Extracting Differential Equations

from the

Generators of Polynomials

Nicholas Wheeler
November 2017

Introduction. The construction

$$\log(1 - 3xh - h^3) = \sum_{n=0}^{\infty} S_n(x)h^n$$

produces a population of polynomials

$$\begin{aligned}
S_0(x) &= 0\\
S_1(x) &= -3x\\
S_2(x) &= -\tfrac{9}{2}x^2\\
S_3(x) &= -1 - 9x^3\\
S_4(x) &= -3x - \tfrac{81}{4}x^4\\
S_5(x) &= -9x^2 - \tfrac{243}{5}x^5\\
S_6(x) &= -\tfrac{1}{2} - 27x^3 - \tfrac{243}{2}x^6\\
S_7(x) &= -3x - 81x^4 - \tfrac{2187}{7}x^7\\
S_8(x) &= -\tfrac{27}{2}x^2 - 243x^5 - \tfrac{6561}{8}x^8\\
S_9(x) &= -\tfrac{1}{3} - 54x^3 - 729x^6 - 2187x^9\\
&\ \vdots
\end{aligned}$$

which are among a quartet of polynomial systems of special interest to Ahmed Sebbar.[1] Sebbar has remarked—without argument, but as is (with the assistance of *Mathematica*) easily confirmed, and as Ray Mayer has by an ingenious argument quite recently deduced—that the $S_n(x)$ are solutions of

[1] Private communications 2014 and 2017.

the 3rd order linear differential equation

$$(4x^3+1)f''' + 18x^2 f'' - (3n^2+3n-10)\,xf' - n^2(n+3)f = 0$$

Mayer's argument, however, draws critically upon features special to the problem at hand, specifically that the generating function (i) is the logarithm of (ii) a cubic, that can be factored: $1-3xh-h^3 = (\alpha-h)(\beta-h)(\gamma-h)$ which by $\alpha(x)\beta(x)\gamma(x)=1$ permits one to write

$$\begin{aligned}\log(1-3xh-h^3) &= \log(1-h/\alpha)+\log(1-h/\beta)+\log(1-h/\gamma)\\ &= \sum_{n=1}^\infty -\frac{1}{n}\Big(\frac{1}{\alpha^n}+\frac{1}{\beta^n}+\frac{1}{\gamma^n}\Big)h^n\end{aligned}$$

with the remarkable implication that

$$S_n(x) = -\frac{1}{n}\Big(\frac{1}{\alpha^n}+\frac{1}{\beta^n}+\frac{1}{\gamma^n}\Big)$$

Mayer's argument leads, moreover, to expressions that are much too enormous[2] to be managed without computer assistance (though the final simplifications, which also require computer assisstance, are dramatic).

Chapter 22 of Abramowitz & Stegun's *Handbook of Mathematical Functions* (1964) provides generating functions (§22.9) and differential equations (§22.6) for all the classic orthogonal polynomials. Those generating functions (with the exception only of $\log(1-2xh+h^2)$, which generates Chebyshev polynomials of the 1st kind) possess none of the special features of which Mayer made use, and the associated differential equations are classical, originally obtained by hand, in days before computer assistance was a possibility.

I have yet to discover in the literature an account of how people "standardly" proceed

generating function ⟶ associated differential equations

My objective here is to describe my own home-grown procedure. We will look first to some representative orthogonal polynomials, then to the more general (non-orthogonal?) of interest to Sebbar. My method employs *Mathematica* as an initially inessential convenience, and I will be quoting details taken from a companion notebook.[3]

I work from the elementary observation that if $G_0(x,h)$ generates polynomials $P_n(x)$

$$G_0(x,h) = \sum_{n=0}^\infty P_n(x)h^n$$

then

$$G_k(x,h) = (\tfrac{d}{dx})^k\, G_0(x,h) \quad : \quad k=1,2,3,\ldots$$

generates the k^{th} derivatives of those polynomials. My method gains essential

[2] See "Ray Mayer's reconstruction of Ahmed Sebbar's DE" (3 November 2017).

[3] "Polynomial DE Worksheet 1" (November 2017).

leverage from the observation (Abramowitz & Stegun, §22.6) that the differential equations satisfied by orthogonal polynomials are in every instance of 2^{nd} order and linear,[4] on which basis we expect to have

$$a\,G_2(x,h)+b\,G_1(x,h)+c\,G_0(x,h)=0$$

We notice also that in every case (i) a and b are n-independent functions of x, and (ii) c is an x-independent function of n. I promote (i) to the status of a working **hypothesis** (one of which Mayer had no need).

Methodological laboratory: the Legendre polynomials. The function

$$G_0(x,h)=\frac{1}{\sqrt{1-2xh+h^2}}$$

generates the Legendre polynomials

$$\begin{aligned}
P_0(x)&=1\\
P_1(x)&=x\\
P_2(x)&=\tfrac{1}{2}(-1+3x^2)\\
P_3(x)&=\tfrac{1}{2}(-3x+5x^3)\\
P_4(x)&=\tfrac{1}{8}(3-30x^2+35x^4)\\
P_5(x)&=\tfrac{1}{8}(15x-70x^3+63x^5)\\
P_6(x)&=\tfrac{1}{16}(-5+10x^2-315x^4)\\
P_7(x)&=\tfrac{1}{16}(-35x+315x^3-693x^5+429x^7)\\
&\;\;\vdots
\end{aligned}$$

Writing

$$a\,G_2(x,h)+b\,G_1(x,h)+c\,G_0(x,h)=\sum_{n=0}^{\infty}Z_n(x;a,b,c_n)h^n$$

with

$$G_1(x,h)=\frac{h}{(1-2xh+h^2)^{\frac{3}{2}}},\qquad G_2(x,h)=\frac{3h^2}{(1-2xh+h^2)^{\frac{3}{2}}}$$

we with computational assistance obtain

$$\begin{aligned}
Z_0&=c_0\\
Z_1&=b+c_1x\\
Z_2&=\tfrac{1}{2}(6a-c_2+6bx+3c_2x^2)\\
Z_3&=\tfrac{1}{2}(-3b+30ax-3c_3x+15bx^2+5c_3x^3)\\
Z_4&=\tfrac{1}{8}(-60a+3c_4-60bx+420ax^2-30c_4x^2)+140bx^3+35c_4x^4)\\
Z_5&=\tfrac{1}{8}(15b-420ax+15c_5x-210bx^2+1260ax^3-70c_5x^3+315bx^4+63c_5x^5)\\
&\;\;\vdots
\end{aligned}$$

[4] The literature must provide an account of why this is necessarily so.

and proceed to solve serially the equations $Z_n = 0$. From $Z_0 = 0$ and $Z_1 = 0$ we have

$$c_0 = 0 \quad \text{and} \quad b = -c_1 x$$

Substitute the latter into Z_2 and obtain $Z_2 = \frac{1}{6}(6a - c_1x^2 - c_2 + 3c_2x^2) = 0$, giving

$$a = \tfrac{1}{6}(6c_1x^2 + c_2 - 3c_2x^2)$$

Substitute those values of a and b into Z_3 and obtain

$$Z_3 = \tfrac{1}{2}(3c_1 + 5c_2 - 3c_3)x + \tfrac{5}{2}(3c_1 - 3c_2 + c_3)x^2$$

Set the coefficients of x and x^2 both equal to 0, solve for $\{c_2, c_3\}$ and get

$$c_2 = 3c_1, \quad c_3 = 6c_1$$

Return with the former to the preceding description of a and get[5]

$$\begin{aligned} a &= \tfrac{1}{2}(1 - x^2)c_1 \\ b &= -xc_1 \end{aligned}$$

Introduce those expressions into Z_4 and get

$$Z_4 = \big(-\tfrac{15}{4}c_1 + \tfrac{3}{8}c_4\big) + \big(\tfrac{75}{2}c_1 - \tfrac{15}{4}c_4\big)x^2 + \big(-\tfrac{175}{4}c_1 + \tfrac{35}{8}c_4\big)x^4$$

in which the requirement that all coefficients vanish gives

$$c_4 = 10c_1$$

Z_5 leads similarly to

$$c_5 = 15c_1$$

So we have

$$\tfrac{1}{2}c_1\big\{(1 - x^2)P_n'' - 2xP_n' + c_nP_n = 0\big\} \quad \text{with} \quad \begin{cases} c_0 = 0 \\ c_1 = 2 \\ c_2 = 6 \\ c_3 = 12 \\ c_4 = 20 \\ c_5 = 30 \end{cases}$$

If we accept the conjecture[6] that $c_n = n(n+1)$ then

$$(1 - x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$$

which is precisely the familiar Legendre differential equation, of which the

[5] This suspended evaluation of a appears to be characteristic of the method.
[6] It would be nice to have an inductive proof.

general solution can be written $\alpha P_n(x) + \beta Q_n(x)$. Here the $Q_n(x)$ are (non-polynomial) "Legendre functions of the 2nd kind," the first few of which are

$$\begin{aligned} Q_0(x) &= P_0(x)\,\text{arctanh}(x)\\ Q_1(x) &= P_1(x)\,\text{arctanh}(x) - 1\\ Q_2(x) &= P_2(x)\,\text{arctanh}(x) - \tfrac{3}{2}x\\ Q_3(x) &= P_3(x)\,\text{arctanh}(x) - \big(\tfrac{15}{6}x^2 - \tfrac{2}{3}\big) \end{aligned}$$

Discussion of the bivariate Legendre functions $P_\nu(x)$, $Q_\nu(x)$ and trivariate Legendre functions $P_\nu^{(\mu)}(x)$, $Q_\nu^{(\mu)}(x)$ can be found in Chapter 59 of Spanier & Oldham's *Atlas of Functions* (1987).

Chebyshev polynomials of the first kind. The function

$$G_0(x,h) = \frac{1 - xh}{1 - 2xh + h^2} = \sum_{n=0}^{\infty} T_n(x)h^n$$

generates Chebyshev polynomials of the 1st kind:

$$\begin{aligned} T_0(x) &= 1\\ T_1(x) &= x\\ T_2(x) &= -1 + 2x^2\\ T_3(x) &= -3x + 4x^3\\ T_4(x) &= 1 - 8x^2 + 8x^4\\ T_5(x) &= 5x - 20x^3 + 16x^5\\ T_6(x) &= -1 + 18x^2 - 48x^4 + 32x^6\\ &\;\;\vdots \end{aligned}$$

Proceeding as before, we write

$$a\,G_2(x,h) + b\,G_1(x,h) + c\,G_0(x,h) = \sum_{n=0}^{\infty} Z_n(x;a,b,c_n)h^n$$

and by assisted computation obtain

$$\begin{aligned} Z_0 &= c_0\\ Z_1 &= b + c_1x\\ Z_2 &= 4a - c_2 + 4bx + 2c_2x^2\\ Z_3 &= -3b + 24ax - 3c_3x + 12bx^2 + 4c_3x^3\\ Z_4 &= -16a + c_4 - 16bx + 96ax^2 - 8c_4x^2 + 32bx^3 + 8c_4x^4\\ Z_5 &= 5b - 120ax + 5c_5x - 60bx^2 + 320ax^3 - 20c_5c^3 + 80bx^4 + 16c_4x^5 \end{aligned}$$

Proceeding again serially to the solution of the equations $Z_n = 0$, we find

$$c_0 = 0 \quad \text{and} \quad b = -c_1 x$$

$$\therefore Z_2 = 4a - 4c_1x^2 - c_2 + 2c_2x^2 \Rightarrow a = \tfrac{1}{4}(4c_1x^2 + c_2 - 2c_2x^2)$$

$$\therefore Z_3 = (3c_1 + 6c_2 - 3c_3)x + (12c_1 - 12c_2 + 4c_3)x^3 \Rightarrow \begin{cases} c_2 = 4c_1 \\ c_3 = 9c_1 \end{cases}$$

$$\therefore a = (1 - x^2)c_1 \quad : \quad \text{Note again the suspended evaluation of } a.$$

$$\therefore Z_4 = (c_4 - 16c_1) - (8c_4 - 128c_1)x^2 + (8c_4 - 128c_1)x^4 \Rightarrow c_4 = 16c_1$$

$$\therefore Z_5 = (5c_5 - 125c_1)x - (20c_5 + 500c_1)x^3 + (16c_5 - 400c_1)x^5 \Rightarrow c_5 = 25c_1$$

These results make plausible the conjecture that $c_n = n^2c_1$. Exercising our option to set $c_1 = 1$, we find that the Chebyshev polynomials $T_n(x)$ satisfy "Chebyshev's differential equation"

$$(1 - x^2)T_n'' - xT_n' + n^2T_n = 0$$

of which the general solution[7] is of the form

$$\begin{aligned} \alpha T_n(x) + \beta\sqrt{1-x^2}\,U_{n-1}(x) \quad &: \quad n = 1, 2, 3, \ldots \\ \alpha + \beta \arcsin(x) \quad &: \quad n = 0 \end{aligned}$$

where the $U_n(x)$ are Chebyshev polynomials of the 2nd kind, generated by $(1-2xh+h^2)^{-1} = \sum_{n=0}^\infty U_n(x)h^n$. Reminiscent of a result mentioned on page 2 is the fact[7] that one can describe the polynomials $T_n(x)$ by expressions

$$T_n(x) = \tfrac{1}{2}(\alpha^n + \beta^n) \quad : \quad \begin{cases} \alpha(x) = x + \sqrt{x^2 - 1} \\ \beta(x) = x - \sqrt{x^2 - 1} \end{cases}$$

that on their face do not *look* much like polynomials. This result is made somewhat less mysterious by the observations[8] that the polynomials $\mathcal{T}_n(x)$ defined

$$\log(1 - 2xh + h^2) = \sum_{n=0}^\infty \mathcal{T}_n(x)h^n$$

are in fact also solutions of Chebyshev's equation, and that

$$1 - 2xh + h^2 = (\alpha - h)(\beta - h) = \Big(1 - \frac{h}{\alpha}\Big)\Big(1 - \frac{h}{\beta}\Big) \text{ by } \alpha\beta = 1$$

entails

$$\begin{aligned} \sum_{n=0}^\infty \mathcal{T}_n(x)h^n &= \log\Big(1 - \frac{h}{\alpha}\Big) + \log\Big(1 - \frac{h}{\beta}\Big) \\ &= \sum_{n=0}^\infty -\frac{1}{n}\Big(\frac{1}{\alpha^n} + \frac{1}{\beta^n}\Big)h^n \end{aligned}$$

---

7 See Spanier & Oldham, page 196.

8 See "Ray Mayer's reconstruction of Ahmed Sebbar's DE," page 7.

We expect $\mathcal{T}_n(x)$ to be a linear combination of $T_n(x)$ and $\sqrt{1-x^2}\,U_{n-1}(x)$, and some *Mathematica*-assisted experimentation indicates that in fact

$$\mathcal{T}_n(x) = -\tfrac{2}{n}T_n(x)$$

This amounts to the statement that

$$\frac{1}{\alpha^n} + \frac{1}{\beta^n} = \alpha^n + \beta^n$$

which follows immediately from the circumstance that $\alpha\beta = 1$.

Hermite polynomials. The Hermite polynomials $H_n(x)$ are generated

$$e^{2xh-h^2} = \sum_{n=0}^{\infty} \tfrac{1}{n!}H_n(x)h^n$$

$$\begin{aligned}
H_0(x) &= 1\\
H_1(x) &= 2x\\
H_2(x) &= -2+4x^2\\
H_3(x) &= -12x+8x^3\\
H_4(x) &= 12-48x^2+16x^4\\
H_5(x) &= 120x-160x^3+32x^5\\
H_6(x) &= -120+720x^2-480x^4+64x^6\\
&\ \ \vdots
\end{aligned}$$

The presence of the factorial requires that we modify our procedure slightly; instead of working from the generaating function we work directly from the polynomials, writing

$$Z_n = aH_n'' + bH_n' + c_nH_n$$

This gives

$$\begin{aligned}
Z_0 &= c_0\\
Z_1 &= 2b+2c_1x\\
Z_2 &= 8a+8bx+c_2(-2+4x^2)\\
Z_3 &= 48ax+b(-12+24x^2)+c_3(-12x+8x^3)\\
Z_4 &= a(-96+192x^2)+b(-96x+64x^3)+c_4(12-48x^2+16x^4)
\end{aligned}$$

From $Z_n = 0$ we obtain

$$c_0 = 0 \quad\text{and}\quad b = -c_1x$$

$$\therefore Z_2 = (8a-2c_2)+(-8c_1+4c_2)x^2 \Rightarrow \begin{cases} c_2 = 2c_1\\ a = \frac{1}{4}c_2 = \frac{1}{2}c_1\end{cases}$$

$$\therefore Z_3 = (36c_1-12c_3)x+(-24c_1+8c_3)x^2 \Rightarrow c_3 = 3c_1$$

$$\therefore Z_4 = (-48c_1+12c_4)+(192c_1-48c_4)x^2+(-64c_1+16c_4)x^4 \Rightarrow c_4 = 4c_1$$

$$\therefore Z_5 = (-600c_1+120c_5)x+(800c_1-160c_5)x^3+(-160c_1+32c_5)x^5 \Rightarrow c_5 = 5c_1$$

We are led thus to Hermite's differential equation

$$H_n'' - 2xH_n' + 2nH_n = 0$$

of which the general solution is

$$\alpha\, H_n(x) + \beta\, \text{Hypergeometric1F1}\big(-\tfrac{n}{2}, \tfrac{1}{2}, x^2\big)$$

Sebbar polynomials of the first kind. The method illustrated above could be used to discover—by hand, with pen and paper—the differential equations satisfied by all of the classic orthogonal polynomials. We look now to the polynomials $S_n(x)$ described on page 1. It will emerge that—though the work could, in principle, still be done by hand—computer-assisted management of the details is almost indispensable.

Since no $\frac{1}{n!}$-factor enters into their construction, we could work either from their generator $G_0(x,h) = \log(1 - 3xh - h^3)$ or directly from the polynomials themselves. I adopt here and henceforth the latter option because it was in those terms that I first approached the problem of reproducing Sebbar's DEs, and was led to the conclusion that the problem is intractable = a problem worthy of the genius of Ray Mayer.

FIRST APPROACH: FAILURE

We attempt to employ unchanged the method that worked when we were discussing orthogonal polynomials, which is to say: we work from

$$Z_n = aS_n'' + bS_n' + c_nS_n$$

which gives

$$\begin{aligned}
Z_0 &= 0\\
Z_1 &= -3b - 3xc_1\\
Z_2 &= -9a - 9bx - \tfrac{9}{2}x^2c_2\\
Z_3 &= -54ax - 27bx^2 - (1 + 9x^3)c_3\\
Z_4 &= -243ax^2 - \tfrac{3}{4}b(4 + 108x^3) - \tfrac{3}{4}(4x + 27x^4)c_45
\end{aligned}$$

Solving $Z_1 = Z_2 = 0$ for $\{a, b\}$ we find

$$\begin{aligned}
a &= \tfrac{1}{2}x^2(2c_1 - c_2)\\
b &= -xc_1
\end{aligned}$$

Which when fed into $Z_3 = 0$ gives

$$\begin{aligned}
c_2 &= c_1\\
c_3 &= 0
\end{aligned}$$

Z_4, having digested all those results, reads $Z_4 = 3(c_1 - c_4)x - \frac{81}{4}(2c_1 + c_4)x^4$. From $Z_4 = 0$ we are led thus to a contradiction

$$\begin{aligned}
c_4 &= c_1\\
c_4 &= -2c_1
\end{aligned}$$

from which $c_1 = 0$ provides the only escape. But then $\{a = b = c_n = 0\}$; the theory has collapsed into vacuous triviality.

SECOND APPROACH: FAILURE

To expand the playing field, give us more parameters to play with, we assume that the S_n satisfy differential equations of 3rd order:

$$Z_n = aS_n''' + bS_n'' + c_n S_n' + d_n S_n$$

We retain—for no better reason that it worked before—the assumption that $\{a, b\}$ are n-indepedent. Then

$$\begin{aligned}
Z_0 &= 0\\
Z_1 &= -3c_1 - 3xd_1\\
Z_2 &= -9b - 9xc_2 - \tfrac{9}{2}x^2 d_2\\
Z_3 &= -54a - 54bx - 27x^2 c_3 - (1 + 9x^3)d_3\\
Z_4 &= -486ax - 243bx^2 - (3 + 81x^3)c_4 - (3x + \tfrac{81}{4}x^4)d_4\\
Z_5 &= -2916ax^2 - b(18 + 972x^3) - (18x + 243x^4)c_5 - (9x^2 + \tfrac{243}{5}x^5)d_5
\end{aligned}$$

Proceeding as before,[9] we are led from $Z_5 = 0$ to another contradiction

$$\begin{aligned}
d_5 &= 37d_2\\
d_5 &= 40d_2
\end{aligned}$$

with familiar catastrophic consequences. The seeds of this development are seen to have been sown at $Z_1 = 0$, which entailed $c_1 = d_1 = 0$.

THIRD APPROACH: SUCCESS

We work now from

$$Z_n = aS_n''' + bS_n'' + c_n xS_n' + d_n S_n$$

where the x has been introduced into the coefficient of S_n' to avoid the fatal result just mentioned. We then find

$$\begin{aligned}
Z_0 &= 0\\
Z_1 &= -3x(c_1 + d_1)\\
Z_2 &= -9b - 9x^2(c_2 + \tfrac{1}{2}d_2)\\
Z_3 &= -54a - 54bx - 27x^3 c_3 - (1 + 9x^3)d_3
\end{aligned}$$

From $Z_1 = 0$ we now have

$$c_1 = -d_1$$

$Z_2 = 0$ gives

$$b = -x^2(c_2 + \tfrac{1}{2}d_2)$$

[9] The details are spelled out in "Polynomial DE Worksheet 1."[3]

and when those results are fed into Z_3 and the result solved for a we have

$$a = -\tfrac{1}{54}d_3 + \tfrac{1}{54}x^3(54c_2 - 27c_3 + 27d_2 - 9d_3)$$

Feeding the results now in hand into $Z_n : n = 4, 5, 6, \dots$ (the Z_n that "lie upstream" from Z_3), we find that

$$\begin{aligned}
Z_4 &= xX_{4,1} + x^4X_{4,4}\\
Z_5 &= x^2X_{5,2} + x^5X_{5,5}\\
Z_6 &= X_{6,0} + x^3X_{6,3} + x^6X_{6,6}\\
Z_7 &= xX_{7,1} + x^4X_{7,4} + x^7X_{7,7}\\
Z_8 &= x^2X_{8,2} + x^5X_{8,50} + x^8X_{8,8}\\
Z_9 &= X_{9,0} + x^3X_{9,3} + x^6X_{9,6} + x^9X_{9,9}\\
Z_{10} &= xX_{10,1} + x^4X_{10,4} + x^7X_{10,7} + x^{10}X_{10,10}\\
Z_{11} &= x^2X_{11,2} + x^5X_{11,5} + x^8X_{11,8} + x^{11}X_{11,11}\\
Z_{12} &= X_{12,0} + x^3X_{12,3} + x^6X_{12,6} + x^9X_{12,9} + x^{12}X_{12,12}
\end{aligned}$$

Here $X_{i,j}$ is linear combination of $\{c_i, d_i\}$ and of those parameters $\{c_2, c_3, d_2, d_3\}$ whose suspended evaluation has not yet been accomplished.[10] Note that Z_n $(n > 3)$ is a polynomial of degree n in which all powers differ by 3, and in this respect mimics a conspicuous property of S_n.[11]

$Z_4 = 0$ is seen to provide two conditions, which we solve for $\{c_4, d_4\}$ and feed upstream. The resulting Z_5 provides two conditions which we solve for $\{c_5, d_5\}$ and again feed upstream. The resulting Z_6 provides *three* conditions, from which we obtain evaluations of $\{c_6, d_6\}$ and a suspended evaluation of (say) c_2, which therefore disappears from all upstream X-factors. Continuing with this "iterative exercise in suspended evaluation," we arrive finally at results that can be expressed

$$\begin{pmatrix} c_6\\ c_7\\ c_8\\ c_9\\ c_{10}\end{pmatrix} = \tfrac{1}{27}d_3\begin{pmatrix} 58\\ 79\\ 103\\ 130\\ 160\end{pmatrix}, \qquad \begin{pmatrix} d_6\\ d_7\\ d_8\\ d_9\\ d_{10}\end{pmatrix} = \tfrac{1}{27}d_3\begin{pmatrix} 162\\ 245\\ 352\\ 486\\ 650\end{pmatrix}$$

and which invite this rescaling:

$$\tfrac{1}{27}d_3 \longrightarrow 1$$

[10] The *Mathematica*-assisted construction of $X_{i,j}$ is instantaneous.

[11] The corresponding power interval in the theory of orthogonal polynomials is not 3 but 2; all such polynomials are either even or odd.

We now pursue the conjecture that c_n (similarly d_n, both rescaled) can be developed as a cubic in n:

$$c_n = pn^3 + qn^2 + rn + s$$

To discover suitable values of $\{p, q, r, s\}$ (= test the conjecture) we construct

$$\mathbb{M} = \begin{pmatrix} 6^3 & 6^2 & 6 & 1 \\ 7^3 & 7^2 & 7 & 1 \\ 8^3 & 8^2 & 8 & 1 \\ 9^3 & 9^2 & 9 & 1 \end{pmatrix}$$

and observe that

$$\mathbb{M}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} 58 \\ 79 \\ 103 \\ 130 \end{pmatrix} \implies \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 3/2 \\ 3/2 \\ -5 \end{pmatrix}$$

from which (and by a similar argument) we obtain

$$\begin{aligned} c_n &= \tfrac{3}{2}n^2 + \tfrac{3}{2}n - 5 \\ d_n &= \tfrac{1}{2}n^3 + \tfrac{3}{2}n^2 \end{aligned}$$

Returning with this information to previous descriptions of $\{a, b\}$ we obtain these (similarly rescaled) suspended evaluations:

$$\begin{aligned} a &= -\tfrac{1}{2}(1 + 4x^3) \\ b &= -9x^2 \end{aligned}$$

A final rescaling (multiply all terms by -2) provides

$$Z_n = (1 + 4x^3)S_n''' + 18x^2 S_n'' - (3n^2 + 3n - 10)xS_n' - n^2(n+3)S_n = 0$$

which (see again page 2) is precisely the result we sought to establish—the equation asserted by Sebbar, and established by Mayer by quite other means.

Sebbar polynomials of the second kind. We look now to the polynomials $R_n(x)$ defined by

$$\log(1 + 3xh^2 - h^3) = \sum_{n=0}^{\infty} R_n(x)h^n$$

These, when spelled out,[12] are seen to lack some of the properties we usually associate with generated sets of polynomials: they are transparently not linearly independent, and the degree of R_n, instead of being equal to n, has become an irregular function of n that never exceets $\frac{1}{2}n$. But they will be found to satisfy

[12] For computational details, see "Polynomial DE Worksheet 2" (November 2017).

$$\begin{aligned}
R_0(x) &= 0\\
R_1(x) &= 0\\
R_2(x) &= 3x\\
R_4(x) &= -\tfrac{9}{2}x^2\\
R_5(x) &= 3x\\
R_6(x) &= \tfrac{1}{6}(-3+54x^3)\\
R_7(x) &= -9x^2\\
R_8(x) &= -\tfrac{3}{4}(-4x+27x^4)\\
R_9(x) &= \tfrac{1}{3}(-1+81x^3)\\
R_{10}(x) &= \tfrac{27}{10}(-5x^2+18x^5)
\end{aligned}$$

differential equations that differ only slightly from those satisfied by the Sebbar polynomials $S_n(x)$. The argument is a straightforward variant of the argument rehearsed in the preceeding section, but—because degree ascents so slowly within $\{R_n(x)\}$, which is so "straggle-toothed"—must be carried to much higher order to produce useful results.

We work again from

$$Z_n = aR_n''' + bR_n'' + c_n xR_n' + d_n R_n$$

where by assisted calculation

$$\begin{aligned}
Z_0 &= 0\\
Z_1 &= 0\\
Z_2 &= 3x(c_2+d_2)\\
Z_3 &= -d_3\\
Z_4 &= -9b - 9x^2(c_4+\tfrac{1}{2}d_4)\\
Z_5 &= 3x(c_5+d_5)\\
Z_6 &= 54a + 54bx + 27x^3c_6 + \tfrac{1}{6}(-3+54x^3)d_6
\end{aligned}$$

From $Z_0 = Z_1 = \cdots = Z_6 = 0$ we obtain temporary valuations of c_2, d_3, b, c_5, a which when fed into Z_7 produce $Z_7 = x^2(18c_4 - 18c_7 + 9d_4 - 9d_7)$ whence $c_7 = \frac{1}{2}(2c_4 + d_4 - d_7)$. From this point the iterative process proceeds straightforwardly, though the irregular slow growth of degree $\leqslant$ order n requires that one proceed all the way to Z_{14} before one has acquired the suspended evaluations that permit one to write

$$\begin{pmatrix} c_{14}\\ c_{15}\\ c_{16}\\ c_{17}\end{pmatrix} = \tfrac{1}{54}d_6\begin{pmatrix} -268\\ -310\\ -355\\ -403\end{pmatrix}, \qquad \begin{pmatrix} d_{14}\\ d_{15}\\ d_{16}\\ d_{17}\end{pmatrix} = \tfrac{1}{54}d_6\begin{pmatrix} 1078\\ 1350\\ 1664\\ 2023\end{pmatrix}$$

Proceeding as before, we construct

$$\mathbb{M} = \begin{pmatrix} 14^3 & 14^2 & 14 & 1 \\ 15^3 & 15^2 & 15 & 1 \\ 16^3 & 16^2 & 16 & 1 \\ 17^3 & 17^2 & 17 & 1 \end{pmatrix}$$

and find that

$$\mathbb{M}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} -268 \\ -310 \\ -355 \\ -403 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ -3/2 \\ +3/2 \\ 5 \end{pmatrix}$$

$$\mathbb{M}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} 1078 \\ 1350 \\ 1664 \\ 2023 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} 1/2 \\ -3/2 \\ 0 \\ 0 \end{pmatrix}$$

We conclude that the $\{c, d\}$ parameters—rescaled by

$$\tfrac{1}{54}d_6 \longrightarrow 1$$

—can be described

$$\begin{aligned} c_n &= -\tfrac{3}{2}n^2 + \tfrac{3}{2}n + 5 \\ d_n &= \tfrac{1}{2}n^3 - \tfrac{3}{2}n^2 \end{aligned}$$

in consequence of which the similarly rescaled parameters $\{a, b\}$ acquire the suspended valuations

$$\begin{aligned} a &= \tfrac{1}{2}(1 + 4x^3) \\ b &= 9x^2 \end{aligned}$$

A final rescaling (multiply all terms by +2) provides

$$\begin{aligned} Z_n &= (1 + 4x^3)R_n''' + 18x^2 R_n'' - (3n^2 - 3n - 10)x\,R_n' - n^2(3 - n)R_n = 0 \\ Z_n &= (1 + 4x^3)\,S_n''' + 18x^2\,S_n'' - (3n^2 + 3n - 10)x\,S_n' - n^2(3 + n)\,S_n = 0 \end{aligned}$$

Here I have repeated the corresponding S_n equation (page 11) to make evident the similarity—remarkable in view of the fact that the polynomials themselves are so dissimilar—of those differential equations; each goes over to the other by simply reversing the sign of n.

Sebbar polynomials of the third kind. Comparison of the generators

$$\log(1 - 3xh - h^3) \quad \text{and} \quad \log(1 + 3xh^2 - h^3)$$

of the Sebbar polynomials of the first and second kinds with the generator[13]

$$\log(1 - 2xh + h^2)$$

of the Chebyshev polynomials $\mathcal{T}_n(x)$ establishes a sense in which "the Sebbar polynomials are Chebyshev-like." The Sebbar polynomials of the third and

[13] See again page 6.

fourth kinds

$$(1-3xh-h^3)^{-\nu} \quad \text{and} \quad (1+3xh^2-h^3)^{-\nu}$$

are in that same sense reminiscent of the generators

$$(1-2xh+h^2)^{-\nu}, \quad (1-2xh+h^2)^{-\frac{1}{2}} \quad \text{and} \quad (1-2xh+h^2)^{-1}$$

of the Gegenbauer, Legendre and Chebyshev polynomials of the 2nd kind: $C_n^{(\nu)}(x)$, $P_n(x)$ and $U_n(x)$. Our objective here will be to discover the differential equations satisfied by Sebbar polynomials of the third kind $Q_{n,\nu}(x)$[14]

$$(1-3xh-h^3)^{-\nu} = \sum_{n=0}^{\infty} Q_{n,\nu}(x)h^n$$

of which the first few are[15]

$$\begin{aligned}
Q_{0,\nu}(x) &= 1\\
Q_{1,\nu}(x) &= 3x\nu\\
Q_{2,\nu}(x) &= \tfrac{9}{2}x^2\nu(\nu+1)\\
Q_{3,\nu}(x) &= \nu + \tfrac{9}{2}x^3\nu(\nu+1)(\nu+2)\\
Q_{4,\nu}(x) &= 3x\nu(\nu+1) + \tfrac{27}{8}x^3\nu(\nu+1)(\nu+2)(\nu+3)\\
Q_{5,\nu}(x) &= \tfrac{9}{2}x^2\nu(\nu+1)(\nu+2) + \tfrac{81}{40}x^5\nu(\nu+1)(\nu+2)(\nu+3)(\nu+5)
\end{aligned}$$

Note that we again have degree = order, and that the exponents again advance by multiples of 3.

Working again from

$$Z_n = aQ_n''' + bQ_n'' + c_n xQ_n' + d_nQ_n$$

by the established suspended evaluation iterative procedure,[16] we arrive finally at

$$\frac{27(2+3\nu+\nu^2)}{d_3}\begin{pmatrix} c_8\\ c_9\\ c_{10}\\ c_{11}\end{pmatrix} = \begin{pmatrix} 206\\ 260\\ 320\\ 386\end{pmatrix} + \nu\begin{pmatrix} 27\\ 33\\ 39\\ 45\end{pmatrix} + \nu^2\begin{pmatrix} -9\\ -9\\ -9\\ -9\end{pmatrix}$$

$$\frac{27(2+3\nu+\nu^2)}{d_3}\begin{pmatrix} d_8\\ d_9\\ d_{10}\\ d_{11}\end{pmatrix} = \begin{pmatrix} 704\\ 972\\ 1300\\ 1694\end{pmatrix} + \nu\begin{pmatrix} 456\\ 567\\ 690\\ 825\end{pmatrix} + \nu^2\begin{pmatrix} 72\\ 81\\ 90\\ 99\end{pmatrix}$$

---

[14] One would honor an ancient convention by writing $Q_n^{(\nu)}(x)$. It is for typographic convenience that I write $Q_{n,\nu}(x)$, and usually will omit explicit reference to the ν-parameter.

[15] Here Pochhammer's notation $(\nu)_p = \nu(\nu+1)(\nu+2)\cdots(\nu+p-1)$ would be of use.

[16] For computational details, see "Polynomial DE Worksheet 3" (November 2017).

We rescale by setting

$$\frac{27(2+3\nu+\nu^2)}{d_3}\longrightarrow 1$$

and with the aid of

$$\mathbb{M}=\begin{pmatrix} 8^3 & 8^2 & 8 & 1\\ 9^3 & 9^2 & 9 & 1\\ 10^3 & 10^2 & 10 & 1\\ 11^3 & 11^2 & 11 & 1\end{pmatrix}$$

obtain

$$\begin{aligned} c_n &= (3n^2+3n-10)+(6n-21)\,\nu-9\,\nu^2\\ d_n &= n^2(3+n)+n(9+6n)\,\nu+9n\,\nu^2\end{aligned}$$

by virtue of which the (similarly rescaled) suspended valuations of $\{a,b\}$ become

$$\begin{aligned} a &= -(1+4x^3)\\ b &= -18x^2(1+\tfrac{2}{3}\nu)\end{aligned}$$

A final sign reversal gives

$$\begin{aligned} Z_n = (1+4x^3)\,Q_n''' + 18x^2(1+\tfrac{2}{3}\nu)\,Q_n''&\\ -[(3n^2+3n-10)+(6n-21)\,\nu-9\,\nu^2]x\,Q_n'&\\ -[n^2(3+n)+n(9+6n)\,\nu+9n\,\nu^2]Q_n = 0&\end{aligned}$$

which agree precisely with the differential equations obtained (somehow!) by Ahmed Sebbar. Remarkably, we at $\nu=0$ recover the previously-encountered equations

$$Z_n = (1+4x^3)\,S_n''' + 18x^2\,S_n'' - (3n^2+3n-10)x\,S_n' - n^2(3+n)\,S_n = 0$$

—this even though the "polynomials" $Q_{n,0}(x) : n>0$ all vanish identically.

Sebbar polynomials of the fourth kind. From

$$(1+3xh^2-h^3)^{-\nu} = \sum_{n=0}^{\infty} P_{n,\nu}(x)h^n$$

one is led to polynomials

$$\begin{aligned} P_{0,\nu}(x) &= 1 & &\\ P_{1,\nu}(x) &= 0 & P_{6,\nu}(x) &= \tfrac{1}{2}(\nu)_2+\tfrac{9}{2}x^3(\nu)_3\\ P_{2,\nu}(x) &= -3x(\nu)_1 & P_{7,\nu}(x) &= \tfrac{9}{2}x^2(\nu)_3\\ P_{3,\nu}(x) &= \nu & P_{8,\nu}(x) &= \tfrac{3}{2}x(\nu)_3+\tfrac{27}{8}x^4(\nu)_4\\ P_{4,\nu}(x) &= \tfrac{9}{2}x^2(\nu)_2 & P_{9,\nu}(x) &= \tfrac{1}{6}x^2(\nu)_3-\tfrac{9}{2}x^3(\nu)_4\\ P_{5,\nu}(x) &= -3x(\nu)_2 & P_{10,\nu}(x) &= \tfrac{9}{4}x^2(\nu)_4-\tfrac{81}{40}x^5(\nu)_5\end{aligned}$$

in which the degree = order property has been lost, as has linear independence; in those respects they stand to the Q polynomials as the R polynomials stand to the S polynomials. The P-population is, like the R-population, unattractively "straggle-toothed," though there is evidence in the preceding short list of the onset of x-exponents advancing by multiples of 3. Nevertheless, Sebbar has observed that—demonstrably—the P polynomials satisfy differential equations quite similar to those satisfied by the Q polynomials. I am satisfied that the suspended evaluation iterative process described above would lead to Sebbar's P-equations, but for purposes of comparison am content simply to borrow from his result. We have

$$\begin{aligned}
\text{for the } P \text{ equations } c_n &= -(3n^2+3n-10)+21\nu-\ \ 6n\nu+9\nu^2\\
\text{for the } Q \text{ equations } c_n &= -(3n^2-3n-10)+30\nu-12n\nu\\
\text{for the } P \text{ equations } d_n &= -n^2(3+n)-9n\nu-6n^2\nu-9n\nu^2\\
\text{for the } Q \text{ equations } d_n &= -n^2(3-n)-9n\nu+3n^2\nu
\end{aligned}$$

where again the ν-independent terms exchange places when the sign of n is reversed, but the ν-depemdemt terms are quite distinct. Remarkably, the ν-independent terms encountered in the P and Q-equations are identical to those encountered in the S and R-equations, respectively. Moreover, a previously remarked property of the P-equations pertains also to the Q-equations:

$$\left.\begin{aligned}
P\text{-equations} &\longrightarrow S\text{-equations}\\
Q\text{-equations} &\longrightarrow R\text{-equations}
\end{aligned}\right\}\text{ in the formal limit } \nu\longrightarrow 0$$

So far as concerns the relationship Sebbar's generating functions, which are of the forms $(expression)^{-\nu}$ and $\log(expression)$, we note that

$$\begin{aligned}
-\nu\int\frac{1}{z^{1+\nu}}dz &= z^{-\nu}\\
\lim_{\nu\to 0}\Big\{-\nu\int\frac{1}{z^{1+\nu}}dz\Big\} &= 1\\
\lim_{\nu\to 0}\Big\{-\int\frac{1}{z^{1+\nu}}dz\Big\} &= \infty\\
\int\frac{1}{z^{1+0}}dz &= \log(z)
\end{aligned}$$

provide a hint of what may be the root of their formal kinship.