Gaussian Integral Evaluation

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To evaluate the Gaussian integral

$$G = \int_{-\infty}^{+\infty} e^{-x^2}\,dx$$

one most commonly looks to

$$G^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dxdy$$

and passes to polar coordinates. This procedure (called "Bethe's trick" by the person who taught it to me, but is certainly much older than Hans Bethe) is, however susceptible to the fussy criticism that it involves replacing integration over an unbounded square by integration over an unbounded circular disk.

Many alternative arguments can be found in mathematical texts and in notes published in the American Mathematical Monthly and such journals. Here I describe one of the simplest of those—acquired from I do not remember what source.

Let

$$I = \int_0^{\infty} e^{-x^2}\,dx.$$

We want to prove that $I = \sqrt{\pi}/2$ and will do that by proving that $I^2 = \pi/4$. Proceeding as before, we multiply the integral by itself

$$I^2 = \int_0^{\infty}\int_0^{\infty} e^{-x^2-y^2}\,dydx.$$

Now comes the key trick in the sense that everything afterwards is purely mechanical. Change variables by setting $y = xu$ to get

$$I^2 = \int_0^{\infty}\int_0^{\infty} e^{-x^2-u^2x^2}\,x\,dudx.$$

Interchange variables (Fubini's theorem) and observe that the presence of the x-factor renders the inner integral doable (it was a similar circumstance that accouned for the success of Bethe's trick). Alternatively, introduce the variable $v = x^2(1+u^2)$ and proceed as follows:

$$\begin{aligned}
I^2 &= \int_{u=0}^{\infty}\int_{x=0}^{\infty} e^{-x^2(1+u^2)}x\,dxdu \\
&= \int_{u=0}^{\infty}\int_{v=0}^{\infty} \frac{e^{-v}}{2(1+u^2)}dvdu \\
&= \tfrac{1}{2}\int_0^{\infty} e^{-v}dv\,\int_0^{\infty}\frac{du}{(1+u^2)} \\
&= \tfrac{1}{2}\cdot 1\cdot \arctan(\infty) \\
&= \pi/4
\end{aligned}$$