Novel approach to the
FRACTIONAL CALCULUS ‡

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Introduction. To describe wave motion on a flexible string or rod one writes the one-dimensional wave equation

$$\partial_t^2 \varphi = +b\,\partial_x^2 \varphi$$

but for a *stiff* rod one has (for complicated rheological reasons, and after certain simplifications)

$$\partial_t^2 \varphi = -\beta\,\partial_x^4 \varphi \tag{1}$$

Similarly, the diffusion equation reads

$$\partial_t \varphi = +a\,\partial_x^2 \varphi$$

but in early 1997 Richard Crandall encountered algorithmic need—the precise connection was never explained to me—of the *biharmonic diffusion equation*

$$\partial_t \varphi = -\alpha\,\partial_x^4 \varphi \tag{2}$$

A question that arises naturally in both cases (and issues from the lips as a *physical* question) is "Why the ∂_x^4?" But the question to which Richard directed my specific attention was "Why the minus sign?"[1] It was my response to Richard's question (which happened to coincide with a freshly-acquired speaking obligation) that set in motion the train of thought that produced the material reported a few weeks later at my "fractional calculus seminar."[2]

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‡ Notes for a Reed College Physics Seminar presented 9 March 2005.

[1] It turns out that the two questions are—not at all surprisingly—intertwined: my attempt to illuminate the latter cast pale light also on the former.

[2] "Construction & physical applications of the fractional calculus," Reed College Physics Seminar presented 5 March 1997. It embarrasses me to report that, of all the things I have written, this is the item for which reprints have been, and continue to be, most frequently requested.

In several dimensions the diffusion equation reads

$$\partial_t \varphi = a\nabla^2\varphi$$

and by mid-1998 Richard's work had led him to wonder about the meaning—if any—that (especially in the 2-dimensional case) might sensibly be assigned to the "fractional Laplacian" that enters into the *fractional diffusion equation*

$$\partial_t \varphi = a\nabla^p\varphi \quad : \quad p \neq 2 \tag{3}$$

This is a question that had been stewing also in the back of my own mind, so in November of 1998 I undertook to explore the issue.[3]

In §2 of those research notes, which is entitled "A novel approach to the fractional calculus," my sole intent was to secure the foundations of, and to explain in the simplest possible terms, the essence of one of my projected lines of attack on the ∇^p problem. Only later did I come to realize that that work was of some independent significance, and that it addressed (among others, and in its modest way) a problem that has long bedeviled the fractional calculus. It is to the substance of that §2 that I today restrict my attention.

1. Short history of the fractional calculus. It is elementary that the operations of differentiation and of indefinite integration

$$\left.\begin{aligned} D \;&:\quad f(x) \longmapsto \frac{df(x)}{dx} \\ {}_aD_x^{-1} \;&:\quad f(x) \longmapsto \int_a^x f(y)\,dy \end{aligned}\right\} \tag{4}$$

can be iterated. And it was notationally obvious to Leibniz (if not to Newton) that iterated differentiation[4] obeys the **law of exponents**:

$$D^m D^n = D^{m+n} \quad : \quad m,n = 0,1,2,\ldots \tag{5}$$

We are therefore not surprised to find Leibniz, in September of 1695, writing to his friend l'Hospital as follows:[5]

> *"Jean Bernoulli seems to have told you of my having mentioned to him a marvelous analogy which makes it possible to say in a way that successive differentials are in geometric progression.*

[3] That work is reported in "'Laplacian operators' of eccentric order" (1998), which runs to some 84 pages. In §1 of those notes, under the head "The sign problem in one dimension," I record my response to the first of Richard's questions.

[4] I find it convenient to suspend for the moment reference to the parallel propeties of the relatively more complicated operator ${}_aD_x^{-1}$.

[5] My source here is B. Mandelbrot (*Fractals: Form, Chance, and Dimension* (1977), p. 299), who claims responsibility for the translation.

> *One can ask what would be a differential having as its exponent a fraction. You see that the result can be expressed by an infinite series, although this seems removed from Geometry, which does not yet know of such fractional exponents. It appears that one day these paradoxes will yield useful consequences, since there is hardly a paradox without utility. Thoughts that mattered little in themselves may give occasion to more beautiful ones."*

Thirty-five years later, Euler expressed a similar thought, and took explicit note of the fact that a kind of interpolation theory comes necessarily into play:

> *"Concerning transcendental progressions whose terms cannot be given algebraically: when n is a positive integer, the ratio $d^n f/dx^n$ can always be expressed algebraically. Now it is asked: what kind of ratio can be made if n be a fraction? . . . the matter may be expedited with the help of the interpolation of series, as explained earlier in this dissertation."*[6]

. . . but I do not know the identity of the "dissertation" to which he refers; the notion of a "fractional calculus" is, so far as I am aware, not mentioned in his monumental *Institutiones calculi differentialis* of 1755, and first public mention of the so-called "Euler integrals"

$$\begin{aligned}\Gamma(z) &= \int_0^\infty x^{z-1}e^{-x}\,dx\\ B(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\end{aligned}$$

—which play such a central role in this story—did not appear until publication of *Institutiones calculi integralis* (3 volumes, 1768–1770).

The first substantive step toward the creation of a fractional calculus was taken in 1819 when S. F. Lacroix—quite casually, and with no evident practical intent—remarked that the familiar formula

$$D^m x^p = p(p-1)(p-2)\cdots(p-m+1)x^{p-m}$$

—which we notate $D^m x^p = \frac{p!}{(p-m)!}\,x^{p-m}$ when m is an integer—can in every case be notated

$$D^m x^p = \frac{\Gamma(p+1)}{\Gamma(p-m+1)}\,x^{p-m} \tag{6.1}$$

and that (6) *makes formal sense even when m is not an integer.* The fact that

$$D^m x^p = 0 \quad \text{when } m \text{ and } p \text{ are integers with } m > p$$

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[6] For discussion of some aspects of an interpolative approach to the fractional calculus—an approach which (in my hands at least) does not work very well because it involves interpolation *in the exponent*—see "Extrapolated interpolation theory" (April 1997).

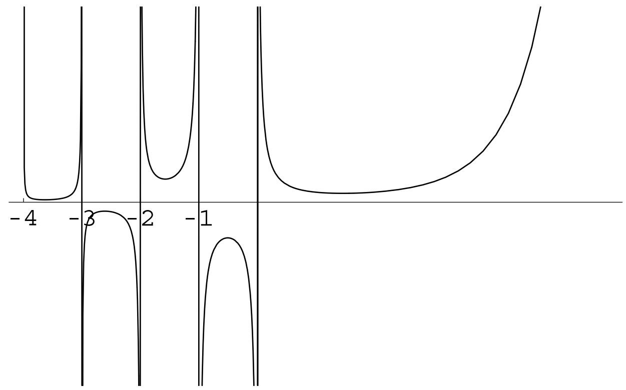

FIGURE 1: *Graph showing the location of the singularies of $\Gamma(x)$.* *By* (1819) *Lacroix was in position to exploit Euler's observation that* $\Gamma(n+1) = n! : n = 0, 1, 2, \ldots$

can then be attributed to the circumstance (see the figure) that $\Gamma(x)$ becomes singular at $x = 0, -1, -2, \ldots$ Proceeding in the other direction, one has

$$\begin{aligned}
D^{-1}x^p &\equiv {}_0D_x^{-1}x^p \equiv \int_0^x y^p\,dy = \tfrac{1}{(p+1)}x^{p+1}\\
D^{-2}x^p &= \tfrac{1}{(p+2)(p+1)}x^{p+2}\\
&\ \ \vdots\\
D^{-n}x^p &= \tfrac{1}{(p+n)\cdots(p+2)(p+1)}x^{p+n} = \tfrac{p!}{(p+n)!}x^{p+n}
\end{aligned}$$

which can in the same spirit be written

$$D^{-n}x^p = \frac{\Gamma(p+1)}{\Gamma(p+n+1)}\,x^{p+n} \tag{6.2}$$

Lacroix found himself in position, therefore, to assign a formally very simple (if computationally intricate) meaning to expressions of the type

$$D^\mu\Big\{\sum_p f_p x^p\Big\} \quad : \quad \mu \text{ any number, real or complex}$$

and to demonstrate that, in consequence of an elementary property of the gamma function, the D operator, thus construed, supports an *unrestricted law of exponents*:

$$D^\mu D^\nu = D^{\mu+\nu} \quad : \quad \text{all real or complex } \mu, \nu \tag{7}$$

Lacroix's construction (which subsumes all of ordinary calculus) survives to this day as a sub-calculus within the full-blown fractional calculus. We note with interest that the interpolative burden of the construction is borne entirely by Euler's Γ function. And that, though it is by entrenched tradition that one speaks of the "fractional calculus," it is not at all necessary that μ, ν be rational.

By the Fundamental Theorem of Calculus

$$D \cdot {}_aD_x^{-1} f(x) \equiv \frac{d}{dx}\int_a^x f(y)\,dy = f(x) \quad : \quad \text{all } f(x), \text{ all } a$$

On the other hand,

$${}_aD_x^{-1} \cdot Df(x) = f(x) - f(a)$$

so

$$\begin{aligned}\big[D \cdot {}_aD_x^{-1} - {}_aD_x^{-1} \cdot D\big] f(x) &= f(a)\\ &= 0 \quad \text{only if } f(x) \text{ vanishes at } a\end{aligned}$$

(which Lacroix tacitly assumed to be the case). The purported "law of exponents" (7) is in this respect deceptive, and in need of repair. To that end...

It is readily verified that

$$\frac{d}{dx}\int_a^x (x-y)f(y)\,d\xi = \int_a^x f(y)\,dy$$

from which it follows straightforwardly that

$$\frac{d^n}{dx^n}\int_a^x (x-y)^n f(y)\,dy = n!\int_a^x f(y)\,dy \quad : \quad n = 0, 1, 2, \ldots$$

Iterated integration of the preceding identity yields[7]

$$\underbrace{\int_a^x\int_a^{y_n}\int_a^{y_{n-1}}\cdots\int_a^{y_2}}_{n\text{-fold iterated}\,:\; n=1,2,3,\ldots} f(y_1)\,dy_1 dy_2\cdots dy_n = \tfrac{1}{(n-1)!}\int_a^x (x-y)^{n-1} f(y)\,dy$$

which is usually attributed to Cauchy, and which we are in position to notate

$${}_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)}\int_a^x (x-y)^{n-1} f(y)\,dy \quad : \quad n = 1, 2, 3, \ldots \tag{8}$$

The fractional calculus was conceived by Riemann and Liouville (who set $a = 0$) and later by Weyl (who didn't) to be a theory of ordinary differentiation and **fractional integration**, based upon this generalization

$${}_aD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)}\int_a^x (x-y)^{\nu-1} f(y)\,dy \quad : \quad \nu > 0 \tag{9}$$

[7] See R. Courant, *Differential & Integral Calculus* (1936), Volume II, p. 221; R. Courant & D. Hilbert, *Methods of Mathematical Physics* (1962), Volume II, p. 523; I. S. Gradshteyn & I. M. Ryzhik, *Tables of Integrals, Series & Products* (1965), **4.631**, p. 620.

of (8).[8] The fractional integrals of $f(x)$ can in this light be said to be *integral transforms* of $f(x)$, and it is as such that they are tabulated on pages 185–212 of A. Erdélyi *et al*, *Tables of Integral Transforms*, Volume II (1954). Fractional derivatives emerge as secondary constructions, from

$$ {}_aD_x^{m-\nu}f(x) = D^m \cdot {}_aD_x^{-\nu}f(x) \tag{10} $$

We would, for example, write

$$ \begin{aligned} {}_aD_x^{\frac{1}{3}}f(x) &= D^1 \cdot {}_aD_x^{-\frac{2}{3}}f(x) \\ {}_aD_x^{\frac{4}{3}}f(x) &= D^2 \cdot {}_aD_x^{-\frac{2}{3}}f(x) \end{aligned} $$

Look in particular to the "semiderivative"

$$ \begin{aligned} {}_aD_x^{\frac{1}{2}}f(x) &= D^1 \cdot {}_aD_x^{-\frac{1}{2}}f(x) \\ &= \frac{d}{dx}\cdot\frac{1}{\Gamma(\frac{1}{2})}\int_a^x \frac{1}{\sqrt{x-y}}f(y)\,dy \end{aligned} $$

which in the simple case $f(x) \equiv 1$ gives

$$ \begin{aligned} {}_aD_x^{\frac{1}{2}}1 &= \frac{d}{dx}\cdot\frac{1}{\sqrt{\pi}}\int_a^x \frac{1}{\sqrt{x-y}}\,dy \\ &= \frac{d}{dx}\cdot\frac{2\sqrt{x-a}}{\sqrt{\pi}} \\ &= \frac{1}{\sqrt{\pi(x-a)}} \end{aligned} $$

and at $a = 0$ gives back the result obtained by Lacroix:

$$ D^{\frac{1}{2}}x^0 = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})}x^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}}x^{-\frac{1}{2}} = \sqrt{\frac{1}{\pi x}} $$

2. Ordinary derivatives as integral transforms. Familiarly

$$ \begin{aligned} \int \delta(y-x)f(y)\,dy &= f(x) \\ \int \delta'(y-x)f(y)\,dy &= \delta(y-x)f(y)\Big| - \int \delta(y-x)f'(y)\,dy \\ &= -f'(x) \\ &\;\vdots \end{aligned} $$

[8] Note that the Riemann-Liouville construction (9) yields nonsense at $\nu = -0, -1, -2, \ldots$ and that these are precisely the points at which ${}_aD_x^{-\nu}$ speaks of the most unexceptionably commonplace objects in the calculus: the derivatives of integral order.

which lead to the general statement

$$D^n f(x) \equiv f^{(n)}(x) = (-)^n \int \delta^{(n)}(y-x) f(y)\,dy \tag{11}$$

Here the n^{th} derivative of $f(x)$ is presented as an integral transform—a "Dirac transform"—of the function in question.

To render more concrete the meaning of (11) we bring *representation theory* into play, writing

$$\delta(y-x) = \lim_{\epsilon\downarrow 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{1}{2\epsilon}(y-x)^2} \tag{12}$$

"Representation of the δ-function" can, of course, be accomplished in infinitely many alternative ways. We have at (12) selected the *Gaussian* representation for this practical reason: derivatives of the Gaussian are well-studied named functions about which a great deal is known. Specifically, we have (compare ADVANCED QUANTUM TOPICS (2000), Chapter 0, page 46)

$$\delta^{(n)}(y-x) = \lim_{\epsilon\downarrow 0} \frac{1}{\sqrt{2\pi\epsilon}} \Big(-\frac{1}{\sqrt{\epsilon}}\Big)^n He_n\Big(\frac{y-x}{\sqrt{\epsilon}}\Big) e^{-\frac{1}{2\epsilon}(y-x)^2} \tag{13}$$

where $He_n(z) \equiv (-)^n e^{\frac{1}{2}z^2} \big(\frac{d}{dz}\big)^n e^{-\frac{1}{2}z^2}$ serves to define the "monic Hermite polynomials"[9]

$$\begin{aligned}
He_0(z) &= 1\\
He_1(z) &= z\\
He_2(z) &= z^2 - 1\\
He_3(z) &= z^3 - 3z\\
He_4(z) &= z^4 - 6z^2 + 3\\
He_5(z) &= z^5 - 10z^3 + 15z\\
&\vdots\\
He_{n+1}(z) &= zHe_n(z) - nHe_{n-1}(z)
\end{aligned}$$

Returning with this information to (11), we have

$$D^n f(x) = \lim_{\epsilon\downarrow 0} \frac{1}{\sqrt{2\pi\epsilon}} \Big(\frac{1}{\epsilon}\Big)^{\frac{n}{2}} \int_{-\infty}^{+\infty} w_n\Big(\frac{y-x}{\sqrt{\epsilon}}\Big) f(y)\,dy \tag{14.1}$$

$$\equiv \lim_{\epsilon\downarrow 0} \int W_n(y-x\,;\epsilon) f(y)\,dy \tag{14.2}$$

with

$$w_n(z) \equiv e^{-\frac{1}{2}z^2} He_n(z) = \big(-\tfrac{d}{dz}\big)^n e^{-\frac{1}{2}z^2} \tag{15}$$

[9] See Magnus & Oberhettinger, *Functions of Mathematical Physics* (1954), page 80; Spanier & Oldham, *Atlas of Functions* (1987), Chapter 24. To produce such functions within *Mathematica* define

$$\texttt{H[n\_,z\_]:=}\,2^{-n/2}\texttt{HermiteH[n, z/}\sqrt{2}\texttt{]}$$

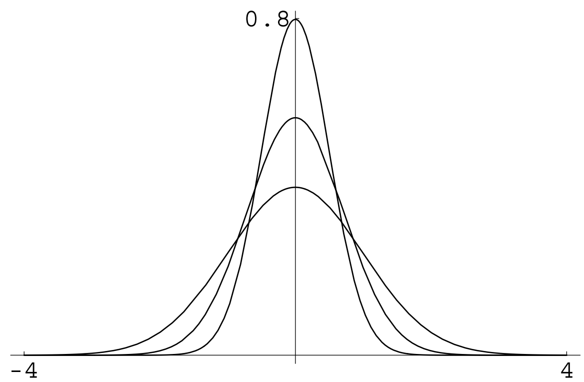

FIGURE 2A: *Graphs of* $W_0(x\,;\epsilon)$ *become sharper and more compact as* ϵ *descends through the values* $1, \frac{1}{2}, \frac{1}{4}$. *The figure illustrates how it comes about that*

$$\lim_{\epsilon\downarrow 0} W_0(x\,;\epsilon) = \delta(x)$$

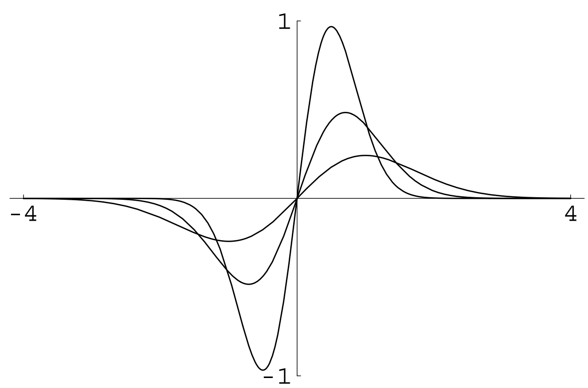

FIGURE 2B: *Graphs of* $W_1(x\,;\epsilon)$ *as* ϵ *descends through the values* $1, \frac{1}{2}, \frac{1}{4}$. *The figure illustrates the sense in which*

$$\lim_{\epsilon\downarrow 0} W_1(x\,;\epsilon) = \lim_{\epsilon\downarrow 0} \frac{\delta(x+\frac{1}{2}\epsilon)-\delta(x-\frac{1}{2}\epsilon)}{\epsilon} = \delta^{(1)}(x)$$

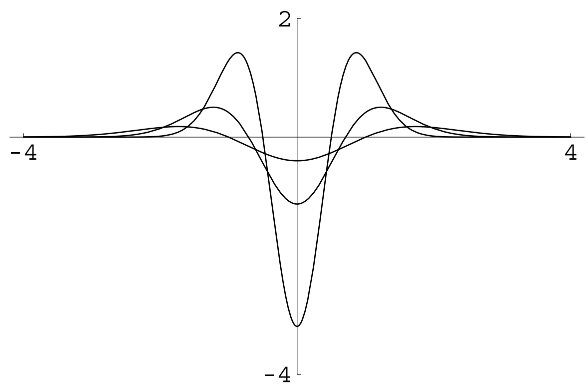

FIGURE 2C: *Graphs of $W_2(x\,;\epsilon)$ as ϵ descends through the values $1, \frac{1}{2}, \frac{1}{4}$. The figure illustrates the sense in which*

$$\begin{aligned}\lim_{\epsilon\downarrow 0} W_2(x\,;\epsilon) &= \lim_{\epsilon\downarrow 0}\frac{\delta^{(1)}(x+\frac{1}{2}\epsilon)-\delta^{(1)}(x-\frac{1}{2}\epsilon)}{\epsilon}\\ &= \lim_{\epsilon\downarrow 0}\frac{\delta(x+\epsilon)-2\delta(x)+\delta(x-\epsilon)}{\epsilon^2} = \delta^{(2)}(x)\end{aligned}$$

Equation (11) can be criticized on the ground that it is excessively formal. At (14) we have removed that defect—have, in fact, *assigned meaning* to (11)—by presenting the n^{th} derivative of $f(x)$ as the **limit of a sequence of integral transforms**.

3. Fractional derivatives as integral transforms. Busy citizens, well-established within the community of higher functions, are the so-called "parabolic cylinder functions" $D_\nu(x)$, often called "Weber functions" and less often a confusing variety of other names. The elaborate theory of such functions is summarized in all the standard handbooks.[10] One has

$$D_n(z) = e^{-\frac{1}{4}z^2} He_n(z) \quad : \quad n = 0, 1, 2, \ldots \tag{16}$$

giving

$$w_n(z) \equiv e^{-\frac{1}{4}z^2} D_n(z)$$

whence

$$W_n(y-x\,;\epsilon) = \frac{1}{\sqrt{2\pi\epsilon}}\Big(\frac{1}{\epsilon}\Big)^{\frac{n}{2}} e^{-\frac{1}{4\epsilon}(\xi-x)^2} D_n\Big(\frac{y-x}{\sqrt{\epsilon}}\Big)$$

[10] See, for example, Erdélyi *et al*, *Higher Transcendental Functions II* (1953), Chapter 8; Abramowitz & Stegun, *Handbook of Mathematical Functions* (1964), Chapter 19; Magnus & Oberhettinger, *Formulas & Theorems for the Functions of Mathematical Physics* (1954), Chapter 6, §3; Spanier & Oldham, *An Atlas of Functions* (1987), Chapter 46.

But the point of interest—fundamental to this entire discussion—is that the Weber functions—which for (*Mathematica*'s) computational purposes are most conveniently described

$$D_\nu(x) = 2^{\nu/2}e^{-\frac{1}{4}x^2}\bigg\{\frac{\sqrt{\pi}\,\texttt{Hypergeometric1F1[}-\tfrac{1}{2}\nu,\tfrac{1}{2},\tfrac{1}{2}x^2\texttt{]}}{\texttt{Gamma[}\tfrac{1}{2}(1-\nu)\texttt{]}} - \frac{\sqrt{2\pi}x\,\texttt{Hypergeometric1F1[}\tfrac{1}{2}(1-\nu),\tfrac{3}{2},\tfrac{1}{2}x^2\texttt{]}}{\texttt{Gamma[}-\tfrac{1}{2}\nu\texttt{]}}\bigg\}$$

—are well-defined *for all* (*positive or negative*) *real values of* ν! They serve ($\nu \geqslant 0$) to interpolate between and ($\nu < 0$) to extrapolate beyond the hermite functions $he_n(x) = e^{-\frac{1}{4}x^2}He_n(x)$—which themselves presume the index to be integer-valued—and do so in what can from many points of view be argued to be the "analytically most natural way." Looking in this light back again to (14.2), it would appear to make tentative good sense to write

$$D^\nu f(x) = \lim_{\epsilon\downarrow 0}\int W_\nu(y-x\,;\epsilon)f(y)\,dy \quad : \quad \text{all } \nu \tag{17.1}$$

with

$$W_\nu(y-x\,;\epsilon) = \frac{1}{\sqrt{2\pi\epsilon}}\Big(\frac{1}{\epsilon}\Big)^{\nu/2}e^{-\frac{1}{4\epsilon}(y-x)^2}D_\nu\Big(\frac{y-x}{\sqrt{\epsilon}}\Big) \tag{17.2}$$

This is the idea I propose to explore.

Look first to the case $\nu = -1$. We are informed by *Mathematica* that

$$W_{-1}(y-x\,;\epsilon) = \tfrac{1}{2}\Big[1-\text{erf}\Big(\frac{y-x}{\sqrt{\epsilon}}\Big)\Big]$$

It is a clear implication of FIGURE 3, and not difficult now to prove,[11] that

$$\lim_{\epsilon\downarrow 0}W_{-1}(y-x\,;\epsilon) = 1-\theta(y-x) = \theta(x-y) = \begin{cases}1 & : \quad y<x\\ 0 & : \quad y>x\end{cases}$$

so (17.1) becomes

$$\begin{aligned} D^{-1}f(x) &= \int_{-\infty}^{+\infty}\theta(x-y)f(y)\,dy\\ &= \int_{-\infty}^{x}f(y)\,dy\end{aligned}$$

while with scarcely more labor we find

$$\begin{aligned} D^{-2}f(x) &= \int_{-\infty}^{x}(x-y)^1 f(y)\,dy\\ D^{-3}f(x) &= \tfrac{1}{2}\int_{-\infty}^{x}(x-y)^2 f(y)\,dy\\ &\;\;\vdots\end{aligned}$$

[11] See page 7 in "'Laplacian operators' of eccentric order."[3]

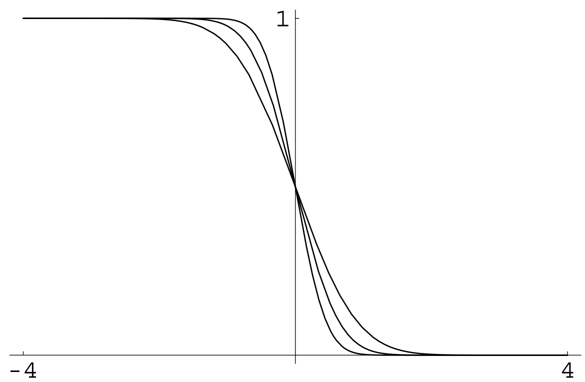

FIGURE 3: *Graphs of* $W_{-1}(y\,;\epsilon)$ *drop ever more abruptly as* ϵ *descends through the values* $1, \frac{1}{2}, \frac{1}{4}$. *The figure illustrates how it comes about that*

$$\lim_{\epsilon\downarrow 0} W_{-1}(y\,;\epsilon) = 1 - \theta(y) = \theta(-y)$$

Demonstration that for all (even fractional) $\nu > 0$

$$\lim_{\epsilon\downarrow 0} W_{-\nu}(y - x\,;\epsilon) = \frac{(x-y)^{\nu-1}}{\Gamma(\nu)}\theta(x-y) \tag{18}$$

presents a formidable analytical assignment—an assignment which I myself have not yet attempted to carry out. In support of my confident claim that (18) must certainly be correct I offer only the evidence of FIGURE 4. To summarize our progress thus far:

Our "interpolated representation-theoretic approach" to the fractional differentiation problem has by extrapolation reproduced a slight variant of the standard theory of fractional integration: it has led us to

$${}_aD_x^{-\nu} \quad \text{in the case} \quad a = -\infty \quad : \quad \nu > 0$$

In standard theory (theory of the Riemann-Liouville transform) it is canonical to set $a = 0$. Our $-\infty$ is a vestige of the circumstance that it is on the entire real line $-\infty < x < +\infty$ that Gaussians live.

Since, as was remarked already at (10), in standard theory one proceeds

$$\text{fractional derivative} = (\text{ordinary derivative}) \cdot (\text{fractional integral})$$

it might appear that our work is done. However...

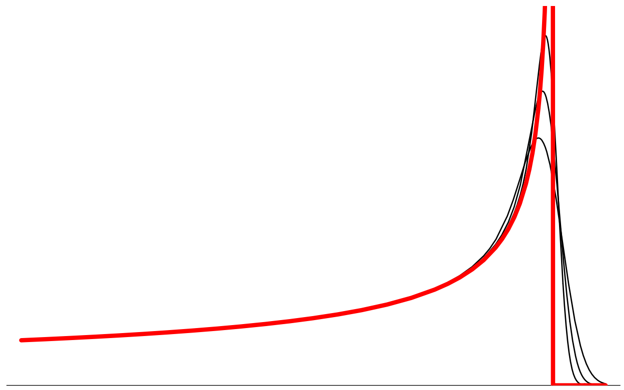

FIGURE 4: *Graphs of* $W_{-\frac{1}{2}}(y\,;\epsilon)$ *are seen to squeeze ever closer to the red graph of*

$$\frac{(-y)^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2})}\theta(-y)$$

as ϵ *descends through the values* $0.12, 0.06, 0.03$. *Similar figures result from*

$$W_{-\nu}(y\,;\epsilon) \qquad \text{compare} \qquad \frac{(-y)^{\nu-1}}{\Gamma(\nu)}\theta(-y) \quad : \quad \nu > 0$$

and serve collectively to secure confidence in the accuracy of (18).

We in FIGURES 2B & 2C saw representations of the integration process (convolution process) that extracts from $f(x)$ its 1^{st} and 2^{nd} derivatives. We are now in position to observe (FIGURES 5A, 5B & 5C) that similar figures serve to describe the process that extracts derivatives of interstitial order:

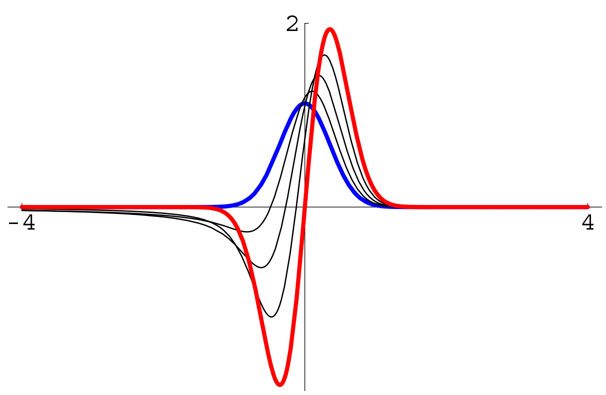

FIGURE 5A: $W_0(y\,;\epsilon)$ *morphs to* $W_1(y\,;\epsilon)$ *as* ν *ranges on the values* $\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$. *Here* $\epsilon = \frac{1}{8}$.

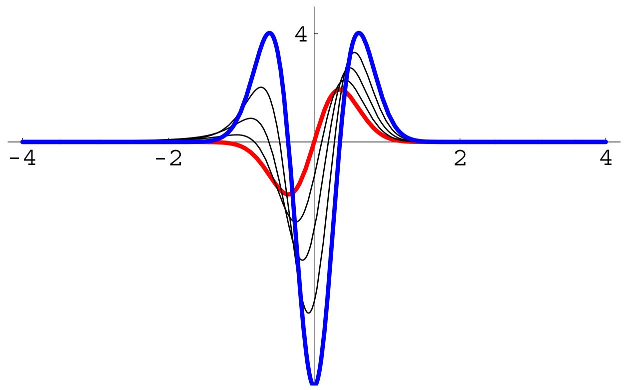

FIGURE 5B: $W_1(y\,;\epsilon)$ *morphs to* $W_2(y\,;\epsilon)$ *as* ν *ranges on the values* $\{1, \frac{5}{4}, \frac{6}{4}, \frac{7}{4}, 2\}$. *Here again* $\epsilon = \frac{1}{8}$. *Note the contracted vertical scale.*

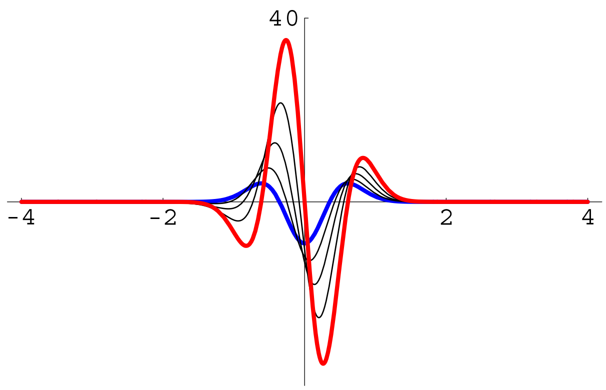

FIGURE 5C: $W_2(y\,;\epsilon)$ *morphs to* $W_3(y\,;\epsilon)$ *as* ν *ranges on the values* $\{2, \frac{9}{4}, \frac{10}{4}, \frac{11}{4}, 3\}$. *Here again* $\epsilon = \frac{1}{8}$ *and the vertical scale has been further contracted.*

It is intended, of course, that $\epsilon \downarrow 0$. The effect of that process is illustrated in FIGURE 6.

4. Some illustrative specific cases. A sense of what we have accomplished—and failed to accomplish—is, in view of the analytical complexity of our results, best conveyed, I think, by study of some illustrative concrete examples. We look first to

$\boxed{\text{SEMI-INTEGRALS/DERIVATIVES OF POWERS}}$ Proceeding from the following instance of (9)

$$ {}_aD_x^{-\frac{1}{2}} f(x) = \frac{1}{\sqrt{\pi}} \int_a^x \frac{1}{\sqrt{x-y}}\, f(y)\, dy $$

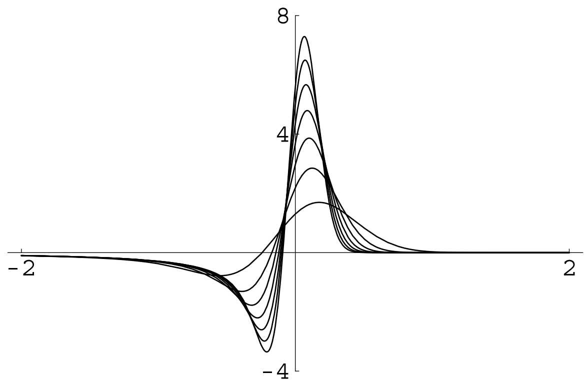

FIGURE 6: *The limiting process $\epsilon \downarrow 0$, illustrated in the case $\nu = \frac{1}{2}$ (semi-differentiation). The graph becomes ever sharper as ϵ ranges on $\{\frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \ldots, \frac{1}{70}\}$.*

in the case $f(x) = x^p$, we with the assistance of *Mathematica* obtain

$$\left.\begin{aligned}
{}_aD_x^{-\frac{1}{2}}x^0 &= \frac{2\sqrt{x-a}}{\sqrt{\pi}}\\
{}_aD_x^{-\frac{1}{2}}x^1 &= \frac{2\sqrt{x-a}\,(2x+a)}{3\sqrt{\pi}}\\
{}_aD_x^{-\frac{1}{2}}x^2 &= \frac{2\sqrt{x-a}\,(8x^2+4ax+3a^2)}{15\sqrt{\pi}}\\
{}_aD_x^{-\frac{1}{2}}x^3 &= \frac{2\sqrt{x-a}\,(16x^3+8ax^2++6a^2x+5a^2)}{35\sqrt{\pi}}\\
{}_aD_x^{-\frac{1}{2}}x^4 &= \frac{2\sqrt{x-a}\,(128x^4+64ax^3+48a^2x^2+40a^3x+35a^4)}{315\sqrt{\pi}}
\end{aligned}\right\} \tag{19}$$

If we set $a = 0$ we have

$$\begin{aligned}
{}_0D_x^{-\frac{1}{2}}x^0 &= \frac{2}{\sqrt{\pi}}\,x^{\frac{1}{2}}\\
{}_0D_x^{-\frac{1}{2}}x^1 &= \frac{4}{3\sqrt{\pi}}\,x^{\frac{3}{2}}\\
{}_0D_x^{-\frac{1}{2}}x^2 &= \frac{16}{15\sqrt{\pi}}\,x^{\frac{5}{2}}\\
{}_0D_x^{-\frac{1}{2}}x^3 &= \frac{32}{35\sqrt{\pi}}\,x^{\frac{7}{2}}\\
{}_0D_x^{-\frac{1}{2}}x^4 &= \frac{256}{315\sqrt{\pi}}\,x^{\frac{9}{2}}\\
&\ \vdots\\
{}_0D_x^{-\frac{1}{2}}x^p &= \frac{\Gamma(p+1)}{\Gamma(p+1+\frac{1}{2})}\,x^{p+\frac{1}{2}}
\end{aligned} \tag{20}$$

as presented at **13.1**(7) in the Erdélyi *Table of Riemann-Liouville fractional integrals* that was cited on page 6. By simple differentiation we obtain the semi-differentiation formula

$$
\begin{aligned}
{}_0D_x^{+\frac{1}{2}}x^p = D\cdot {}_0D_x^{-\frac{1}{2}}x^p &= \frac{\Gamma(p+1)}{\Gamma(p+1+\frac{1}{2})}\big(p+\tfrac{1}{2}\big)x^{p-\frac{1}{2}} \\
&= \frac{\Gamma(p+1)}{\Gamma(p+\frac{1}{2})}\,x^{p-\frac{1}{2}} \qquad (21)
\end{aligned}
$$

first obtained (by other means: compare (6.1)) by Lacroix. The formalism developed in preceding pages requires, however, that we set $a=-\infty$. Equations (19) then supply

$$D^{-\frac{1}{2}}x^p = \infty \quad : \quad p = 0,1,2,\ldots$$

and it would appear to be pointless to attempt to assign valuation to

$$\text{“}D^{+\frac{1}{2}}x^p = D\cdot D^{-\frac{1}{2}}x^p = D\infty\text{”}$$

The present formalism supplies, however, a *direct* valuation

$$D^{\frac{1}{2}}x^p = \lim_{\epsilon\downarrow 0}\int_{-\infty}^{+\infty} W_{\frac{1}{2}}(y-x\,;\epsilon)\,y^p\,dy$$

For computational purposes we assign ϵ a “sufficiently small test value” and—in view of the localized structure assumed by $W_\nu(y-x\,;\epsilon)$ when $\nu>0$ (see FIGURES 5 and especially FIGURE 6)—restrict the range of the integral, writing (say)

$$\approx \int_{x-3}^{x+3} W_{\frac{1}{2}}(y-x\,;\tfrac{1}{100})\,y^p\,dy \qquad (22)$$

So complicated is the integrand (see again (17.2)) that we have no alternative but to proceed numerically. The following figures provide comparative displays

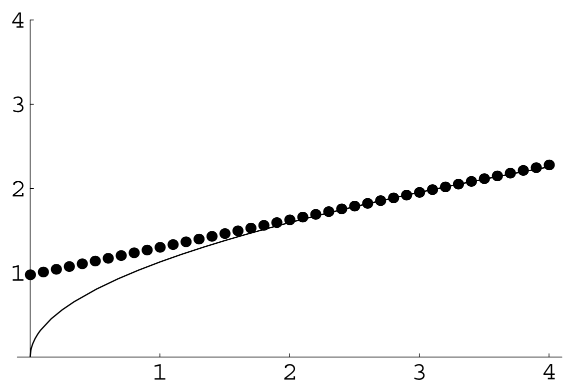

FIGURE 7A: *Semi-derivative of x^1, derived first—solid curve—from* (21) *and then—dotted curve—from* (22). *The two appear to be in asymptotic agreement.*

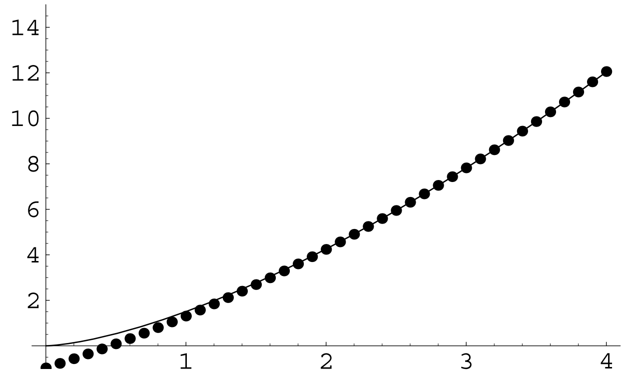

FIGURE 7B: *Semi-derivatives of* x^2, *derived from* (21) *and* (22). *Again we appear to see asymptotic agreement.*

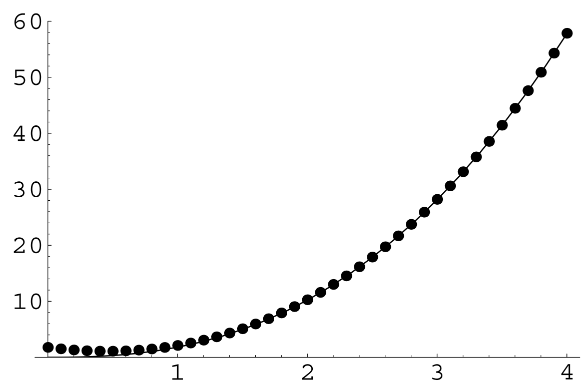

FIGURE 7C: *Semi-derivatives of* x^3, *computed again in the same two alternative ways.*

of some of the representative implications of (21) and (22). Remarkably, we appear to have precise agreement for $x \gg 0$; *i.e.*, for x much larger than the value to which the Riemann-Liouville method assigns a distinguished place. Asymptotic agreement serves to underscore the **local character of fractional differentiation**—a point that is obscured when (as is standardly done) one construes fractional derivatives to be ordinary derivatives of fractional integrals. Figure 8 provides evidence supportive of the claim that the fractional calculus advocated here precisely reproduces the ordinary calculus when asked to make statements about derivatives of integral order.

$\boxed{\text{SEMI-INTEGRALS/DERIVATIVES OF EXPONENTIALS}}$ Proceeding as before from an instance of (9)

$$ {}_aD_x^{-\frac{1}{2}}e^{px} = \frac{1}{\sqrt{\pi}}\int_a^x \frac{1}{\sqrt{x-y}}\,e^{py}\,dy $$

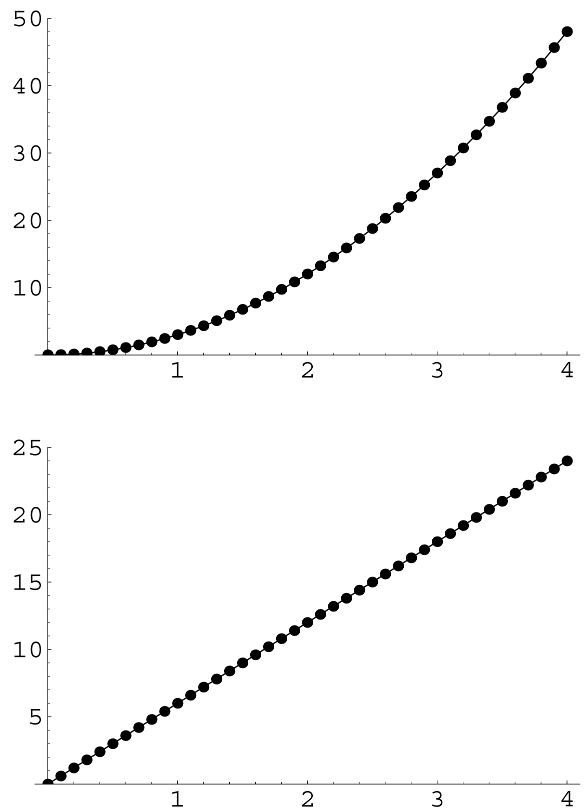

FIGURE 8: *Shown above is the first derivative—and below the second derivative—of x^3. The solid curve was obtained by ordinary calculus, the dotted points computed by numerical appeal to the obvious variants of* (22)*. We take this to be evidence that formulæ analogous to* (22) *could be used to reproduce the content of ordinary differential calculus.*

we are informed by *Mathematica* that

$$ {}_aD_x^{-\frac{1}{2}}e^{px} = \frac{e^{px}\,\mathrm{erf}(\sqrt{p(x-a)}\,)}{\sqrt{p}} $$

where the notation $\mathrm{erf}(z)$ refers to the "error function."[12] If we set $a = 0$, as recommended by Riemann-Liouville, we obtain

$$ {}_0D_x^{-\frac{1}{2}}e^{px} = \frac{e^{px}\,\mathrm{erf}(\sqrt{px}\,)}{\sqrt{p}} $$

[12] See Chapter 7 in Abramowitz & Stegun.

which when hit with D gives

$$ {}_0D_x^{+\frac{1}{2}}e^{px} = \frac{1}{\sqrt{\pi x}} + \sqrt{p}\,e^{px}\,\mathrm{erf}(\sqrt{px}\,) \tag{23}$$

We, on the other hand, are motivated to set $a = -\infty$, which leads to

$$D^{-\frac{1}{2}}e^{px} = \frac{e^{px}}{\sqrt{p}}$$

whence

$$D^{+\frac{1}{2}}e^{px} = \sqrt{p}\,e^{px} \tag{24}$$

It is easy to show (FIGURE 9) that

$$\lim_{x\uparrow\infty}\frac{\text{right side of (24)}}{\text{right side of (23)}} = 1$$

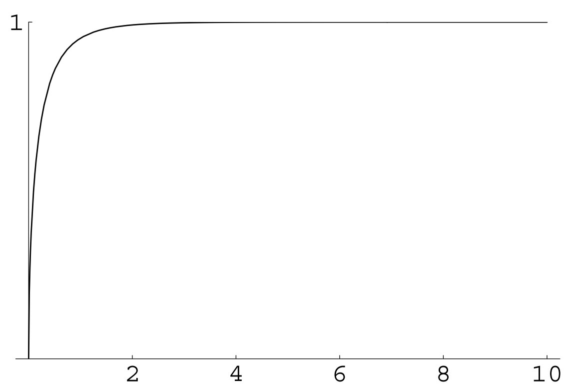

FIGURE 9: *Demonstration that the alternative semi-derivatives of e^x become asymptotically identical, in the sense that their ratio approaches unity.*

which is again symptomatic of the general fact that **asymptotically the two formalisms become identical**. Alternatively to (24) we expect, by adjustment of (22), to have

$$D^{+\frac{1}{2}}e^{px} \approx \int_{x-3}^{x+3} W_{\frac{1}{2}}(y-x;\tfrac{1}{100})\,e^{py}\,dy \tag{25}$$

In support of that expectation, see FIGURE 10. The present formalism leads to a striking general result

$$D^\nu e^{px} = p^\nu e^{px} \quad : \quad \nu > 0$$

that appears on its face to be simpler and more natural than the corresponding statement within the Riemann-Liouville formalism, though the latter is shown in §11 of some notes already cited[2] to give rise to some fascinating mathematics.

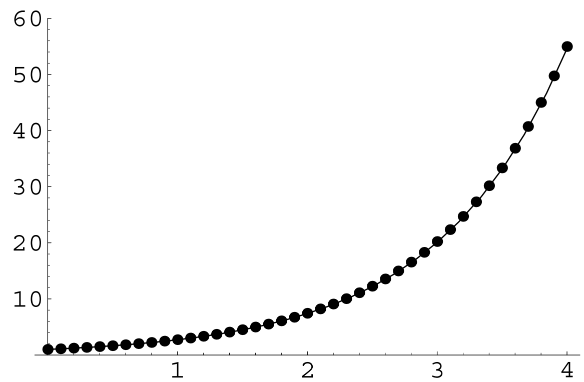

FIGURE 10: *Superposition of the valuations of the semi-derivatives* $D^{\frac{1}{2}}e^{px}$ *that derive from* (24) *and* (25), *respectively.*

5. Law of exponents. Gaussian representation theory led us at (17.1) to an equation which, when notationally stripped of its epsilonic details, reads

$$D^\nu f(x) = \int W_\nu(y-x)f(y)\,dy$$

and from which we expect to obtain

$$\big(\tfrac{d}{dx}\big)^n D^\nu f(x) = \int W_{n+\nu}(y-x)f(y)\,dy \tag{26}$$

—the formal design of which we expect to recur *whatever* the "representation of the delta function" upon which we elect to base our formalism. It is, in this light, gratifying to discover that at **46:10:3** in their *Atlas of Functions* (1987) J. Spanier & K. B. Oldham report what they call "the elegant relationship"

$$\big(-\tfrac{d}{dz}\big)^n\Big\{e^{-\frac{1}{4}z^2}D_\nu(z)\Big\} = e^{-\frac{1}{4}z^2}D_{n+\nu}(z) \quad : \quad n = 0,1,2,\ldots$$

from which, by (17.2), it readily follows that

$$\big(\tfrac{d}{dx}\big)^n W_\nu(y-x;\epsilon) = W_{n+\nu}(y-x;\epsilon)$$

holds for all ϵ, and therefore also in the limit $\epsilon\downarrow 0$. The implication is that (26) is valid for all ν, whether positive or negative.[13]

[13] Spanier & Oldham were themselves the co-authors of the first book devoted to the fractional calculus (*The Fractional Calculus*: *Theory & Applications of Differentiation and Integration to Arbitrary Order* (1974)), but apparently failed to notice that their "elegant relationship" had any relevance to that subject. So also did Erdélyi, who cultivated an interest in the fractional calculus, and presents the identity in question on page 119 of *Higher Transcendental Functions* (1953).

To establish the law of exponents as it relates to compounded fractional integration we might proceed

$$
\begin{aligned}
D^{-\mu}D^{-\nu}f(x) &= \int W_{-\mu}(z-x)\Big\{\int W_{-\nu}(y-z)f(y)\,dy\Big\}\,dz \\
&= \int\Big\{\int W_{-\mu}(z-x)W_{-\nu}(y-z)\,dz\Big\}f(y)\,dy \qquad (27)
\end{aligned}
$$

Since both μ and ν are assumed presently to be positive, we can on appeal to (18) write

$$
\int W_{-\mu}(z-x)W_{-\nu}(y-z)\,dz = \frac{1}{\Gamma(\mu)\Gamma(\nu)}(x-z)^{\mu-1}(z-y)^{\nu-1}\theta(x-z)\theta(z-y)
$$

for which *Mathematica*—after a good deal of thought—supplies the valuation

$$
\begin{aligned}
&= \frac{1}{\Gamma(\mu+\nu)}(x-y)^{\mu+\nu-1}\theta(x-y) \\
&= W_{-(\mu+\nu)}(y-x)
\end{aligned}
$$

Returning with this information to (27) we have the integrative law of exponents

$$
D^{-\mu}D^{-\nu} = D^{-(\mu+\nu)}
$$

6. Geometrical interpretation of fractional derivatives. K. S. Miller & B. Ross—authors of *An Introduction to the Fractional Calculus & Fractional Differential Equations* (1993)—remark in §8 of their Chapter I: Historical Survey that "Some of the still-open questions are intriguing. For example: Is it possible to find a geometrical interpretation for a fractional derivative of noninteger order?" The question is ancient—recall Leibniz' lament that the subject "seems removed from Geometry, which does not yet know of such fractional exponents"—and it is, in view of the diagram we traditionally draw when we explain what it means to construct $Df(x)$, quite natural. The "geometrical meaning" of $D^{-1}f(x)$ is similarly direct. But when we look to $D^{\pm n}f(x)$ our geometrical intuition becomes progressively more tenuous as n increases; in such situations we find it entirely natural to abandon geometrical representations, and to adopt a more formal, a more algorithmic mode of thought... and are seldom or never heard to fuss about the loss of geometrical immediacy.

The absence within the fractional calculus of a figure directly analogous to FIGURE 11 has, however, tended to marginalize our subject. Calculus $101\frac{1}{2}$ courses do not exist, and for the most part only desperate eccentrics attempt to exploit the little-known resources of the fractional calculus—attempt, that is to say, to follow in the tradition established almost two centuries ago (1823) by Abel.[14] It is, therefore, perhaps of some importance that the present formalism

[14] See Miller & Ross, pages 4 and 255–260.

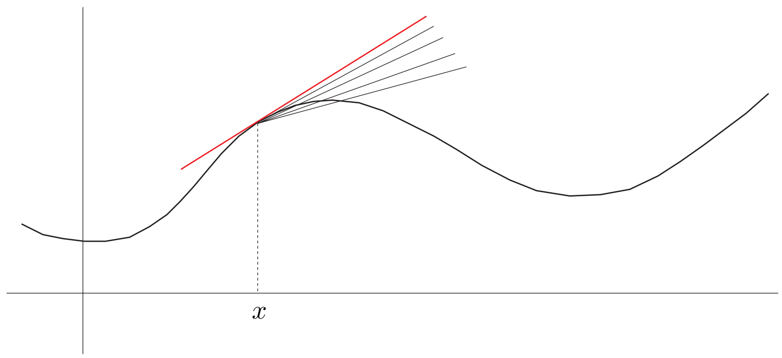

FIGURE 11: *Representation of the most commonly understood “geometrical meaning” (slope of the tangent) of the derivative $Df(x)$. It will be appreciated that the tangent has meaning only as the limit of a convergent sequence of chords.*

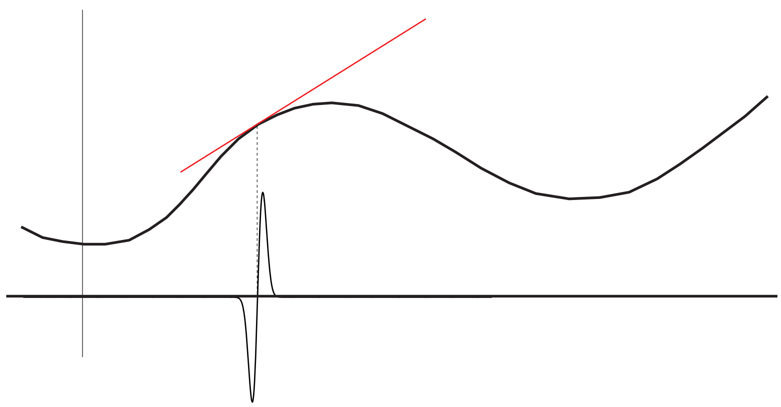

FIGURE 12: *In the present formalism $Df(x)$ emerges as the* **limit of the area under the product** *of the two functions shown. The lower function is representative of a ϵ-parameterized class of functions, which in the limit $\lim_{\epsilon\downarrow 0}$ provide a representation of $-\delta'(y-x)$.*

does provide a direct “geometrical interpretation” of $D^\nu f(x)$ for all real ν, positive as well as negative. For the formalism invites us to consider an alternative (FIGURE 12) to the standard interpretation (FIGURE 11) of $D^1 f(x)$, the point being that the alternative construction “morphs” in a natural way (FIGURES 13 & 14) to provide equally direct interpretations of $D^\nu f(x)$ for *all* values of ν. Underlying this advance is the fact tha $D^\nu f(x)$ has been presented at (17.1) as the **limit of a sequence of convolutions**, and convolutions—whatever the context in which they are encountered—admit of elementary diagramatic representation.

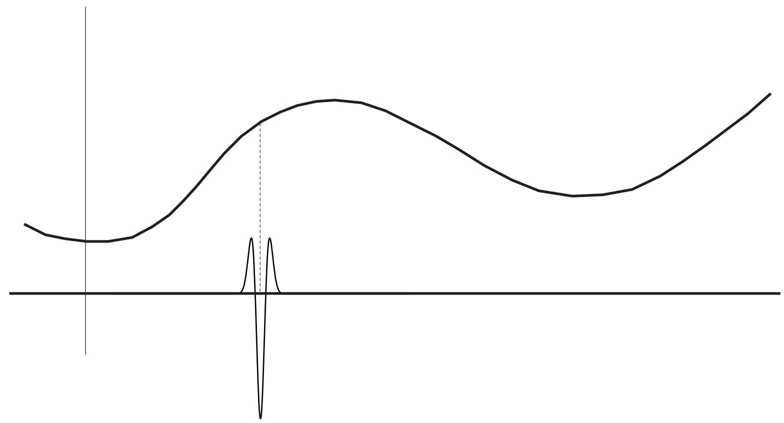

FIGURE 13: *Representation within the present formalism of* $D^2 f(x)$, *to be understood in the sense described in the preceding caption. The lower function is representative of a* ϵ*-parameterized class of functions which in the limit provide a representation of* $+\delta''(y-x)$.

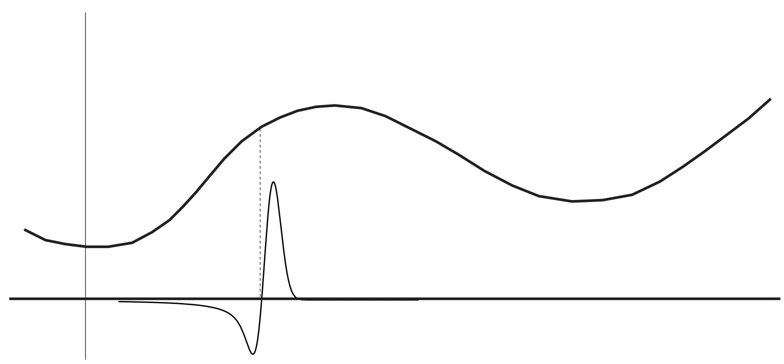

FIGURE 14: *Representation within the present formalism of* $D^{\frac{1}{2}} f(x)$, *to be understood in the sense already described* (*in which connection see again* FIGURE 6).

7. Concluding remarks. The formalism sketched in preceding pages proceeds from the Gaussian representation (12) of the Dirac delta function, and it is because "the Gaussian lives on the entire line" that we were—at a cost which was shown to be

- major as it relates to fractional integration, but
- often vanishingly slight as it relates to fractional differentiation

—obliged to set $a = -\infty$. Other representations would lead to variants of the formalism—some more workable than others. If one were willing to restrict one's attention to the positive half line one could arrange to achieve $a = 0$, and thus to reproduce the Riemann-Liouville formalism.

The formalism described above manages to detach the concept of fractional differentiation from that of fractional integration: it becomes possible to speak of $D^{\frac{1}{2}}f(x)$ even in contexts where $D \cdot D^{-\frac{1}{2}}f(x)$ does not exist. This is a capability absent from the Riemann-Liouville formalism.

Though we owe to (17.1)—*i.e.*, to

$$D^{\nu}f(x) = \lim_{\epsilon\downarrow 0}\int W_{\nu}(y-x\,;\epsilon)f(y)\,dy$$

—the diagramatic transparency of our results, the function $W_{\nu}(y-x\,;\epsilon)$ is so complicated that only exceptionally does it become possible to evaluate the integral in analytically closed form. We have demonstrated, however, that it is often quite possible to obtain useful information by numerical means. The preceding discussion does, in any event, serve to reenforce the view that "fractional calculus" is but the seductive name given to what is, in the end, simply an integral transform theory, like any other.[15]

I am mindful that this is a *physics* seminar, and that discussion of this subject before such an audience can be justified only by reference to the importance of the *physical applications of the fractional calculus*. Some typical applications were sketched in a previous seminar,[2] the text of which is available at Reed College on the courses server. To gain a sense of present activity in the field consult Google,[16] where it becomes evident (as it has to me from the identity of those who have requested reprints[17] of my seminar notes) that some of the heaviest consumers of fractional calculus these days are biophysicists and—of all things—soil mechanics. Reportedly the dispersal of pollutants in ground water, as of water in porus soil, is most accurately modeled[18] not by the standard diffusion equation but by an equation of the form (compare (3))

$$(\partial_t)^{\nu}\varphi = a\nabla^2\varphi$$

It is my pleasure to thank R. Crandall for the conversations that stimulated this effort, and A. Pellegrini for inspiration during the writing.

[15] This is the view tacitly embraced in A. I. Zayed, *Handbook of Function & Generalized Function Transformations* (1996). Chapter 20 provides a fairly sophisticated account of the Riemann-Liouville and Weyl transforms.

[16] To "fractional calculus" Google responds with about 83,100 hits, many of which are quite interesting: see, for example, Adam Loverro, "Fractional Calculus: History, Definitions and Applications for the Engineer" (2004) at www.nd.edu/~msen/Teaching/UnderRes/FracCalc.pdf.

[17] Most of those requests have been responsive to complementary remarks in Marcia Kleinz & Thomas J. Osler, "A child's garden of fractional derivatives," The College Mathematics Journal **31**, 82 (2000), which can be found at http.//www.rowan.edu/mars/depts/math/osler/Childs_garden.

[18] See, for example, R. Schumer, D. A. Benson & B. Baeumer, "Multiscaling fractional advection-dispersion equations and their solutions," Water Resources Research **39**, 1022 (2003) at unr.edu/homepage/mcubed/MultifADEwrr.pdf.