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Ray Mayer's Reconstruction of Ahmed Sebbar's DE

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▪ Introduction

Ahmed Sebbar has interest in the polynomials

```
SebbarSeries = Series[Log[1 - 3 x h - h^3], {h, 0, 10}];
Table[{n, 𝒫n = SeriesCoefficient[SebbarSeries, n]}, {n, 0, 10}] // TableForm
```

0	0
1	$-3x$
2	$-\frac{9x^2}{2}$
3	$-1-9x^3$
4	$-3x-\frac{81x^4}{4}$
5	$-9x^2-\frac{243x^5}{5}$
6	$-\frac{1}{2}-27x^3-\frac{243x^6}{2}$
7	$-3x-81x^4-\frac{2187x^7}{7}$
8	$-\frac{27x^2}{2}-243x^5-\frac{6561x^8}{8}$
9	$-\frac{1}{3}-54x^3-729x^6-2187x^9$
10	$-3x-\frac{405x^4}{2}-2187x^7-\frac{59049x^{10}}{10}$

the nth of which he correctly asserts is killed by the following n-dependent differential operator:

```
SebbarDE[f_, n_] :=
  (4 x^3 + 1) D[f, {x, 3}] + 18 x^2 D[f, {x, 2}] - (3 n^2 + 3 n - 10) x D[f, {x, 1}] - n^2 (n + 3) f
Table[{n, Simplify[SebbarDE[𝒫n, n]]}, {n, 0, 10}] // TableForm
```

0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	0
10	0

Sebbar, however, has thus far provided no indication—beyond a cryptic "it can be shown"—of how he was led to those differential equations. On 19 October I asked Ray Mayer for suggestions in that regard. On 2 November he informed me that he had delivered some relavant material to the Physics Office, which today I fetched, and which is is my business here to digest.

Ray's preparatory exercise

To describe the essence of his method in simplest terms, Ray looks to the polynomials defined

```
RaySeries1 = Series[Log[1 - 2 x h + h^2], {h, 0, 10}];
Table[{n, P_n = SeriesCoefficient[RaySeries1, n]}, {n, 0, 10}] // TableForm
```

0	0
1	$-2x$
2	$1-2x^2$
3	$-\frac{2}{3}\left(-3x+4x^3\right)$
4	$\frac{1}{2}\left(-1+8x^2-8x^4\right)$
5	$-\frac{2}{5}\left(5x-20x^3+16x^5\right)$
6	$\frac{1}{3}\left(1-18x^2+48x^4-32x^6\right)$
7	$-\frac{2}{7}\left(-7x+56x^3-112x^5+64x^7\right)$
8	$\frac{1}{4}\left(-1+32x^2-160x^4+256x^6-128x^8\right)$
9	$-\frac{2}{9}\left(9x-120x^3+432x^5-576x^7+256x^9\right)$
10	$\frac{1}{5}\left(1-50x^2+400x^4-1120x^6+1280x^8-512x^{10}\right)$

- **Digression**

The construction

$$R = \sqrt{1 - 2\,x\,h + h^2}$$

figures in the generating functions of many diverse orthogonal polynomials (see Abramowitz & Stegun, page 783). In particular, the Chebyshev polynomials of the 1st kind are produced

```
Chebyshev1 = Series[(1 - x h)/(1 - 2 x h + h^2), {h, 0, 10}];
Table[{n, T_n = SeriesCoefficient[Chebyshev1, n]}, {n, 0, 9}] // TableForm
```

0	1
1	x
2	$-1+2x^2$
3	$-3x+4x^3$
4	$1-8x^2+8x^4$
5	$5x-20x^3+16x^5$
6	$-1+18x^2-48x^4+32x^6$
7	$-7x+56x^3-112x^5+64x^7$
8	$1-32x^2+160x^4-256x^6+128x^8$
9	$9x-120x^3+432x^5-576x^7+256x^9$

NOTE: The list of such polynomials that appears on page 196 of Spanier & Oldham appears to contain some sign errors.

I have discovered by tinkering that this different generating function

```
Table[{n, n SeriesCoefficient[Series[1 - Log[Sqrt[1 - 2 h x + h^2]], {h, 0, 10}], n]}, {n, 0, 9}] //
 TableForm
```

0	0
1	x
2	$-1+2x^2$
3	$-3x+4x^3$
4	$1-8x^2+8x^4$
5	$5x-20x^3+16x^5$
6	$-1+18x^2-48x^4+32x^6$
7	$-7x+56x^3-112x^5+64x^7$
8	$1-32x^2+160x^4-256x^6+128x^8$
9	$9x-120x^3+432x^5-576x^7+256x^9$

produces a set of polynomials that differs only in the case n = 0. This indicates that Ray's polynomials are closely related to Chebyshev polynomials of the 1st kind. **End of digression**.

- **Back to Ray's argument**

Write

$$1-2xh+h^2=(\alpha-h)(\beta-h)$$

Then

```
Solve[{α β == 1, α + β == 2 x}, {α, β}]
```

$$\left\{\left\{\alpha\to x-\sqrt{-1+x^2},\ \beta\to x+\sqrt{-1+x^2}\right\},\ \left\{\alpha\to x+\sqrt{-1+x^2},\ \beta\to x-\sqrt{-1+x^2}\right\}\right\}$$

With Ray we elect to work with the second solution

$$\alpha=x+\sqrt{x^2-1}$$
$$\beta=x-\sqrt{x^2-1}$$

Dividing $1-2xh+h^2=(\alpha-h)(\beta-h)$ by $\alpha\beta=1$ gives

$$1-2xh+h^2=\left(1-\frac{h}{\alpha}\right)\left(1-\frac{h}{\beta}\right)$$

$$\text{Log}\left[1-2xh+h^2\right]=\text{Log}\left[\left(1-\frac{h}{\alpha}\right)\right]+\text{Log}\left[\left(1-\frac{h}{\beta}\right)\right]$$

$$\text{Series}\left[\text{Log}\left[\left(1-\frac{h}{\alpha}\right)\right]+\text{Log}\left[\left(1-\frac{h}{\beta}\right)\right],\ \{h,0,7\}\right]$$

$$\left(-\frac{1}{\alpha}-\frac{1}{\beta}\right)h+\left(-\frac{1}{2\alpha^2}-\frac{1}{2\beta^2}\right)h^2+\left(-\frac{1}{3\alpha^3}-\frac{1}{3\beta^3}\right)h^3+\left(-\frac{1}{4\alpha^4}-\frac{1}{4\beta^4}\right)h^4+\left(-\frac{1}{5\alpha^5}-\frac{1}{5\beta^5}\right)h^5+\left(-\frac{1}{6\alpha^6}-\frac{1}{6\beta^6}\right)h^6+\left(-\frac{1}{7\alpha^7}-\frac{1}{7\beta^7}\right)h^7+O[h]^8$$

which can be written

$$\sum_{n=1}^{\infty}\left(-\frac{1}{n}\left(\frac{1}{\alpha^n}+\frac{1}{\beta^n}\right)\right)h^n=\sum_{n=1}^{\infty}P_n\,h^n$$

NOTE: The construction $\left(\frac{1}{\alpha^n}+\frac{1}{\beta^n}\right)$ does not *look* much like a polynomial, but it is reminiscent of this construction (Spamier & Oldham, page 196) of the Chebyshev polynomials:

$$T_n[x]=\frac{1}{2}\left(x+\sqrt{x^2-1}\right)^n+\frac{1}{2}\left(x-\sqrt{x^2-1}\right)^n=\frac{1}{2}\left(\alpha^n+\beta^n\right)$$

Granted the validity of our anticipation that the differential equations that we seek are linear, we can drop the $-\frac{1}{n}$ factor, and look to the polynomials

$$Q_n=\left(\frac{1}{\alpha^n}+\frac{1}{\beta^n}\right)$$

With Ray we observe that

$$D\left[\frac{1}{\alpha[x]^n},\{x,1\}\right]$$

$$-n\,\alpha[x]^{-1-n}\,\alpha'[x]$$

can be written

$$-\frac{n}{\alpha^n}\left(\frac{\alpha'}{\alpha}\right)$$

which by

$$\frac{D\left[x+\sqrt{x^2-1},x\right]}{x+\sqrt{x^2-1}}\ //\ \text{Simplify}$$

$$\frac{1}{\sqrt{-1+x^2}}$$

gives

$$D\left[\left(\frac{1}{\alpha^n}\right),\{x,1\}\right]=\frac{1}{\alpha^n}\left(\frac{-n}{\sqrt{x^2-1}}\right)$$

Similarly

$$D\left[\frac{1}{\alpha[x]^n},\{x,2\}\right]$$

$$-(-1-n)\,n\,\alpha[x]^{-2-n}\,\alpha'[x]^2-n\,\alpha[x]^{-1-n}\,\alpha''[x]$$

which can be written

$$\frac{1}{\alpha^n}\left(n\ (n+1)\ \left(\frac{\alpha\,'}{\alpha}\right)^2 - n\ \left(\frac{\alpha\,''}{\alpha}\right)\right)$$

But

$$\frac{\mathbf{D}\left[\mathbf{x} + \sqrt{\mathbf{x}^2 - 1}\,,\ \{\mathbf{x},\ 2\}\right]}{\mathbf{x} + \sqrt{\mathbf{x}^2 - 1}}\ \mathbf{//\ Simplify}$$

$$-\frac{x}{\left(-1 + x^2\right)^{3/2}} + \frac{1}{-1 + x^2}$$

so we have

$$\frac{\mathbf{1}}{\boldsymbol{\alpha}^{\mathbf{n}}}\left(\mathbf{n}\ (\mathbf{n}+\mathbf{1})\ \left(\frac{\mathbf{1}}{\sqrt{-\mathbf{1}+\mathbf{x}^2}}\right)^2 - \mathbf{n}\ \left(-\frac{\mathbf{x}}{\left(-\mathbf{1}+\mathbf{x}^2\right)^{3/2}} + \frac{\mathbf{1}}{-\mathbf{1}+\mathbf{x}^2}\right)\right)\ \mathbf{//\ Simplify}$$

$$\frac{n\left(x + n\sqrt{-1 + x^2}\right)\alpha^{-n}}{\left(-1 + x^2\right)^{3/2}}$$

giving

$$D\left[\left(\frac{1}{\alpha^n}\right),\ \{x,\ 2\}\right] = \frac{1}{\alpha^n}\left(\frac{n\,x}{\left(x^2 - 1\right)^{3/2}} + n^2\ \frac{1}{x^2 - 1}\right)$$

Simliarly for β except for some sign reversals: compare

$$\frac{\mathbf{D}\left[\mathbf{x} + \sqrt{\mathbf{x}^2 - 1}\,,\ \{\mathbf{x},\ 1\}\right]}{\mathbf{x} + \sqrt{\mathbf{x}^2 - 1}}\ \mathbf{//\ Simplify}$$

$$\frac{\mathbf{D}\left[\mathbf{x} - \sqrt{\mathbf{x}^2 - 1}\,,\ \{\mathbf{x},\ 1\}\right]}{\mathbf{x} - \sqrt{\mathbf{x}^2 - 1}}\ \mathbf{//\ Simplify}$$

$$\frac{1}{\sqrt{-1 + x^2}}$$

$$-\frac{1}{\sqrt{-1 + x^2}}$$

$$\frac{\mathbf{D}\left[\mathbf{x}+\sqrt{\mathbf{x}^2-1}\,,\ \{\mathbf{x},\ 2\}\right]}{\mathbf{x}+\sqrt{\mathbf{x}^2-1}}\ \texttt{// Simplify}$$

$$\frac{\mathbf{D}\left[\mathbf{x}-\sqrt{\mathbf{x}^2-1}\,,\ \{\mathbf{x},\ 2\}\right]}{\mathbf{x}-\sqrt{\mathbf{x}^2-1}}\ \texttt{// Simplify}$$

$$-\frac{x}{\left(-1+x^2\right)^{3/2}}+\frac{1}{-1+x^2}$$

$$\frac{x}{\left(-1+x^2\right)^{3/2}}+\frac{1}{-1+x^2}$$

So for β we have

$$D\left[\left(\frac{1}{\beta^n}\right),\ \{x,\ 1\}\right]=\frac{1}{\beta^n}\left(\frac{n}{\sqrt{x^2-1}}\right)$$

$$D\left[\left(\frac{1}{\beta^n}\right),\ \{x,\ 2\}\right]=\frac{1}{\beta^n}\left(-\frac{n\,x}{\left(x^2-1\right)^{3/2}}+n^2\,\frac{1}{x^2-1}\right)$$

So we have

$$\begin{pmatrix} Q_n \\ Q_n{}' \\ Q_n{}'' \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ \left(-\frac{n}{\sqrt{x^2-1}}\right) & \left(\frac{n}{\sqrt{x^2-1}}\right) \\ \left(\frac{n\,x}{\left(x^2-1\right)^{3/2}}+n^2\,\frac{1}{x^2-1}\right) & \left(-\frac{n\,x}{\left(x^2-1\right)^{3/2}}+n^2\,\frac{1}{x^2-1}\right) \end{pmatrix}\begin{pmatrix} \frac{1}{\alpha^n} \\ \frac{1}{\beta^n} \end{pmatrix}$$

which shows that the three objects Q_n, $Q_n{}'$, $Q_n{}''$ live in a 2-space, so must be linearly dependent. From

```
Clear[n, x, p, q]
```

$$\mathbf{Solve}\left[\left\{\left(\frac{n\,x}{\left(x^2-1\right)^{3/2}}+n^2\,\frac{1}{x^2-1}\right)+p+q\left(-\frac{n}{\sqrt{x^2-1}}\right)=0,\right.\right.$$
$$\left.\left.\left(-\frac{n\,x}{\left(x^2-1\right)^{3/2}}+n^2\,\frac{1}{x^2-1}\right)+p+q\left(\frac{n}{\sqrt{x^2-1}}\right)=0\right\},\ \{p,\ q\}\right]$$

$$\left\{\left\{p\to-\frac{n^2}{-1+x^2},\ q\to\frac{x}{-1+x^2}\right\}\right\}$$

we conclude that the polynomials

$$\mathbf{Table}\left[Q_n=\left(\frac{1}{x+\sqrt{x^2-1}}\right)^n+\left(\frac{1}{x-\sqrt{x^2-1}}\right)^n,\ \{n,\ 1,\ 10\}\right];$$

```
Table[{n, Simplify[Qn]}, {n, 1, 8}] // TableForm
```

1	$2x$
2	$-2+4x^2$
3	$-6x+8x^3$
4	$2(1-8x^2+8x^4)$
5	$2x(5-20x^2+16x^4)$
6	$-2+36x^2-96x^4+64x^6$
7	$2x(-7+56x^2-112x^4+64x^6)$
8	$2-64x^2+320x^4-512x^6+256x^8$

are solutions of

$$f'' + \frac{x}{x^2-1} f' - \frac{n^2}{x^2-1} f = 0$$

or—which is the same except at $x^2 = 1$ —

$$(x^2-1) f'' + x f' - n^2 f = 0$$

```
MayerDE1[f_, n_] := (x^2 - 1) D[f, {x, 2}] + x D[f, {x, 1}] - n^2 f
```

Which we check:

```
Table[{n, Simplify[MayerDE1[Qn, n]]}, {n, 1, 8}] // TableForm
```

1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	0

Spanier & Oldham assert (page 196) that Mayer's DE is precisely "Chebyshev's DE," of which for n = 1, 2, 3, ... the most general solution is

```
fn[x] = C[1] ChebyshevT[n, x] + C[2] Sqrt[1 - x^2] ChebyshevU[n - 1, x]
```

which we verify. Defining symbols T_n and U_n

```
Table[{n, Tn = ChebyshevT[n, x], Un = ChebyshevU[n - 1, x]}, {n, 1, 10}];

T4
U4
```

$1-8x^2+8x^4$

$1-12x^2+16x^4$

we have (as claimed)

```
Table[{n, Simplify[MayerDE1[Tn, n]], Simplify[MayerDE1[Sqrt[1 - x^2] Un-1, n]]}, {n, 1, 8}] //
 TableForm
```

1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	0	0
7	0	0
8	0	0

Mathematica presents the most general solution if the Chebyshev-Mayer DE in quite a different (but presumably equivalent) form:

```
DSolve[(x^2 - 1) f''[x] + x f'[x] - n^2 f[x] == 0, f[x], x]
```

$$\left\{\left\{f[x] \to C[1] \operatorname{Cosh}\left[n\left(\operatorname{Log}[2] + \operatorname{Log}\left[x + \sqrt{-1 + x^2}\right]\right)\right] + i\, C[2] \operatorname{Sinh}\left[n\left(\operatorname{Log}[2] + \operatorname{Log}\left[x + \sqrt{-1 + x^2}\right]\right)\right]\right\}\right\}$$

Ray's approach to the construction of Sebbar's DE

We are interested now in the polynomials $S_n[x]$ defined

```
RaySeries2 = Series[Log[1 - 3 x h - h^3], {h, 0, 10}];
Table[{n, Sn = SeriesCoefficient[RaySeries2, n]}, {n, 0, 10}] // TableForm
```

0	0
1	$-3x$
2	$-\frac{9x^2}{2}$
3	$-1 - 9x^3$
4	$-3x - \frac{81x^4}{4}$
5	$-9x^2 - \frac{243x^5}{5}$
6	$-\frac{1}{2} - 27x^3 - \frac{243x^6}{2}$
7	$-3x - 81x^4 - \frac{2187x^7}{7}$
8	$-\frac{27x^2}{2} - 243x^5 - \frac{6561x^8}{8}$
9	$-\frac{1}{3} - 54x^3 - 729x^6 - 2187x^9$
10	$-3x - \frac{405x^4}{2} - 2187x^7 - \frac{59049x^{10}}{10}$

We note that the powers augment in each instance by multiples of 3.

Write

```
1 - 3 x h - h^3 = (α - h) (β - h) (γ - h)
```

where $\{\alpha, \beta, \gamma\}$ are the roots of $1 - 3\,x\,h - h^3 = 0$. From

```
Series[(α - h) (β - h) (γ - h), {h, 0, 3}] // Normal
```

$$-h^3 + \alpha\beta\gamma + h^2(\alpha + \beta + \gamma) + h(-\alpha\beta - \alpha\gamma - \beta\gamma)$$

we see that they satisfy the system

$$\alpha\beta\gamma == 1$$
$$\alpha\beta + \alpha\gamma + \beta\gamma == -3x$$
$$\alpha + \beta + \gamma == 0$$

and note that if $\{\alpha, \beta, \gamma\}$ is a solution then so is each of the 6 permutations of $\{\alpha, \beta, \gamma\}$; for that reason the command

```
Solve[{αβγ == 1, αβ + αγ + βγ == 3 x, α + β + γ == 0}, {α, β, γ}]
```

produces an unworkable 6-fold redundant mess. We work therefore from

```
Solve[1 - 3 x h - h^3 == 0, h]
```

$$\left\{\left\{h \to \frac{2^{1/3}\,x}{\left(-1+\sqrt{1+4x^3}\right)^{1/3}} - \frac{\left(-1+\sqrt{1+4x^3}\right)^{1/3}}{2^{1/3}}\right\},\right.$$

$$\left\{h \to -\frac{\left(1+i\sqrt{3}\right)x}{2^{2/3}\left(-1+\sqrt{1+4x^3}\right)^{1/3}} + \frac{\left(1-i\sqrt{3}\right)\left(-1+\sqrt{1+4x^3}\right)^{1/3}}{2\;2^{1/3}}\right\},$$

$$\left.\left\{h \to -\frac{\left(1-i\sqrt{3}\right)x}{2^{2/3}\left(-1+\sqrt{1+4x^3}\right)^{1/3}} + \frac{\left(1+i\sqrt{3}\right)\left(-1+\sqrt{1+4x^3}\right)^{1/3}}{2\;2^{1/3}}\right\}\right\}$$

If we set

$$A = \frac{2^{1/3}}{\left(-1+\sqrt{1+4x^3}\right)^{1/3}}$$

we have

$$\alpha = A\,x - \frac{1}{A}$$

$$\beta = -\left(\frac{1+i\sqrt{3}}{2}\right)A\,x + \left(\frac{1-i\sqrt{3}}{2}\right)\frac{1}{A}$$

$$\gamma = -\left(\frac{1-i\sqrt{3}}{2}\right)A\,x + \left(\frac{1+i\sqrt{3}}{2}\right)\frac{1}{A}$$

in which connection we notice that

$$\omega = \left(\frac{1+i\sqrt{3}}{2}\right);$$

and its conjugate

$$\Omega = \left(\frac{1-i\sqrt{3}}{2}\right);$$

are conjugate cube roots of -1:

```
ω^3 // Simplify
Ω^3 // Simplify
```

-1

-1

REMARK: I would like to write $\overline{\omega}$ in place of Ω, but don't because *Mathematica* complains, when I have occasion to write **Clear[ω̄]**, that "$\bar{\omega}$ is not a symbol or a string."

Dividing $1 - 3\,x\,h - h^3 = (\alpha - h)(\beta - h)(\gamma - h)$ by $\alpha\,\beta\,\gamma = 1$, we use

$$\frac{(\alpha - h)\ (\beta - h)\ (\gamma - h)}{\alpha\,\beta\,\gamma} == \left(1 - \frac{h}{\alpha}\right)\left(1 - \frac{h}{\beta}\right)\left(1 - \frac{h}{\gamma}\right) \text{ // Simplify}$$

True

to obtain

$$\text{Log}\left[1 - 3\,x\,h - h^3\right] = \text{Log}\left[\left(1 - \frac{h}{\alpha}\right)\right] + \text{Log}\left[\left(1 - \frac{h}{\beta}\right)\right] + \text{Log}\left[\left(1 - \frac{h}{\gamma}\right)\right]$$

$$= \sum_{n=1}^{\infty} -\frac{1}{n}\left(\frac{1}{\alpha^n} + \frac{1}{\beta^n} + \frac{1}{\gamma^n}\right) h^n$$

giving

$$S_n = -\frac{1}{n}\left(\frac{1}{\alpha^n} + \frac{1}{\beta^n} + \frac{1}{\gamma^n}\right)$$

which we check:

$$\text{Table}\left[\left\{n, \text{Factor}[\text{Together}[S_n]], \text{FullSimplify}\left[-\frac{1}{n}\left(\left(\frac{1}{\alpha^n} + \frac{1}{\beta^n} + \frac{1}{\gamma^n}\right)\right)\right]\right\}, \{n, 1, 8\}\right] \text{ // TableForm}$$

1	$-3\,x$	$-3\,x$
2	$-\frac{9\,x^2}{2}$	$-\frac{9\,x^2}{2}$
3	$-1 - 9\,x^3$	$-1 - 9\,x^3$
4	$-\frac{3}{4}\,x\,(4 + 27\,x^3)$	$-\frac{3}{4}\,x\,(4 + 27\,x^3)$
5	$-\frac{9}{5}\,x^2\,(5 + 27\,x^3)$	$-\frac{9}{5}\,x^2\,(5 + 27\,x^3)$
6	$\frac{1}{2}\,(-1 - 54\,x^3 - 243\,x^6)$	$\frac{1}{2}\,(-1 - 54\,x^3 - 243\,x^6)$
7	$-\frac{3}{7}\,x\,(7 + 189\,x^3 + 729\,x^6)$	$-\frac{3}{7}\,x\,(7 + 189\,x^3 + 729\,x^6)$
8	$-\frac{27}{8}\,x^2\,(2 + 9\,x^3)\,(2 + 27\,x^3)$	$-\frac{27}{8}\,x^2\,(4 + 72\,x^3 + 243\,x^6)$

With Mayer, we elect to work with the slightly simpler "Mayer polynomials"

$$M_n = \left(\frac{1}{\alpha^n} + \frac{1}{\beta^n} + \frac{1}{\gamma^n}\right)$$

which, since they differ only by numeric factors, will satisfy the same linear DEs.

$$\mathbf{D}\left[\frac{1}{\alpha[x]^n}, \{x, 1\}\right]$$

$$-n\,\alpha[x]^{-1-n}\,\alpha'[x]$$

$$\% == \frac{1}{\alpha[x]^n}\left(-n\,\frac{\alpha'[x]}{\alpha[x]}\right) \text{ // Simplify}$$

True

$$\mathbf{D}\left[\frac{1}{\alpha[x]^n}, \{x, 2\}\right]$$

$$-(-1-n)\,n\,\alpha[x]^{-2-n}\,\alpha'[x]^2 - n\,\alpha[x]^{-1-n}\,\alpha''[x]$$

$$\% == \frac{1}{\alpha[x]^n}\left(n\,(n+1)\left(\frac{\alpha'[x]}{\alpha[x]}\right)^2 - n\,\frac{\alpha''[x]}{\alpha[x]}\right) \text{ // Simplify}$$

True

$$\mathbf{D}\left[\frac{1}{\alpha[x]^n}, \{x, 3\}\right]$$

$$-(-2-n)\,(-1-n)\,n\,\alpha[x]^{-3-n}\,\alpha'[x]^3 - 3\,(-1-n)\,n\,\alpha[x]^{-2-n}\,\alpha'[x]\,\alpha''[x] - n\,\alpha[x]^{-1-n}\,\alpha^{(3)}[x]$$

$$\% == \frac{1}{\alpha[x]^n}\left(-n\,(1+n)\,(2+n)\left(\frac{\alpha'[x]}{\alpha[x]}\right)^3 + 3\,n\,(1+n)\,\frac{\alpha'[x]}{\alpha[x]}\,\frac{\alpha''[x]}{\alpha[x]} - n\,\frac{\alpha^{(3)}[x]}{\alpha[x]}\right) \text{ // Simplify}$$

True

Formally identical results pertain to β and γ; distinctions arise when we undertake to evaluate the ratios

$$\frac{\alpha'[x]}{\alpha[x]},\quad \frac{\alpha''[x]}{\alpha[x]},\quad \frac{\alpha^{(3)}[x]}{\alpha[x]}$$

and their analogs, and this is where things begin to get messy. We have ultimately to build upon these definitions:

$$A = \frac{2^{1/3}}{\left(-1+\sqrt{1+4\,x^3}\right)^{1/3}};$$

$$\omega = \left(\frac{1+i\sqrt{3}}{2}\right);$$

$$\Omega = \left(\frac{1-i\sqrt{3}}{2}\right);$$

but in an attempt to avoid intractable complexities will work so long as we can from these generic constructions:

```
Clear[A, ω, Ω, α, β, γ]
```

$$\alpha[x\_] := A[x]\,x - \frac{1}{A[x]}$$

$$\beta[x\_] := -\omega\, A[x]\,x + \Omega\,\frac{1}{A[x]}$$

$$\gamma[x\_] := -\Omega\, A[x]\,x + \omega\,\frac{1}{A[x]}$$

- **The case α**

$$C_{\alpha 1} = \left(-n\,\frac{D[\alpha[x], \{x, 1\}]}{\alpha[x]}\right) \text{ // Together}$$

$$C_{\alpha 2} = \left(n\,(n+1)\left(\frac{D[\alpha[x], \{x, 1\}]}{\alpha[x]}\right)^2 - n\,\frac{D[\alpha[x], \{x, 2\}]}{\alpha[x]}\right) \text{ // Together}$$

$$C_{\alpha 3} = \left(-n\,(1+n)\,(2+n)\left(\frac{D[\alpha[x], \{x, 1\}]}{\alpha[x]}\right)^3 + 3\,n\,(1+n)\,\frac{D[\alpha[x], \{x, 1\}]}{\alpha[x]}\,\frac{D[\alpha[x], \{x, 2\}]}{\alpha[x]} - n\,\frac{D[\alpha[x], \{x, 3\}]}{\alpha[x]}\right) \text{ // Together}$$

$$-\frac{n\left(A[x]^3 + A'[x] + x\,A[x]^2\,A'[x]\right)}{A[x]\left(-1 + x\,A[x]^2\right)}$$

$$\Big(n\,A[x]^6 + n^2\,A[x]^6 + 4\,n\,A[x]^3\,A'[x] + 2\,n^2\,A[x]^3\,A'[x] + 2\,n^2\,x\,A[x]^5\,A'[x] - n\,A'[x]^2 + n^2\,A'[x]^2 + 4\,n\,x\,A[x]^2\,A'[x]^2 + 2\,n^2\,x\,A[x]^2\,A'[x]^2 + n\,x^2\,A[x]^4\,A'[x]^2 + n^2\,x^2\,A[x]^4\,A'[x]^2 + n\,A[x]\,A''[x] - n\,x^2\,A[x]^5\,A''[x]\Big) \Big/ \left(A[x]^2\left(-1 + x\,A[x]^2\right)^2\right)$$

$$\Big(-2\,n\,A[x]^9 - 3\,n^2\,A[x]^9 - n^3\,A[x]^9 - 12\,n\,A[x]^6\,A'[x] - 15\,n^2\,A[x]^6\,A'[x] - 3\,n^3\,A[x]^6\,A'[x] - 3\,n^2\,x\,A[x]^8\,A'[x] - 3\,n^3\,x\,A[x]^8\,A'[x] - 6\,n\,A[x]^3\,A'[x]^2 - 9\,n^2\,A[x]^3\,A'[x]^2 - 3\,n^3\,A[x]^3\,A'[x]^2 - 18\,n\,x\,A[x]^5\,A'[x]^2 - 24\,n^2\,x\,A[x]^5\,A'[x]^2 - 6\,n^3\,x\,A[x]^5\,A'[x]^2 - 3\,n^2\,x^2\,A[x]^7\,A'[x]^2 - 3\,n^3\,x^2\,A[x]^7\,A'[x]^2 - 2\,n\,A'[x]^3 + 3\,n^2\,A'[x]^3 - n^3\,A'[x]^3 + 6\,n\,x\,A[x]^2\,A'[x]^3 - 9\,n^2\,x\,A[x]^2\,A'[x]^3 - 3\,n^3\,x\,A[x]^2\,A'[x]^3 - 18\,n\,x^2\,A[x]^4\,A'[x]^3 - 15\,n^2\,x^2\,A[x]^4\,A'[x]^3 - 3\,n^3\,x^2\,A[x]^4\,A'[x]^3 - 2\,n\,x^3\,A[x]^6\,A'[x]^3 - 3\,n^2\,x^3\,A[x]^6\,A'[x]^3 - n^3\,x^3\,A[x]^6\,A'[x]^3 - 6\,n\,A[x]^4\,A''[x] - 3\,n^2\,A[x]^4\,A''[x] + 6\,n\,x\,A[x]^6\,A''[x] + 3\,n^2\,x^2\,A[x]^8\,A''[x] + 3\,n\,A[x]\,A'[x]\,A''[x] - 3\,n^2\,A[x]\,A'[x]\,A''[x] - 15\,n\,x\,A[x]^3\,A'[x]\,A''[x] - 3\,n^2\,x\,A[x]^3\,A'[x]\,A''[x] + 9\,n\,x^2\,A[x]^5\,A'[x]\,A''[x] + 3\,n^2\,x^2\,A[x]^5\,A'[x]\,A''[x] + 3\,n\,x^3\,A[x]^7\,A'[x]\,A''[x] + 3\,n^2\,x^3\,A[x]^7\,A'[x]\,A''[x] - n\,A[x]^2\,A^{(3)}[x] + n\,x\,A[x]^4\,A^{(3)}[x] + n\,x^2\,A[x]^6\,A^{(3)}[x] - n\,x^3\,A[x]^8\,A^{(3)}[x]\Big) \Big/ \left(A[x]^3\left(-1 + x\,A[x]^2\right)^3\right)$$

- **The case β**

NOTE: Since these results are so complicated, I display only the simplest of them.

$$C_{\beta 1} = \left(-n\,\frac{D[\beta[x],\{x,1\}]}{\beta[x]}\right)\ //\ \text{Together}$$

$$C_{\beta 2} = \left(n\,(n+1)\left(\frac{D[\beta[x],\{x,1\}]}{\beta[x]}\right)^2 - n\,\frac{D[\beta[x],\{x,2\}]}{\beta[x]}\right)\ //\ \text{Together;}$$

$$C_{\beta 3} = \left(-n\,(1+n)\,(2+n)\left(\frac{D[\beta[x],\{x,1\}]}{\beta[x]}\right)^3 + 3\,n\,(1+n)\,\frac{D[\beta[x],\{x,1\}]}{\beta[x]}\,\frac{D[\beta[x],\{x,2\}]}{\beta[x]} - n\,\frac{D[\beta[x],\{x,3\}]}{\beta[x]}\right)\ //\ \text{Together;}$$

$$-\frac{n\left(\omega A[x]^3 + \Omega A'[x] + x\,\omega A[x]^2 A'[x]\right)}{A[x]\left(-\Omega + x\,\omega A[x]^2\right)}$$

- **The case γ**

$$C_{\gamma 1} = \left(-n\,\frac{D[\gamma[x],\{x,1\}]}{\gamma[x]}\right)\ //\ \text{Together}$$

$$C_{\gamma 2} = \left(n\,(n+1)\left(\frac{D[\gamma[x],\{x,1\}]}{\gamma[x]}\right)^2 - n\,\frac{D[\gamma[x],\{x,2\}]}{\gamma[x]}\right)\ //\ \text{Together;}$$

$$C_{\gamma 3} = \left(-n\,(1+n)\,(2+n)\left(\frac{D[\gamma[x],\{x,1\}]}{\gamma[x]}\right)^3 + 3\,n\,(1+n)\,\frac{D[\gamma[x],\{x,1\}]}{\gamma[x]}\,\frac{D[\gamma[x],\{x,2\}]}{\gamma[x]} - n\,\frac{D[\gamma[x],\{x,3\}]}{\gamma[x]}\right)\ //\ \text{Together;}$$

$$-\frac{n\left(\Omega A[x]^3 + \omega A'[x] + x\,\Omega A[x]^2 A'[x]\right)}{A[x]\left(-\omega + x\,\Omega A[x]^2\right)}$$

- **Assembly and implications**

We have now

$$\begin{pmatrix} M_n \\ M_n{}' \\ M_n{}'' \\ M_n{}''' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ C_{\alpha 1} & C_{\beta 1} & C_{\gamma 1} \\ C_{\alpha 2} & C_{\beta 2} & C_{\gamma 2} \\ C_{\alpha 3} & C_{\beta 3} & C_{\gamma 3} \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha^n} \\ \frac{1}{\beta^n} \\ \frac{1}{\gamma^n} \end{pmatrix}$$

and infer that there must exist a linear relationship among $M_n$and its first three derivatives:

$$M_n{}^{(3)} + p\,M_n{}'' + q\,M_n{}' + r\,M_n = 0$$

NOTE: This DE is of 3rd degree because

$$1 - 3\,x\,h - h^3 = (\alpha - h)\,(\beta - h)\,(\gamma - h)$$

is cubic in h.

We look now to the evaluation of p, q and r:

```
Solve[{Cα3 + p Cα2 + q Cα1 + r == 0, Cβ3 + p Cβ2 + q Cβ1 + r == 0, Cγ3 + p Cγ2 + q Cγ1 + r == 0}, {p, q, r}] //
  Simplify
```

```
{{p → (-3 (2 + n) x^2 ω Ω A[x]^8 - 6 (1 + n) x ω Ω A'[x]^2 +
      2 (-1 + n) x^2 (ω^2 + ω Ω + Ω^2) A[x]^2 A'[x]^2 + (ω^2 + ω Ω + Ω^2) A[x]^4 (-2 - n + 2 (5 + n) x^3 A'[x]^2) +
      2 x A[x]^6 ((2 + n) (ω^2 + ω Ω + Ω^2) - 3 (3 + n) x^3 ω Ω A'[x]^2) - x (ω^2 + ω Ω + Ω^2) A[x]^3
       ((-4 + n) A'[x] - 6 x A''[x]) + x^2 (ω^2 + ω Ω + Ω^2) A[x]^5 ((4 + 5 n) A'[x] - 6 x A''[x]) -
      3 ω Ω A[x] ((4 + n) A'[x] + 2 x A''[x]) + 3 x^3 ω Ω A[x]^7 (-(4 + 3 n) A'[x] + 2 x A''[x])) /
    (A[x] (-1 + x A[x]^2) (-Ω + x ω A[x]^2) (ω - x Ω A[x]^2) (A[x] + 2 x A'[x])),
  q → (-3 (2 + 3 n + n^2) x ω Ω A[x]^10 + 12 (1 + 3 n + n^2) x^2 ω Ω A'[x]^4 +
      12 x ω Ω A[x] A'[x]^2 ((3 + 6 n + n^2) A'[x] + (1 + n) x A''[x]) + A[x]^2 (3 (14 + 9 n + n^2) ω Ω A'[x]^2 +
         4 (1 + n + n^2) x^3 (ω^2 + ω Ω + Ω^2) A'[x]^4 + 12 x^2 ω Ω A''[x]^2 + 2 x ω Ω A'[x] (15 A''[x] - 2 x A^(3)[x])) +
      A[x]^3 (12 (1 + n^2) x^2 (ω^2 + ω Ω + Ω^2) A'[x]^3 - 4 (-1 + n) x^3 (ω^2 + ω Ω + Ω^2) A'[x]^2 A''[x] -
         ω Ω (3 (2 + n) A''[x] + 2 x A^(3)[x])) - (ω^2 + ω Ω + Ω^2) A[x]^5 (-2 (2 + n)^2 A'[x] +
         4 (-1 - 6 n + n^2) x^3 A'[x]^3 + 4 (5 + n) x^4 A'[x]^2 A''[x] - x (5 (2 + n) A''[x] + 2 x A^(3)[x])) -
      x A[x]^7 (-4 n (2 + n) (ω^2 + ω Ω + Ω^2) A'[x] + 12 (7 + 12 n + 3 n^2) x^3 ω Ω A'[x]^3 -
         12 (3 + n) x^4 ω Ω A'[x]^2 A''[x] + x (ω^2 + ω Ω + Ω^2) (7 (2 + n) A''[x] + 2 x A^(3)[x])) +
      x^2 ω Ω A[x]^9 (-6 (4 + 8 n + 3 n^2) A'[x] + x (9 (2 + n) A''[x] + 2 x A^(3)[x])) -
      x (ω^2 + ω Ω + Ω^2) A[x]^4 (-3 (-2 + 5 n + 3 n^2) A'[x]^2 + 4 (-3 - n + n^2) x^3 A'[x]^4 + 12 x^2 A''[x]^2 -
         2 x A'[x] ((-7 + 4 n) A''[x] + 2 x A^(3)[x])) + A[x]^8 ((2 + 3 n + n^2) (ω^2 + ω Ω + Ω^2) -
         39 (2 + 3 n + n^2) x^3 ω Ω A'[x]^2 - 12 x^5 ω Ω A''[x]^2 + 2 x^4 ω Ω A'[x] (3 (3 + 4 n) A''[x] + 2 x A^(3)[x])) +
      x^2 A[x]^6 (3 (6 + 5 n + n^2) (ω^2 + ω Ω + Ω^2) A'[x]^2 - 12 (5 + 5 n + n^2) x^3 ω Ω A'[x]^4 +
         12 x^2 (ω^2 + ω Ω + Ω^2) A''[x]^2 - 2 x (ω^2 + ω Ω + Ω^2) A'[x] ((1 + 8 n) A''[x] + 2 x A^(3)[x]))) /
    (A[x]^2 (-1 + x A[x]^2) (-Ω + x ω A[x]^2) (ω - x Ω A[x]^2) (A[x] + 2 x A'[x])^2), r →
   -(n ((2 + 3 n + n^2) ω Ω A[x]^11 + 2 (37 + 21 n + 2 n^2) x ω Ω A[x] A'[x]^4 + 4 (8 + 6 n + n^2) x^2 ω Ω A'[x]^5 + A[x]^2
         A'[x]^2 ((56 + 15 n + n^2) ω Ω A'[x] + 4 n (2 + n) x^3 (ω^2 + ω Ω + Ω^2) A'[x]^3 - 6 (1 + n) x ω Ω A''[x]) +
        A[x]^4 ((-8 + 21 n + 5 n^2) x (ω^2 + ω Ω + Ω^2) A'[x]^3 + 4 n (2 + n) x^4 (ω^2 + ω Ω + Ω^2) A'[x]^5 +
           2 (-1 + n) x^2 (ω^2 + ω Ω + Ω^2) A'[x]^2 A''[x] + ω Ω A^(3)[x]) +
        A[x]^6 ((8 + 15 n + 13 n^2) x^2 (ω^2 + ω Ω + Ω^2) A'[x]^3 + 4 (8 + 6 n + n^2) x^5 ω Ω A'[x]^5 +
           2 (5 + n) x^3 (ω^2 + ω Ω + Ω^2) A'[x]^2 A''[x] - (ω^2 + ω Ω + Ω^2) ((2 + n) A''[x] + x A^(3)[x])) +
        A[x]^8 ((2 + 3 n + n^2) (ω^2 + ω Ω + Ω^2) A'[x] + (104 + 117 n + 25 n^2) x^3 ω Ω A'[x]^3 -
           6 (3 + n) x^4 ω Ω A'[x]^2 A''[x] + x (ω^2 + ω Ω + Ω^2) (2 (2 + n) A''[x] + x A^(3)[x])) +
        x ω Ω A[x]^10 (7 (2 + 3 n + n^2) A'[x] - x (3 (2 + n) A''[x] + x A^(3)[x])) +
        (ω^2 + ω Ω + Ω^2) A[x]^5 ((10 + 7 n + n^2) A'[x]^2 + 6 (-1 + 3 n + 2 n^2) x^3 A'[x]^4 +
           6 x^2 A''[x]^2 - x A'[x] ((-13 + n) A''[x] + 2 x A^(3)[x])) +
        x A[x]^7 (2 (-2 + 5 n + 3 n^2) (ω^2 + ω Ω + Ω^2) A'[x]^2 + 2 (55 + 45 n + 8 n^2) x^3 ω Ω A'[x]^4 -
           6 x^2 (ω^2 + ω Ω + Ω^2) A''[x]^2 + x (ω^2 + ω Ω + Ω^2) A'[x] (5 (-1 + n) A''[x] + 2 x A^(3)[x])) +
        A[x]^3 (2 (-1 + 9 n + 4 n^2) x^2 (ω^2 + ω Ω + Ω^2) A'[x]^4 - 6 x ω Ω A''[x]^2 +
           ω Ω A'[x] (-3 (7 + n) A''[x] + 2 x A^(3)[x])) +
        x^2 ω Ω A[x]^9 ((62 + 69 n + 19 n^2) A'[x]^2 + 6 x^2 A''[x]^2 - x A'[x] ((3 + 9 n) A''[x] + 2 x A^(3)[x])))) /
     (A[x]^3 (-1 + x A[x]^2) (-Ω + x ω A[x]^2) (ω - x Ω A[x]^2) (A[x] + 2 x A'[x])^2)}}
```

So we have

$$
\begin{aligned}
p = {} & \big(-3\,(2+n)\,x^2\,\omega\,\Omega\,A[x]^8 - 6\,(1+n)\,x\,\omega\,\Omega\,A'[x]^2 + \\
& 2\,(-1+n)\,x^2\,(\omega^2+\omega\,\Omega+\Omega^2)\,A[x]^2\,A'[x]^2 + (\omega^2+\omega\,\Omega+\Omega^2)\,A[x]^4\,(-2-n+2\,(5+n)\,x^3\,A'[x]^2) + \\
& 2\,x\,A[x]^6\,((2+n)\,(\omega^2+\omega\,\Omega+\Omega^2) - 3\,(3+n)\,x^3\,\omega\,\Omega\,A'[x]^2) - x\,(\omega^2+\omega\,\Omega+\Omega^2)\,A[x]^3 \\
& ((-4+n)\,A'[x] - 6\,x\,A''[x]) + x^2\,(\omega^2+\omega\,\Omega+\Omega^2)\,A[x]^5\,((4+5\,n)\,A'[x] - 6\,x\,A''[x]) - \\
& 3\,\omega\,\Omega\,A[x]\,((4+n)\,A'[x] + 2\,x\,A''[x]) + 3\,x^3\,\omega\,\Omega\,A[x]^7\,(-(4+3\,n)\,A'[x] + 2\,x\,A''[x])\big) \,/ \\
& \big(A[x]\,(-1+x\,A[x]^2)\,(-\Omega+x\,\omega\,A[x]^2)\,(\omega-x\,\Omega\,A[x]^2)\,(A[x]+2\,x\,A'[x])\big);
\end{aligned}
$$

$$
\begin{aligned}
q = {} & \big(-3\,(2+3\,n+n^2)\,x\,\omega\,\Omega\,A[x]^{10} + 12\,(1+3\,n+n^2)\,x^2\,\omega\,\Omega\,A'[x]^4 + \\
& 12\,x\,\omega\,\Omega\,A[x]\,A'[x]^2\,((3+6\,n+n^2)\,A'[x] + (1+n)\,x\,A''[x]) + A[x]^2\,(3\,(14+9\,n+n^2)\,\omega\,\Omega\,A'[x]^2 + \\
& \quad 4\,(1+n+n^2)\,x^3\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]^4 + 12\,x^2\,\omega\,\Omega\,A''[x]^2 + 2\,x\,\omega\,\Omega\,A'[x]\,(15\,A''[x] - 2\,x\,A^{(3)}[x])) + \\
& A[x]^3\,(12\,(1+n^2)\,x^2\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]^3 - 4\,(-1+n)\,x^3\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]^2\,A''[x] - \\
& \quad \omega\,\Omega\,(3\,(2+n)\,A''[x] + 2\,x\,A^{(3)}[x])) - (\omega^2+\omega\,\Omega+\Omega^2)\,A[x]^5\,(-2\,(2+n)^2\,A'[x] + \\
& \quad 4\,(-1-6\,n+n^2)\,x^3\,A'[x]^3 + 4\,(5+n)\,x^4\,A'[x]^2\,A''[x] - x\,(5\,(2+n)\,A''[x] + 2\,x\,A^{(3)}[x])) - \\
& x\,A[x]^7\,(-4\,n\,(2+n)\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x] + 12\,(7+12\,n+3\,n^2)\,x^3\,\omega\,\Omega\,A'[x]^3 - \\
& \quad 12\,(3+n)\,x^4\,\omega\,\Omega\,A'[x]^2\,A''[x] + x\,(\omega^2+\omega\,\Omega+\Omega^2)\,(7\,(2+n)\,A''[x] + 2\,x\,A^{(3)}[x])) + \\
& x^2\,\omega\,\Omega\,A[x]^9\,(-6\,(4+8\,n+3\,n^2)\,A'[x] + x\,(9\,(2+n)\,A''[x] + 2\,x\,A^{(3)}[x])) - \\
& x\,(\omega^2+\omega\,\Omega+\Omega^2)\,A[x]^4\,(-3\,(-2+5\,n+3\,n^2)\,A'[x]^2 + 4\,(-3-n+n^2)\,x^3\,A'[x]^4 + \\
& \quad 12\,x^2\,A''[x]^2 - 2\,x\,A'[x]\,((-7+4\,n)\,A''[x] + 2\,x\,A^{(3)}[x])) + \\
& A[x]^8\,((2+3\,n+n^2)\,(\omega^2+\omega\,\Omega+\Omega^2) - 39\,(2+3\,n+n^2)\,x^3\,\omega\,\Omega\,A'[x]^2 - 12\,x^5\,\omega\,\Omega\,A''[x]^2 + \\
& \quad 2\,x^4\,\omega\,\Omega\,A'[x]\,(3\,(3+4\,n)\,A''[x] + 2\,x\,A^{(3)}[x])) + \\
& x^2\,A[x]^6\,(3\,(6+5\,n+n^2)\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]^2 - 12\,(5+5\,n+n^2)\,x^3\,\omega\,\Omega\,A'[x]^4 + \\
& \quad 12\,x^2\,(\omega^2+\omega\,\Omega+\Omega^2)\,A''[x]^2 - 2\,x\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]\,((1+8\,n)\,A''[x] + 2\,x\,A^{(3)}[x]))\big) \,/ \\
& \big(A[x]^2\,(-1+x\,A[x]^2)\,(-\Omega+x\,\omega\,A[x]^2)\,(\omega-x\,\Omega\,A[x]^2)\,(A[x]+2\,x\,A'[x])^2\big);
\end{aligned}
$$

$$
\begin{aligned}
r = {} & -\big(n\,((2+3\,n+n^2)\,\omega\,\Omega\,A[x]^{11} + 2\,(37+21\,n+2\,n^2)\,x\,\omega\,\Omega\,A[x]\,A'[x]^4 + 4\,(8+6\,n+n^2)\,x^2\,\omega\,\Omega\,A'[x]^5 + A[x]^2 \\
& \quad A'[x]^2\,((56+15\,n+n^2)\,\omega\,\Omega\,A'[x] + 4\,n\,(2+n)\,x^3\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]^3 - 6\,(1+n)\,x\,\omega\,\Omega\,A''[x]) + \\
& \quad A[x]^4\,((-8+21\,n+5\,n^2)\,x\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]^3 + 4\,n\,(2+n)\,x^4\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]^5 + \\
& \qquad 2\,(-1+n)\,x^2\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]^2\,A''[x] + \omega\,\Omega\,A^{(3)}[x]) + \\
& \quad A[x]^6\,((8+15\,n+13\,n^2)\,x^2\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]^3 + 4\,(8+6\,n+n^2)\,x^5\,\omega\,\Omega\,A'[x]^5 + \\
& \qquad 2\,(5+n)\,x^3\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]^2\,A''[x] - (\omega^2+\omega\,\Omega+\Omega^2)\,((2+n)\,A''[x] + x\,A^{(3)}[x])) + \\
& \quad A[x]^8\,((2+3\,n+n^2)\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x] + (104+117\,n+25\,n^2)\,x^3\,\omega\,\Omega\,A'[x]^3 - \\
& \qquad 6\,(3+n)\,x^4\,\omega\,\Omega\,A'[x]^2\,A''[x] + x\,(\omega^2+\omega\,\Omega+\Omega^2)\,(2\,(2+n)\,A''[x] + x\,A^{(3)}[x])) + \\
& \quad x\,\omega\,\Omega\,A[x]^{10}\,(7\,(2+3\,n+n^2)\,A'[x] - x\,(3\,(2+n)\,A''[x] + x\,A^{(3)}[x])) + \\
& \quad (\omega^2+\omega\,\Omega+\Omega^2)\,A[x]^5\,((10+7\,n+n^2)\,A'[x]^2 + 6\,(-1+3\,n+2\,n^2)\,x^3\,A'[x]^4 + \\
& \qquad 6\,x^2\,A''[x]^2 - x\,A'[x]\,((-13+n)\,A''[x] + 2\,x\,A^{(3)}[x])) + \\
& \quad x\,A[x]^7\,(2\,(-2+5\,n+3\,n^2)\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]^2 + 2\,(55+45\,n+8\,n^2)\,x^3\,\omega\,\Omega\,A'[x]^4 - \\
& \qquad 6\,x^2\,(\omega^2+\omega\,\Omega+\Omega^2)\,A''[x]^2 + x\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]\,(5\,(-1+n)\,A''[x] + 2\,x\,A^{(3)}[x])) + \\
& \quad A[x]^3\,(2\,(-1+9\,n+4\,n^2)\,x^2\,(\omega^2+\omega\,\Omega+\Omega^2)\,A'[x]^4 - 6\,x\,\omega\,\Omega\,A''[x]^2 + \\
& \qquad \omega\,\Omega\,A'[x]\,(-3\,(7+n)\,A''[x] + 2\,x\,A^{(3)}[x])) + \\
& \quad x^2\,\omega\,\Omega\,A[x]^9\,((62+69\,n+19\,n^2)\,A'[x]^2 + 6\,x^2\,A''[x]^2 - x\,A'[x]\,((3+9\,n)\,A''[x] + 2\,x\,A^{(3)}[x])))\big) \,/ \\
& \big(A[x]^3\,(-1+x\,A[x]^2)\,(-\Omega+x\,\omega\,A[x]^2)\,(\omega-x\,\Omega\,A[x]^2)\,(A[x]+2\,x\,A'[x])^2\big);
\end{aligned}
$$

- **Final simplifications**

We have gone now as far as we can go with the assignment of generic meanings to A[x], ω and Ω. We look now to the simplifications that result when specific values—appropriate to the problem at hand—are assigned to those expressions:

```
A[x_] := 2^(1/3) / (-1 + Sqrt[1 + 4 x^3])^(1/3);
```

$$\texttt{A[x\_]} := \frac{2^{1/3}}{\left(-1+\sqrt{1+4x^3}\right)^{1/3}};$$

$$\omega = \left(\frac{1+i\sqrt{3}}{2}\right);$$

$$\Omega = \left(\frac{1-i\sqrt{3}}{2}\right);$$

```
p // FullSimplify
```

$$\frac{18x^2}{1+4x^3}$$

```
q // FullSimplify
```

$$\frac{(10-3n(1+n))x}{1+4x^3}$$

```
r // FullSimplify
```

$$-\frac{n^2(3+n)}{1+4x^3}$$

NOTE: Each of those simplifications took *Mathematica* not not more than a couple of seconds.

We are brought thus to the conclusion that the Mayer polynomials $M_n[x]$ are solutions of

$$f^{(3)} + \frac{18x^2}{1+4x^3} f'' + \frac{(10-3n(1+n))x}{1+4x^3} f' - \frac{n^2(3+n)}{1+4x^3} f = 0$$

or

$$(1+4x^3) f^{(3)} + 18x^2 f'' + (10-3n-n^2)\, x f - n^2(3+n) f = 0$$

which presents us with PRECISELY THE 3rd ORDER DIFFERENTIAL OPERATOR WE SOUGHT:

```
SebbarDE[f_, n_] :=
  (4 x^3 + 1) D[f, {x, 3}] + 18 x^2 D[f, {x, 2}] - (3 n^2 + 3 n - 10) x D[f, {x, 1}] - n^2 (n + 3) f
```

■ Concluding comments

Much hinged on the simplicity of the roots produced by the command

```
Solve[1 - 3 x h - h^3 == 0, h]
```

Had the h^3 been replaced by a higher power of h we would have been stuck at the outset, though we might expect already-factored expressions

$(\alpha - h)\,(\beta - h)\,(\gamma - h) \cdots (\lambda - h)$

to lead, for suitably constructed functions $\alpha, \beta, \gamma, \ldots, \lambda$, to tractable results.

The Log in

$$\text{Log}\left[1 - 3\,x\,h - h^3\right]$$

served also in an important way to lend tractability to the argument. Sebbar has interest also in the polynomials generated by

$$\sqrt{1 - 3\,x\,h - h^3}$$

It is not obvious that Ray's argumnent can be so modified as to produce the DE satisfied by such polynomials, though according to Sebbar it differs not much from the DE obtained above.

It was the h^3 that resulted finally in a 3rd-order DE. Ray's argument suggests that polynomials with generating functions of the form

Log[polynomial of degree μ in h]

satisfy DEs of degree μ, but pushing the argument through in cases $\mu > 3$ (almost certainly in cases $\mu > 4$) would appear to present great difficulties.