by the requirement that $\sigma(s,t)$ be in fact a solution of the sine-Gordon equation:

$$\sigma_{st} = \sin\sigma = 2\,\mathrm{sech}(s+t)\tanh(s+t)$$

It is, however, distinct from the solution encountered earlier, which (after obvious notational adjustment of (10)) can be written

$$\omega(s,t) = \arccos\big[1 - 2\,\mathrm{sech}^2(s+t)\big]$$

and was observed on page 5 to give

$$\omega_{st} = \sin\omega = 2\sqrt{\mathrm{sech}^2(s+t)\tanh^2(s+t)}$$

The two functions are, however, very closely related: graphic experimentation leads to the realization that[14]

$$\omega(s,t) = \begin{cases} \sigma(s,t) & : \quad s+t<0 \\ 2\pi - \sigma(s,t) & : \quad s+t>0 \end{cases}$$

The function $\sigma(s,t)$ is even more closely related to the a-parameterized class of functions[15]

$$\omega(s,t;a) = 4\arctan\big(e^{as+t/a}\big)$$

that result from Bäcklund transformation of the trivial sine-Gordon function 0. Setting $a=1$ to obtain

$$\Omega(s,t) = 4\arctan\big(e^{s+t}\big)$$

we discover graphically (and confirm alytically) that[16]

$$\Omega(s,t) = \sigma(s,t) = \begin{cases} \omega(s,t) & : \quad s+t<0 \\ 2\pi - \omega(s,t) & : \quad s+t>0 \end{cases}$$

The literature[17] describes methods for constructing curves with constant torsion, but one cannot expect the curvature of those curves to lead via $\kappa = \sigma_s$

[14] Note in this regard that $\omega \mapsto -\omega$ and $\omega \mapsto \omega \bmod 2\pi$ both send solutions to solutions of the sine-Gordon equation.

[15] See "Rectilinear congruences" (February 2016), page 14.

[16] See "Some remarks concerning the sine-Gordon equation" (November 2015), page 14.

[17] See Luther Eisenhart, *A Treatise on the Differential Geometry of Curves and Survaces* (1909), Exercise 9, page 50; Larry M. Bates & O. Michael Melko, "On curves of constant torsion," arXiv [Math.DG] (29 June 2012).

to functions $\sigma(s,t)$ that satisfy the sine-Gordon equation. Those curves, as illustrated, have no relation to the pseudosphere. The literature provides also methods for constructing curves with constant curvature and variable torsion,[18] and to them similar remarks pertain.

On the other hand, given any solution $\sigma(s,t)$ of the sine-Gordon equation one could construct $\kappa = \sigma_s$ and (setting $\tau = \pm 1$) integrate the Frenet-Serret equations to obtain what might be called a "sine-Gordon curve." It would be of interest to know what such curves look like in illustrative cases. The theory of such curves and their interrelationships would presumably be distinct from the theory of "sine-Gordon surfaces" (transformed pseudospheres), and might prove to be of independent interest.

[18] See, for example, J. Monterde, "Salkowski curves revisited: A family of curves with constant curvature and non-constant torsion," (2008), available on the web. In an appendix the author discusses also "A family of curves with constant torsion and non-constant curvature." E. Salkowski was a differential geometer active during the first decades of the 19^{th} Century, who published on this subject in 1909, and later pioneered development of the affine differential geometry of hypersurfaces.