“LAPLACIAN OPERATORS” OF ECCENTRIC ORDER

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Introduction. The one-dimensional wave equation reads

$$\ddot{\varphi} = +bD^2\varphi$$

but for a *stiff* rod one has (for complicated rheological reasons, and after certain simplifications)

$$\ddot{\varphi} = -\beta D^4\varphi \tag{1}$$

Similarly, the diffusion equation reads

$$\dot{\varphi} = +aD^2\varphi$$

but in recent work Richard Crandall has encountered algorithmic need of the *biharmonic diffusion equation*

$$\dot{\varphi} = -\alpha D^4\varphi \tag{2}$$

A question that arises naturally in both cases (and issues from the lips as a *physical* question) is “Why the D^4?” But the question to which Richard directed my specific attention is “Why the minus sign?” It turns out that the two questions are—not at all surprisingly—intertwined: my attempt to illuminate the latter cast pale light also on the former.

In several dimensions the diffusion equation reads

$$\dot{\varphi} = a\nabla^2\varphi$$

and Richard’s more recent work has led him to ask: “What meaning (especially in the two-dimensional case) can be assigned to the ‘fractional Laplacian’ ∇^p that enters into the *fractional diffusion equation*

$$\dot{\varphi} = a\nabla^p\varphi \tag{3}$$

On these pages I provide an account of my initial response (21 January 1997) to Richard's first question, and explore one possible approach to his (relatively more interesting) second question.

1. The sign problem in one dimension. By definition

$$\begin{aligned} Df(x) &= \frac{e^{+\frac{1}{2}hD} - e^{-\frac{1}{2}hD}}{h} f(x) \\ &= \frac{f(x+\frac{1}{2}h) - f(x-\frac{1}{2}h)}{h} \end{aligned}$$

with $\lim h \downarrow 0$ understood. Similarly

$$\begin{aligned} D^2 f(x) &= \Big[\frac{e^{+\frac{1}{2}hD} - e^{-\frac{1}{2}hD}}{h}\Big]^2 f(x) \\ &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \\ &= -\frac{2}{h^2}\Big\{f(x) - \langle f(x)\rangle\Big\} \qquad (4.1) \\ &\qquad\qquad \langle f(x)\rangle \equiv \tfrac{1}{2}\big[f(x-h) + f(x+h)\big] \end{aligned}$$

and by the same argument

$$\begin{aligned} D^4 f(x) &= \Big[\frac{e^{+\frac{1}{2}hD} - e^{-\frac{1}{2}hD}}{h}\Big]^4 f(x) \\ &= \frac{f(x+2h) - 4f(x+h) + 6f(x) - 4f(x-h) + f(x-2h)}{h^4} \\ &= +\frac{6}{h^4}\Big\{f(x) - \langle f(x)\rangle\Big\} \qquad (4.2) \\ &\qquad\qquad \langle f(x)\rangle \equiv \tfrac{1}{6}\big[-f(x-2h) + 4f(x-h) \\ &\qquad\qquad\qquad + 4f(x+h) - f(x+2h)\big] \end{aligned}$$

The point of immediate interest is that the leading sign has flipped, then flopped, in accordance with the general pattern

$$D^{2n} f(x) = (-)^n h^{-2n} \tbinom{2n}{n} \Big\{f(x) - \langle f(x)\rangle\Big\} \quad : \quad n = 1, 2, 3, \ldots \qquad (4.n)$$

where $\langle f(x)\rangle$ becomes progressively more complicated, yet is never really very complicated; it uses binomial weights to favor nearer neighbors over more remote neighbors, but does so subject to a principle of sign alternation.

I describe now an alternative approach to the same problem. I find the following line of argument to be of some intrinsic interest, and will show in §2 that it serves to open some surprising doors. From

$$f(x) = \int \delta(y - x) f(y)\, dy$$

it follows that

$$\begin{aligned} f'(x) &= \int \delta(y-x) f'(y)\,dy \\ &= -\int \delta'(y-x) f(y)\,dy \quad \text{after integrating by parts} \\ f''(x) &= (-)^2 \int \delta''(y-x) f(y)\,dy \\ &\;\vdots \\ f^{(n)}(x) &= (-)^n \int \delta^{(n)}(y-x) f(y)\,dy \end{aligned} \tag{5}$$

The $(-)^n$ factor (which becomes invisible when n is even, and therefore cannot account for the "sign alternation phenomenon" to which Richard directed my attention) is a characteristic artifact of iterated integration-by-parts; in §2 we will find it convenient (and easy) to achieve its elimination.

Adopt (as a matter of analytical convenience) the Gaussian representation[1] of the delta function

$$\delta(y-x) = \lim_{\epsilon\downarrow 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{1}{2\epsilon}(y-x)^2} \tag{6.0}$$

Then

$$\delta^{(n)}(y-x) = \lim_{\epsilon\downarrow 0} \frac{1}{\sqrt{2\pi\epsilon}} \Big(-\frac{1}{\sqrt{\epsilon}}\Big)^n He_n\Big(\frac{y-x}{\sqrt{\epsilon}}\Big) e^{-\frac{1}{2\epsilon}(y-x)^2} \tag{6.n}$$

where $He_n(z) \equiv (-)^n e^{\frac{1}{2}z^2}\big(\frac{d}{dz}\big)^n e^{-\frac{1}{2}z^2}$ serves to define the monic Hermite polynomials

$$\begin{aligned} He_0(z) &= 1 \\ He_1(z) &= z \\ He_2(z) &= z^2 - 1 \\ He_3(z) &= z^3 - 3z \\ He_4(z) &= z^4 - 6z^2 + 3 \\ He_5(z) &= z^5 - 10z^3 + 15z \\ He_6(z) &= z^6 - 15z^4 - 45z^2 - 15 \\ &\;\vdots \\ He_{n+1}(z) &= yHe_n(z) - nHe_{n-1}(z) \end{aligned} \tag{7}$$

Returning with this information to (5), we have

$$D^n f(x) = \lim_{\epsilon\downarrow 0} \frac{1}{\sqrt{2\pi\epsilon}} \Big(\frac{1}{\epsilon}\Big)^{\frac{n}{2}} \int_{-\infty}^{+\infty} f(y)\, w_n\Big(\frac{y-x}{\sqrt{\epsilon}}\Big)\,dy \tag{8}$$

with

$$w_n(z) \equiv e^{-\frac{1}{2}z^2} He_n(z) = \big(-\tfrac{d}{dz}\big)^n e^{-\frac{1}{2}z^2} \tag{9}$$

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[1] Other representations (see below) give rise to qualitatively similar results, but easily lead to nameless functions which lie outside the established canon of higher analysis.

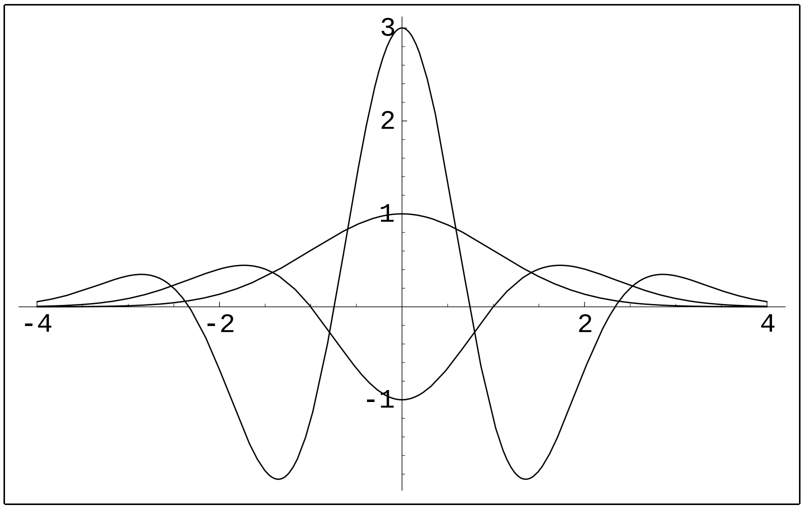

FIGURE 1: *Superimposed graphs of $w_0(z)$,$w_2(z)$ and $w_4(z)$. One can look to (10.2) to figure out which is which*

$$He_0(0) = +1$$
$$He_2(0) = -1$$
$$He_4(0) = +3$$

or one can simply count count axis crossings.

We have a lively interest at present only in the even cases $n = 2m$. Graphs of the weight functions $w_0(z)$,$w_2(z)$ and $w_4(z)$ are superimposed in Figure 1. It is an implication of (9) that

$$w_{2m}(0) = He_{2m}(0) \tag{10.1}$$

and follows from (7) that $He_{n+2}(0) = -(n+1)He_n(0)$ which (since $He_0(0) = 1$) entails

$$He_{2m}(0) = (-)^m\, 1 \cdot 3 \cdot 5 \cdots (2m-1) \tag{10.2}$$

The "sign alternation phenomenon" can, according to this result, be attributed —in Gaussian representation (but *only* in that context)—to an elementary property of the Hermite polynomials.

I have in Figure 2 attempted to capture the essence of the process

$$\lim_{\epsilon\downarrow 0}\Big(\frac{1}{\epsilon}\Big)^{\frac{n+1}{2}} w_n\Big(\frac{y-x}{\sqrt{\epsilon}}\Big)$$

Figures 1 and 2 jointly serve to establish the sense in which (8) provides a continuous analog of (4); the weight function $w_{2m}(\frac{y-x}{\sqrt{\epsilon}})$ favors neighborhoods near to x over neighborhoods more remote, but does so subject to a principle of sign alternation.

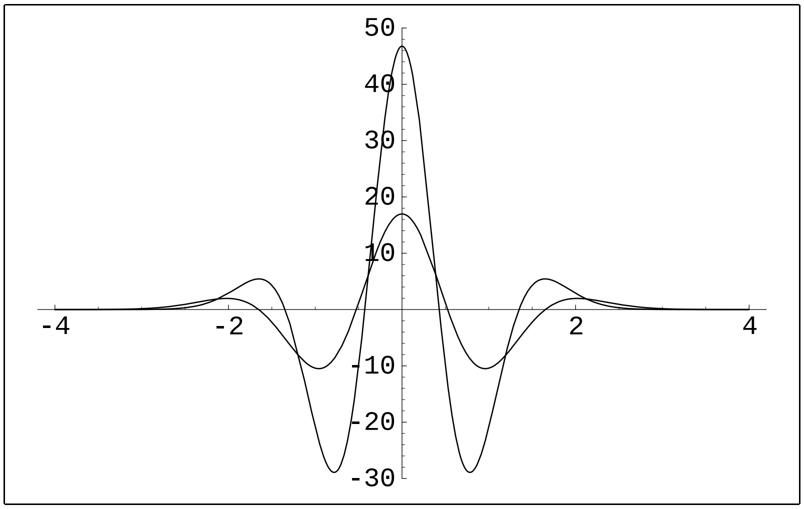

FIGURE 2: *Graphs of the function* $\left(\frac{1}{\epsilon}\right)^{\frac{n+1}{2}} w_n\left(\frac{y-0}{\sqrt{\epsilon}}\right)$ *in the case* $n=4$ *with* $\epsilon = \frac{1}{2}$ *and* $\epsilon = \frac{1}{3}$. *The latter is more compact, has enhanced extremal values, and illustrates the trend associated with the limiting process* $\epsilon \downarrow 0$.

2. A novel approach to the fractional calculus. Though our interest in (8) was initially specific to cases in which n is even, the formula makes good sense for *all* integral values of n. My intention here will be to explore implications of the observation that it can be *assigned* a natural meaning *even when* n *is non-integral*.

Citizens well-established within the community of higher functions are the so-called "parabolic cylinder functions" $D_\nu(x)$, often called "Weber functions" and less often a confusing variety of other names. The elaborate theory of such functions is summarized in all the standard handbooks.[2] One has

$$D_n(z) = e^{-\frac{1}{4}z^2} He_n(z) \quad : \quad n = 0, 1, 2, \ldots \tag{11}$$

giving

$$w_n(z) \equiv e^{-\frac{1}{4}z^2} D_n(z) \tag{12}$$

Returning with this information to (8), we have

$$D^n f(x) = \lim_{\epsilon\downarrow 0} \tfrac{1}{\sqrt{2\pi}} \left(\tfrac{1}{\epsilon}\right)^{\frac{n+1}{2}} \int_{-\infty}^{+\infty} f(y)\, e^{-\frac{1}{4}[\frac{y-x}{\sqrt{\epsilon}}]^2} D_n\left(\tfrac{y-x}{\sqrt{\epsilon}}\right) dy \tag{13}$$

[2] See, for example, Erdélyi *et al*, *Higher Transcendental Functions II* (1953) Chapter 8; Abramowitz & Stegun, *Handbook of Mathematical Functions* (1964) Chapter 19; Magnus & Oberhettinger, *Formulas & Theorems for the Functions of Mathematical Physics* (1954) Chapter 6, §3; Spanier & Oldham, *An Atlas of Functions* (1987) Chapter 46.

The interesting point to which I would draw attention is that $D_\nu(z)$ is well-defined for *all* real values of ν. Does it therefore make sense to write (say)

$$D^{\frac{1}{2}}f(x) = \lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\epsilon}\big)^{\frac{3}{4}}\int_{-\infty}^{+\infty} f(y)\,e^{-\frac{1}{4}[\frac{y-x}{\sqrt{\epsilon}}]^2}D_{\frac{1}{2}}\big(\tfrac{y-x}{\sqrt{\epsilon}}\big)\,dy \qquad ? \quad (14.1)$$

$$D^{-1}f(x) = \lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\epsilon}\big)^{0}\int_{-\infty}^{+\infty} f(y)\,e^{-\frac{1}{4}[\frac{y-x}{\sqrt{\epsilon}}]^2}D_{-1}\big(\tfrac{y-x}{\sqrt{\epsilon}}\big)\,dy \qquad ? \quad (14.2)$$

$$\vdots$$

Is it perhaps possible to recover (some natural variant of) the entire "fractional calculus"[3] from (some natural variant of) (13)?

To test the merit of the idea, we look to (14.2). One has

$$D_{-1}(z) = \sqrt{\tfrac{\pi}{2}}e^{+\frac{1}{4}z^2}\,\text{erfc}\big(\tfrac{z}{\sqrt{2}}\big) \quad \text{with} \quad \text{erfc}\,(x) \equiv \tfrac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

giving

$$D^{-1}f(x) = \int_{-\infty}^{+\infty} f(y)\Big\{\lim_{\epsilon\downarrow 0}\tfrac{1}{2}\text{erfc}\big(\tfrac{y-x}{\sqrt{2\epsilon}}\big)\Big\}\,dy \tag{15}$$

and it is to make sense of the expression within brackets that I now digress:

I had occasion recently to draw attention[4] to the computational power of certain techniques made available by the intimacy of the relationship between the Heaviside step function $\theta(x-a)$ and the Dirac spike $\delta(x-a)$:

$$\theta(x-a) = \int_{-\infty}^{x}\delta(\xi-a)\,d\xi = \begin{cases} 0 & \text{for } x < a \\ \frac{1}{2} & \text{at } \ x = a \\ 1 & \text{for } x > a \end{cases}$$
$$\Updownarrow$$
$$\tfrac{d}{dx}\theta(x-a) = \delta(x-a)$$

It is widely appreciated that the δ-function admits in principle of infinitely many representations (in the sense $\delta(x-a) = \lim_{\epsilon\downarrow 0}\delta(x-a;\epsilon)$), and that some of these are actually/indispensably useful; among the representations most frequently encountered in practical work are

$$\delta(x-a;\epsilon) = \begin{cases} \frac{1}{2\epsilon} \text{ if } a-\epsilon < x < a+\epsilon, \text{ and 0 otherwise; else} \\ \frac{1}{\sqrt{2\pi\epsilon}}\exp\big\{-\frac{1}{2\epsilon}(x-a)^2\big\}; \qquad \text{else} \\ \frac{1}{\pi x}\sin(x/\epsilon); \qquad \text{else} \ldots \end{cases}$$

[3] For an account of this subject, and references to its small literature, see "Construction & physical application of the fractional calculus" (1997).

[4] "Formal theory of singular functions" (1997).

Less widely appreciated is the elementary fact that each such representation gives rise to an associated "representation of the θ-function," in the sense

$$\theta(x-a) = \lim_{\epsilon \downarrow 0} \theta(x-a;\epsilon) \quad \text{with} \quad \theta(x-a;\epsilon) \equiv \int_{-\infty}^{x} \delta(\xi - a;\epsilon)\, d\xi \tag{16}$$

From the Gaussian representation we are led, for example, to

$$\begin{aligned}
\theta(x-a) &= \lim_{\epsilon \downarrow 0} \int_{-\infty}^{x} \tfrac{1}{\sqrt{2\pi\epsilon}}\, e^{-\frac{1}{2\epsilon}(\xi-a)^2} d\xi \\
&= \lim_{\epsilon \downarrow 0} \left\{ \int_{-\infty}^{\infty} - \int_{x}^{\infty} \right\} \tfrac{1}{\sqrt{2\pi\epsilon}}\, e^{-\frac{1}{2\epsilon}(\xi-a)^2} d\xi \\
&= \lim_{\epsilon \downarrow 0} \left\{ 1 - \tfrac{1}{\sqrt{\pi}} \int_{\frac{x-a}{\sqrt{2\epsilon}}}^{\infty} e^{-t^2} dt \right\} \\
&= \lim_{\epsilon \downarrow 0} \left\{ 1 - \tfrac{1}{2}\operatorname{erfc}\big(\tfrac{x-a}{\sqrt{2\epsilon}}\big) \right\}
\end{aligned}$$

which by appeal to the "reflection formula" $\operatorname{erfc}(-x) = 2 - \operatorname{erfc}(x)$ becomes

$$= \lim_{\epsilon \downarrow 0} \tfrac{1}{2}\operatorname{erfc}\big(\tfrac{a-x}{\sqrt{2\epsilon}}\big) \tag{17}$$

The meaning of this striking result is illustrated in Figure 3. On the right side of (17) we encounter a particular "parameterized sigmoid family" of functions $\theta(x-a;\epsilon)$. The interesting point—formerly too obvious to mention, but now brought emphatically to our attention—is that we could start with *any* such family and (by simple differention) recover an associated representation of $\delta(x-a)$; in short, the association

$$\text{representation of } \delta(x-a) \iff \text{representation of } \theta(x-a)$$

is a two-way street, most easily traversed in the reverse ($\Leftarrow$) of the standard direction.

Returning now with (17) to (15), we have

$$\begin{aligned}
D^{-1}f(x) &= \int_{-\infty}^{+\infty} f(y)\theta(x-y)\, dy \\
&= \int_{-\infty}^{x} f(y)\, dy
\end{aligned}$$

which is gratifying, since application of D gives

$$D^{+1}D^{-1}f(x) = f(x)$$

as required by the Fundamental Theorem of the Calculus.

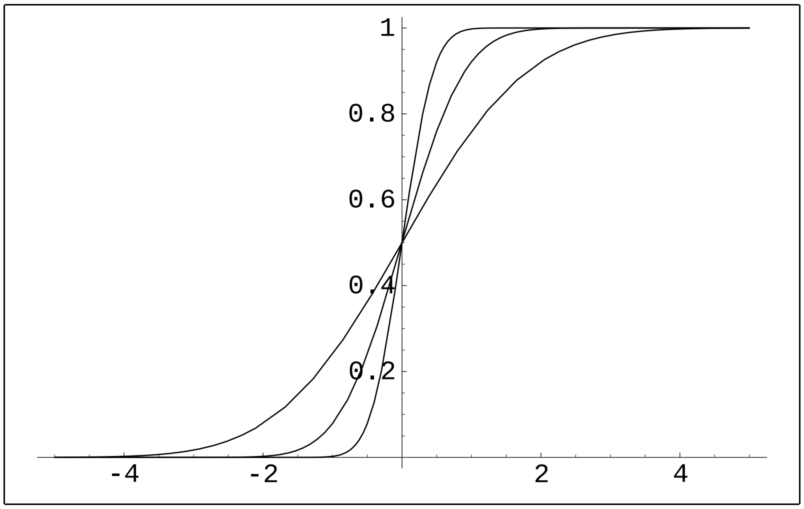

FIGURE 3: *Representation of $\theta(x)$ as the limit of a sequence of complementary error functions $\frac{1}{2}\,erfc\big(\frac{a-x}{\sqrt{2\epsilon}}\big)$. In the figure I have set $\epsilon = 2, 1, \frac{1}{2}$. The sigmoid curve rises ever more abruptly as $\epsilon \downarrow 0$.*

Thus encouraged, I step back now to contemplate the broad outlines of a formally *unified theory of differentiation/integration*, then step again closer to the easel to examine the feasibility of using parabolic cylinder functions to construct an *interpolating realization* of such a theory. By way of preparation for the former undertaking—taking motivation from the observation that while sign alternation is a "characteristic artifact of iterated integration-by-parts" it is not a characteristic artifact of iterated integration—I show how one can expunge the $(-)^n$-factor from (5).

If $\delta(x)$ is considered to be an even function (limit of a sequence of even functions) then $\delta^{(n)}(x)$ is odd/even according as n is odd/even, from which it follows that (5) can be rewritten

$$\begin{aligned} D^n f(x) &= \int f(y)\,\delta^{(n)}(x-y)\,dy \qquad (18)\\ &\equiv f\ast\delta^{(n)} \ \ : \ \text{convolution (or "Faltung"), in standard form} \end{aligned}$$

We are led thus to the scheme summarized in Table 1, the substance of which (compare (8)) can be notated

$$D^n f = f\ast W_n \quad \text{with} \quad W_n(x) = \begin{cases} \delta^{(2)}(x) & : \quad n = +2 \\ \delta^{(1)}(x) & : \quad n = +1 \\ \delta(x) & : \quad n = \ \ 0 \\ \theta(x) & : \quad n = -1 \\ x^1\theta(x) & : \quad n = -2 \\ \frac{1}{2!}\,x^2\theta(x) & : \quad n = -3 \end{cases} \qquad (19)$$

$$
\begin{aligned}
&\vdots\\
D^2 f(x) &= \int \delta^{(2)}(x-y) f(y)\,dy\\
D^1 f(x) &= \int \delta^{(1)}(x-y) f(y)\,dy\\
D^0 f(x) &= \int \quad \delta(x-y) f(y)\,dy = f(x)\\
D^{-1} f(x) &= \int \theta(x-y_1) f(y_1)\,dy_1\\
&= \int^x f(y)\,dy = \int^x (x-y)^0 f(y)\,dy\\
D^{-2} f(x) &= \iint \theta(x-y_2)\theta(y_2-y_1) f(y_1)\,dy_1 dy_2\\
&= \int^x\!\!\int^{y_2} f(y_1)\,dy_1 dy_2 = \int^x (x-y)^1 f(y)\,dy\\
D^{-3} f(x) &= \iiint \theta(x-y_3)\theta(y_3-y_2)\theta(y_2-y_1) f(y_1)\,dy_1 dy_2 dy_3\\
&= \int^x\!\!\int^{y_3}\!\!\int^{y_2} f(y_1)\,dy_1 dy_2 dy_3 = \tfrac{1}{2!}\int^x (x-y)^2 f(y)\,dy\\
&\vdots
\end{aligned}
$$

TABLE 1: *Inverted refinement and extension of the equations which culminated in (5). The upper/lower implicit limits are $\pm\infty$, but this is forced upon us only because we have (arbitrarily) adopted a Gaussian representation of the δ-function. The differentiation operator D, when applied to any entry, yields the entry next higher in the table, while the integration operator D^{-1} yields the entry next lower. The elegant (but elementary) identity*

$$\int^x\!\!\int^{y_n}\!\!\int^{y_{n-1}}\!\!\cdots\int^{y_2} f(y_1)\,dy_1 dy_2\cdots dy_n = \tfrac{1}{(n-1)!}\int^x (x-y)^{n-1} f(y)\,dy$$

is due to Cauchy.

Of course, functions are only *conditionally* differentiable/integrable; it is easy to think up functions which make nonsense of any of the preceding statements. Sooner or later, one must characterize the class of functions to which purported formulæ are claimed to pertain. My procedure will be to proceed formally, and to address the class-characterization problem (if at all) only after the fact. That understood...

It is striking that the theory of integration appears on these representation-independent formal grounds to be simpler—in the sense that it provides more obvious means of escape from the "tyranny of the integers"—than the theory of differentiation; it becomes natural, in view of the foregoing, to write

$$\left.\begin{aligned} D^{-\nu}f(x) &= \tfrac{1}{\Gamma(\nu)}\int^x (x-y)^{\nu-1}f(y)\,dy \\ &= f{\ast}W_{-\nu} \quad\text{with}\quad W_{-\nu}(x)=\tfrac{1}{\Gamma(\nu)}\,x^{\nu-1}\theta(x)\end{aligned}\right\} \;:\; \nu>0 \tag{20}$$

By formal extension

$$\downarrow$$

$$D^{\nu}f(x) = f{\ast}W_{\nu} \quad\text{with}\quad W_{\nu}(x)=\tfrac{1}{\Gamma(-\nu)}\,x^{-\nu-1}\theta(x) \tag{21}$$

which (on account of a familiar property of the gamma function) becomes singular at precisely the ν-values for which the meaning of $D^\nu f(x)$ should be least exceptionable. Look, however, to the associated (Gaussian) representation theory:

We have seen[5] it to be an implication of

$$\delta(y-x) = \lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{2\pi\epsilon}}e^{-\frac{1}{2\epsilon}(y-x)^2}$$

—i.e., of the representation

$$W_0(x) = \lim_{\epsilon\downarrow 0}\tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\epsilon}\big)^{\frac{1}{2}}e^{-\frac{1}{4}[-\frac{x}{\sqrt{\epsilon}}]^2}D_0\big(-\tfrac{x}{\sqrt{\epsilon}}\big)$$

$$D_0\big(-\tfrac{x}{\sqrt{\epsilon}}\big) = e^{-\frac{1}{4}[-\frac{x}{\sqrt{\epsilon}}]^2}$$

—that

$$\begin{aligned}\delta^{(n)}(x-y) &= \lim_{\epsilon\downarrow 0}\tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\epsilon}\big)^{\frac{n+1}{2}}e^{-\frac{1}{2}[\frac{y-x}{\sqrt{\epsilon}}]^2}He_n\big(\tfrac{y-x}{\sqrt{\epsilon}}\big)\\ &= \lim_{\epsilon\downarrow 0}\tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\epsilon}\big)^{\frac{n+1}{2}}e^{-\frac{1}{4}[\frac{y-x}{\sqrt{\epsilon}}]^2}D_n\big(\tfrac{y-x}{\sqrt{\epsilon}}\big) \quad:\quad n=0,1,2,\ldots\end{aligned} \tag{22}$$

which give

$$\left.\begin{aligned} W_n(x) &= \lim_{\epsilon\downarrow 0}W_n(x;\epsilon)\\ W_n(x;\epsilon) &\equiv \lim_{\epsilon\downarrow 0}\tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\epsilon}\big)^{\frac{n+1}{2}}e^{-\frac{1}{4}[-\frac{x}{\sqrt{\epsilon}}]^2}D_n\big(-\tfrac{x}{\sqrt{\epsilon}}\big)\end{aligned}\right\} \tag{23}$$

By formal extension

$$\left.\begin{aligned} D^{\nu}f(x) &= \int f(y)W_\nu(x-y)\,dy\\ W_\nu(x-y) &\equiv \lim_{\epsilon\downarrow 0}\underbrace{\tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\epsilon}\big)^{\frac{\nu+1}{2}}e^{-\frac{1}{4}[-\frac{y-x}{\sqrt{\epsilon}}]^2}D_\nu\big(\tfrac{y-x}{\sqrt{\epsilon}}\big)}_{W_\nu(x-y;\epsilon)}\end{aligned}\right\} \tag{24}$$

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[5] I take my information from (6.n) and (13).

One expects it to be an implication of (24)—as, indeed, of its analog in *any* representation—that

$$\left(\tfrac{d}{dx}\right)^n D^\nu f(x) = \int f(y) W_{n+\nu}(x-y)\,dy \tag{25}$$

and is therefore gratified to observe that (25) follows directly from what Spanier & Oldham (at **46:10:3**) call the "elegant relationship"

$$\left(-\tfrac{d}{dz}\right)^n \left\{e^{-\frac{1}{4}z^2} D_\nu(z)\right\} = e^{-\frac{1}{4}z^2} D_{n+\nu}(z) \quad : \quad n = 0, 1, 2, \ldots \tag{26}$$

For (26), when brought to (24), gives

$$\left(-\tfrac{d}{dz}\right)^n W_\nu(x;\epsilon) = W_{n+\nu}(x;\epsilon) \text{ for all } \epsilon, \text{ therefore also in the limit } \epsilon \downarrow 0$$

One can approach the same issue also from another angle, but before I consider the instructive details I interpose this important remark: Generally (as in the elementary theory of δ-functions) one understands $\int f(y)W_\nu(x-y)\,dy$ to be by intent an

$$\text{abbreviated allusion to the process } \lim_{\epsilon\downarrow 0} \int f(y) W_\nu(x-y;\epsilon)\,dy \tag{27}$$

Integrate, then proceed to the limit. But in some favorable cases the sequence can be reversed, and it makes *literal* sense to write $\int f(y)W_\nu(x-y)\,dy$. That understood... we expect to be able to extract a statement of the form

$$\int \left\{ \int f(y) W_\nu(z-y;\epsilon_1)\,dy \right\} W_n(x-z;\epsilon_2)\,dz = \int f(y) W_{n+\nu}(x-y;\epsilon)\,dy$$

or again

$$\int W_n(x-z;\epsilon_1) W_\nu(z-y;\epsilon_2)\,dz = W_{n+\nu}(x-y;\epsilon)$$

from the theory of D-functions, but the following calculation shows that the preceding equation makes literal *good sense even in the limit* in an identifiable class of cases. Working from (21), we have

$$\begin{aligned} &\int W_\mu(x-z) W_\nu(z-y)\,dz \\ &\qquad = \tfrac{1}{\Gamma(-\mu)\Gamma(-\nu)} \int (x-z)^{-\mu-1}(z-y)^{-\nu-1}\theta(x-z)\theta(z-y)\,dz \\ &\qquad = \tfrac{1}{\Gamma(-\mu)\Gamma(-\nu)} \int_y^x (x-z)^{-\mu-1}(z-y)^{-\nu-1}\,dz \end{aligned}$$

for which *Mathematica* reports the pretty result

$$= \tfrac{1}{\Gamma(-\mu-\nu)} (x-y)^{-\mu-\nu-1}\theta(x-y) \tag{28}$$

subject only to the requirement that $\Re(\mu) < 0$ and $\Re(\nu) < 0$. It is, in particular, not required that either μ or ν be the negative of an integer. The implication is that (21) gives rise to a theory of fractional integration in which the anticipated iteration law $D^{-\mu}D^{-\nu} = D^{-(\mu+\nu)}$ is satisfied (here μ and ν are taken to be positive, and the minus signs emphasize that we refer to iterated *integration*).

So we are brought again to the perception that "it is easier to fractionally integrate than to fractionally differentiate," and that the latter operation is most naturally given meaning by one or the other (which? does it make a difference? [6]) of the schemes

$$\text{fractional derivative} = \begin{cases} \text{ordinary derivative of fractional integral, else} \\ \text{fractional integral of ordinary derivative} \end{cases}$$

Thus the "semiderivative:"

$$D^{\frac{1}{2}}f(x) = \begin{cases} D\Big\{D^{-\frac{1}{2}}f(x)\Big\} = D\Big\{\frac{1}{\Gamma(\frac{1}{2})}\int^x (x-y)^{-\frac{1}{2}}f(y)\,dy\Big\}, \text{ else} \\ D^{-\frac{1}{2}}\Big\{Df(x)\Big\} = \Big\{\frac{1}{\Gamma(\frac{1}{2})}\int^x (x-y)^{-\frac{1}{2}}f'(y)\,dy\Big\} \end{cases}$$

That we confront real options, and have acquired an obligation to recognize some delicate distinctions, becomes clear when we look to simple examples. Suppose $f(x) = x^0$: then

$$D^{\frac{1}{2}}x^0 = \begin{cases} D\Big\{\frac{1}{\sqrt{\pi}}\int_a^x \frac{1}{\sqrt{x-y}}\,dy = \frac{2\sqrt{x-a}}{\sqrt{\pi}}\Big\} = \frac{1}{\sqrt{\pi(x-a)}}, \text{ but} \\ D^{-\frac{1}{2}}\Big\{\text{------}\,0\,\text{------}\Big\} = 0 \end{cases}$$

which could not be more different; note also that the former expression becomes meaningless when we set $a = -\infty$, but at $a = 0$ reproduces a statement first obtained by Lacroix (1819) by the simplest of formal means. In the next simplest case $f(x) = x^1$ we obtain

$$D^{\frac{1}{2}}x = \begin{cases} D\Big\{\frac{1}{\sqrt{\pi}}\int_a^x \frac{y}{\sqrt{x-y}}\,dy = \frac{2\sqrt{x-a}(2x+a)}{3\sqrt{\pi}}\Big\} = \frac{2x-a}{\sqrt{\pi(x-a)}}, \text{ but} \\ D^{-\frac{1}{2}}\Big\{\text{------}\,1\,\text{------}\Big\} = \frac{2\sqrt{x-a}}{\sqrt{\pi}} \end{cases}$$

which are again meaningless at $a = -\infty$ and distinct for most values of a, but become coincident at $a = 0$, where they reproduce Lacroix' $D^{\frac{1}{2}}x = 2\sqrt{\frac{x}{\pi}}$.

How can it be that $D\cdot D^{-\frac{1}{2}}$ and $D^{-\frac{1}{2}}\cdot D$ yield different results? An answer to that question is provided in §1 of an essay already cited.[3] Quite generally,

$$\begin{aligned} (\text{differentiation})\cdot(\text{differentiation}) &= D^{\text{law of exponents}} \\ (\text{differentiation})\cdot(\text{integration}) &= D^{\text{law of exponents}} \\ (\text{integration})\cdot(\text{integration}) &= D^{\text{law of exponents}} \end{aligned}$$

but

$$(\text{integration})\cdot(\text{differentiation}) = D^{\text{law of exponents}} + \text{extra terms}$$

[6] See immediately below.

It is for this reason that the fractional calculus is standardly considered to rest upon the principle

$$\text{(differentiation)} \cdot \text{(fractional integration)} = D^{\text{law of exponents}} \tag{29}$$

That principle can be detected in the design of (25), which I am now motivated to write in the more emphatic form

$$\left(\tfrac{d}{dx}\right)^n D^{-\nu} f(x) = \int f(y) W_{n-\nu}(x-y)\,dy \quad : \quad \nu > 0 \tag{30}$$

The phrase "fractional integration" is susceptible, as we have just noticed, to an infinite range of interpretations, depending upon the value assigned to the "fiducial point" a in the formula

$$D^{-\nu} f(x) = \tfrac{1}{\Gamma(\nu)} \int_a^x (x-y)^{\nu-1} f(y)\,dy$$

There is much to recommend the common practice of setting $a = 0$; then

$$\begin{aligned} D^{-\nu} f(x) &= \tfrac{1}{\Gamma(\nu)} \int_0^x (x-y)^{\nu-1} f(y)\,dy \\ &\equiv \text{"Riemann-Liouville transform" of } f(x) \\ &= \tfrac{1}{\Gamma(\nu)} \int_{-\infty}^{+\infty} f(y) \Big\{ \tfrac{1}{\Gamma(\nu)} (x-y)^{\nu-1} \theta\big(y(x-y)\big) \Big\}\,dy \end{aligned}$$

where $g(y) \equiv y(x-y)$ is, by design, positive only on the interval $0 < y < x$.[7] But we have (at (20)) been led by our interest in Gaussian representation theory to set $a = -\infty$, giving

$$\begin{aligned} D^{-\nu} f(x) &= \tfrac{1}{\Gamma(\nu)} \int_{-\infty}^x (x-y)^{\nu-1} f(y)\,dy \\ &\equiv \text{"Liouville transform" of } f(x) \\ &= \tfrac{1}{\Gamma(\nu)} \int_{-\infty}^{+\infty} f(y) \Big\{ \tfrac{1}{\Gamma(\nu)} (x-y)^{\nu-1} \theta(x-y) \Big\}\,dy \end{aligned} \tag{31}$$

My examples serve to demonstrate that *functions which are Riemann-Liouville transformable may not be Liouville transformable*; the extent of the class of fractionally integrable functions is contingent upon specification of the fiducial point. In principle one has

$$\int_{-\infty}^x - \int_{-\infty}^a = \int_a^x$$

but this is of limited help in practice: it is not usually satisfactory to write

$$\big\{\text{finite} + \infty\big\} - \big\{\infty\big\} = \big\{\text{finite}\big\}$$

[7] Note that the integral is *not* quite of convolutory form.

Liouville's equation (31) was acquired (as were its cousins) by fractional generalization of (19), and recently I have, in that historical tradition, proceeded as though the distributions $W_\nu(x)$ were known. Now I ask whether (31) can be acquired from (24); i.e., from the generalized Gaussian representation theory I have elsewhere been at such pains to construct. Can one show that if

$$W_\nu(x;\epsilon) \equiv \tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\epsilon}\big)^{\frac{\nu+1}{2}}e^{-\frac{1}{4}[-\frac{x}{\sqrt{\epsilon}}]^2}D_\nu\big(-\tfrac{x}{\sqrt{\epsilon}}\big)$$

then

$$W_\nu(x) \equiv \lim_{\epsilon\downarrow 0} W_\nu(x;\epsilon) = \text{distributions reported at (19)?}$$

This we have accomplished thus far only in the cases $\nu \in \big\{-1, 0, 1, 2, \dots\big\}$. For reasons implicit in (29), I look here only to the fractional *integral* aspects of the problem (and for emphasis adjust my ν-sign convention). I want to show that if

$$W_{-\nu}(x;\epsilon) \equiv \tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\epsilon}\big)^{\frac{1-\nu}{2}}e^{-\frac{1}{4}[-\frac{x}{\sqrt{\epsilon}}]^2}D_{-\nu}\big(-\tfrac{x}{\sqrt{\epsilon}}\big) \quad : \quad \nu > 0 \tag{32}$$

then $W_{-\nu}(x) \equiv \lim_{\epsilon\downarrow 0} W_{-\nu}(x;\epsilon) = \frac{1}{\Gamma(\nu)}\,x^{\nu-1}\,\theta(x)$. At (15) I was able in the case $\nu = 1$ to extract a proof from the leading member of the series of integral representations supplied by Spanier & Oldham (**46:4:4** and **40:13:1/2/3**):

$$D_{-1-n}(x) = 2^{\frac{n+1}{2}}e^{\frac{1}{4}x^2}\int_x^\infty \tfrac{1}{n!}\big(t - \tfrac{x}{\sqrt{2}}\big)^n e^{-t^2}\,dt \quad : \quad n = 0, 1, 2, \dots$$

Returning with this more elaborate information to (32), we have

$$W_{-\nu}(x;\epsilon) = \tfrac{1}{\sqrt{\pi}}(2\epsilon)^{\frac{\nu-1}{2}}\int_{-\frac{x}{\sqrt{2\epsilon}}}^\infty \tfrac{1}{(\nu-1)!}\big(t + \tfrac{x}{\sqrt{2\epsilon}}\big)^{\nu-1}e^{-t^2}\,dt \tag{33}$$

at least when ν is an integer: $\nu = 1, 2, \dots$ At $\nu = 1$ *Mathematica* returns

$$\begin{aligned} W_{-1}(x;\epsilon) = \tfrac{1}{2}\big[1 - \text{erf}\big(-\tfrac{x}{\sqrt{2\epsilon}}\big)\big] &= \tfrac{1}{2}\text{erfc}\big(-\tfrac{x}{\sqrt{2\epsilon}}\big)\\ &\downarrow\\ &= \theta(x) \ \text{ as } \ \epsilon \downarrow 0\end{aligned}$$

while at $\nu = 2$ and $\nu = 3$ *Mathematica* gives

$$\begin{aligned} W_{-2}(x;\epsilon) &= \tfrac{1}{2}x\big[1 - \text{erf}\big(-\tfrac{x}{\sqrt{2\epsilon}}\big)\big] + \sqrt{2\epsilon}\,e^{-[\frac{x}{\sqrt{2\epsilon}}]^2}\\ &\downarrow\\ &= x\,\theta(x)\end{aligned}$$

$$\begin{aligned} W_{-3}(x;\epsilon) &= \tfrac{1}{4}\big(x^2 + \epsilon\big)\big[1 - \text{erf}\big(-\tfrac{x}{\sqrt{2\epsilon}}\big)\big] + \tfrac{1}{4\sqrt{\pi}}x\sqrt{2\epsilon}\,e^{-[\frac{x}{\sqrt{2\epsilon}}]^2}\\ &\downarrow\\ &= \tfrac{1}{2!}x^2\theta(x)\end{aligned}$$

These results are encouraging, but make clear that more powerful methods would be required to establish the general result; since my present effort is only exploratory, I will decline the invitation—posed by (32)—to invest major time snooping through the handbooks in quest of servicable identities, and instead look to few more accessible details. After all, it is from imagery provided by the process (27)—therefore from the functions $W_\nu(x - y;\epsilon)$ rather than from their singular limits—that representation theory acquires much of its interest.

It is to bring attention to a puzzle implicit in some of the preceding material that I look now again to the semiderivative:

$$D^{\frac{1}{2}}f(x) \equiv D\int_a^x f(y)\cdot \tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x-y}}\,dy \tag{34}$$

In a couple of illustrative cases we have integrated-then-differentiated, and enjoyed seeming success; it is by that procedure that (for future reference) I have generated the following short table:

$$D^{\frac{1}{2}}x^0 = \tfrac{1}{\sqrt{\pi(x-a)}} \tag{35.0}$$

$$D^{\frac{1}{2}}x^1 = \tfrac{1}{\sqrt{\pi(x-a)}}\big\{2x-a\big\} \tag{35.1}$$

$$D^{\frac{1}{2}}x^2 = \tfrac{1}{3\sqrt{\pi(x-a)}}\big\{8x^2-4ax-a^2\big\} \tag{35.2}$$

$$D^{\frac{1}{2}}x^3 = \tfrac{1}{5\sqrt{\pi(x-a)}}\big\{16x^3-8ax^2-2a^2x-a^3\big\} \tag{35.3}$$

$$D^{\frac{1}{2}}x^4 = \tfrac{1}{35\sqrt{\pi(x-a)}}\big\{128x^4-64ax^3-16a^2x^2-8s^3x-5a^4\big\} \tag{35.4}$$

$$D^{\frac{1}{2}}x^5 = \tfrac{1}{63\sqrt{\pi(x-a)}}\big\{256x^2-128ax^4-32a^2x^3-16a^3x^2-10a^4x-7a^5\big\}$$

$$\vdots$$

$$\begin{aligned} D^{\frac{1}{2}}x^p &= \tfrac{\Gamma(p+1)}{\Gamma(p+1-\frac{1}{2})}x^{p-\frac{1}{2}} \\ &\qquad + \tfrac{a^{p+1}}{x}\Big\{\tfrac{1}{\sqrt{\pi(x-a)}} - \tfrac{1+2p}{2(1+p)}\tfrac{1}{\sqrt{\pi x}}\,{}_2\mathsf{F}_1\big[\tfrac{1}{2},1+p,p+2,\tfrac{a}{x}\big]\Big\} \end{aligned} \tag{35.p}$$

The expressions on the right blow up as $a\downarrow -\infty$ (Liouville transform), but conform to Lacroix' formula

$$D^m x^p = \tfrac{\Gamma(p+1)}{\Gamma(p+1-m)}x^{p-m}$$

at $a=0$ (Riemann-Liouville transform).

Looking back again to (34), I ask: Can one do the differentiation first, *prior* to actual specification of $f(\bullet)$? The prospect of success might appear to be dim, since the right side of

$$\tfrac{d}{dx}\int_a^x f(y)\cdot\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x-y}}\,dy = f(y)\cdot\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x-y}}\Big|_{y\to x} - \int_a^x f(y)\cdot\tfrac{1}{2\sqrt{\pi}}\tfrac{1}{(x-y)^{3/2}}\,dy$$

is singular nonsense on its face, and makes only precarious good sense when examined more closely: look again to the examples of the form $f(x)=x^p$; we have

$$\begin{aligned} \tfrac{d}{dx}\int_a^x y^0\cdot\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x-y}}\,dy &= \tfrac{1}{\sqrt{\pi(x-y)}}\Big|^{y\uparrow x} - \tfrac{1}{\sqrt{\pi(x-y)}}\Big|_{y\downarrow a}^{y\uparrow x} \\ &= \tfrac{1}{\sqrt{\pi(x-a)}} + \lim_{y\uparrow x}\Big\{\tfrac{1-1}{\sqrt{\pi(x-y)}}\Big\} \end{aligned} \tag{36.0}$$

$$\begin{aligned}\tfrac{d}{dx}\int_a^x y^1\cdot\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x-y}}\,dy &= \frac{y}{\sqrt{\pi(x-y)}}\bigg|^{y\uparrow x} - \frac{2x-y}{\sqrt{\pi(x-y)}}\bigg|^{y\uparrow x}_{y\downarrow a}\\ &= \frac{2x-a}{\sqrt{\pi(x-a)}} - \lim_{y\uparrow x}\Big\{\frac{2x-2y}{\sqrt{\pi(x-y)}}\Big\} \end{aligned}\tag{36.1}$$

$$\begin{aligned}\tfrac{d}{dx}\int_a^x y^2\cdot\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x-y}}\,dy &= \frac{y^2}{\sqrt{\pi(x-y)}}\bigg|^{y\uparrow x} - \frac{8x^2-4xy-y^2}{3\sqrt{\pi(x-y)}}\bigg|^{y\uparrow x}_{y\downarrow a}\\ &= \frac{8x^2-4ax-a^2}{3\sqrt{\pi(x-a)}} - \lim_{y\uparrow x}\Big\{\frac{8x^2-4xy-4y^2}{3\sqrt{\pi(x-y)}}\Big\} \\ &\;\;\vdots\end{aligned}\tag{36.2}$$

The expressions lim $\{$etc.$\}$ come to us in the form $\{\infty-\infty\}$, but can be assigned unambiguous values by rearrangement: at (36.0) we encounter $\lim_{y\uparrow x}\big\{\frac{0}{\sqrt{x-y}}\big\}$, which clearly vanishes. At (36.1) we encounter $\lim_{y\uparrow x}\big\{\sqrt{x-y}\big\}$ and at (36.2) $\lim_{y\uparrow x}\big\{(2x+y)\sqrt{x-y}\big\}$, both of which also vanish. So the results to which we are led are in fact finite, and in agreement with (35).

The element of delicacy can be removed from the preceding line of argument by a simple regularization procedure[8] Write

$$\begin{aligned}\tfrac{d}{dx}\int_a^x f(y)\cdot\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x-y}}\,dy &= \lim_{\epsilon\downarrow 0}\tfrac{d}{dx}\int_a^x f(y)\cdot\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x+\epsilon-y}}\,dy\\ &= \lim_{\epsilon\downarrow 0}\Big\{f(x)\cdot\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{\epsilon}} - \int_a^x f(y)\cdot\tfrac{1}{2\sqrt{\pi}}\tfrac{1}{(x+\epsilon-y)^{3/2}}\,dy\Big\}\end{aligned}\tag{37}$$

which *Mathematica* finds quite palatable; for example, at $f(x)=x^2$ we are told that

$$\begin{aligned}D^{\frac{1}{2}}x^2 &= \lim_{\epsilon\downarrow 0}\Big\{\tfrac{x^2}{\sqrt{\pi\epsilon}} - \tfrac{3x^2+12\epsilon x+8\epsilon^2}{3\sqrt{\pi\epsilon}} + \tfrac{8(x+\epsilon)^2-4a(x+\epsilon)-a^2}{3\sqrt{\pi(x+\epsilon-a)}}\Big\}\\ &= \tfrac{1}{3\sqrt{\pi(x-a)}}\big\{8x^2-4ax-a^2\big\}\end{aligned}$$

in precise agreement with (35.2). In the general case *Mathematica* gives

$$\begin{aligned}D^{\frac{1}{2}}x^p = \lim_{\epsilon\downarrow 0}\Big\{\tfrac{x^p}{\sqrt{\pi\epsilon}} &- \tfrac{x^{1+p}(\frac{\epsilon}{x+\epsilon})^{3/2}{}_2\mathsf{F}_1[1+p,\frac{3}{2},2+p,\frac{x}{x+\epsilon}]}{2(1+p)\sqrt{\pi}\epsilon^{3/2}}\\ &+ \tfrac{a^{1+p}(\frac{x+\epsilon-a}{x+\epsilon})^{3/2}{}_2\mathsf{F}_1[1+p,\frac{3}{2},2+p,\frac{a}{x+\epsilon}]}{2(1+p)\sqrt{\pi}(x+\epsilon-a)^{3/2}}\Big\}\end{aligned}$$

but warns that it has been "unable to check for convergence;" it does *not* give back (35.p) in the limit, but when p is set equal to an integer does (after a vast amount of simplification) eventually reproduce (35.1/2/3/4/5/$\cdots$).

---

[8] The idea here is a variant of that used standardly to assign value to the "principal part" (when it exists) of an integral with a singular integrand.

It is as a first step back toward representation theory that I undertake now to re-orchestrate the preceeding material. Write

$$\begin{aligned}\tfrac{d}{dx}\int_a^x f(y)\cdot\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x-y}}\,dy &= \tfrac{d}{dx}\int_{-\infty}^{+\infty} f(y)\cdot\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x-y}}\theta\big((y-a)(x-y)\big)\,dy\\ &= \int f(y)\cdot\tfrac{\partial}{\partial x}\Big\{\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x-y}}\theta\big((y-a)(x-y)\big)\Big\}\,dy\end{aligned}$$

where $g(y;a,x)\equiv(y-a)(x-y)$ describes a down-turned parabola which crosses the axis at obvious points, and is positive only if $a<y<x$. Evidently

$$D^{\frac{1}{2}}f(x)=\int f(y)\Big\{\Big[-\tfrac{1}{2\sqrt{\pi}(x-y)^{3/2}}+\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x-y}}\tfrac{\partial}{\partial x}\Big]\theta\big((y-a)(x-y)\big)\Big\}\,dy$$

which is just a fancy way of saying some things already said; from

$$\begin{aligned}\theta\big((y-a)(x-y)\big)&=\theta(y-a)-\theta(y-x)\\ &\Downarrow\\ \tfrac{\partial}{\partial x}\theta\big((y-a)(x-y)\big)&=\delta(y-x)\end{aligned}$$

we obtain

$$D^{\frac{1}{2}}f(x)=\int f(y)\Big\{\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x-y}}\delta(y-x)-\tfrac{1}{2\sqrt{\pi}(x-y)^{3/2}}\theta\big((y-a)(x-y)\big)\Big\}$$

which is simply a disguised rewrite of the equation that led to (36). And susceptible to the same fragility of meaning. Which we render more robust by the same mechanism, writing

$$D^{\frac{1}{2}}f(x)=\lim_{\epsilon\downarrow 0}\int f(y)\tfrac{\partial}{\partial x}\Big\{\tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{x+\epsilon-y}}\theta\big((y-a)(x-y)\big)\Big\}\,dy \tag{38}$$

Which—and this is the point—serves to clarify the delicate meaning of the statement

$$D^{\frac{1}{2}}f(x)=\int f(y)W_{\frac{1}{2}}(y;a,x)\,dy \tag{39}$$

and to place semidifferentiation more nearly on the same formal footing as semiintegration than was accomplished by (30).

Pretty clearly, the calculus sketched above is in no essential respect special to *semi*differentiation, but can be used to lend sharpened meaning to the notion of a fractional derivative of *any* positive order. Look, for example, to the case $\nu=\frac{8}{5}$: the procedure would be to write

$$\begin{aligned}D^{\frac{8}{5}}f(x)&=D^2D^{-\frac{2}{5}}f(x)\\ &=\big(\tfrac{d}{dx}\big)^2\int_a^x f(y)\tfrac{1}{\Gamma(\frac{2}{5})}(x-y)^{\frac{2}{5}-1}\,dy\\ &=\lim_{\epsilon\downarrow 0}\int f(y)\cdot\big(\tfrac{\partial}{\partial x}\big)^2\Big\{\tfrac{1}{\Gamma(\frac{2}{5})}\tfrac{1}{(x+\epsilon-y)^{3/5}}\theta\big((y-a)(x-y)\big)\Big\}\,dy\end{aligned}$$

...and so it goes in general. We recover (20) as $a\downarrow-\infty$.

Go to that limit, where we have waiting for us the rich resources of a "Gaussian representation theory."[9] We expect that theory to be (when $\epsilon \neq 0$) "function-theoretically nice," and to *spontaneously supply* means for dealing with the fussy points discussed above. But our faithful guinea pigs x^p abandon us in the limit $a \downarrow -\infty$ (x^p remains integrally differentiable but not fractionally, because non-integrable); their place, for the purposes of this discussion, will be assigned to the functions $f(x) = (e^x)^p = e^{px}$. Trivially

$$D^n e^{px} = p^n e^{px} \quad : \quad n = 0, 1, 2, \ldots \tag{40}$$

while (20) entails

$$D^{-\nu} e^{px} = \tfrac{1}{\Gamma(\nu)} \int_{-\infty}^{x} (x-y)^{\nu-1} e^{py}\, dy = p^{-\nu} e^{px}$$

giving

$$D^{n-\nu} e^{px} = p^{n-\nu} e^{px} \tag{41}$$

which serves in the simplest imaginable way to provide a natural "fractional generalization" of (39).[10]

I pause to confirm that our regularization procedure still works, even in this slightly modified setting. Proceeding in mimicry of (37), we have

$$\begin{aligned} D^{\frac{1}{2}} e^{px} &= \tfrac{d}{dx}\int_{-\infty}^{x} e^{py} \cdot \tfrac{1}{\sqrt{\pi}} \tfrac{1}{\sqrt{x-y}}\, dy = \lim_{\epsilon \downarrow 0} \tfrac{d}{dx}\int_{-\infty}^{x} e^{py} \cdot \tfrac{1}{\sqrt{\pi}} \tfrac{1}{\sqrt{x+\epsilon-y}}\, dy \\ &= \lim_{\epsilon\downarrow 0} \Big\{ e^{px} \cdot \tfrac{1}{\sqrt{\pi}} \tfrac{1}{\sqrt{\epsilon}} - \int_{-\infty}^{x} e^{py} \cdot \tfrac{1}{2\sqrt{\pi}} \tfrac{1}{(x+\epsilon-y)^{3/2}}\, dy \Big\} \\ &= \lim_{\epsilon\downarrow 0} \sqrt{p}\, e^{p(x+\epsilon)} \Big\{ 1 - \text{erf}(\sqrt{p\epsilon}\,) \Big\} \\ &= \sqrt{p}\, e^{px} \end{aligned}$$

—in precise agreement with (41).

---

[9] Representation theory appropriate to arbitrary a values (most particularly to the case $a = 0$) requires special discussion, and will be taken up on another occasion.

[10] It is perhaps well, in view of the striking simplicity of these equations, to notice that more complicated formulæ would have been obtained had we set $a = 0$: (40) is preserved unchanged, but

$$\begin{aligned} D^{-\nu} e^{px} &= p^{\ -\nu} e^{px} \Big\{ 1 - \tfrac{\Gamma(\nu, px)}{\Gamma(\nu)} \Big\} \\ D^1 D^{-\nu} e^{px} &= p^{1-\nu} e^{px} \Big\{ 1 - \tfrac{\Gamma(\nu, px)}{\Gamma(\nu)} \Big\} + p^{1-\nu} (px)^{-1+\nu} \tfrac{1}{\Gamma(\nu)} \\ &\ \ \vdots \end{aligned}$$

where the incomplete gamma function $\Gamma(\nu, z) \equiv \int_z^\infty t^{\nu-1} e^{-t}\, dt$ gives back the Euler gamma function $\Gamma(\nu)$ at $z = 0$.

We have acquired interest (see again (14)) in the following generalization of (13):

$$D^\nu e^{px} = \lim_{\epsilon\downarrow 0} \tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\epsilon}\big)^{\frac{\nu+1}{2}}\int_{-\infty}^{+\infty} e^{py}\, e^{-\frac{1}{4}[\frac{y-x}{\sqrt{\epsilon}}]^2} D_\nu\big(\tfrac{y-x}{\sqrt{\epsilon}}\big)\, dy$$

Erdélyi (in his §**8.3**) lists no fewer than ten integral representations of the functions $D_\nu(z)$, of which Spanier & Oldham (at **46:3:1** & **2**) quote two; by slight adjustment of those two, we have

$$e^{-\frac{1}{4}z^2}D_\nu(z) = \begin{cases} \sqrt{\tfrac{2}{\pi}}\displaystyle\int_0^\infty t^\nu e^{-\frac{1}{2}t^2}\cos\big(zt-\nu\tfrac{\pi}{2}\big)\,dt &:\quad \nu>-1\\ \tfrac{1}{\Gamma(-\nu)}e^{-\frac{1}{2}z^2}\displaystyle\int_0^\infty \tfrac{1}{t^{\nu+1}}\exp\big(-\tfrac{1}{2}t^2-zt\big)\,dt &:\quad \nu<0 \end{cases}$$

Returning with this information to (the following expanded version of) the definition (32)

$$W_\nu(z;\epsilon) \equiv \tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\epsilon}\big)^{\frac{\nu+1}{2}} e^{-\frac{1}{4}(z/\sqrt{\epsilon})^2} D_\nu(-z/\sqrt{\epsilon})$$

and asking *Mathematica* to perform the integrals, we obtain

$$W_\nu(z;\epsilon) = \begin{cases} \tfrac{1}{\pi}2^{\frac{\nu-1}{2}}\big(\tfrac{1}{\epsilon}\big)^{\frac{\nu+1}{2}}\Big[\cos\tfrac{\nu\pi}{2}\,\Gamma(\tfrac{\nu+1}{2})M(\tfrac{\nu+1}{2};\tfrac{1}{2};-\tfrac{z^2}{2\epsilon})\\ \qquad -\sqrt{2}\,\tfrac{z}{\sqrt{\epsilon}}\sin\tfrac{\nu\pi}{2}\,\Gamma(\tfrac{\nu+2}{2})M(\tfrac{\nu+2}{2};\tfrac{3}{2};-\tfrac{z^2}{2\epsilon})\Big] \;:\; \nu>-1\\ \\ 2^{\frac{\nu-1}{2}}\big(\tfrac{1}{\epsilon}\big)^{\frac{\nu+1}{2}}e^{-\frac{1}{2}(z/\sqrt{\epsilon})^2}\Big[\tfrac{1}{\Gamma(\frac{1-\nu}{2})}M(-\tfrac{\nu}{2};\tfrac{1}{2};+\tfrac{z^2}{2\epsilon})\\ \qquad +\sqrt{2}\,\tfrac{z}{\sqrt{\epsilon}}\,\tfrac{1}{\Gamma(-\frac{\nu}{2})}M(\tfrac{1-\nu}{2};\tfrac{3}{2};+\tfrac{z^2}{2})\Big] \;:\; \nu<\;\;0 \end{cases} \tag{42}$$

where the "confluent hypergeometric function" (or "Kummer function") can be defined

$$M(a;b;z) \equiv {}_1\mathsf{F}_1(a;b;z) = 1 + \tfrac{az}{b} + \tfrac{a(1+a)z^2}{2b(1+b)} + \tfrac{a(1+a)(2+a)z^3}{6b(1+b)(2+b)} + \cdots$$

I shall—with meaning made clear by the right side of (42)—find it convenient to distinguish the "positive branch" from the "negative branch" of $W_\nu(\bullet,\bullet)$. The claim has been made by *Mathematica* (from whom I have borrowed all ensuing details) that the two branches actually overlap on the interval $-1<\nu<0$; I will not attempt to prove the implied identity, but submit Figure 4 in evidence of the truth of the point at issue. We are gratified (but not surprised) to discover that the complicated expressions on the right side of (42) simplify greatly when $\nu = \ldots,+2,+1,0,-1,-2,\ldots$:

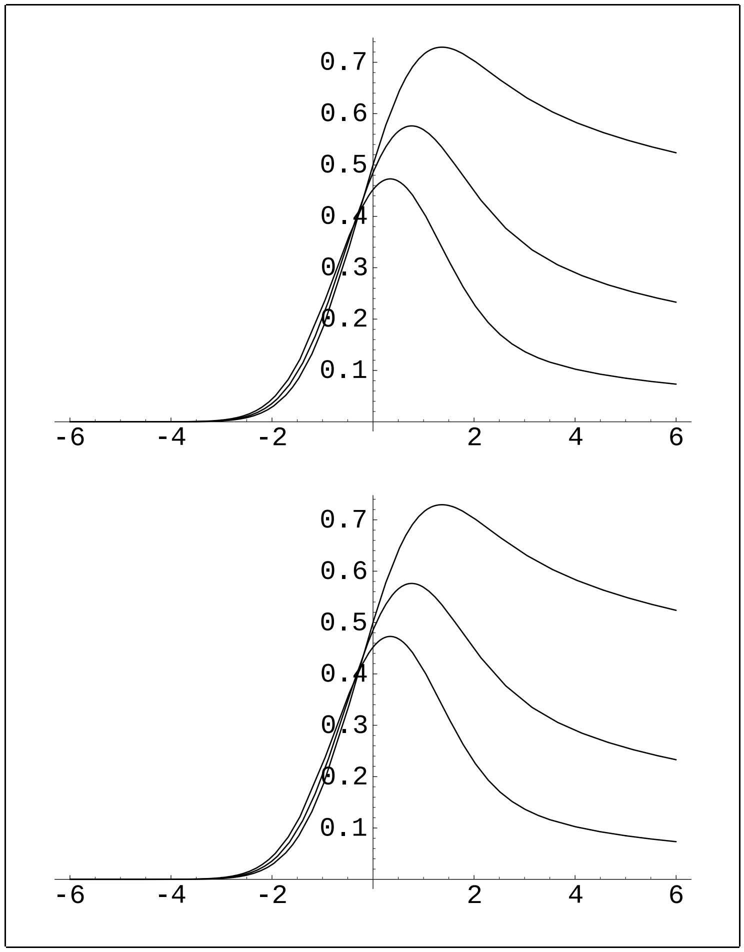


FIGURE 4: *Graphs of* $W_{-\frac{1}{4}}(z,1)$, $W_{-\frac{2}{4}}(z,1)$ *and* $W_{-\frac{3}{4}}(z,1)$ *computed from the positive (top) and negative (bottom) branches of (42). The former function is the most nearly Gauss-like, the latter the most nearly Heaviside-like. From the evident identity of the figures we infer that on the region of overlap* $(-1 < \nu < 0)$ *the functions associated with the two branches are linked by some obscure identity, but I am not motivated to undertake analytical exploration of the issue.*

$$
\begin{aligned}
W_{+2}(z;\epsilon) &= -\tfrac{1}{\sqrt{2\pi\epsilon}}e^{-z^2/2\epsilon}\tfrac{1}{\epsilon}\big[1-\tfrac{z^2}{\epsilon}\big]\\
W_{+1}(z;\epsilon) &= -\tfrac{1}{\sqrt{2\pi\epsilon}}e^{-z^2/2\epsilon}\big[\tfrac{z}{\epsilon}\big]\\
W_{+0}(z;\epsilon) &= +\tfrac{1}{\sqrt{2\pi\epsilon}}e^{-z^2/2\epsilon}\\
W_{-1}(z;\epsilon) &= \tfrac{1}{2}\big[1+\mathrm{erf}\big(\tfrac{z}{\sqrt{2\epsilon}}\big)\big]\\
W_{-2}(z;\epsilon) &= \tfrac{1}{2}z\big[1+\mathrm{erf}\big(\tfrac{z}{\sqrt{2\epsilon}}\big)\big]+\sqrt{\tfrac{\epsilon}{2\pi}}e^{-z^2/2\epsilon}
\end{aligned}
$$

Working from

$$D^\nu f(x) = \lim_{\epsilon\downarrow 0}\int_{-\infty}^{+\infty} F(y)W_\nu(x-y;\epsilon)\,dy \tag{43}$$

with the aid of (42), *Mathematica* computes

$$\begin{aligned}
D^0 e^{px} &= \lim_{\epsilon\downarrow 0}\int_{-\infty}^{+\infty} e^{py}\,W_0(x-y;\epsilon)\,dy\\
&= \lim_{\epsilon\downarrow 0}\Big\{e^{px+\frac{1}{2}p^2\epsilon}\Big\}\\
&= e^{px}\\
D^1 e^{px} &= \lim_{\epsilon\downarrow 0}\int_{-\infty}^{+\infty} e^{py}\,W_1(x-y;\epsilon)\,dy\\
&= \lim_{\epsilon\downarrow 0}\Big\{\text{ horrendously complicated expression }\Big\}\\
&= \lim_{\epsilon\downarrow 0}\Big\{p\,e^{px+\frac{1}{2}p^2\epsilon}\quad\text{when asked to }\texttt{Simplify[\%]}\Big\}\\
&= p\,e^{px}\\
D^{-1} e^{px} &= \lim_{\epsilon\downarrow 0}\int_{-\infty}^{+\infty} e^{py}\,W_{-1}(x-y;\epsilon)\,dy\\
&= \lim_{\epsilon\downarrow 0}\int_{-\infty}^{+\infty} e^{py}\,\tfrac{1}{2}\big[1+\text{erf}\big(\tfrac{x-y}{\sqrt{2\epsilon}}\big)\big]\,dy\\
&= \int_{-\infty}^{+\infty} e^{py}\theta(x-y)\,dy\ \ :\ \text{known to us, but not to } \textit{Mathematica}\\
&= \int_{-\infty}^{x} e^{py}\,dy\\
&= \tfrac{1}{p}e^{px}
\end{aligned}$$

But (to pick only the most familiar from an infinitude of potential examples) the semiderivative $D^{\frac{1}{2}}e^{px} = \lim\int e^{py}W_{\frac{1}{2}}(x-y;\epsilon)\,dy$ appears to lie beyond *Mathematica*'s capability, even though the anticipated result ($\sqrt{p}\,e^{px}$) is very simple.

So the method latent in (42/43) is computationally almost useless: it struggles—and I do mean *struggles!*—to achieve the evaluation even of $\frac{d}{dx}e^{px}$. But it has much to recommend it from some other points of view. It achieves its results-in-principle *without recourse to a regularization procedure* (is "already regularized"). Moreover, it *places (fractional) differentiation and integration on the same formal footing*, and thus obviates the asymmetry of the standard procedure (29). Finally, it yields (see Figures 5–9) a clear—and intuitively very satisfying—*diagramatic interpretation of the "interpolative deformation" process by which the ordinary calculus gives rise to the fractional calculus.*[11]

[11] It should, however, be emphasized that precise imagery is *specific to the*

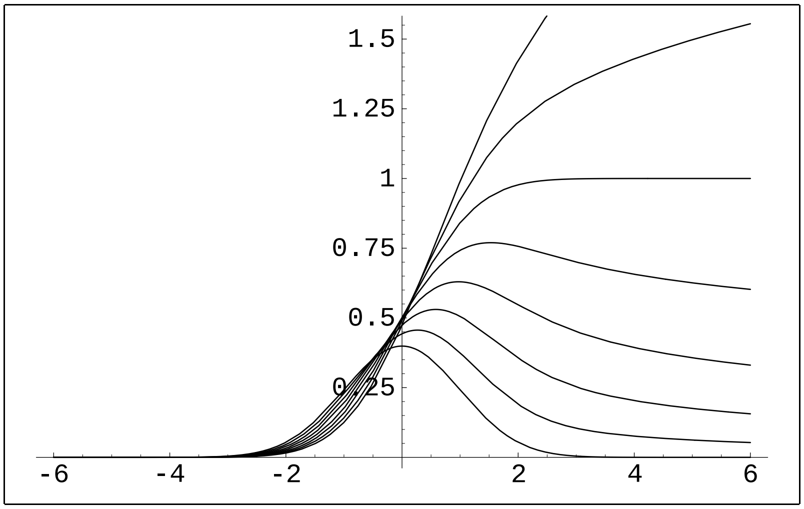


FIGURE 5: *Morphing $W_{-1}(z;\epsilon)$ into $W_0(z;\epsilon)$: graphs of the weight functions $W_\nu(z;\epsilon)$ with $\nu = -\frac{14}{10}$ (at top), $-\frac{12}{10}, -1, -\frac{8}{10}, \dots, -\frac{2}{10}, 0$ (bottom) and ϵ set equal to unity. In the limit $\epsilon \downarrow 0$ the Gaussian $W_0(z;\epsilon)$ sharpens to become $\delta(z)$, while $W_{-1}(z;\epsilon)$ (note its flat top) steepens to become $\theta(z)$. Semiintegration is achieved by $W_{-\frac{1}{2}}(z;\epsilon)$.*

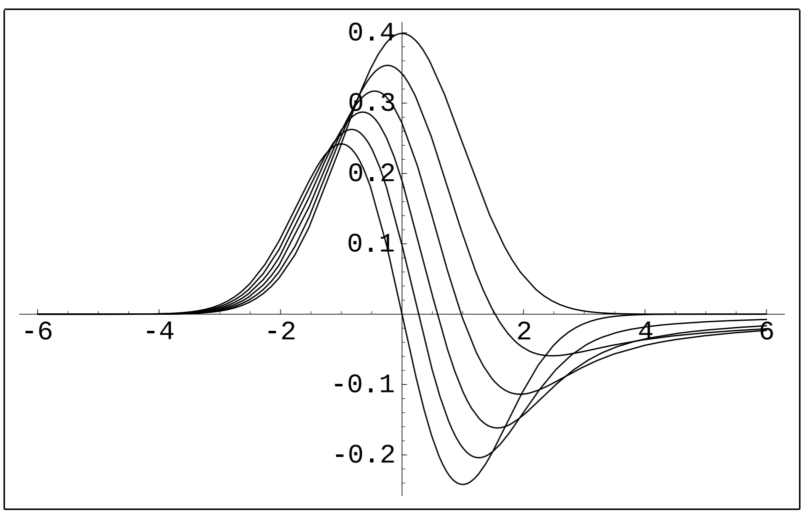


FIGURE 6: *Morphing $W_0(z;\epsilon)$ into $W_{+1}(z;\epsilon)$: graphs of the weight functions $W_\nu(z;\epsilon)$ with $\nu = 0$ (even, with highest max), $\frac{2}{10}, \dots, \frac{8}{10}, 1$ (odd, with lowest min) and ϵ set equal to unity. Semidifferentiation is achieved by $W_{\frac{1}{2}}(z;\epsilon)$.*

[continued from the preceding page] *representation*: I work in Gaussian representation; other representations (of which there are infinitely many) would lead to variants of the same basic conception.

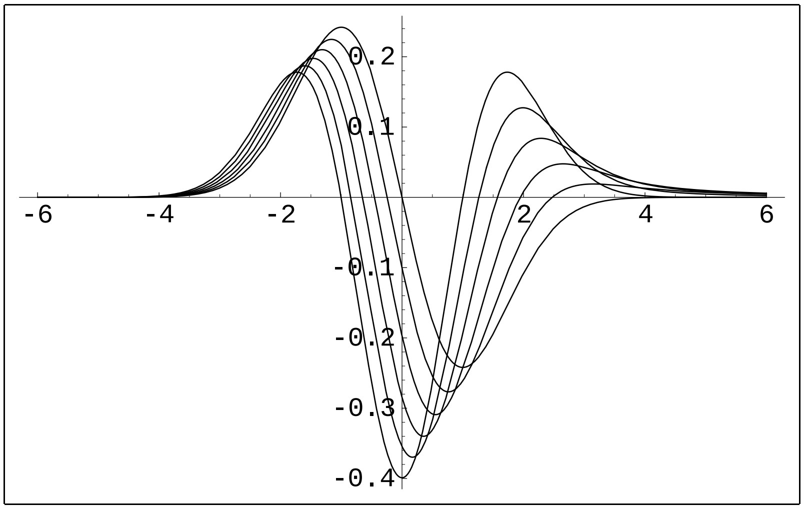


FIGURE 7: *Morphing $W_1(z;\epsilon)$ into $W_2(z;\epsilon)$: graphs of the weight functions $W_\nu(z;\epsilon)$ with $\nu = 1$ (odd, with highest max), $\frac{12}{10}, \ldots, \frac{18}{10}, 2$ (even, with lowest min) and ϵ set equal to unity. Note that the vertical scale changes from figure to figure.*

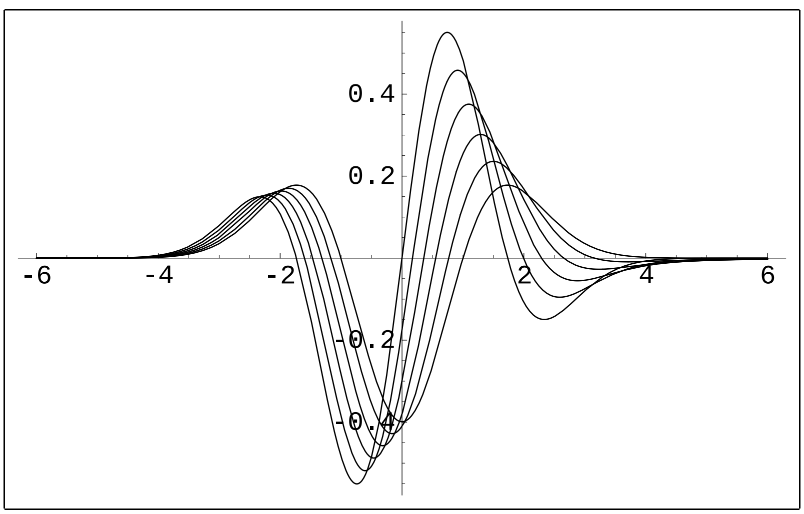


FIGURE 8: *Morphing $W_2(z;\epsilon)$ into $W_3(z;\epsilon)$: graphs of the weight functions $W_\nu(z;\epsilon)$ with $\nu = 2$ (even), $\frac{22}{10}, \ldots, \frac{28}{10}, 3$ (odd, with lowest min and highest max) and ϵ set equal to unity.*

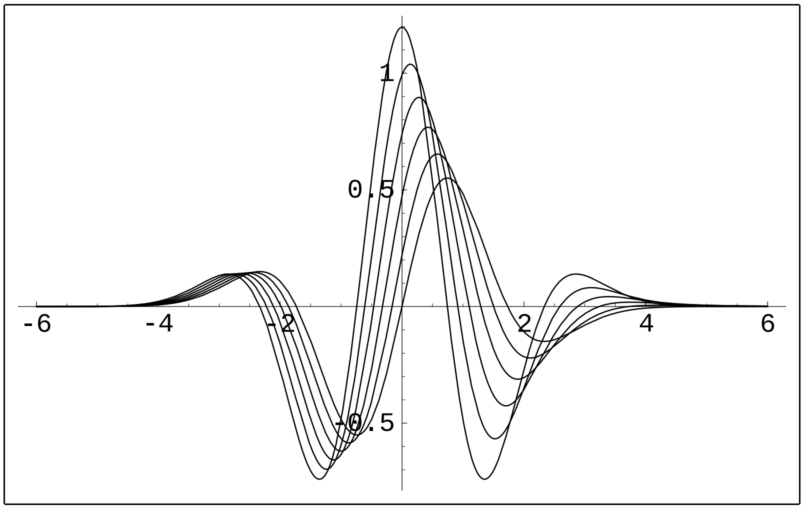


FIGURE 9: *Morphing* $W_3(z;\epsilon)$ *into* $W_4(z;\epsilon)$*: graphs of the weight functions* $W_\nu(z;\epsilon)$ *with* $\nu = 3$ *(odd),* $\frac{32}{10}, \ldots, \frac{38}{10}, 4$ *(even, with lowest min and highest max) and* ϵ *set equal to unity. Notice that the central extremum is*

- *maximal in Figure 6 (achieved by* W_0*),*
- *minimal in Figure 7 (achieved by* W_2*),*
- *maximal in Figure 9 (achieved by* W_4*);*

we have encountered here a manifestation of the "sign alternation" phenomenon which engaged our attention in §1.

So everything remains, from a function-theoretic point of view, as smooth and nice as it can be, so long as ϵ is greater than zero. But—and we come now to the point of this entire discussion—*at* $\epsilon = 0$ we achieve

$$W_\nu(z;0) = \begin{cases} \text{(singular stuff at origin) if } \nu = 0,1,2,\ldots \\ \text{(singular stuff at origin)} + \text{(modulated step function) otherwise} \end{cases}$$

with the consequence that

$$D^\nu \begin{cases} \text{is a local operator if } \nu = 0,1,2,\ldots \\ \text{is non-local for all other real values of } \nu \end{cases}$$

The story is told in Figure 10 and its caption.

The preceding material fell unbidden into my lap, and its exploration has taken me away from my main subject matter, so—though it is easy to think of topics yet unaddressed, of further elaborations—I draw this discussion to an arbitrary close. The "representation-theoretic approach to the factional calculus" springs from such a simple/natural idea that I find it difficult to suppose that it has not already been developed somewhere, by somebody, but my limited familarity with the sparce literture permits me to cite no reference of direct relevance. I would be interested to know whether the preceding material is, in particular, "well-known" to Spanier & Oldham: their *Atlas of Functions*

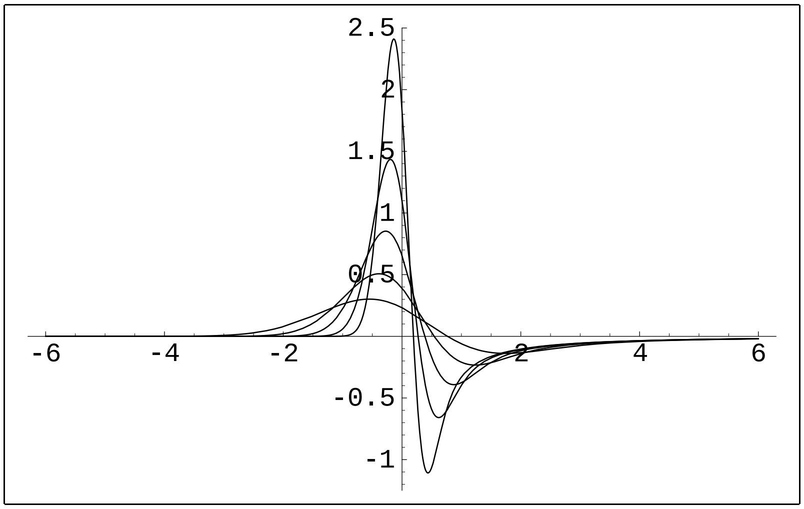


FIGURE 10: *Superimposed graphs of* $W_{\frac{1}{2}}(z;\epsilon)$ *as* ϵ *descends through the values* $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ *A* δ*-function is assembling itself at the origin, but it is evident from the figure that*

$$\lim_{\epsilon\downarrow 0} W_{\frac{1}{2}}(z;\epsilon) = \begin{cases} 0 & : \quad z < 0 \\ \text{negative-valued function of } z & : \quad z > 0 \end{cases}$$

This conforms to the upshot of equations developed just prior to (38) in the text, from which we learn that the "negative-valued function of z*" can be described* $-\frac{1}{2\sqrt{\pi}\,z^{3/2}}$. *The figure illustrates how it happens that*

$$D^\nu \begin{cases} \text{is a local operator if } \nu = 0, 1, 2, \ldots \\ \text{is non-local for all other real values of } \nu \end{cases}$$

(which has served me as a primary resource) remains the only handbook in which attention is given routinely to the fractional integral/derivative properties of the functions treated. And concerning the fractional calculus itself, they literally "wrote the book." Yet in Chapter 8 of their *Atlas*—which is given over (remarkably!) to discussion of the step function $\theta(x-a)$—one finds only passing allusion (**8:9**) to an associated representation theory, and only oblique reference (end of **8:10**) to the role played by $\theta(x-a)$ within the fractional calculus. Their Chapter 10 treats the delta function $\delta(x-a)$, which (see **10:3**) they reasonably consider to be *defined* by its representation theory, and in **10:12** one encounters the remark that

> *"By using the concept of differintegration it is possible to define a continuum of functions of which the unit-step function and the Dirac delta function are respectively the* $\nu = 0$ *and* $\nu = 1$ *instances. The general definition is*
>
> $$\frac{d^\nu}{[d(x+\infty)]^\nu}\theta(x-a)$$

which evaluates to

$$\theta(x-a)\frac{(x-a)^{-\nu}}{\Gamma(1-\nu)} \quad : \quad \nu \neq 1, 2, 3, \ldots$$

except when ν is a positive integer. The $\nu = 2$ case, symbolized $\delta'(x-a)$, may be regarded as the limit

$$\delta'(x-a) = \lim_{h\to 0} \frac{\delta\big(x-a-\frac{h}{2}\big) - \delta\big(x-a+\frac{h}{2}\big)}{h}$$

and is named the unit-moment function. *It satisfies the integral identity*

$$\int_{-\infty}^{+\infty} \delta'(t-a)f(t)\,dt = -f'(a)$$

But that's as far as they take it. I cannot determine whether Spanier & Oldham —who held all the pieces in their hands—considered the subject too obvious to write out (or perhaps too tediously unimportant), or simply neglected to seize the opportunity.

3. General remarks concerning Laplace's operator. My objective here is to describe the theory of ∇^2 in such a way as to open doors to generalization, both with regard to *dimension* and to *differential order*.

In one dimension the Laplacian is just the second derivative

$$\nabla^2 \text{ means } \partial_x^2 \text{ in } E^1$$

and generalization with respect to differential order

$$\nabla^2 \longrightarrow \nabla^{2\mu}$$

can be accomplished by methods standard to the fractional calculus. But it is dimensional generalization which serves in the first instance to make things more interesting.

Working initially in E^3: To declare an interest in *differentiation procedures which respect the realities of transformation theory* is, by the simplest interpretation, to declare an interest in the life and times of $\boldsymbol{\nabla}$. Let $\varphi(\boldsymbol{x})$ be a scalar field. Then $\boldsymbol{\nabla}\varphi$ transforms as a vector field, and $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\varphi$ is again a scalar field. We arrive thus already at the Laplacian—the simplest differential operator

$$\nabla^2 = \text{div}\cdot\text{grad} = (\,\partial_1 \quad \partial_2 \quad \partial_3\,)\begin{pmatrix}\partial_1\\ \partial_2\\ \partial_3\end{pmatrix} = \partial_1^2 + \partial_2^2 + \partial_3^2$$

with the property that it sends

$$\text{scalar field} \longrightarrow \text{derived scalar field} \tag{44}$$

∇^2 is (as the notation was designed to emphasize) a differential operator of 2^{nd} order. Operators of 4^{th},6^{th}, ... order which also achieve (44) can be obtained by iteration:

$$\begin{aligned}\nabla^{2m} \equiv (\text{div}\cdot\text{grad})^m &= (\partial_1^2 + \partial_2^2 + \partial_3^2)^m \\ &= (\partial_1^{2m} + \partial_2^{2m} + \partial_3^{2m}) + \text{cross terms}\end{aligned}$$

For an occurrence of the so-called "biharmonic equation"

$$\nabla^4 \varphi = 0$$

(in connection with the theory of elastic media) see Morse & Feshbach.[12] Preceeding remarks extend straightforwardly from E^3 to E^N.

Looking now to operators which achieve

$$\text{vector field} \longrightarrow \text{derived vector field} \tag{45}$$

we in 1^{st} order have at our disposal only

$$\text{curl} = \boldsymbol{\nabla}\times = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}$$

but in 2^{nd} order encounter a pair of such operators:

$$\begin{aligned}\text{curl curl} &= \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} \partial_1\partial_1 & \partial_1\partial_2 & \partial_1\partial_3 \\ \partial_2\partial_1 & \partial_2\partial_2 & \partial_2\partial_3 \\ \partial_3\partial_1 & \partial_3\partial_2 & \partial_3\partial_3 \end{pmatrix} - \begin{pmatrix} \nabla^2 & 0 & 0 \\ 0 & \nabla^2 & 0 \\ 0 & 0 & \nabla^2 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\text{grad div} &= \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \end{pmatrix} \\ &= \begin{pmatrix} \partial_1\partial_1 & \partial_1\partial_2 & \partial_1\partial_3 \\ \partial_2\partial_1 & \partial_2\partial_2 & \partial_2\partial_3 \\ \partial_3\partial_1 & \partial_3\partial_2 & \partial_3\partial_3 \end{pmatrix}\end{aligned}$$

The operator

$$\text{grad div} - \text{curl curl} = \begin{pmatrix} \nabla^2 & 0 & 0 \\ 0 & \nabla^2 & 0 \\ 0 & 0 & \nabla^2 \end{pmatrix} \tag{46}$$

---

[12] *Methods of Theoretical Physics* (1953), p. 1786.

is, for obvious reasons, often called the "vector Laplacian." That

$$
\begin{aligned}
(\text{grad div} - \text{curl curl})^m &= \begin{pmatrix} \nabla^2 & 0 & 0 \\ 0 & \nabla^2 & 0 \\ 0 & 0 & \nabla^2 \end{pmatrix}^m \\
&= \begin{pmatrix} \nabla^{2m} & 0 & 0 \\ 0 & \nabla^{2m} & 0 \\ 0 & 0 & \nabla^{2m} \end{pmatrix}
\end{aligned}
\tag{47}
$$

is obvious if one works from the expression on the right, but less obvious if one works from the expression on the left; in the case $m = 2$ we have

$$
\begin{aligned}
(\text{grad div} - \text{curl curl})^2 &= \text{grad div grad div} + \text{curl curl curl curl} \\
&\qquad - \big\{\text{grad div curl curl} + \text{curl curl grad div}\big\}
\end{aligned}
$$

where the cross terms $\big\{\text{etc.}\big\}$ vanish because

$$
\text{div curl} = \text{curl grad} = 0 \tag{48}
$$

i.e., because $\partial_i \partial_j = \partial_j \partial_i$ entails

$$
\underbrace{\begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \end{pmatrix} \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}}_{\text{operates on vector fields}} = \underbrace{\begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}}_{\text{operates on scalar fields}} = \mathbb{O}
$$

In the general case one has (since (48) serves generally to kill all cross terms)

$$
(\text{grad div} - \text{curl curl})^m = (\text{grad div})^m + (-)^m (\text{curl curl})^m \tag{49}
$$

Extension of the preceeding remarks from E^3 to E^N is possible but—to the extent "curl" has been involved—*not* straightforward. Nor can one (unless familiar with the theory of dyadics) proceed straightforwardly to the formation of differential operators which achieve

$$
\text{tensor field} \longrightarrow \text{derived tensor field} \tag{50}
$$

The high road to the simultaneous solution of *both* problems is provided by the exterior calculus, which I have described in fairly elaborate detail (both in general, and as it relates to the matters at hand) elsewhere.[13] The progression (44) $\rightarrow$ (45) $\rightarrow$ (50) $\rightarrow \cdots$ is achieved by *rank* generalization, which serves usefully to place individual results in context, but will be of no further interest to me here; in service of expository simplicity I restrict my remarks to ∇^2-like operators which act upon *scalar* fields.

[13] "Electrodynamnical applications of the exterior calculus" (1996), §§2 & 3.

I have concentrated thus far the upon the transformational/algebraic basis of the claim that ∇^2 is a "natural object." Rather more convincing is the argument which proceeds by dimensional generalization of the argument which gave (4.1), and which in the case $N = 2$ reads

$$\begin{aligned}\nabla^2 f(x,y) = &\quad \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2}\\ &+ \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2}\\ &= -\tfrac{4}{h^2}\big\{f(x) - \langle f(x)\rangle\big\}\end{aligned} \tag{51.1}$$

with

$$\begin{aligned}\langle f(x)\rangle &\equiv \tfrac{1}{4}\big[f(x+h,y) + f(x-h,y) + f(x,y+h) + f(x,y-h)\big]\\ &\equiv \tfrac{1}{4}\big[f_{1,0} + f_{-1,0} + f_{0,1} + f_{0,-1}\big]\\ &= \text{average over nearest-neighbor lattice points}\end{aligned}$$

By the same argument[14] we are led to

$$\begin{aligned}\nabla^4 f(x,y) &= \Big\{\partial_x^4 + 2\partial_x^2\partial_y^2 + \partial_y^4\Big\}f\\ &= \tfrac{1}{h^4}\Big\{20f_{0,0} - 8\big[f_{1,0} + f_{0,1} + f_{-1,0} + f_{0,-1}\big]\\ &\qquad\qquad + 2\big[f_{1,1} + f_{-1,1} + f_{-1,-1} + f_{1,-1}\big]\\ &\qquad\qquad + \ \ \big[f_{2,0} + f_{0,2} + f_{-2,0} + f_{0,-2}\big]\Big\}\\ &= +\tfrac{20}{h^4}\big\{f(x) - \langle f(x)\rangle\big\}\end{aligned} \tag{51.2}$$

$$\begin{aligned}\langle f(x)\rangle \equiv &+ \tfrac{8}{20}\big[f_{1,0} + f_{0,1} + f_{-1,0} + f_{0,-1}\big]\\ &- \tfrac{2}{20}\big[f_{1,1} + f_{-1,1} + f_{-1,-1} + f_{1,-1}\big]\\ &- \tfrac{1}{20}\big[f_{2,0} + f_{0,2} + f_{-2,0} + f_{0,-2}\big]\end{aligned}$$

$$\nabla^6 f = -\tfrac{112}{h^6}\big\{f(x) - \langle f(x)\rangle\big\} \tag{51.3}$$

where in the final equation

$$\begin{aligned}\langle f(x)\rangle \equiv &+ \tfrac{57}{112}\big[f_{1,0} + f_{0,1} + f_{-1,0} + f_{0,-1}\big]\\ &- \tfrac{24}{112}\big[f_{1,1} + f_{-1,1} + f_{-1,-1} + f_{1,-1}\big]\\ &- \tfrac{12}{112}\big[f_{2,0} + f_{0,2} + f_{-2,0} + f_{0,-2}\big]\\ &+ \tfrac{3}{112}\big[f_{2,1} + f_{1,2} + f_{-1,2} + f_{-2,1} + f_{-2,-1} + f_{-1,-2} + f_{1,-2} + f_{2,-1}\big]\\ &+ \tfrac{1}{112}\big[f_{3,0} + f_{0,3} + f_{-3,0} + f_{0,-3}\big]\end{aligned}$$

---

[14] I define

$$\texttt{Lap[m]} := \left(\left[\frac{e^{+\frac{1}{2}hX} - e^{-\frac{1}{2}hX}}{h}\right]^2 + \left[\frac{e^{+\frac{1}{2}hY} - e^{-\frac{1}{2}hY}}{h}\right]^2\right)^m$$

and ask *Mathematica* to `Expand[Lap[2]]`, etc. I am borrowing my subscripted notation from Abramowitz & Stegun's §**25.3**.

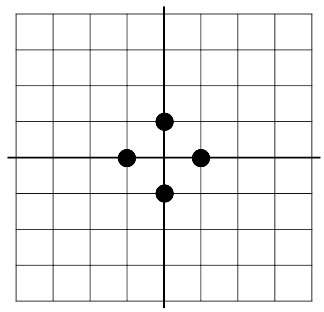
That we can continue even in higher order to embrace the notion that

$$\langle f(x)\rangle = \text{signed proximity-weighted average over a near neighborhood}$$

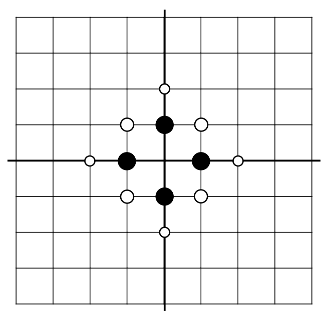
is supported by the following observations:

$$\begin{aligned} \tfrac{8}{20}\times 4 - \tfrac{2}{20}\times 4 - \tfrac{1}{20}\times 4 &= 1 \\ \tfrac{57}{112}\times 4 - \tfrac{24}{112}\times 4 - \tfrac{12}{112}\times 4 + \tfrac{3}{112}\times 8 + \tfrac{1}{112}\times 4 &= 1 \end{aligned}$$

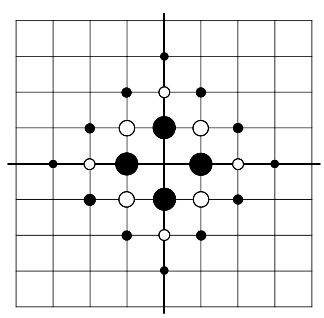
The specific neighborhoods in question are shown in the following diagrams:



Harmonic neighborhood of the central point.



Biharmonic neighborhood.



Triharmonic neighborhood.

Dimensional generalization of the "lattice argument" is straightforward,[15] and the method does bring naturally into play the notion of "proximity-weighted neighborhood." The method is, however, susceptible to the criticism that it assigns importance only to a discrete subset of the continuum of points which stand in given proximity to any given point; it introduces a "graininess" into the analysis of a subject which is itself devoid of intrinsic graininess. Removal of that formal defect stands as a precondition to arrival at my intended objective, and will be accomplished by extension of the "spherical averaging technique" which I have employed on a couple of previous occasions.[16]

Looking first to the case $N = 2$: to describe the values assumed by $f(x,y)$ in the neighborhood of some generic point, center a polar coordinate system upon that point and (by multivariate Taylor expansion) write

$$\begin{aligned} f(x + r\cos\phi, y + r\sin\phi) &= e^{r\cos\phi\frac{\partial}{\partial x} + r\sin\phi\frac{\partial}{\partial y}} f(x,y) \\ &= \sum \tfrac{1}{n!} r^n (\cos\phi\tfrac{\partial}{\partial x} + \sin\phi\tfrac{\partial}{\partial y})^n f(x,y) \end{aligned}$$

Let $\langle f(x,y;r)\rangle$ denote the average of the values assumed by $f(\bullet,\bullet)$ on the circle of points which stand in proximity r to (x,y):

$$\begin{aligned} \langle f(x,y;r)\rangle &\equiv \tfrac{1}{2\pi r}\int_0^{2\pi} \sum \tfrac{1}{n!} r^n (\cos\phi\tfrac{\partial}{\partial x} + \sin\phi\tfrac{\partial}{\partial y})^n f(x,y)\cdot r d\phi \\ &= \sum \tfrac{1}{n!} r^n \Big\{ \tfrac{1}{2\pi}\int_0^{2\pi} (\cos\phi\tfrac{\partial}{\partial x} + \sin\phi\tfrac{\partial}{\partial y})^n \, d\phi \Big\} f(x,y) \end{aligned}$$

Asking *Mathematica* to perform the integrals, we are reminded that $\{\text{etc.}\}$ necessarily vanishes unless n is even, and informed that in those cases it assumes such values as to yield[17]

$$\begin{aligned} = \Big\{ 1 + 0 + \tfrac{1}{2!}\tfrac{1}{2} r^2 \nabla^2 + 0 + \tfrac{1}{4!}\tfrac{3}{8} r^4 \nabla^4 + 0 + \tfrac{1}{6!}\tfrac{5}{16} r^6 \nabla^6 \\ + 0 + \tfrac{1}{8!}\tfrac{35}{128} r^8 \nabla^8 + 0 + \tfrac{1}{10!}\tfrac{63}{256} r^{10}\nabla^{10} + 0 + \cdots \Big\} f(x,y) \end{aligned} \tag{52.2}$$

---

[15] In view of the sign alternation evident in (51) one could on this basis argue that the phenomenon discussed in §1 (case $N = 1$) is in fact dimensionally persistent.

[16] The technique, as it relates to ∇^2 in the 2-dimensional case, is described in the introduction to "Algebraic theory of spherical harmonics" (1996), and at pp. 84–88 in CLASSICAL FIELD THEORY (1996) is used in its N-dimensional formulation ($N \to \infty$) to carry the notion of a "Laplacian" over into the calculus of functionals. Here I have N-dimensional interest in the generalization $\nabla^2 \to \nabla^{2m}$.

[17] The calculation produces multinomials $\sum_{a+b=2m}(\text{numeric})\big(\frac{\partial}{\partial x}\big)^a\big(\frac{\partial}{\partial y}\big)^b$ of ascending complexity which, upon the instruction `Simplify[%]`, are reassembled to read $(\text{numeric})\big[\big(\frac{\partial}{\partial x}\big)^2 + \big(\frac{\partial}{\partial y}\big)^2\big]^m$. The process is a wonder to witness.

In the case $N = 3$ one proceeds similarly, but averages over the sphere of proximity r:

$$\begin{aligned}
&\langle f(x,y,z;r)\rangle\\
&\equiv \sum \tfrac{1}{n!}r^n\Big\{\tfrac{1}{4\pi r^2}\int_0^\pi\!\!\int_0^{2\pi}(\sin\theta\cos\phi\tfrac{\partial}{\partial x}+\sin\theta\sin\phi\tfrac{\partial}{\partial y}+\cos\theta\tfrac{\partial}{\partial z})^n\\
&\hspace{6cm}\cdot r^2\sin\theta\, d\phi d\theta\Big\}f(x,y,z)\\
&=\Big\{1+0+\tfrac{1}{2!}\tfrac{1}{3}r^2\nabla^2+0+\tfrac{1}{4!}\tfrac{1}{5}r^4\nabla^4+0+\tfrac{1}{6!}\tfrac{1}{7}r^6\nabla^6+\cdots\Big\}f(x,y,z) \qquad (52.3)
\end{aligned}$$

In the case $N = 4$ one averages over a hypersphere:[18]

$$\begin{aligned}
&\langle f(x,y,z,u;r)\rangle\\
&\equiv \sum \tfrac{1}{n!}r_n\Big\{\tfrac{1}{2\pi^2r^3}\int_0^\pi\!\!\int_0^\pi\!\!\int_0^{2\pi}\\
&(r\sin\theta_2\sin\theta_1\sin\phi\tfrac{\partial}{\partial y}+r\sin\theta_2\sin\theta_1\cos\phi\tfrac{\partial}{\partial x}+r\sin\theta_2\cos\theta_1\tfrac{\partial}{\partial z}+r\cos\theta_2\tfrac{\partial}{\partial u})^n\\
&\hspace{6cm}\cdot r^3\sin^2\theta_2\sin\theta_1\, d\phi d\theta_1 d\theta_2\Big\}f(x,y,z)\\
&=\Big\{1+0+\tfrac{1}{2!}\tfrac{1}{4}r^2\nabla^2+0+\cdots\Big\}f(x,y,z,u) \qquad (52.4)
\end{aligned}$$

But beyond this point (or some point) direct "averaging over the hypersphere" becomes unfeasible. More elegantly efficient methods are developed in the CLASSICAL FIELD THEORY notes already cited; there I show that

$$\begin{aligned}
\langle x^p\rangle_N &\equiv x^p \text{ averaged over surface of an } N\text{-sphere of radius } r\\
&= \frac{S_{N-1}}{S_N}r^p\begin{cases} 0 & \text{when } p \text{ is odd}\\ 2\displaystyle\int_0^{\frac{1}{2}\pi}\cos^p\theta\sin^{N-2}\theta\, d\theta & \text{when } p \text{ is even}\end{cases}
\end{aligned}$$

where the numbers S_N acquire meaning from the statements

$$\text{area of } N\text{-sphere of radius } r \text{ is given by } S_N r^{N-1}$$

$$S_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma\big(\frac{N}{2}\big)}$$

---

[18] I take my definition of the hyperspheric coordinate system, the description of the associated Jacobian, etc. from §4 of the first of the sources cited in Footnote 16. It is, by the way, one thing to speak of averaging over a hypersphere, and quite another to do it; *Mathematica* kept exhausting its (specially expanded) memory when attempting direct evaluation even of the ∇^2 coefficient in the 4-dimensional case, and to obtain the next term in that same series would have to evaluate $4^4 = 64$ triple integrals.

which entail

$$\frac{S_{N-1}}{S_N} = \frac{\Gamma\left(\frac{N}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{N-1}{2}\right)} \quad : \quad N = 2, 3, 4, \ldots$$

The integral is tabulated, and known also to *Mathematica*:

$$2\int_0^{\frac{1}{2}\pi} \cos^p\theta \sin^q\theta\, d\theta = \frac{\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(\frac{1+q}{2}\right)}{\Gamma\left(\frac{2+p+q}{2}\right)} \quad : \quad \Re[q] > -1$$

So we have

$$\langle x^p\rangle_N = \begin{cases} 0 & : \quad p \text{ odd} \\ \dfrac{\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(\frac{N}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{p+N}{2}\right)} & : \quad p \text{ even} \end{cases}$$

Building upon this information, I show in the source most recently cited that—consistently with the evidence of (52)—

$$\begin{aligned} \langle f(\boldsymbol{x}; r)\rangle &= \left\{\langle x^0\rangle_N + \tfrac{1}{2!}\langle x^2\rangle_N r^2\nabla^2 + \cdots\right\} f(\boldsymbol{x}) \\ &= \left\{1 + \tfrac{1}{2!}\tfrac{1}{N} r^2\nabla^2 + \cdots\right\} f(\boldsymbol{x}) \quad : \quad N = 2, 3, \ldots \end{aligned}$$

But the argument can be extended,[19] to give

$$\langle f(\boldsymbol{x}; r)\rangle_N = \left\{\sum_{m=0}^{\infty} \tfrac{1}{(2m)!}\langle x^{2m}\rangle_N r^{2m}\nabla^{2m}\right\} f(\boldsymbol{x}) \tag{53}$$

The generating function technique serves usefully to expose the detailed meaning of this strong result; form

$$F_N(R) \equiv \sum_{m=0}^{\infty} \langle x^{2m}\rangle_N R^{2m} \tag{54.1}$$

and with the assistance of *Mathematica* obtain

$$\begin{aligned} F_2(R) &= \tfrac{1}{\sqrt{1-R^2}} \\ &= 1 + \tfrac{1}{2}R^2 + \tfrac{3}{8}R^4 + \tfrac{5}{16}R^6 + \tfrac{35}{128}R^8 + \tfrac{63}{256}R^{10} + \cdots \\ F_3(R) &= \tfrac{\text{arctanh}(R)}{R} \\ &= 1 + \tfrac{1}{3}R^2 + \tfrac{1}{5}R^4 + \tfrac{1}{7}R^6 + \tfrac{1}{9}R^8 + \tfrac{1}{11}R^{10} + \cdots \\ F_4(R) &= 2\,\tfrac{1-\sqrt{1-R^2}}{R^2} \\ &= 1 + \tfrac{1}{4}R^2 + \tfrac{1}{8}R^4 + \tfrac{5}{64}R^6 + \tfrac{7}{128}R^8 + \tfrac{21}{512}R^{10} + \cdots \\ F_5(R) &= \tfrac{3}{2}\,\tfrac{R-(1-R^2)\text{arctanh}(R)}{R^3} \\ &= 1 + \tfrac{1}{5}R^2 + \tfrac{3}{35}R^4 + \tfrac{1}{21}R^6 + \tfrac{1}{33}R^8 + \tfrac{3}{143}R^{10} + \cdots \\ &\ \vdots \end{aligned}$$

[19] "Unfortunately, there is not room enough in this margin to permit me to write out the demonstration," which is so pretty as to merit independent discussion on some other occasion.

which precisely reproduce (and vastly extend) the laborously achieved data displayed in (52).

It is pertinent to notice that generating functions of the "exponential" design[20]

$$G_N(R) = \sum_{m=0}^{\infty} \tfrac{1}{(2m)!}\langle x^{2m}\rangle_N R^{2m} \tag{54.2}$$

are also tractable; *Mathematica* supplies

$$\begin{aligned}
G_2(R) &= \text{BesselI}[0, R]\\
G_3(R) &= \tfrac{\text{Sinh}(R)}{R} = \sqrt{\tfrac{\pi}{2R}}\ \text{BesselI}[\tfrac{1}{2}, R]\\
G_4(R) &= \tfrac{2}{R}\,\text{BesselI}[1, R]\\
G_5(R) &= 3\tfrac{R\text{Cosh}(R)-\text{Sinh}(R)}{R^3} = 3\sqrt{\tfrac{\pi}{2R^3}}\ \text{BesselI}[\tfrac{3}{2}, R]\\
&\ \ \vdots\\
G_N(R) &= \Gamma\big(\tfrac{N}{2}\big)\big(\tfrac{2}{R}\big)^{\frac{N-2}{2}}\text{BesselI}\,[\tfrac{N-2}{2}, R]\\
&= \tfrac{1}{\sqrt{\pi}}\frac{\Gamma\big(\frac{N}{2}\big)}{\Gamma\big(\frac{N-1}{2}\big)}\int_{-1}^{+1}(1-t^2)^{\frac{N-3}{2}}e^{Rt}dt \quad : \quad N>1
\end{aligned} \tag{55}$$

where the functions `BesselI[n,z]` are more commonly denoted $I_n(z)$, and are known as "hyperbolic Bessel functions;" their properties are summarized in Spanier & Oldham's Chapter 50. In the notation of (54.2) our fundamental identity (53) becomes

$$\langle f(\boldsymbol{x};r)\rangle_N = \Big\{G_N(r\nabla)\Big\}f(\boldsymbol{x}) \tag{56}$$

The integral representation (55)—adapted from a formula presented by Spanier & Oldham (of which several variants appear in §8.431 of Gradshteyn & Ryzhik) and confirmed by *Mathematica*—becomes exceptionally simple in the case $N=3$, but is in all cases remarkable for the simplicity with which R enters into the integrand. Expansion of the exponential leads back again to (52).

To summarize: $\langle f(\boldsymbol{x};r)\rangle_N$ is, by construction, *invariant with respect to rotations about the point* $\boldsymbol{x}$. The function $f(\boldsymbol{x})$ supports lots of local derivative structure $\partial_1^{a_1}\partial_2^{a_2}\cdots\partial_N^{a_N}f(\boldsymbol{x})$, but the only such structure which enters into the description of $\langle f(\boldsymbol{x};r)\rangle_N$ is, as we have established by explicit calculation, that associated with the expressions $\nabla^{2m}f(\boldsymbol{x})$,[21] and such expressions contribute with weights to which we can now assign precise values.

[20] I borrow my terminology from H. S. Wilf, *Generatingfunctionology*, (1994).

[21] One might argue on transformation-theoretic grounds that *it could not be otherwise*: if $\varphi(\boldsymbol{a})$ is a scalar-valued rotationally-invariant function of a vector $\boldsymbol{a}$ then (in the absence of auxiliary apparatus) φ must be expressible as a function of $\boldsymbol{a}\cdot\boldsymbol{a}$.

As final preparation for the work ahead I turn now to a different aspect of our topic: One has

$$f(\boldsymbol{x}) = \int f(\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{x})\,d^N y \tag{57}$$

giving

$$\nabla^2 f(\boldsymbol{x}) = \begin{cases} \displaystyle\int \nabla_y^2 f(\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{x})\,d^N y & \text{on the one hand, but} \\ \displaystyle\int f(\boldsymbol{y})\nabla_x^2 \delta(\boldsymbol{y}-\boldsymbol{x})\,d^N y & \text{on the other} \end{cases}$$

Equivalence is established by appeal jointly to Green's theorem[22] and to the circumstance that $\nabla_x^2\delta(\boldsymbol{y}-\boldsymbol{x}) = (-)^2\nabla_y^2\delta(\boldsymbol{y}-\boldsymbol{x})$. More generally, we have

$$\nabla^{2m} f(\boldsymbol{x}) = \iint\cdots\int f(\boldsymbol{y})\nabla^{2m}\delta(\boldsymbol{x}-\boldsymbol{y})\,d^N y$$

To perform the integration, install hyperspherical coordinates at $\boldsymbol{x}$ and, after integrating with respect to all angles, obtain

$$\nabla^{2m} f(\boldsymbol{x}) = \int_0^\infty \langle f(\boldsymbol{x};r)\rangle_N \nabla^{2m}\delta(r)\cdot S_N r^{N-1}\,dr \tag{58}$$

The result to which we have been led is from one point of view so formal as to be nearly devoid of meaning, yet from another too nearly tautologous to give serious offense. It has been designed to prefigure the representation theory to which I now turn; when thus transmogrified it will be deprived simultaneously of both defects.

4. Multivariate Gaussian representation theory. Occupying a place of distinction among the representations of $\delta(\boldsymbol{x})$ are the *rotationally invariant* representations, and of those we concentrate upon one in particular: the Gaussian representation

$$\delta(\boldsymbol{x}) = \lim_{\epsilon\downarrow 0}\left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{1}{2}N} e^{-\frac{1}{2\epsilon}r^2} \quad \text{where} \quad \begin{aligned} r^2 &\equiv \boldsymbol{x}\cdot\boldsymbol{x} \\ &= x_1^2 + x_2^2 + \cdots + x_N^2 \end{aligned} \tag{59}$$

which gives back (6.0) in the case $N = 1$. In the latter case I at (42) introduced a notational convention which entails

$$W_0(z;\epsilon) = \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{1}{2}} e^{-\frac{1}{2\epsilon}z^2}$$

---

[22] By which term mathematicians and physicists tend to understand distinct things. I allude to the statement (see, for example, H. Lass, *Vector and Tensor Analysis* (1950), p. 118) which in the 3-dimensional case reads

$$\iiint_R u\nabla^2 v\,d(\text{volume}) = \iiint_R v\nabla^2 u\,d(\text{volume}) + \iint_{\partial R}\big\{u\boldsymbol{\nabla} v - v\boldsymbol{\nabla} u\big\}\boldsymbol{\cdot}\,\boldsymbol{ds}$$

and achieves (compare (5)) a multidimensional analog of "integration by parts."

By extension of that convention I will write

$$W_0(r;\epsilon;N) = \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{1}{2}N} e^{-\frac{1}{2\epsilon}r^2} \tag{60}$$

but will frequently omit some of the cumbersome detail when confusion seems unlikely to result. Notice that *no Jacobian has been absorbed into the definition of* $W_0(r;\epsilon;N)$; one has

$$f(\boldsymbol{x}) = \lim_{\epsilon\downarrow 0} f_\epsilon(\boldsymbol{x}) \tag{61}$$

with

$$f_\epsilon(\boldsymbol{x}) \equiv \iint\!\cdots\!\int f(\boldsymbol{y})W_0(r;\epsilon;N)\,dy_1dy_2\dots dy_N$$
$$r \equiv \sqrt{(\boldsymbol{y}-\boldsymbol{x})\cdot(\boldsymbol{y}-\boldsymbol{x})}$$

which upon the introduction of $\boldsymbol{x}$-centered hyperspherical coordinates becomes

$$\begin{aligned} &= \int_0^\infty\!\!\iint\!\cdots\!\int f(\boldsymbol{x}+\boldsymbol{r})W_0(r;\epsilon;N)\cdot\text{Jacobian(angles}, r)\,d(\text{angles})dr \\ &= \int_0^\infty \langle f(\boldsymbol{x};r)\rangle_N W_0(r;\epsilon;N)\cdot S_N r^{N-1}\,dr \end{aligned} \tag{62}$$

Equation (58) can now be representated/interpreted as the limit $\epsilon\downarrow 0$ of the statement

$$\nabla^{2m} f_\epsilon(\boldsymbol{x}) = \int_0^\infty \langle f(\boldsymbol{x};r)\rangle_N \nabla^{2m}W_0(r;\epsilon;N)\cdot S_N r^{N-1}\,dr \tag{63}$$

at which point we acquire an interest in the description of expressions of the form

$$\nabla^{2m} f(x_1,x_2,\dots,x_N) \quad\text{when}\quad f(x_1,x_2,\dots,x_N) = \varphi(r)$$

An elementary argument[23] gives

$$\nabla^2 f = \left\{\left[r\tfrac{d}{dr}+N\right]\tfrac{1}{r}\tfrac{d}{dr}\right\}\varphi \tag{64.1}$$

in connection with which it will sometimes be useful to

Write $\varphi(r)\equiv\phi(s)$ with $s\equiv\frac{1}{2}r^2$ which entails $\frac{1}{r}\frac{d}{dr}=\frac{d}{ds}$

[23]
$$\begin{aligned} \boldsymbol{\nabla} f &= \varphi'\,\boldsymbol{\nabla} r = \boldsymbol{r}\cdot\tfrac{1}{r}\tfrac{d}{dr}\varphi \\ &\downarrow \\ \nabla^2 f &= \boldsymbol{r}\cdot\boldsymbol{\nabla}\left(\tfrac{1}{r}\tfrac{d}{dr}\varphi\right) + (\boldsymbol{\nabla}\cdot\boldsymbol{r})\tfrac{1}{r}\tfrac{d}{dr}\varphi \quad\text{with}\quad \boldsymbol{\nabla}\cdot\boldsymbol{r} = N \end{aligned}$$

and thus to obtain

$$\begin{aligned}\nabla^2 f &= \big\{\big[2s\tfrac{d}{ds} + N\big]\tfrac{d}{ds}\big\}\phi \\ &= \big\{2\big[\tfrac{d}{ds}s + \tfrac{N-2}{2}\big]\tfrac{d}{ds}\big\}\phi\end{aligned} \tag{64.2}$$

Here I digress to visit some facts of independent interest, since we happen to be in the neighborhood. If $N = 1$ then

$$\nabla^2 f = \big\{2s\big(\tfrac{d}{ds}\big)^2 + 1\tfrac{d}{ds}\big\}\phi$$

so if $\phi(s)$ is harmonic then

$$\phi(s) = a\sqrt{s} + b$$

If $N = 2$ then

$$\nabla^2 f = \big\{2s\big(\tfrac{d}{ds}\big)^2 + 2\tfrac{d}{ds}\big\}\phi$$

so if $\phi(s)$ is harmonic then

$$\phi(s) = a\log s + b$$

For $N > 2$ harmonicity entails

$$\phi(s) = as^{\frac{2-N}{2}} + b$$

Enforcement of the boundary condition $\phi(\infty) = 0$ is not possible (except trivially) if $N = 1$ or $N = 2$, but for $N \geq 3$ entails $b = 0$; look therefore to the functions

$$\phi(s) = \begin{cases} a\sqrt{s} & : \quad \text{1-dimensional case} \\ a\log s & : \quad \text{2-dimensional case} \\ as^{\frac{2-N}{2}} & : \quad N\text{-dimensional case}\end{cases}$$

which in r-language become (after simplifications)

$$\varphi(r) = \begin{cases} (1/r)^{-1} = r & : \quad \text{1-dimensional case} \\ \log(1/r) & : \quad \text{2-dimensional case} \\ (1/r)^{+1} & : \quad \text{3-dimensional case} \\ (1/r)^{+2} & : \quad \text{4-dimensional case} \\ \quad\vdots & \\ (1/r)^{N-2} & : \quad N\text{-dimensional case}\end{cases}$$

These functions are—except in the case $N = 1$—harmonic *except at the origin*. To assign measure to the *strength* of the singularity I proceed non-standardly:[24]

[24] I proceed, that is to say, by adaptation of a "regularization trick" borrowed from p. 17 of my ELECTRODYNAMICS (1980). The distinguishing merit of the method—slight though it is—is that it achieves its result without appeal to Gauss' divergence theorem.

Introduce functions

$$
\begin{aligned}
g_2(r;\epsilon) &\equiv \log\left(\tfrac{1}{r+\epsilon}\right)\\
g_3(r;\epsilon) &\equiv \tfrac{1}{(r+\epsilon)^1}\\
g_4(r;\epsilon) &\equiv \tfrac{1}{(r+\epsilon)^2}\\
&\vdots\\
g_N(r;\epsilon) &\equiv \tfrac{1}{(r+\epsilon)^{N-2}}
\end{aligned}
$$

which are *non*-singular except in the limit $\epsilon\downarrow 0$. Working from (64.1), compute

$$
\begin{aligned}
\nabla^2 g_2(r;\epsilon) &= -\tfrac{\epsilon}{r(r+\epsilon)^2}\\
\nabla^2 g_3(r;\epsilon) &= -\tfrac{2\epsilon}{r(r+\epsilon)^3}\\
\nabla^2 g_4(r;\epsilon) &= -\tfrac{6\epsilon}{r(r+\epsilon)^4}\\
&\vdots\\
\nabla^2 g_N(r;\epsilon) &= -\tfrac{(N-1)(N-2)\epsilon}{r(r+\epsilon)^N} \quad : \quad N\geq 3
\end{aligned}
$$

These functions vanish in the limit $\epsilon\downarrow 0$, but for $\epsilon\neq 0$ become singular at $r=0$. To measure the strength of the singularity we compute

$$I_N \equiv \int_0^R \nabla^2 g_N(r;\epsilon)\cdot S_N r^{N-1}\,dr$$

to obtain

$$
\begin{aligned}
I_2 &= -\ \ S_2 + \epsilon S_2\tfrac{1}{R+\epsilon}\\
I_3 &= -1S_3 + \epsilon S_3\tfrac{2R+\epsilon}{(R+\epsilon)^2}\\
I_4 &= -2S_4 + \epsilon S_4\big\{\text{factor that vanishes as } R\uparrow\infty\big\}\\
I_5 &= -3S_5 + \epsilon S_5\big\{\text{factor that vanishes as } R\uparrow\infty\big\}\\
I_6 &= -4S_6 + \epsilon S_6\big\{\text{factor that vanishes as } R\uparrow\infty\big\}\\
&\vdots
\end{aligned}
$$

from which we conclude that

$$
\begin{aligned}
\lim_{\epsilon\downarrow 0}\nabla^2\log\left(\tfrac{1}{r+\epsilon}\right) &= -\ \ S_2\delta(\boldsymbol{x}) \quad : \quad \text{2-dimensional, with } S_2 = 2\pi\\
\lim_{\epsilon\downarrow 0}\nabla^2\tfrac{1}{(r+\epsilon)^1} &= -\ \ S_3\delta(\boldsymbol{x}) \quad : \quad \text{3-dimensional, with } S_3 = 4\pi\\
\lim_{\epsilon\downarrow 0}\nabla^2\tfrac{1}{(r+\epsilon)^2} &= -2S_4\delta(\boldsymbol{x}) \quad : \quad \text{4-dimensional, with } S_4 = 2\pi^2\\
\lim_{\epsilon\downarrow 0}\nabla^2\tfrac{1}{(r+\epsilon)^3} &= -3S_5\delta(\boldsymbol{x}) \quad : \quad \text{5-dimensional, with } S_5 = \tfrac{8}{3}\pi^2\\
\lim_{\epsilon\downarrow 0}\nabla^2\tfrac{1}{(r+\epsilon)^4} &= -4S_6\delta(\boldsymbol{x}) \quad : \quad \text{6-dimensional, with } S_6 = \ \ \pi^3\\
&\vdots\\
\lim_{\epsilon\downarrow 0}\nabla^2\tfrac{1}{(r+\epsilon)^{N-2}} &= -(N-2)S_N\delta(\boldsymbol{x}) \quad \text{with} \quad S_N = \tfrac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \qquad (65)
\end{aligned}
$$

With (65) we are placed in position to do in N dimensions what physicists are often called upon to do in the case $N = 3$: given an instance of the Poisson equation

$$\nabla^2 A(\boldsymbol{x}) = B(\boldsymbol{x}) \tag{66.1}$$

we have the formal solution

$$A(\boldsymbol{x}) = \nabla^{-2} B(\boldsymbol{x}) \tag{66.2}$$

to which we can ascribe the non-local meaning

$$= \begin{cases} -\frac{1}{S_2}\iint B(\boldsymbol{y})\log\Big[\frac{1}{r(\boldsymbol{y}-\boldsymbol{x})}\Big]\,dy_1dy_2 \\ -\frac{1}{(N-2)S_N}\iint\!\cdots\!\int B(\boldsymbol{y})\Big[\frac{1}{r(\boldsymbol{y}-\boldsymbol{x})}\Big]^{N-2}dy_1dy_2\cdots dy_N \;:\; N\geq 3 \end{cases} \tag{66.3}$$

We have at this point established contact with the theory of Green's functions. Notice that many of the equations presented above make sense even when N is *not an integer*, but to complete that line of generalization we must be in position to assign useful meaning to the notion

$$f(\boldsymbol{x}) \text{ with } \boldsymbol{x}\in E^{\text{non-integer}}$$

With that cake in the oven, I return now to Gaussian representation theory:

At (60) I introduced

$$W_0(r;\epsilon;N) = \big(\tfrac{1}{2\pi\epsilon}\big)^{\frac{1}{2}N} e^{-\frac{1}{2\epsilon}r^2} \;:\; \text{Gaussian representation of } \delta(\boldsymbol{x})$$

and seek now to construct

$$W_{2m}(r;\epsilon;N) = \nabla^{2m} W_0 \;:\; \text{Gaussian representation of } \nabla^{2m}\delta(\boldsymbol{x})$$

It proves convenient to make the familiar adjustment $r \to s \equiv \frac{1}{2}r^2$, writing

$$W_0(r;\epsilon;N) \equiv w_0(s;\epsilon;N) = \big(\tfrac{1}{2\pi\epsilon}\big)^{\frac{1}{2}N} e^{-s/\epsilon}$$

and working from (64.2):

$$\nabla^2 W_0 = \big\{\big[2s\tfrac{d}{ds} + N\big]\tfrac{d}{ds}\big\}w_0(s;\epsilon;N)$$

I begin by recording the results of some experiments inspired by the fact that my nose has picked up the scent[25] of a herd of associated Laguerre polynomials $L_n^a(s)$ —known to *Mathematica* as `LaguerreL[n,a,s]`—grazing not too far away:

---

[25] My nose had been pre-sensitized by exposure to the identities (see, for example, Magnus & Oberhettinger, p. 84)

$$\begin{aligned} He_{2n}(x) &= (-2)^n n! \quad L_n^{-\frac{1}{2}}\big(\tfrac{x^2}{2}\big) \\ He_{2n+1}(x) &= (-2)^n n!\, x\, L_n^{+\frac{1}{2}}\big(\tfrac{x^2}{2}\big) \end{aligned}$$

$$
\begin{aligned}
\big\{\big[2s\tfrac{d}{ds}+N\big]\tfrac{d}{ds}\big\}^1 e^{-s} &= e^{-s}\big[-N+2s\big]\\
&= (-2)^1 1!\, e^{-s} L_1^{\frac{1}{2}(N-2)}(s)\\
\big\{\big[2s\tfrac{d}{ds}+N\big]\tfrac{d}{ds}\big\}^2 e^{-s} &= e^{-s}\big[N^2+N(2-4s)+4(-2+s)s\big]\\
&= (-2)^2 2!\, e^{-s} L_2^{\frac{1}{2}(N-2)}(s)\\
\big\{\big[2s\tfrac{d}{ds}+N\big]\tfrac{d}{ds}\big\}^3 e^{-s} &= e^{-s}\big[-N^3+6N^2(-1+s)\\
&\qquad -4N(2-9s+3s^2)+8s(6-6s+s^2)\big]\\
&= (-2)^3 3!\, e^{-s} L_3^{\frac{1}{2}(N-2)}(s)
\end{aligned}
$$

Evidently we have come upon an identity

$$
\begin{aligned}
\big\{\big[s\tfrac{d}{ds}+(a+1)\big]\tfrac{d}{ds}\big\}^n e^{-s} &= (-)^n n!\, e^{-s} L_n^a(s)\\
&= (-)^n s^{-a}\big(\tfrac{d}{ds}\big)^n\big\{e^{-s}s^{n+a}\big\}
\end{aligned}
$$

which conforms precisely to our needs, but is of such curious design that formal proof still eludes me; I present

$$
\big\{\big[2s\tfrac{d}{ds}+N\big]\tfrac{d}{ds}\big\}^n e^{-s} = (-2)^n n!\, e^{-s} L_n^{\frac{1}{2}(N-2)}(s) \tag{67}
$$

as an "experimental" fact, but a fact nonetheless. I note that (67) appears to be subject to none of the familiar dimensionality restrictions, but to work for *all* real values of N; for example, if we set $n=3$ then (whether we work from the expression on the left or the expression on the right) we obtain

$$
\begin{array}{rcl}
e^{-s}[\;\;3+\;18s-\;36s^2+\;8s^3] & : & N=-1\\
\frac{1}{8}e^{-s}[\;21+252s-336s^2+64s^3] & : & N=-\frac{1}{2}\\
8e^{-s}[\;\;0+\;\;6s-\;\;6s^2+\;\;s^3] & : & N=\;\;0\\
\frac{1}{8}e^{-s}[-45+540s-432s^2+64s^3] & : & N=\;\;\frac{1}{2}\\
e^{-s}[-15+\;90s-\;60s^2+\;8s^3] & : & N=\;\;1
\end{array}
$$

Reinstating now the details which distinguish $w_0(s;\epsilon;N)$ from e^{-s}, we on the basis of (67) obtain

$$
\begin{aligned}
w_{2m}(s;\epsilon;N) &= \big\{\big[2s\tfrac{d}{ds}+N\big]\tfrac{d}{ds}\big\}^m w_0(s;\epsilon;N) && (68.1)\\
&= \big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N\big(-\tfrac{2}{\epsilon}\big)^m \Gamma(m+1)e^{-s/\epsilon}L_m^{\frac{1}{2}(N-2)}(s/\epsilon) && (68.2)
\end{aligned}
$$

of which I give now some examples:[26]

[26] In each case I have, with the assistance of *Mathematica*, used *both* variants of (68) to obtain the result quoted.

Results special to the case $N = 2$

I look to this case first because it lends itself so uniquely well to graphical representation. We have

$$w_0(s;\epsilon;2) = \left(\tfrac{1}{2\pi\epsilon}\right)e^{-s/\epsilon}$$

which is normalized in the sense (use $rdr = ds$ and $S_2 = 2\pi$) that

$$\int_0^\infty w_0(s;\epsilon;2)\cdot S_2\,ds = 1 \quad : \quad (\text{all } \epsilon > 0)$$

Working from (68) we compute

$$\begin{aligned}
w_2(s;\epsilon;2) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{2}{2}}\tfrac{1}{\epsilon^2}e^{-s/\epsilon}\,2[s-\epsilon]\\
w_4(s;\epsilon;2) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{2}{2}}\tfrac{1}{\epsilon^4}e^{-s/\epsilon}\,4[s^2-4s\epsilon+2\epsilon^2]\\
w_6(s;\epsilon;2) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{2}{2}}\tfrac{1}{\epsilon^6}e^{-s/\epsilon}\,8[s^3-9s^2\epsilon+18s\epsilon^2-6\epsilon^3]\\
&\;\;\vdots
\end{aligned}$$

where I have refrained from making obvious simplifications in order to expose more clearly the *pattern* of events. These results are illustrated in Figures 11–13. Comparative study of the figures shows that the height/depth of the central spike is a rapidly increasing function of ascending order, and that the central spikes conform to the "principle of sign alternation" discussed in §1. The surrounding terrain gives rotationally invariant smooth meaning to the notion of "neighborhood" which was encountered (in lattice approximation) on p. 30.

(See the figures, after which the text resumes.)

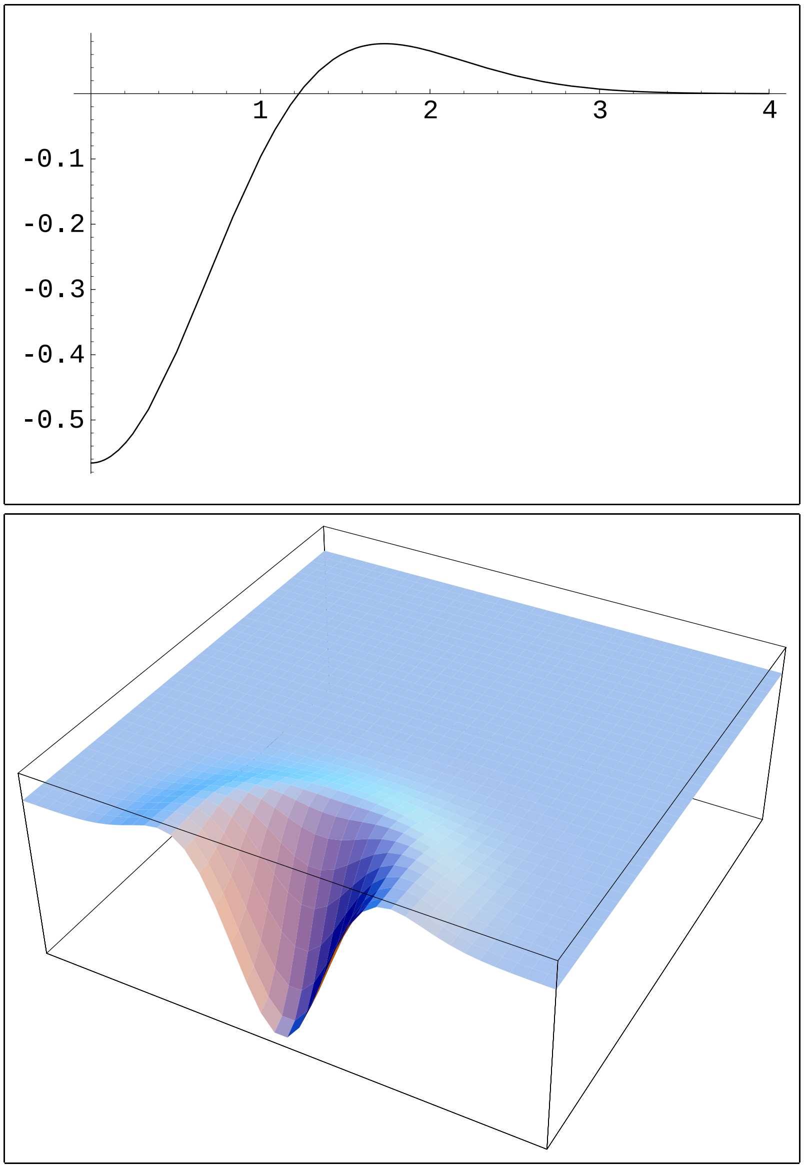


FIGURE 11: *Gaussian representation of the harmonic weight function in the 2-dimensional case. Should be regarded as frames (frame number* $\epsilon = 3/4$*) from the movies*

$$\lim_{\epsilon\downarrow 0} W_2(r;\epsilon;2)_{\text{above}} \quad \textit{and} \quad \lim_{\epsilon\downarrow 0} W_2(\sqrt{x^2+y^2};\epsilon;2)_{\text{below}}$$

As the film progresses, the design—in the tradition of Figure 2—becomes ever more compacted about the origin.

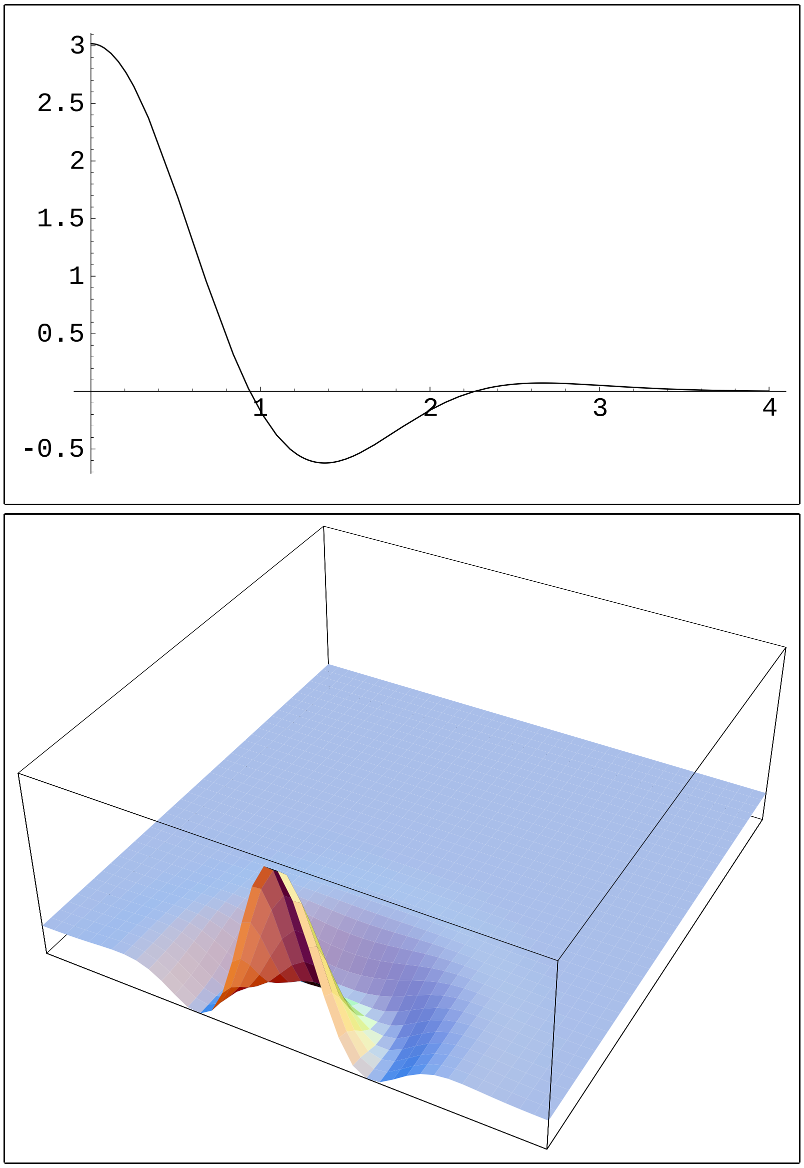


Figure 12: *Gaussian representation of the biharmonic weight function in the 2-dimensional case. Should be regarded as frames (frame number* $\epsilon = 3/4$*) from the movies*

$$\lim_{\epsilon \downarrow 0} W_4(r;\epsilon;2)_{\text{above}} \quad \textit{and} \quad \lim_{\epsilon \downarrow 0} W_4(\sqrt{x^2+y^2};\epsilon;2)_{\text{below}}$$

As the film progresses, the design becomes ever more compacted about the origin.

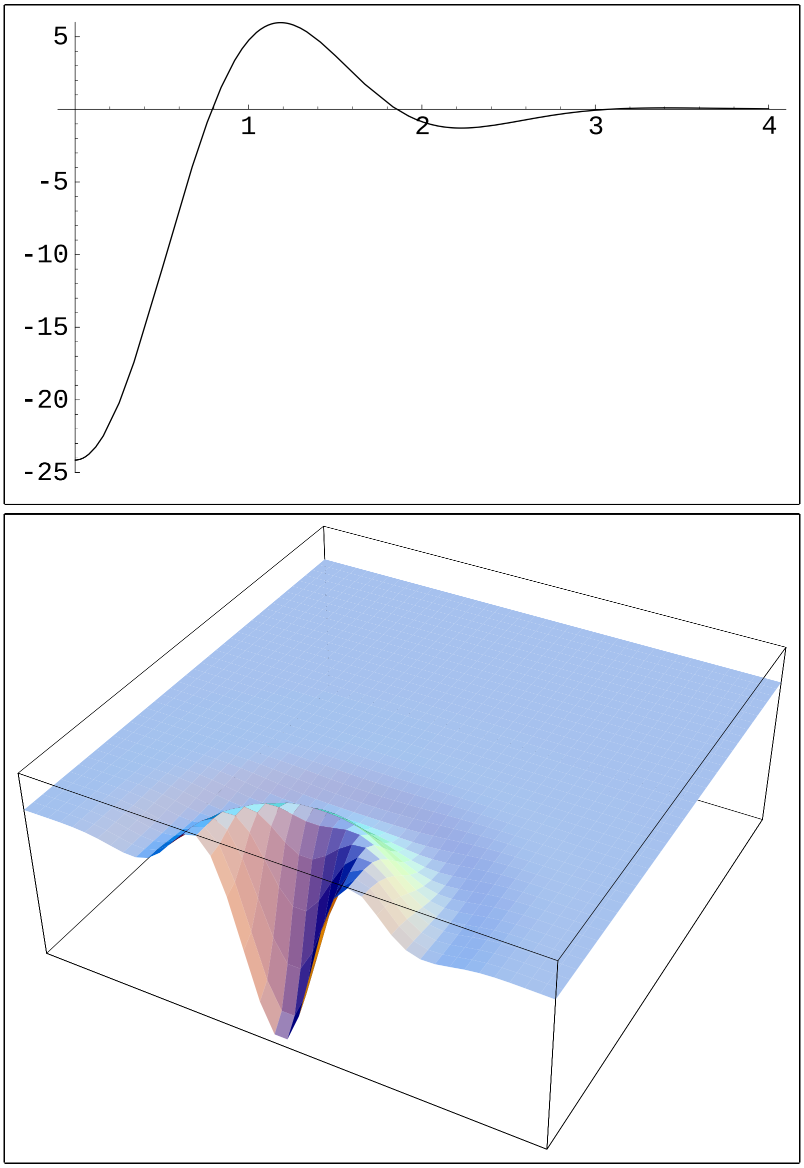


Figure 13: *Gaussian representation of the triharmonic weight function in the 2-dimensional case. Should be regarded as frames (frame number* $\epsilon = 3/4$*) from the movies*

$$\lim_{\epsilon\downarrow 0} W_6(r;\epsilon;2)_{\text{above}} \quad \textit{and} \quad \lim_{\epsilon\downarrow 0} W_6(\sqrt{x^2+y^2};\epsilon;2)_{\text{below}}$$

As the film progresses, the design becomes ever more compacted about the origin.

Results special to the case $N = 3$

Here our results acquire a latently familiar look because we inhabit this case; we have

$$w_0(s;\epsilon;3) = \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{3}{2}} e^{-s/\epsilon}$$

which is normalized in the sense (use $r^2 dr = \sqrt{2s}\,ds$ and $S_3 = 4\pi$) that

$$\int_0^\infty w_0(s;\epsilon;3)\cdot S_3\sqrt{2s}\,ds = 1 \quad : \quad (\text{all } \epsilon > 0)$$

Working from (68) we compute

$$\begin{aligned}
w_2(s;\epsilon;3) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{3}{2}} \tfrac{1}{\epsilon^2} e^{-s/\epsilon}[2s - 3\epsilon] \\
w_4(s;\epsilon;3) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{3}{2}} \tfrac{1}{\epsilon^4} e^{-s/\epsilon}[4s^2 - 20s\epsilon + 15\epsilon^2] \\
w_6(s;\epsilon;3) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{3}{2}} \tfrac{1}{\epsilon^6} e^{-s/\epsilon}[8s^3 - 84s^2\epsilon + 210s\epsilon^2 - 105\epsilon^3] \\
&\ \ \vdots
\end{aligned}$$

Results special to the case $N = 4$

We have

$$w_0(s;\epsilon;4) = \left(\tfrac{1}{2\pi\epsilon}\right)^2 e^{-s/\epsilon}$$

which is normalized in the sense (use $r^3 dr = 2s\,ds$ and $S_4 = 2\pi^2$) that

$$\int_0^\infty w_0(s;\epsilon;4)\cdot S_4 2s\,ds = 1 \quad : \quad (\text{all } \epsilon > 0)$$

Working from (68) we compute

$$\begin{aligned}
w_2(s;\epsilon;4) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{4}{2}} \tfrac{1}{\epsilon^2} e^{-s/\epsilon}\, 2[s - 2\epsilon] \\
w_4(s;\epsilon;4) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{4}{2}} \tfrac{1}{\epsilon^4} e^{-s/\epsilon}\, 4[s^2 - 6s\epsilon + 6\epsilon^2] \\
w_6(s;\epsilon;4) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{4}{2}} \tfrac{1}{\epsilon^6} e^{-s/\epsilon}\, 8[s^3 - 12s^2\epsilon + 36s\epsilon^2 - 24\epsilon^3] \\
&\ \ \vdots
\end{aligned}$$

Results appropriate to the general case

We have

$$w_0(s;\epsilon;N) = \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{1}{2}N} e^{-s/\epsilon}$$

which is (as *Mathematica* confirms) normalized in the sense—use

$$S_N \cdot r^{N-1}dr = S_N \cdot (2s)^{\frac{N-2}{2}}\, ds = \tfrac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \cdot (2s)^{\frac{N-2}{2}}\, ds$$

—that

$$\int_0^\infty w_0(s;\epsilon;N) \cdot \tfrac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \cdot (2s)^{\frac{N-2}{2}}\, ds = 1 \quad : \quad \Re[N\,] > 0 \;\; \& \;\; \Re[\epsilon\,] > 0$$

and (working most conveniently from (68.2)) we compute

$$\begin{aligned}
w_2(s;\epsilon;N) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{N}{2}} \tfrac{1}{\epsilon^2} e^{-s/\epsilon} \left[2s - N\epsilon\right] \\
w_4(s;\epsilon;N) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{N}{2}} \tfrac{1}{\epsilon^4} e^{-s/\epsilon} \left[4s^2 - 8s\epsilon - 4Ns\epsilon + 2N\epsilon^2 + N^2\epsilon^2\right] \\
w_6(s;\epsilon;N) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{N}{2}} \tfrac{1}{\epsilon^6} e^{-s/\epsilon} \left[8s^3 - 48s^2\epsilon - 12Ns^2\epsilon + 48s\epsilon^2 + 36Ns\epsilon^2\right. \\
&\qquad\qquad \left. + 6N^2s\epsilon^2 - 8N\epsilon^3 - 6N^2\epsilon^3 - N^3\epsilon^3\right] \\
&\;\;\vdots
\end{aligned}$$

from which all previous results can be recovered as particular instances. Look finally to the simplest case of all—the case from which we started:

Results special to the case $N = 1$

We have

$$w_0(s;\epsilon;1) = \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{1}{2}} e^{-s/\epsilon}$$

which is normalized in this non-standard formulation (use $dr = (2s)^{-1}\, ds$ and $S_1 = 2$) of the familiar Gaussian sense:

$$\int_0^\infty w_0(s;\epsilon;1) \cdot S_1 \tfrac{1}{\sqrt{2s}}\, ds = 1 \quad : \quad (\text{all } \epsilon > 0)$$

Working from (68) we compute

$$\begin{aligned}
w_2(s;\epsilon;1) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{1}{2}} \tfrac{1}{\epsilon^2} e^{-s/\epsilon} \left[2s - \epsilon\right] \\
w_4(s;\epsilon;1) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{1}{2}} \tfrac{1}{\epsilon^4} e^{-s/\epsilon} \left[4s^2 - 12s\epsilon + 3\epsilon^2\right] \\
w_6(s;\epsilon;1) &= \left(\tfrac{1}{2\pi\epsilon}\right)^{\frac{1}{2}} \tfrac{1}{\epsilon^6} e^{-s/\epsilon} \left[8s^3 - 60s\epsilon + 90s\epsilon^2 - 15\epsilon^3\right] \\
&\;\;\vdots
\end{aligned}$$

giving

$$\begin{aligned}
W_2(r;\epsilon;1) &= w_2(\tfrac{1}{2}r^2;\epsilon;1) = \tfrac{1}{\sqrt{2\pi\epsilon}}\tfrac{1}{\epsilon^2}e^{-\frac{1}{2\epsilon}r^2}\,[r^2-\epsilon]\\
W_4(r;\epsilon;1) &= w_4(\tfrac{1}{2}r^2;\epsilon;1) = \tfrac{1}{\sqrt{2\pi\epsilon}}\tfrac{1}{\epsilon^4}e^{-\frac{1}{2\epsilon}r^2}\,[r^4-6r^2\epsilon+3\epsilon^2]\\
W_6(r;\epsilon;1) &= w_6(\tfrac{1}{2}r^2;\epsilon;1) = \tfrac{1}{\sqrt{2\pi\epsilon}}\tfrac{1}{\epsilon^6}e^{-\frac{1}{2\epsilon}r^2}\,\underbrace{[r^6-15r^4\epsilon+45r^2\epsilon^2-15\epsilon^3]}_{\text{Hermite polynomials}}\\
&\;\;\vdots
\end{aligned}$$

at which point we have, in effect, recovered (6.2/6.4/6.6/...).

The effect of dimensional adjustment

$$N \longrightarrow N+1 \longrightarrow N+2 \longrightarrow \cdots$$

is illustrated in Figures 14–16; as N becomes large the contribution of outlying suburbs of the central concentration diminishes—and seems ultimately to vanish altogether; it becomes in this sense difficult to distinguish ∇^4 or $\nabla^6\ldots$ from $\delta(\boldsymbol{x})$. I cannot claim to understand this phenomenon, which seems counterintuitive; it is associated, I presume, with the more familiar fact that in high dimension "most of the points interior to a hypersphere lie very near to its surface."

The oppressive detail from which we now emerge is of little interest in itself: it was presented only to illustate the pattern of events—only, that is to say, to make plain the meaning of the integrand which enters at (63) into the integral representation of $\nabla^{2m}f(\boldsymbol{x})$. Even the integral representation itself—though it does provide insight into "what kind of a thing ∇^{2m} is"—might be dismissed as "uninteresting" by anyone willing simply to sit down and *compute* $\nabla^{2m}f(\boldsymbol{x})$. Or, working from (53), one could attempt to compute

$$(\langle x^{2m}\rangle_N)^{-1}\big(\tfrac{\partial}{\partial r}\big)^{2m}\langle f(\boldsymbol{x};r)\rangle_N\Big|_{r=0}$$

The Gaussian representation theorem (63) will acquire real interest only if it can be shown to bring formal unity to notions which seemed otherwise distinct. Or to permit us to formulate thoughts which were otherwise unthinkable. Or to undertake computations which we were otherwise powerless to undertake. To those ends...

Recall that we have already at (42) made the acquaintance of the confluent hypergeometric "Kummer function"

$$\begin{aligned}
M(a;b;z) &= {}_1\mathsf{F}_1(a,b,z) = \texttt{Hypergeometric1F1[a,b,z]}\\
&= 1+\tfrac{az}{b}+\tfrac{a(1+a)z^2}{2!b(1+b)}+\tfrac{a(1+a)(2+a)z^3}{3!b(1+b)(2+b)}+\cdots
\end{aligned}$$

to which Spanier & Oldham devote their Chapter 47. Of more immediate interest now will be its close relative, the "Tricomi function" $U(a;b;z)$[27] which

[27] Spanier & Oldham, Chapter 48. The functions are treated simultaneously in Abramowitz & Stegun, Chapter 13.

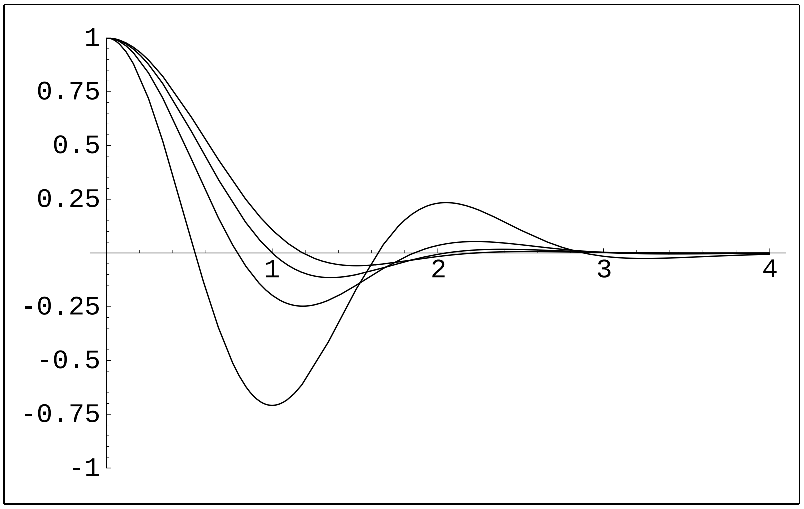


FIGURE 14: *Graphs of*

$$\frac{W_6(r;\epsilon;N)}{W_6(0;\epsilon;N)} \quad : \quad N = 1, 2, 3, 4$$

intended to provide normalized illustration of the fact that weight functions of given order become progressively flatter as the space dimension N *increases.*

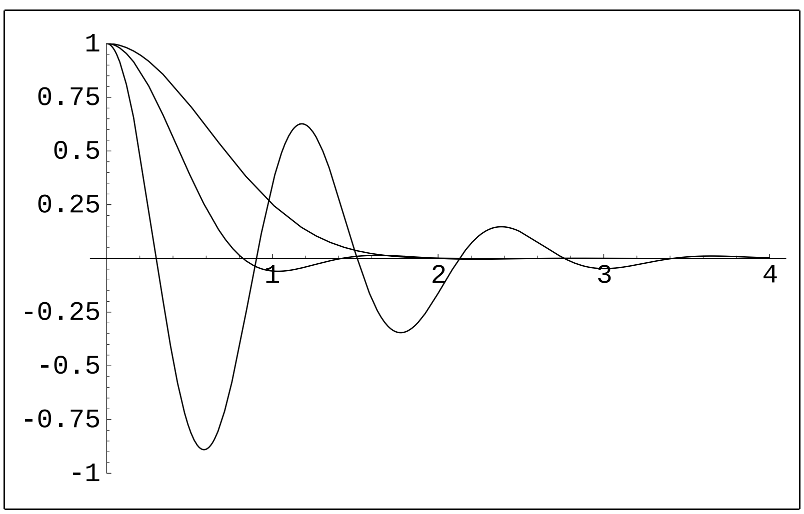


FIGURE 15: *Graphs of the ratio* $W_{20}(r;\epsilon;N)/W_{20}(0;\epsilon;N)$ *with* $N = 1, 5$ *& 20, intended to provide more vivid evidence of the same phenomenon.*

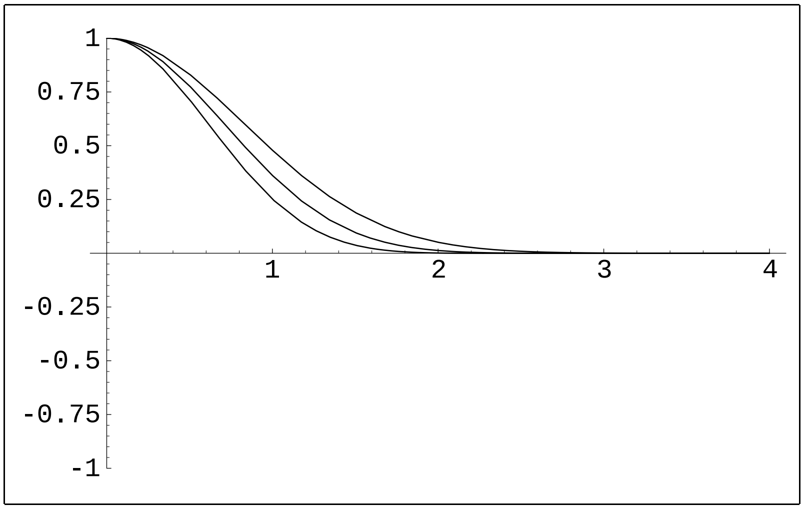


FIGURE 16: *Graphs of*

$$\frac{W_{2m}(r;\epsilon;20)}{W_{2m}(0;\epsilon;20)} \quad : \quad m = 1, 5 \textit{ and } 10$$

The concentric "suburban rings" have become invisible. The central radius grows smaller as m increases.

$$\begin{aligned} U(a;b;z) &= z^{-a}{}_2\mathsf{F}_0(a;1+a-b;-1/z) = \texttt{HypergeometricU[a,b,z]} \\ &= \tfrac{\Gamma(1-b)}{\Gamma(1+a-b)}M(a;b;z) + z^{1-b}\cdot\tfrac{\Gamma(b-1)}{\Gamma(a)}M(1+a-b;2-b;z) \end{aligned} \tag{69}$$

—are linearly independent solutions of "Kemmer's equation;" i.e., of the self-adjoint differential equation which can be written

$$z\tfrac{d^2w}{dz^2} + (b-z)\tfrac{dw}{dz} - aw = 0 \quad ; \text{equivalently} \quad \tfrac{d}{dz}\big[z^b e^{-z}\tfrac{dw}{dz}\big] = az^{b-1}e^{-z}w$$

The latter equation has a very "Laguerre-ish" look about it, and indeed: the associated Laguerre polynomial

$$L_n^\alpha(z) = \texttt{LaguerreL[n,}\alpha\texttt{,z]} = \tfrac{1}{n!}z^{-\alpha}\big(\tfrac{d}{dz}\big)^n\big[z^{n+\alpha}e^{-z}\big]$$

satisfies

$$z\tfrac{d^2w}{dz^2} + (1+\alpha-z)\tfrac{dw}{dz} + nw = 0$$

which is Kummer's equation in the case $b \mapsto 1+\alpha$ and $a \mapsto -n$. We are not surprised, therefore, to be informed (Spanier & Oldham, **23:12:14**; Abramowitz & Stegun, **13.6.9** and **13.6.27**) that

$$L_n^\alpha(z) = \tfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}M(-n;1+\alpha;z) \tag{70.1}$$

$$(-)^n\Gamma(n+1)L_n^\alpha(z) = U(-n;1+\alpha;z) \tag{70.2}$$

My desire is to enlarge upon this slight variant of (68)

$$w_{2m}(s;\epsilon;N) = \left(\tfrac{1}{\sqrt{2\pi\epsilon}}\right)^N \left(\tfrac{2}{\epsilon}\right)^m \cdot e^{-t}(-)^m \Gamma(1+m) L_m^{\frac{1}{2}(N-2)}(t) \qquad t \equiv s/\epsilon \tag{71}$$

in such a way as to achieve escape from the "tyranny of the integers" without incurring the unwelcome complexification implicit in the observation that

$$(-)^m \text{ is complex except when } m = \ldots, -2, -1, 0, +1, +2, \ldots$$

Equation (70.2) seems in this light particularly attractive, but lessons learned from arduous exploration of that pathway—particularly the observation that

$$(-)^m = \cos m\pi$$

—lead me back now again to a renewed interest in (70.1). I propose to study the implications of writing[28]

$$\tilde{w}_{2m}(s;\epsilon;N) = \left(\tfrac{1}{\sqrt{2\pi\epsilon}}\right)^N \left(\tfrac{2}{\epsilon}\right)^m \cdot e^{-t}\cos m\pi \tfrac{\Gamma(\frac{N}{2}+m)}{\Gamma(\frac{N}{2})} M(-m;\tfrac{N}{2};t) \tag{72.1}$$

which by "Kummer's transformation"[29] $M(a;b;z) = e^z M(b-a;b;-z)$ can also be written

$$= \left(\tfrac{1}{\sqrt{2\pi\epsilon}}\right)^N \left(\tfrac{2}{\epsilon}\right)^m \cdot \cos m\pi \tfrac{\Gamma(\frac{N}{2}+m)}{\Gamma(\frac{N}{2})} M(\tfrac{N}{2}+m;\tfrac{N}{2};-t) \tag{72.2}$$

Additional variants are made possible by this consequence

$$\begin{aligned} \cos m\pi \cdot \Gamma(\tfrac{N}{2}+m) &= \pi \frac{\cos m\pi}{\sin(\frac{N}{2}+m)\pi} \frac{1}{\Gamma(1-\frac{N}{2}-m)} \\ \sin(\tfrac{N}{2}+m)\pi &= \sin\tfrac{N}{2}\pi\cos m\pi + \cos\tfrac{N}{2}\pi\sin m\pi \end{aligned} \tag{73}$$

of the gamma reflection formula.[30] It is to gain a better sense of where we now stand that I look to special cases:

In the one-dimensional case we have

$$\tilde{w}_{2m}(s;\epsilon;1) = \left(\tfrac{1}{\sqrt{2\pi\epsilon}}\right)\left(\tfrac{2}{\epsilon}\right)^m \tfrac{1}{\sqrt{\pi}} \begin{cases} \cos m\pi\,\Gamma(\frac{1}{2}+m) M(\frac{1}{2}+m;\frac{1}{2};-t) \\ \\ \pi e^{-t}\frac{1}{\Gamma(\frac{1}{2}-m)} M(-m;\frac{1}{2};t) \end{cases} \tag{74.1}$$

---

[28] I will write now $\tilde{w}_{2m}$ instead of w_{2m} to emphasize that I have adopted a modified stance, and to facilitate comparing the new with the old.

[29] See Abramowitz & Stegun, **13.1.27**; Spanier & Oldham, **47:5:1**. This important discovery of Gauss-Kummer is precisely the "obscure identity" which in the caption to Figure 4 I declined to think about!

[30] Abramawitz & Stegun, **6.1.17**.

On the other hand, (42) yields expressions which, when adapted to present notational conventions, read

$$\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)\big(\tfrac{2}{\epsilon}\big)^m\tfrac{1}{\sqrt{\pi}}\begin{cases}\cos m\pi\,\Gamma(\tfrac{1}{2}+m)M(\tfrac{1}{2}+m;\tfrac{1}{2};-t)\\ \quad -2\sqrt{t}\sin m\pi\,\Gamma(1+m)M(1+m;\tfrac{3}{2};-t)\\ \\ \pi e^{-t}\tfrac{1}{\Gamma(\frac{1}{2}-m)}M(-m;\tfrac{1}{2};t)\\ \quad 2\pi\sqrt{t}\,e^{-t}\tfrac{1}{\Gamma(-m)}M(\tfrac{1}{2}-m;\tfrac{3}{2};t)\end{cases}$$

—of which (74.1) captures only a fragment,[31] yet a fragment sufficient to reproduce the $\boxed{N=1}$ data presented on p. 46. That data derived from (68.2); we have

$$\tilde{w}_{2m}(s;\epsilon;1)=w_{2m}(s;\epsilon;1)\quad:\quad m=0,1,2,\ldots$$

According to (68.2), $w_{2m}(s;\epsilon;1)$ becomes imaginary for $m=\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{3}{2},\ldots$ (and *Mathematica* is unable to assign meaning to $L_m^\alpha(t)$ for $t<0$), but (74.1) gives

$$\tilde{w}_{2m}(s;\epsilon;1)=0\quad:\quad m=+\tfrac{1}{2},+\tfrac{3}{2},+\tfrac{5}{2},\ldots$$

Though equation $(74.1)^{\text{upper}}$ (for evident reasons) becomes indeterminate at $m=-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\ldots$, $(74.1)_{\text{lower}}$ assigns meaning to those cases; for example, we have

$$\begin{aligned}\tilde{w}_{-1}(s;\epsilon;1)&=\tfrac{1}{2}\\ \tilde{w}_{-3}(s;\epsilon;1)&=\tfrac{1}{4}(2s+\epsilon)\\ \tilde{w}_{-5}(s;\epsilon;1)&=\tfrac{1}{48}(4s^2+12s\epsilon+3\epsilon^2)\end{aligned}$$

And whether we work from $(74.1)^{\text{upper}}$ or $(74.1)_{\text{lower}}$ we find

$$\tilde{w}_{-2}(s;\epsilon;1)=\tfrac{1}{\sqrt{2\pi}}e^{-s/\epsilon}\sqrt{\epsilon}\Big\{1+e^{s/\epsilon}\sqrt{\pi s/\epsilon}\ \text{erf}\sqrt{s/\epsilon}\Big\}$$

To summarize: the Laplacian theory implicit in (72), when pulled back to one dimension, reproduces only a fraction of the full fractional calculus supported by (42),[32] but—insofar as it permits m to become fractional/negative—does extend the reach of (68.2).

In the two-dimensional case (72) gives

$$\tilde{w}_{2m}(s;\epsilon;2)=\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^2\big(\tfrac{2}{\epsilon}\big)^m\begin{cases}\cos m\pi\,\Gamma(1+m)M(1+m;1;-t)\\ \\ -\pi e^{-t}\cot m\pi\,\tfrac{1}{\Gamma(-m)}M(-m;1;t)\end{cases}\tag{74.2}$$

We find that $(74.2)^{\text{upper}}$ reproduces the $\boxed{N=2}$ data presented on p. 41, though $(74.2)_{\text{lower}}$ becomes indeterminate at for $m=0,1,2,\ldots$. One has, whether

[31] Equations (74.1) capture only the terms which are "even in $\sqrt{t}$."

[32] I will have more to say about how that comes to be so.

working from the upper variant or the lower,

$$\tilde{w}_{2m}(s;\epsilon;2) = 0 \qquad : \quad m = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \ldots$$

and

$$\tilde{w}_{2m}(s;\epsilon;2) = \texttt{ComplexInfinity} \quad : \quad m = -1, -2, -3, \ldots$$

Look finally to the three-dimensional case: (72) gives

$$\tilde{w}_{2m}(s;\epsilon;3) = \big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^3 \big(\tfrac{2}{\epsilon}\big)^m \tfrac{2}{\sqrt{\pi}} \begin{cases} \cos m\pi\,\Gamma(\tfrac{3}{2}+m) M(\tfrac{3}{2}+m; \tfrac{3}{2}; -t) \\ \\ -\pi e^{-t} \frac{1}{\Gamma(-\frac{1}{2}-m)} M(-m; \tfrac{3}{2}; t) \end{cases} \tag{74.3}$$

At $m = 0, 1, 2, \ldots$ equations $(74.3)^{\text{upper}}$ and $(74.3)_{\text{lower}}$ serve equally well to reproduce the $\boxed{N=3}$ data presented on p. 45. Both variants give

$$\tilde{w}_{2m}(s;\epsilon;3) = 0 \quad : \quad m = -\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{3}{2}, \ldots$$

At other negative half-integral values $(74.3)^{\text{upper}}$ becomes interminate, but $(74.3)_{\text{lower}}$ gives (for example)

$$\begin{aligned} \tilde{w}_{-3}(s;\epsilon;3) &= -\tfrac{1}{4\pi} \\ \tilde{w}_{-5}(s;\epsilon;3) &= -\tfrac{1}{24\pi}(2s+3\epsilon) \\ \tilde{w}_{-7}(s;\epsilon;3) &= -\tfrac{1}{480\pi}(4s^2+20s\epsilon+15\epsilon^2) \end{aligned}$$

working from either variant we find

$$\tilde{w}_{-2}(s;\epsilon;3) = -\tfrac{1}{4\pi\sqrt{2s}}\,\mathrm{erf}\sqrt{s/\epsilon}$$

and that $\tilde{w}_{-4}(s;\epsilon;3)$, $\tilde{w}_{-6}(s;\epsilon;3)$, etc. are given by similar expressions of ascending complexity.

5. Contact with the theory of Green's functions. The result just achieved bears so directly upon the interpretation of (66), and therefore upon what has served throughout this discussion as a point of primary motivation, that I digress now to place it in the company of some of the more immediate of its variously-dimensioned siblings. We work from (72.2)—more specifically, from

$$\begin{aligned} \tilde{w}_{-2}(s;\epsilon;N) &= -\tfrac{\epsilon}{2}\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N \frac{\Gamma(\frac{N}{2}-1)}{\Gamma(\frac{N}{2})} M(\tfrac{N}{2}-1; \tfrac{N}{2}; -t) \\ &= -\tfrac{\epsilon}{2}\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N \tfrac{2}{N-2} \cdot M(\tfrac{N}{2}-1; \tfrac{N}{2}; -t) \end{aligned} \tag{75}$$

where to obtain the latter simplification I have made use of the functional equation $\Gamma(z+1) = z\Gamma(z)$—and compute

$$\left.\begin{array}{ll}
\blacktriangleright & \tilde{w}_{-2}(s;\epsilon;1) = +\epsilon\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^1 \quad\cdot \big[t^{+\frac{1}{2}}\sqrt{\pi}\,\text{erf}\sqrt{t} + e^{-t}\big] \\
\triangleright & \tilde{w}_{-2}(s;\epsilon;2) = \pm\epsilon\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^2\tfrac{1}{0}\cdot 1 \qquad\qquad \texttt{ComplexInfinity} \\
\blacktriangleright & \tilde{w}_{-2}(s;\epsilon;3) = -\epsilon\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^3 \quad\cdot \tfrac{1}{2}t^{-\frac{1}{2}}\big[\sqrt{\pi}\,\text{erf}\sqrt{t}\,\big] \\
\triangleright & \tilde{w}_{-2}(s;\epsilon;4) = -\epsilon\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^4\tfrac{1}{2}\cdot \quad t^{-1}\big[1-e^{-t}\big] \\
\blacktriangleright & \tilde{w}_{-2}(s;\epsilon;5) = -\epsilon\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^5\tfrac{1}{3}\cdot\tfrac{3}{4}t^{-\frac{3}{2}}\big[\sqrt{\pi}\,\text{erf}\sqrt{t} - 2\sqrt{t}e^{-t}\,\big] \\
\triangleright & \tilde{w}_{-2}(s;\epsilon;6) = -\epsilon\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^6\tfrac{1}{4}\cdot\tfrac{2}{1}t^{-2}\big[1-e^{-t}(1+t)\big] \\
\blacktriangleright & \tilde{w}_{-2}(s;\epsilon;7) = -\epsilon\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^7\tfrac{1}{5}\cdot\tfrac{5}{8}t^{-\frac{5}{2}}\big[3\sqrt{\pi}\,\text{erf}\sqrt{t} - 2\sqrt{t}e^{-t}(3+2t)\big] \\
\triangleright & \tilde{w}_{-2}(s;\epsilon;8) = -\epsilon\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^8\tfrac{1}{6}\cdot\tfrac{3}{1}t^{-3}\big[2-e^{-t}(2+2t+t^2)\big] \\
 & \qquad\vdots
\end{array}\right\} \tag{76}$$

Clearly, the even entries (▷) are of a type, the odd entries (►) are of a distinct type. The members of either type are interconnected in a very simple way, as I now show: one has (see Abramowitz & Stegun, **133.4.7**)

$$M(a+1;b+1;t) = \tfrac{b}{a}\cdot\tfrac{d}{dt}M(a;b;t)$$

from which it follows in particular that

$$M(\tfrac{N+2}{2}-1;\tfrac{N+2}{2};-t) = -\tfrac{N}{N-2}\tfrac{d}{dt}M(\tfrac{N}{2}-1;\tfrac{N}{2};t)$$

Returning with this information to (75) we obtain

$$\begin{aligned}
\tilde{w}_{-2}(s;\epsilon;N+2) &= \tfrac{1}{2\pi\epsilon}\tfrac{N-2}{N}\Big\{-\tfrac{\epsilon}{2}\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N\tfrac{2}{N-2}\cdot M(\tfrac{N+2}{2}-1;\tfrac{N+2}{2};-t)\Big\} \\
&= -\tfrac{1}{2\pi\epsilon}\tfrac{d}{dt}\Big\{-\tfrac{\epsilon}{2}\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N\tfrac{2}{N-2}\cdot M(\tfrac{N}{2}-1;\tfrac{N}{2};-t)\Big\} \\
&= -\tfrac{1}{2\pi}\tfrac{d}{ds}\tilde{w}_{-2}(s;\epsilon;N)
\end{aligned} \tag{77}$$

which can be (has been) used in a scheme of the form

$$\begin{array}{ll}
\blacktriangleright & \tilde{w}_{-2}(s;\epsilon,1)\to\tilde{w}_{-2}(s;\epsilon,3)\to\tilde{w}_{-2}(s;\epsilon,5)\to\tilde{w}_{-2}(s;\epsilon,7)\to\cdots \\
\triangleright & \qquad\qquad\qquad\quad\;\; \tilde{w}_{-2}(s;\epsilon,4)\to\tilde{w}_{-2}(s;\epsilon,6)\to\tilde{w}_{-2}(s;\epsilon,8)\to\cdots
\end{array}$$

to regenerate the entire list (76) on the basis only of the information written into the"seed functions" $\tilde{w}_{-2}(s;\epsilon;1)$ and $\tilde{w}_{-2}(s;\epsilon;4)$. Inversely, one has

$$\begin{aligned}
\tilde{w}_{-2}(s;\epsilon;N) &= \tilde{w}_{-2}(0;\epsilon;N) - 2\pi\int_0^s \tilde{w}_{-2}(\sigma;\epsilon;N+2)\,d\sigma \\
\tilde{w}_{-2}(0;\epsilon;N) &= -\epsilon\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N\tfrac{1}{N-2}
\end{aligned} \tag{78}$$

but (essentially because $\tilde{w}_{-2}(0;\epsilon;N)\neq 0$) I have failed in my attempts to achieve $\tilde{w}_{-2}(0;\epsilon;N)\longleftarrow\tilde{w}_{-2}(0;\epsilon;N+1)$ by *semi*-integration, and thus to construct an analog of the elegant Hadamard-Ritsz semi-differentiation scheme

$$\blacktriangleright\rightarrow\triangleright\rightarrow\blacktriangleright\rightarrow\triangleright\rightarrow\blacktriangleright\rightarrow\ \cdots$$

described in §7 of "Construction & Physical Application of the Fractional Calculus (1997). It is, however, interesting in connection with

$$\tilde{w}_{-2}(s;\epsilon;2)\longleftarrow\tilde{w}_{-2}(s;\epsilon;4)$$

that $\int_0^t\tau^{-1}(1-e^{-\tau})\,d\tau=$ `EulerGamma+Gamma[0,s]+Log[s]`, so $\tilde{w}_{-2}(s;\epsilon;2)$ fails to exist mainly for the elementary (meaning easily remedied?) reason that $(N-2)^{-1}$ is undefined.

The function $M(\frac{N}{2}-1;\frac{N}{2};-t)$ satisfies (as *Mathematica* confirms) this instance of Kummer's equation

$$\Big\{t\tfrac{d^2}{dt^2}+\big(\tfrac{N}{2}+t\big)\tfrac{d}{dt}+\tfrac{N-2}{2}\Big\}M(t)=0$$

The implication, by (75), is[33] that

$$\Big\{\epsilon\big[s\tfrac{d}{ds}+\tfrac{N}{2}\big]\tfrac{d}{ds}+\big[s\tfrac{d}{ds}+\tfrac{N}{2}\big]-1\Big\}\tilde{w}_{-2}(s;\epsilon;N)=0$$

But the operator $2\big[s\frac{d}{ds}+\frac{N}{2}\big]\frac{d}{ds}$ was found at (64.2) to refer to the action of ∇^2, and we are informed by *Mathematica* that

$$\begin{aligned}2\big[s\tfrac{d}{ds}+\tfrac{N}{2}\big]\tfrac{d}{ds}\tilde{w}_{-2}(s;\epsilon;N)&=A_N\cdot\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N e^{-s/\epsilon}\\ A_N&\equiv(N-2)\tfrac{\Gamma(\frac{N}{2}-2)}{2^1\Gamma(\frac{N}{2})}\\ &=\delta(\boldsymbol{x})\text{ in }N\text{-dimensional Gaussian representation!}\end{aligned}\tag{79}$$

The factor A_N is indeterminate in the case $N=2$, but assumes unit value in all other positive integral cases. In the singular case *Mathematica* supplies

$$\lim_{N\to2}\tfrac{2}{N-2}M(\tfrac{N}{2}-1;\tfrac{N}{2};-t)=\lim_{N\to2}\Big\{t^{\frac{N}{2}-1}\big[\Gamma(\tfrac{N}{2}-1)-\Gamma(\tfrac{N}{2}-1,t)\big]\Big\}$$

and if one were to accept the following reinterpretation of (75)

$$\tilde{w}_{-2}(s;\epsilon;2)\equiv+\tfrac{\epsilon}{2}\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^2\Gamma(0,s/\epsilon)\tag{80}$$

then computation would give

$$2\big[s\tfrac{d}{ds}+\tfrac{2}{2}\big]\tfrac{d}{ds}\tilde{w}_{-2}(s;\epsilon;2)=\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^2e^{-s/\epsilon}$$

—consistently with (79).

[33] Use $t=s/\epsilon$ and $\frac{d}{dt}=\epsilon\frac{d}{ds}$.

The functions $\tilde{w}_\nu(s;\epsilon;N)$ are of interest to us for what they accomplish in the limit $\epsilon\downarrow 0$. Reverting to a prior notation

$$W_\nu(r;\epsilon;N)=\tilde{w}_\nu(s;\epsilon;N)\quad\text{by}\quad t\mapsto \tfrac{s}{\epsilon}\mapsto\tfrac{r^2}{2\epsilon}$$

to emphasize that we have now (partially) reinstalled the "physical" variable r, and recalling (from §3) that

$$S_N\equiv\text{area of unit }N\text{-sphere}=2\pi^{\frac{N}{2}}/\Gamma(\tfrac{N}{2})$$

(for $N=1,2,3,\ldots$ we have $S_N=2,\pi,4\pi,2\pi^2,\frac{8}{3}\pi^2,\pi^3,\frac{16}{15}\pi^3,\frac{1}{3}\pi^4,\ldots$) we look back again to (76) and obtain

$$\left.\begin{aligned}
&\blacktriangleright\quad W_{-2}(r;\epsilon;1)=+\tfrac{1}{1S_1}r^{+1}\cdot\big[\mathrm{erf}\sqrt{t}+\tfrac{1}{\sqrt{\pi t}}e^{-t}\big]\\
&\triangleright\quad W_{-2}(r;\epsilon;2)=\quad\tfrac{1}{4\pi}\,r^{0}\quad\cdot\big\{\Gamma(0,t)\sim t^{-1}e^{-t}\big\}\\
&\blacktriangleright\quad W_{-2}(r;\epsilon;3)=-\tfrac{1}{1S_3}r^{-1}\cdot\big[\mathrm{erf}\sqrt{t}\,\big]\\
&\triangleright\quad W_{-2}(r;\epsilon;4)=-\tfrac{1}{2S_4}r^{-2}\cdot\big[1-e^{-t}\big]\\
&\blacktriangleright\quad W_{-2}(r;\epsilon;5)=-\tfrac{1}{3S_5}r^{-3}\cdot\big[\mathrm{erf}\sqrt{t}-\tfrac{2}{\sqrt{\pi}}\sqrt{t}e^{-t}\big]\\
&\triangleright\quad W_{-2}(r;\epsilon;6)=-\tfrac{1}{4S_6}r^{-4}\cdot\big[1-e^{-t}(1+t)\big]\\
&\blacktriangleright\quad W_{-2}(r;\epsilon;7)=-\tfrac{1}{5S_7}r^{-5}\cdot\big[\mathrm{erf}\sqrt{t}-\tfrac{2}{3\sqrt{\pi}}\sqrt{t}e^{-t}(3+2t)\big]\\
&\triangleright\quad W_{-2}(r;\epsilon;8)=-\tfrac{1}{6S_8}r^{-6}\cdot\big[1-e^{-t}(1+t+\tfrac{1}{2}t^2)\big]\\
&\qquad\qquad\vdots
\end{aligned}\right\}\tag{81}$$

where $\{\text{etc.}\}$ vanishes (the case $N=2$ remains exceptional to the end) but all the terms $[\text{etc.}]$ approach unity in the limit $\epsilon\downarrow 0$. By extrapolation we appear to have

$$\lim_{\epsilon\downarrow 0}W_{-2}(r;\epsilon;N)=-\tfrac{1}{(N-2)S_N}r^{-(N-2)}\tag{82}$$

which is nicely consonant with (65).

Equation (79) can be expressed

$$\begin{aligned}\nabla^2W_{-2}(r;\epsilon;N)&=W_0(r;\epsilon;N)\\&=\delta_N(\boldsymbol{r})\quad\text{in the limit }\epsilon\downarrow 0\end{aligned}\tag{83}$$

This is a lovely result: it reproduces the N-dimensional theory of harmonic Green's functions (so far as it can proceed without reference to "vanishing on a boundary") and *makes sense even when N is not an integer*. But (83) is a special instance of a much broader class of even more wonderful results—most of which would certainly remain conjectural were we denied the computational power of an engine such as *Mathematica*. For example, we compute

$$\begin{aligned}
\big[2s\tfrac{d^2}{ds^2}+N\tfrac{d}{ds}\big]^1\tilde{w}_{-4}(s;\epsilon;N)&=-B_N\cdot\tfrac{1}{2}\big(\tfrac{1}{\sqrt{2\pi s}}\big)^N\Big\{\Gamma\big(\tfrac{N-2}{2}\big)-\Gamma\big(\tfrac{N-2}{2},t\big)\Big\}s\\
B_N&\equiv(N-2)(N-4)\tfrac{\Gamma(\frac{N-4}{2})}{2^2\Gamma(\frac{N}{2})}\\
\big[2s\tfrac{d^2}{ds^2}+N\tfrac{d}{ds}\big]^2\tilde{w}_{-4}(s;\epsilon;N)&=\quad B_N\cdot\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N e^{-s/\epsilon}
\end{aligned}\tag{84.1}$$

and similarly

$$\begin{aligned}
\big[2s\tfrac{d^2}{ds^2}+N\tfrac{d}{ds}\big]^1\tilde w_{-6}(s;\epsilon;N) &= \text{complicated expression (see below)}\\
\big[2s\tfrac{d^2}{ds^2}+N\tfrac{d}{ds}\big]^2\tilde w_{-6}(s;\epsilon;N) &= -C_N\cdot\tfrac{1}{2}\big(\tfrac{1}{\sqrt{2\pi s}}\big)^N\Big\{\Gamma\big(\tfrac{N-2}{2}\big)-\Gamma\big(\tfrac{N-2}{2},t\big)\Big\}s\\
C_N &\equiv (N-2)(N-4)(N-6)\tfrac{\Gamma(\frac{N-6}{2})}{2^3\Gamma(\frac{N}{2})}\\
\big[2s\tfrac{d^2}{ds^2}+N\tfrac{d}{ds}\big]^3\tilde w_{-6}(s;\epsilon;N) &= \quad C_N\cdot\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N e^{-s\epsilon}
\end{aligned}\tag{84.2}$$

The factor B_N is, according to *Mathematica*, "indeterminate" if $N=2$ or 4, but for all other (integral or non-integral) values of $N>0$ assumes unit value; it makes sense therefore to write

$$\lim_{N\to 2}B_N=\lim_{N\to 4}B_N=1$$

A similar remark pertains to C_N, which is indeterminate if $N=2$ or 4 or 6, but of which it can be asserted that $C_N=1$: (all $N>0$). Equations (84) can therefore be expressed

$$\left.\begin{aligned}
\nabla^4 W_{-4}(r;\epsilon;N) &= W_0(r;\epsilon;N) \quad : \quad \text{all } N>0\\
\nabla^6 W_{-6}(r;\epsilon;N) &= W_0(r;\epsilon;N)\\
&\;\vdots\\
&= \delta_N(\boldsymbol{r}) \quad \text{in the limit } \epsilon\downarrow 0
\end{aligned}\right\}\tag{85}$$

The evident conclusion—that $W_{-4}(r;\epsilon;N)$ supports a theory of *biharmonic Green's functions*, and $W_{-6}(r;\epsilon;N)$ a theory of *triharmonic Green's functions*—is of deep interest in itself, but of even greater is a result exposed along the way to that conclusion: when asked what it has to say about the function

$$W[s\_,N\_,m\_,\epsilon\_] :=$$
$$\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N\big(\tfrac{2}{\epsilon}\big)^m \text{Cos}[m\pi]\frac{\text{Gamma}[\frac{N}{2}+m]}{\text{Gamma}[\frac{N}{2}]}\text{Hypergeometric1F1}[\tfrac{N}{2}+m,\tfrac{N}{2},-\tfrac{s}{\epsilon}]$$

in the cases $m=-1$ and $m=-2$, *Mathematica* responds[34]

$$\begin{aligned}
\tilde w_{-4}(s;N;\epsilon) &= -\tfrac{1}{2}\big(\tfrac{1}{\sqrt{2\pi s}}\big)^N\Big\{\Gamma\big(\tfrac{N-2}{2}\big)-\Gamma\big(\tfrac{N-2}{2},t\big)\Big\}s\\
\tilde w_{-6}(s;N;\epsilon) &= \quad\tfrac{1}{2}\big(\tfrac{1}{\sqrt{2\pi s}}\big)^N\Big\{\Gamma\big(\tfrac{N-2}{2}\big)-\Gamma\big(\tfrac{N-2}{2},t\big)\Big\}(s-\tfrac{N-4}{2}\epsilon)\\
&\quad+\epsilon^2\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N e^{-t}\\
&= \text{precisely the "complicated expression" encountered above}
\end{aligned}$$

[34] I allow myself here to bring this simplification

$$B_N=C_N=1$$

to the results actually reported by *Mathematica*.

Evidently the statements which led to (84.2) can now be notated

$$\left.\begin{aligned}\nabla^2 W_{-6} &= \quad W_{-4}\\ \nabla^4 W_{-6} &= \nabla^2 W_{-4} = \quad W_{-2}\\ \nabla^6 W_{-6} &= \nabla^4 W_{-4} = \nabla^2 W_{-2} = W_0\end{aligned}\right\}\tag{86}$$

and if we omit all references to W_{-6} we recover the statements which led to (84.1).

It becomes at this point natural to conjecture that

$$\nabla^2 W_{2m} = W_{2m+2} \quad \text{even if } m \text{ is not an integer}\tag{87}$$

To establish such a result it proves efficient to elaborate upon a remark make just prior to (79): the function $M(\frac{N}{2}+m;\frac{N}{2};-t)$ satisfies this instance of Kummer's equation

$$\Big\{t\tfrac{d^2}{dt^2} + \big(\tfrac{N}{2}+t\big)\tfrac{d}{dt} + \tfrac{N}{2} + m\Big\}M(t) = 0$$

The implication, by (72.2), is that

$$\Big\{2\big[s\tfrac{d}{ds} + \tfrac{N}{2}\big]\tfrac{d}{ds} + \tfrac{2}{\epsilon}\big[s\tfrac{d}{ds} + \big(\tfrac{N}{2}+m\big)\big]\Big\}\tilde{w}_{2m}(s;\epsilon;N) = 0$$

The operator $2\big[s\frac{d}{ds} + \frac{N}{2}\big]\frac{d}{ds}$ was found at (64.2) to refer to the action of ∇^2, and can in the present context be replaced now by this operator of lower order: $-\frac{2}{\epsilon}\big[s\frac{d}{ds} + \big(\frac{N}{2}+m\big)\big]$. Quick calculation gives

$$\begin{aligned}-\tfrac{2}{\epsilon}&\big[s\tfrac{d}{ds} + \big(\tfrac{N}{2}+m\big)\big]\tilde{w}_{2m}(s;\epsilon;N)\\ &= \big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N\big(\tfrac{2}{\epsilon}\big)^{m+1}\cos(m+1)\pi\frac{\Gamma(\frac{N}{2}+m+1)}{\Gamma(\frac{N}{2})}\cdot\\ &\qquad\cdot\Big\{M(\tfrac{N}{2}+m;\tfrac{N}{2};-t) - \tfrac{2}{N}tM(\tfrac{N}{2}+m+1;\tfrac{N}{2}+1;-t)\Big\}\end{aligned}\tag{88}$$

By Kummer transformation

$$\Big\{\text{etc.}\Big\} = e^{-t}\Big\{M(-m;\tfrac{N}{2};t) - \tfrac{2}{N}\,t\,M(-m;\tfrac{N}{2}+1;t)\Big\}$$

which by appeal to one of the several recurrence properties of the Kummer functions[35] becomes

$$\begin{aligned}&= e^{-t}M(-m-1;\tfrac{N}{2};t)\\ &= M(\tfrac{N}{2}+m+1;\tfrac{N}{2};-t)\end{aligned}$$

So we have

$$2\big[s\tfrac{d}{ds} + \tfrac{N}{2}\big]\tfrac{d}{ds}\tilde{w}_{2m}(s;\epsilon;N) = \tilde{w}_{2(m+1)}(s;\epsilon;N)$$

---

[35] Use (Abramowitz & Stegun, **13.4.4**)

$$M(a-1;b;z) = M(a;b;z) - \tfrac{1}{b}\,t\,M(a;b+1;z)$$

with $a = -m$ and $b = \frac{1}{2}N$.

which can be expressed

$$\nabla^2 W_\nu = W_{\nu+2} \tag{89}$$

Here W_ν refers to the weight function which (in the limit $\epsilon \downarrow 0$) lends meaning to the operator ∇^ν, so symbolically we have

$$\nabla^2\nabla^\nu = \nabla^{2+\nu} \tag{90}$$

This is the analog, within the present multivariate formalism, of the statement

$$D^1 D^\nu = D^{1+\nu}$$

which within the ordinary fractional calculus is used to erect upon the theory of fractional integration a theory of fractional differentiation: thus

$$D^{\frac{1}{2}} = D\cdot D^{-\frac{1}{2}}$$

Within our representational approach to the factional calculus we can assign also *direct* meaning to $D^{\frac{1}{2}}$, and so it is within the present formalism: (89) is not a definition, but a statement relating individually meaningful expressions. Equation (90) entails

$$\nabla^{2m}\nabla^\nu = \nabla^{2m+\nu} \quad : \quad m = 0, 1, 2, \ldots \tag{91}$$

which in the case $\nu = -2m$ becomes a *theory of Green's functions*. More generally we expect to have (but have as yet not quite established) a "law of exponents"

$$\nabla^\mu\nabla^\nu = \nabla^{\mu+\nu} \quad : \quad \text{conditional} \tag{92}$$

where "conditonal" means that (92) is *not* to be asserted in cases where μ or *nu* or their sum is "exceptional," in the sense soon to be explained. Thus will we be proscribed from writing (for example) $\nabla^2\nabla^{-1} = \nabla = \sqrt{\nabla^2}$; my theory (in its present form) therefore does *not* create the possibility of an alternative to Dirac's famous trickery. But it does, by lending meaning to (92), clear the way toward development of a *fractional harmonic analysis on spaces of fractional dimension*, which seems to me to be a fairly wonderous accomplishment.

6. Exceptional cases: a portfolio of figures. We are in position now (see again (63) and (72)) to write

$$\begin{aligned}\nabla^\nu f(\boldsymbol{x}) = \lim_{\epsilon\downarrow 0}\int_0^\infty \langle f(\boldsymbol{x}; r)\rangle_{N} W_\nu(r;\epsilon;N)\cdot S_N r^{N-1}\,dr \\ S_N = 2\pi^{\frac{N}{2}}/\Gamma(\tfrac{N}{2})\end{aligned}$$

$$W_\nu = \big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N\big(\tfrac{2}{\epsilon}\big)^{\frac{\nu}{2}}\cos\big(\nu\tfrac{\pi}{2}\big)\frac{\Gamma(\frac{N+\nu}{2})}{\Gamma(\frac{N}{2})}M\big(\tfrac{N+\nu}{2};\tfrac{N}{2};-\tfrac{1}{2\epsilon}r^2\big)$$

and possess a good intuitive understanding of what $\langle f(\boldsymbol{x}; r)\rangle_N$ means. My objective here will be to make the meaning of W_ν equally plain. Formulæ such as (76)—which involve only (complicated combinations of) elementary functions—become available in special cases,[36] but even they don't speak sharply to intuition. To explore smooth trends in typical regions of parameter space the graphical technique seems optimal.

It has been remarked (p. 51) that "the Laplacian theory implicit in (72), when pulled back to one dimension, reproduces only a fraction of the full factional calculus supported by (42);" Figures 1–10 are therefore not directly relevant to the theory here at issue, though they illustrate points of persistent qualitative significance. Figures 11-13 do arise from our theory of generalized Laplacians, but are special to the cases

$$\{N,\nu\} = \{2,2\}, \{2,4\}, \{2,6\}$$

and therefore convey little sense of the *smoothly interpolative* aspects of that theory. The same can be said of Figures 14-16, where N and ν range on an expanded set of positive integers. None of those figures pertain to cases of the type $\nu < 0$ (i.e., to those parts of our formalism which support an associated theory of generalized Green's functions), nor do any cast light on the location and nature of the "exceptional cases." Our work is therefore cut out for us.

I will find it useful to write

$$\text{weight function} = (\text{trigamma factor})\cdot(\text{Kummer factor})$$

with

$$\begin{aligned} \text{trigamma factor } J(N,\nu) &\equiv \cos\big(\nu\tfrac{\pi}{2}\big)\frac{\Gamma(\frac{N+\nu}{2})}{\Gamma(\frac{N}{2})} \\ \text{Kummer factor } K(r,\epsilon;N,\nu) &\equiv \big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N\big(\tfrac{2}{\epsilon}\big)^{\frac{\nu}{2}} M\big(\tfrac{N+\nu}{2};\tfrac{N}{2};-\tfrac{1}{2\epsilon}r^2\big) \end{aligned}$$

and to distinguish effects due to the one from effects due to the other. Looking first to the former (which is the source of the "exceptional case" phenomenon), we have

$$J(N,\nu) = \begin{cases} \infty \text{ on the lines } N+\nu = 0,-2,-4,\ldots \\ \;0\; \text{ on the lines } \nu = \ldots,-5,-3,-1,+1,+3,+5,\ldots \\ \# \text{ where the above lines intersect} \end{cases}$$

where # refers to certain finite numbers (essentially residues of the gamma function), as explained below. Figure 17 maps the primary structural features of $J(N,\nu)$.

---

[36] I invite my reader to engage in a little experimentation at this point. Insert various values of N and ν into the function defined

$$W[r\_,N\_,\nu\_,\epsilon\_] :=$$

$$\big(\tfrac{1}{\sqrt{2\pi\epsilon}}\big)^N\big(\tfrac{2}{\epsilon}\big)^{\frac{\nu}{2}}\text{Cos}[\nu\tfrac{\pi}{2}]\frac{\text{Gamma}[\frac{N+\nu}{2}]}{\text{Gamma}[\frac{N}{2}]}\text{Hypergeometric1F1}[\tfrac{N+\nu}{2},\tfrac{N}{2},-\tfrac{1}{2\epsilon}r^2]$$

and discover what *Mathematica* has to say. Typically ν (though not N) will have to be an integer, else *Mathematica* will simply return the definition.

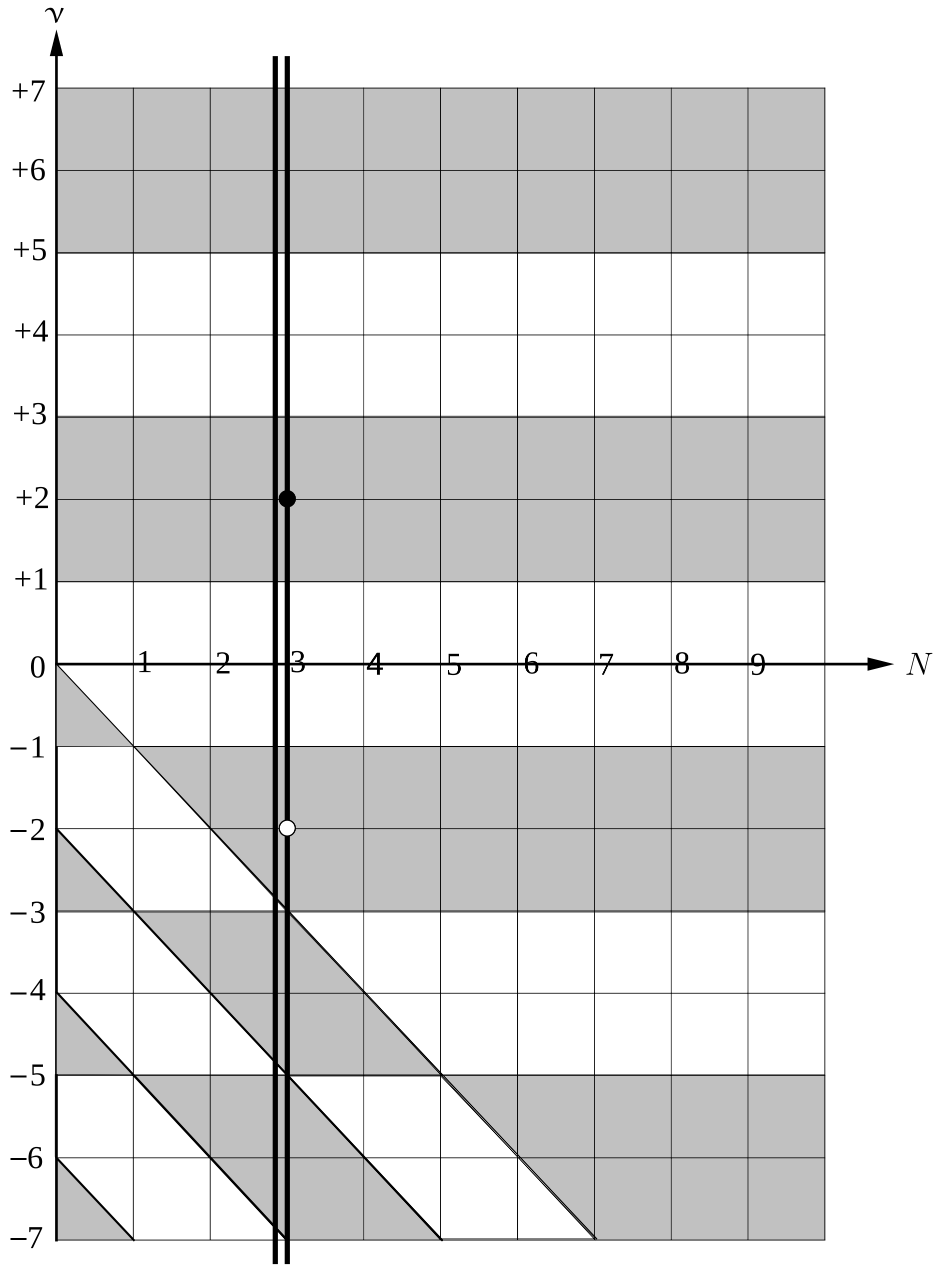


FIGURE 17: *The factor $J(N,\nu)$ is positive at white points, negative at shaded points. It becomes infinite on the slant sides, is zero on the flat tops/bottoms, but assumes finite values on the corners of the parallelograms.* • *marks the location of the familiar 3-dimensional ∇^2, and* ∘ *the location of the associated Green's function ∇^{-2}. Heavy lines at $N = 3$ and $N = 2.9$ mark locations of the sections shown in Figure 18.*

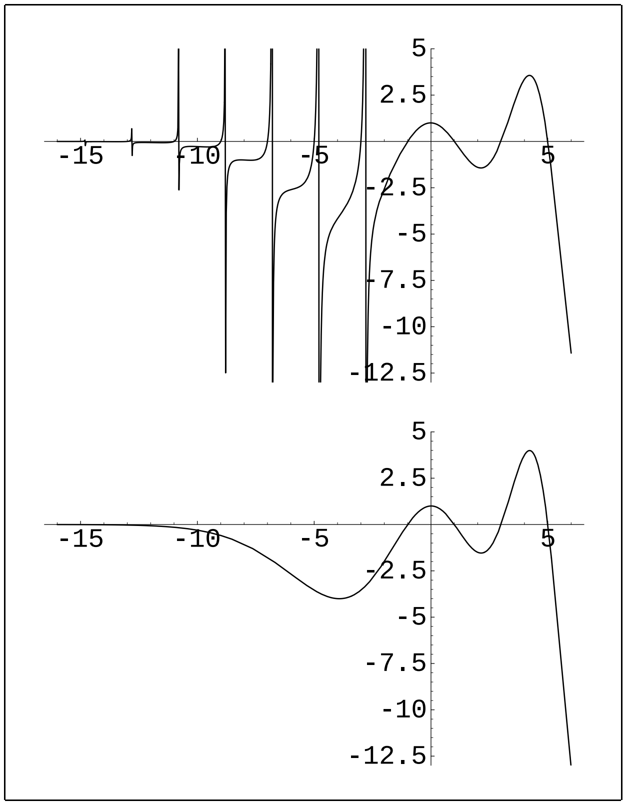


FIGURE 18: *Graphs of* $J(2.9, \nu)$ *(above) and* $J(3, \nu)$*. In the former figure, the zeros at* $\nu = \pm 1, \pm 3, \pm 5, \ldots$ *are those of* $\cos \nu \frac{\pi}{2}$*, while the singularities arise from the simple poles of the gamma function. Their relative placement is that implied by the* $N = 2.9$ *section-line inscribed on Figure 17. In the lower figure, zeros and singularities have come into confluence (the* $N = 3$ *section-line passes through corners of the parallelograms) and "cancelled each other out," for reasons discussed in connection with Figure 19.*

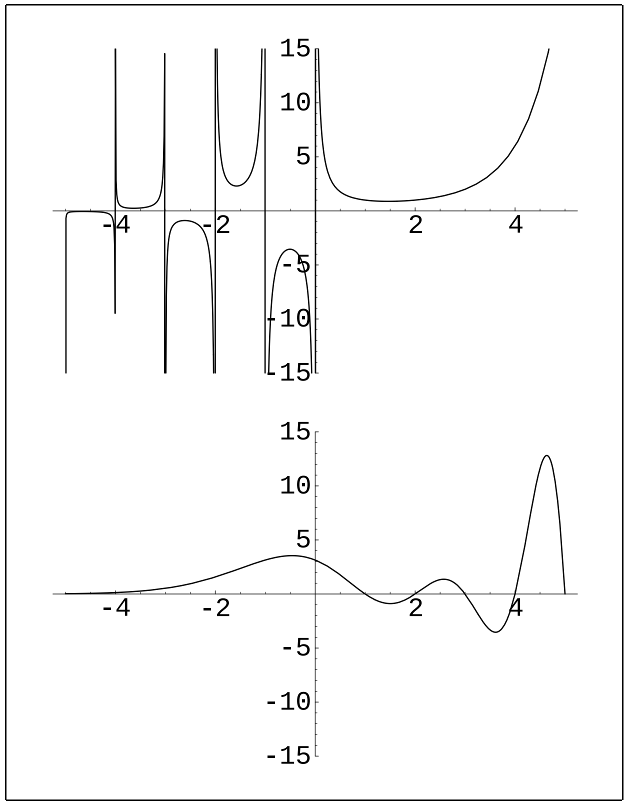


FIGURE 19: *The gamma function $\Gamma(x)$ has (see p. 1 of Magnus & Oberhettinger) simple poles at $x = 0, -1, -2, \ldots$, and its residue at $-n$ is $(-)^n \frac{1}{n!}$. The singular points of $\Gamma(x)$ coincide with zeros of the function $\sin x\pi$, which can be expanded*

$$\sin x\pi = (-)^n \pi(x+n) + \cdots$$

So $\sin x\pi \cdot \Gamma(x)$ is regular (lower figure): it assumes the value $\pi/n!$ at $x = -n$ and vanishes at $x = +n : n = 0, 1, 2, \ldots$

It is clear from Figure 17 that the "order advancement sequence"

$$J(3,2) \to J(3,2+2) \to J(3,2+4) \to J(3,2+6) \to \cdots$$

exhibits sign alternation, but that the "order reduction sequence"

$$\cdots \leftarrow J(3,2-6) \leftarrow J(3,2-4) \leftarrow J(3,2-2) \leftarrow J(3,2)$$

—for entries subsequent to $J(3,2-4)$—does not; every entry is positive. From the figure it becomes evident that interesting variants of this remark arise, in

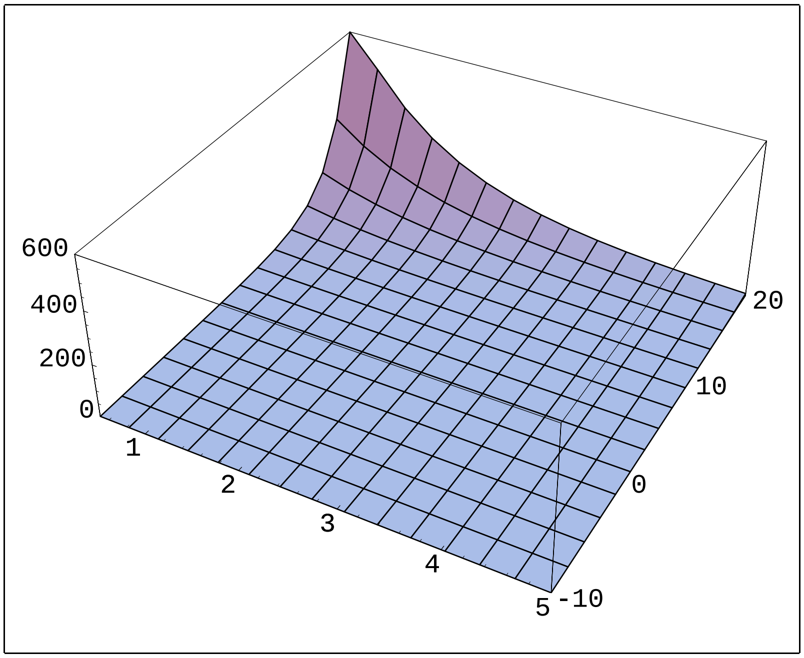


FIGURE 20: *The central value $K(0,\epsilon;N,\nu)$ of the Kummer factor $K(r,\epsilon;N,\nu)$—shown here on the domain $0<N<5$; $-10<\nu<20$ with ϵ set to unity—appears to be everywhere positive.*

fact, from *every* "seed" $J(N,\nu)$. The "sign alternation phenomenon," first mentioned in the Introduction and encountered several times subsequently,[37] refers (in the language of our representation formalism) to the order-dependence of the sign of $W_\nu(0;N;\epsilon)$; i.e., to the sign reversals which associate with the process

$$W_\nu(0;N;\epsilon)\to W_{\nu+2}(0;N;\epsilon)$$

The sign structure of J would be inherited by $W=JK$ if the sign of $K(0,\bullet;\bullet,\nu)$ were ν-independent. Which, on the evidence of the above figure, it is. We infer that *the principle of sign alternation is compromised* when one steps so far into "Green's domain" as to render $\nu<-N$.

Spanier & Oldham's Figure 47–2 bears a superficial resemblance to my own Figure 17, though it refers to a property not of J but of $M(a;b;z)$. It asserts, in effect, that the the Kummer factor $K(r,\epsilon;\frac{N+\nu}{2},\frac{N}{2})$, thought of as a function of $r\geq 0$, exhibits (for all $N>0$)

$$\begin{array}{rl} \text{no zero crossing} & \text{if} \quad\quad \nu<0\\ \text{1 zero crossing} & \text{if } 0<\nu<2\\ \text{2 zero crossings} & \text{if } 2<\nu<4\\ & \quad\quad\vdots \end{array}\tag{94}$$

[37] It becomes most vividly evident when one compares Figures 11, 12 & 13.

Using instructions on the pattern

```
Plot[W[r,N,ν,3/4]/Abs[W[0,N,ν,3/4]],{r,0,5},PlotRange->{-1,+0.2}]
```

to study sign structure, and

```
Plot[W[r,N,ν,3/4]/Abs[W[0,N,ν,3/4]],{r,0,5},PlotRange->{-0.01,+0.01}]
```

to count zero crossings, I have ranged widely on the $\{N, \nu\}$-plane, and found exact conformity with the assertions made in Figure 17 and at (94). I supply below a single example of the product of such exploration:

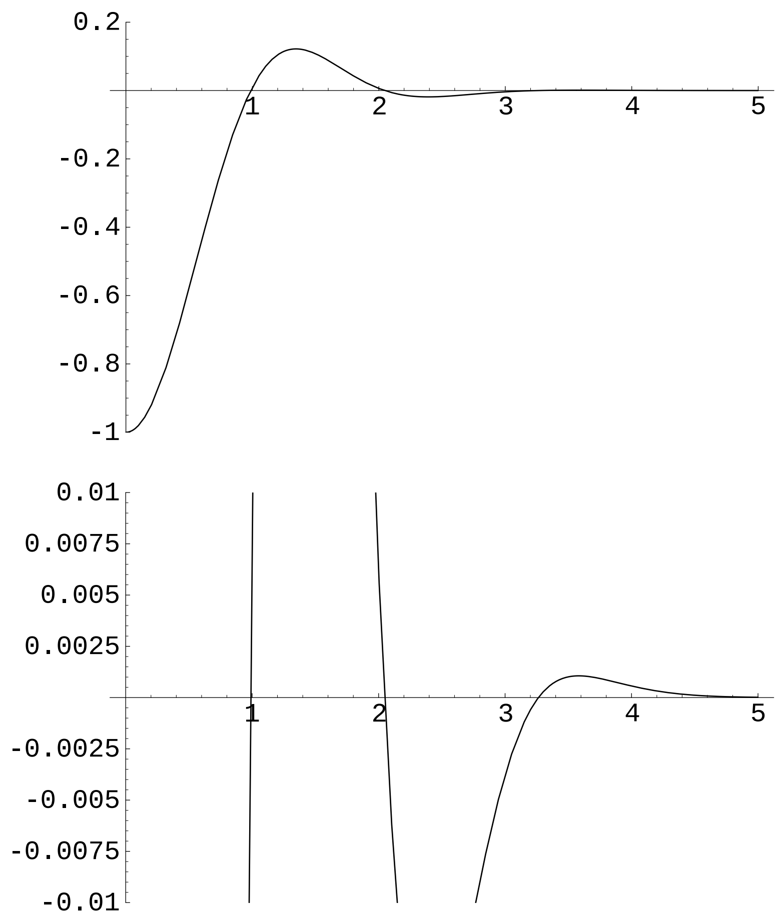


FIGURE 21: *Illustrative figures in which I have set $N = 2.9$ and $\nu = 5.9$. Exploiting the continuity of $W_\nu(r; N; \epsilon)$ in N and ν, I have learned to assign fractional values to those parameters in order to avoid the nulls and infinities which can occur when either assumes an integral value.*

7. The composition problem for generalized Laplacians. In the ordinary calculus of a single variable, differentiation operators $D \equiv \frac{d}{dx}$

compose by the law of exponents

$$D^m D^n = D^{m+n} \quad : \quad m \text{ and } n \in \{0, 1, 2, \ldots\}$$

and so do the integral operators defined $D^{-1}f(x) \equiv \int_a^x f(y)\,dy$:

$$D^{-m}D^{-n} = D^{-m-n}$$

Mixed composition is a bit more complicated, since one has (see again the bottom of p. 12)

$$DD^{-1}f(x) = f(x) \quad \text{but} \quad D^{-1}Df(x) = f(x) - f(a)$$

An unrestricted law of exponents pertains only to functions with the special property that $f(a) = 0$.

In a representation-theoretic approach to the *fractional* calculus of a single variable (such as that developed in §2) one has

$$D^\nu f(x) = \lim_{\epsilon\downarrow 0} \int f(y) W_\nu(x-y;\epsilon)\,dy \tag{95}$$

which—as was remarked already in the discussion subsequent to (25)—affords two distinct approaches to establishment of a law of exponents: one might undertake to show that

$$\left(\tfrac{d}{dx}\right)^m D^\nu f(x) = \lim_{\epsilon\downarrow 0} \int f(y) \left(\tfrac{d}{dx}\right)^m W_\nu(x-y;\epsilon)\,dy$$

$$\left(\tfrac{d}{dx}\right)^m W_\nu(x-y;\epsilon) = W_{m+\nu}(x-y;\epsilon)$$

but alternatively one might undertake to show that

$$\int W_\mu(x-z;\epsilon) W_\nu(z-y;\epsilon)\,dz = W_{\mu+\nu}(x-y;\epsilon)$$

The latter procedure is latently more general, because one is released from the requirement that μ be a (positive or negative) integer. In the derivation of (91) we used the multivariate analog of the former procedure. My objective here will be to explore the potentialities of the latter.

The ordinary fractional calculus springs from a procedure (95) which might be diagramed

$$f(x) \xrightarrow[\text{convolve}]{} f_\nu(x;\epsilon) \xrightarrow[\epsilon\downarrow 0]{} D^\nu f(x)$$

The fractional Laplacian springs, on the other hand, from a procedure (93) which involves a preparatory step

$$f(\boldsymbol{x}) \xrightarrow[\text{average}]{} \langle f(\boldsymbol{x};r)\rangle \xrightarrow[\text{convolve}]{} f_\nu(\boldsymbol{x};\epsilon) \xrightarrow[\epsilon\downarrow 0]{} \nabla^\nu f(\boldsymbol{x})$$

and is therefore relatively more complicated, even in the one-dimensional case. I look now, by way of orientation, to details of the associated composition problem in the case $N = 2$.

We have[38]

$$\nabla^\nu f(x,y) = \tfrac{1}{2\pi}\int_0^\infty\!\!\int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)W_\nu(r;\epsilon)\,d\theta dr$$

giving

$$\begin{aligned}\nabla^\mu\nabla^\nu f(x,y) = \big(\tfrac{1}{2\pi}\big)^2\int_0^\infty\!\!\int_0^{2\pi}\Big\{\int_0^\infty\!\!\int_0^{2\pi} &\\ f(x+r_1\cos\theta_1+r_2\cos\theta_2, y+r_1\sin\theta_1+r_2\sin\theta_2)&\\ \cdot W_\nu(r_1;\epsilon_1)\,d\theta_1 dr_1\Big\}W_\mu(r_2;\epsilon_2)\,d\theta_2 dr_2&\end{aligned}$$

In view of our intended destination it becomes natural to write

$$\begin{aligned}r\cos\theta &= r_1\cos\theta_1 + r_2\cos\theta_2\\ r\sin\theta &= r_1\sin\theta_1 + r_2\sin\theta_2\end{aligned}$$

Then

$$\begin{aligned}r_2 &= \sqrt{r^2+r_1^2-2rr_2\cos(\theta-\theta_1)}\\ \theta_2 &= \arctan\Big[\frac{r\sin\theta - r_1\sin\theta_1}{r\cos\theta - r_1\cos\theta_1}\Big]\end{aligned}$$

and with *Mathematica*'s assistance we find the Jacobian of the transformation $\{r_1,\theta_1,r_2,\theta_2\}\to\{r_1,\theta_1,r,\theta\}$ to be given by

$$\Big|\frac{\partial(r_1,\theta_1,r_2,\theta_2)}{\partial(r_1,\theta_1,r\;\;,\theta\;\;)}\Big| = \frac{r}{\sqrt{r^2+r_1^2-2rr_1\cos(\theta-\theta_1)}}$$

So we have

$$\begin{aligned}\nabla^{2m}\!\cdot\!\nabla^{2n}f(x,y) = \tfrac{1}{2\pi}\int_0^\infty\!\!\int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)&\\ \cdot\Big\{\tfrac{r}{2\pi}\int_0^\infty\!\!\int_0^{2\pi}\frac{W_{2n}(s;\epsilon_1)W_{2m}(\sqrt{r^2+s^2-2rs\cos\varphi};\epsilon_2)}{\sqrt{r^2+s^2-2rs\cos\varphi}}\,d\varphi ds\Big\}d\theta dr&\end{aligned}$$

where I have found it convenient to adopt the notational simplifications $r_2\mapsto s$ and $\theta-\theta_2\mapsto\varphi$. Reading from (93), we have[39]

$$\begin{aligned}W_{2m}(s;\epsilon;2) &= \tfrac{1}{2\pi\epsilon}\big(\tfrac{2}{\epsilon}\big)^m\cos(m\pi)\Gamma(1+m)\cdot M(1+m;1;-\tfrac{1}{2\epsilon}s^2)\\ &= \text{———— ditto ————}\cdot e^{-\frac{1}{2\epsilon}s^2}M(-m;1;\tfrac{1}{2\epsilon}s^2)\end{aligned}$$

---

[38] Here and henceforth $\lim_{\epsilon\downarrow 0}$ will be understood but not notated, and when no confusion can result I will write $W_\nu(r)$ for $W_\nu(r;\epsilon)$, $W_\nu(r;\epsilon)$ for $W_\nu(r;\epsilon;N)$.

[39] It is only because W_{2m} is easier to write out than W_μ that I speak now of $\nabla^{2m}\nabla^{2n}$ rather than of $\nabla^\mu\nabla^\nu$; the notation carries no presumption that $2m$ and $2n$ are integers.

and would like to show that $\{\text{etc.}\} = W_{2m+2n}(r;\epsilon)$. *But in no case—by any trickery I have in a full day of effort been able to devise—have I been able to do so; the integrals appear to be intractable.* In cases $N > 2$ one would have to construct a relatively intricate geometrical argument before arriving at integrals that can be expected to be no less intractable, but I see no point in pursuing that train of thought.

The purported "law of exponents" (92) remains, therefore, at this point unproven, and the following circumstance

$$\cos m\pi \cos n\pi = \tfrac{1}{2}\cos(m+n)\pi + \underbrace{\tfrac{1}{2}\cos(m-n)\pi}_{\text{unwelcome term}}$$

leads me to suspect it may not even be correct. The point at issue, so long as it remains unsettled, will remain a blemish on the face of my formalism, but a blemish of (so far as I can see) no practical consequence.

8. Alternatives to the Gaussian representation. All of our results (and, more recently, non-results) are—so far as concerns their function-theoretic details, and except (presumably) in the limit $\epsilon \downarrow 0$—artifacts of our having elected (at (59), in imitation of (6)) to work in Gaussian representation. Infinitely many alternatives to the Gaussian representation could, in principle, be devised, and of those a handful are of occasional practical importance. One might plausibly suppose that our theory could equally well be supported by *any* of those alternative representations. But so far as I am presently aware, the Gaussian represention is in fact the *only* representation which serves *all* of our formal needs, and leads to a theory which can be carried through to completion. My purpose here will be to survey some of the more familiar alternatives, pursuing each to the point where the *reasons for its failure* become evident. From the resulting "catalog of failures" we stand to gain a sharpened sense of the qualities from which the Gaussian representation acquires its exceptional power.

Each of the δ-representations discussed below will be introduced in its one-dimensional formulation—generically

$$\delta(x) = \lim_{\epsilon\downarrow 0} W(x;\epsilon)$$

—from which its rotationally-invariant N-dimensional generalization will be constructed

$$\delta(\boldsymbol{x}) = \lim_{\epsilon\downarrow 0} Z^{-1}W(r;\epsilon)$$
$$Z = Z(\epsilon;N) \equiv \int_0^\infty W(r;\epsilon)\cdot S_N r^{N-1}\,dr$$

Here $r = \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$ and S_N retains its familiar geometrical meaning.[40]

[40] See again p. 32.

As a first example, let us, in place of (6), write[41]

$$\begin{aligned}\delta(y-x) &= \lim_{\epsilon\downarrow 0}\tfrac{1}{2\epsilon}\,\text{sech}^2\big(\tfrac{y-x}{\epsilon}\big) \\ &= \tfrac{d}{dy}\theta(y-x) \quad \text{with} \quad \theta(y-x) = \lim_{\epsilon\downarrow 0}\tfrac{1}{2}\Big[1+\tanh\big(\tfrac{y-x}{\epsilon}\big)\Big]\end{aligned} \tag{96}$$

and notice that if $F(z) \equiv \text{sech}^2 z = 1 - \tanh^2 z$ is notated

$$F(z) = -(T^2-1) \quad : \quad \text{here } T \equiv \tanh(z)$$

then

$$\begin{aligned}\big(\tfrac{d}{dz}\big)^1 F(z) &= +2(T^3 - T)\\ \big(\tfrac{d}{dz}\big)^2 F(z) &= -2(3T^4 - 4T^2 + 1)\\ \big(\tfrac{d}{dz}\big)^3 F(z) &= +8(3T^5 - 5T^3 + 2T)\\ \big(\tfrac{d}{dz}\big)^4 F(z) &= -8(15T^6 - 30T^4 + 17T^2 - 2)\\ \big(\tfrac{d}{dz}\big)^5 F(z) &= +16(45T^7 - 105T^5 + 77T^3 - 17T)\\ \big(\tfrac{d}{dz}\big)^6 F(z) &= -16(315T^8 - 840T^6 + 756T^4 - 284T^2 + 17)\\ &\ \ \vdots\end{aligned}$$

These results could, in principle, be used to lend representation-theoretic meaning to the ordinary differential operators $D^{\pm\,\text{integer}}$, but do not appear to support a workable *fractional* calculus, for this reason: I am unable to discover a population of well-studied higher functions $\mathcal{F}_\nu(z)$ with the property that

$$\begin{aligned}\mathcal{F}_0(z) &= \text{sech}^2 z\\ \mathcal{F}_n(z) &= n^{\text{th}} \text{ entry in preceding list: } n = 1, 2, 3, \ldots\end{aligned}$$

which would permit me to assign non-integral value to the index n, therefore unable to play again the interpolative game we played at (11). Looking to the higher-dimensional generalization of (96), we encounter a heightened version of the same problem; the normalization factors $Z(\epsilon; N)$ are given by awkward expressions

$$\begin{aligned}Z(\epsilon;2) &= \tfrac{1}{2}\epsilon^1 S_2 \log 2\\ Z(\epsilon;3) &= -\ \epsilon^2 S_3\big[\texttt{PolyLog[2,-I]}+\texttt{PolyLog[2,+I]}\big] = \tfrac{\pi^2}{24}\epsilon^2 S_3\\ Z(\epsilon;4) &= -3\epsilon^3 S_4\big[\texttt{PolyLog[3,-I]}+\texttt{PolyLog[3,+I]}\big]\\ &\ \ \vdots\end{aligned}$$

[41] The following equations have been designed to emphasize that it is, for many purposes, a matter of indifference whether one imagines oneself to be working from a representation of $\delta(z)$ or of $\theta(z)$. The sech^2 distribution is, in relation to the Gaussian distribution, examined in fair detail in §10 of "Gaussian wavepackets" (1998).

(*Mathematica* defines `PolyLog[n,z]`$\equiv \sum_{k=1}^{\infty} \frac{z^k}{k^n}$, which makes it a close relative of the zeta function), and when we look to the evaluation[42] of $\{$etc.$\}$ in

$$\nabla^{2m}\delta(\boldsymbol{x}) = \lim_{\epsilon\downarrow 0} \frac{1}{Z(\epsilon;N)}\Big\{\big[\big(\tfrac{d}{dr}\big)^2 + \tfrac{N-1}{r}\tfrac{d}{dr}\big]^m W(r;\epsilon)\Big\} \tag{97}$$

$$W(r;\epsilon) = \tfrac{1}{2\epsilon}\,\mathrm{sech}^2\big(\tfrac{r}{\epsilon}\big)$$

we are led to expressions whose m-dependence is much too complicated to permit the interpolation and backward extrapolation which lie at the formal heart of my fractional Laplacian concept. The representation fails, therefore, for reasons of "analytical recalcitrance." But the complicated functions to which it leads are found, when plotted, to resemble closely their Gaussian counterparts; it there were a plauible way to "morph figures" which did not rely upon analytical interpolation/extrapolation then the representation might be salvaged.

Look to the representation

$$\delta(z) = \lim_{\epsilon\downarrow 0} \frac{\sin(z/\epsilon)}{\pi z} \tag{98}$$

and notice that in this case $\theta(z) = \lim_{\epsilon\downarrow 0}\int_{-\infty}^{z} \frac{1}{\pi u}\sin(u/\epsilon)\,du$ does *not* admit of description in terms of simple functions. The representation stands in close proximity to a family of well-studied functions

$$\frac{\sin z}{z} = j_0(z) \equiv \sqrt{\tfrac{1}{2}\pi/z}\,J_{\frac{1}{2}}(z)$$

and the functions in question (the "spherical Bessel functions," see Abramowitz & Stegun, **10.1**) possess some lovely derivative properties—for example, one has

$$\big(\tfrac{1}{z}\tfrac{d}{dz}\big)^m[z^{-n}j_n(z)] = (-)^n z^{-(m+n)} j_{m+n}(z)$$

—but none of that appears to help much, for

$$\begin{aligned}
\big(\tfrac{d}{dz}\big)^0\tfrac{\sin z}{z} &= 1 - \tfrac{z^2}{6} + \tfrac{z^4}{120} - \tfrac{z^6}{5040} + \tfrac{z^8}{362880} - \cdots\\
\big(\tfrac{d}{dz}\big)^1\tfrac{\sin z}{z} &= +\tfrac{z\cos z - \sin z}{z^2}\\
&= -\tfrac{z}{3} + \tfrac{z^3}{30} - \tfrac{z^5}{840} - \tfrac{z^7}{45360} + \cdots\\
\big(\tfrac{d}{dz}\big)^2\tfrac{\sin z}{z} &= -\tfrac{2z\cos z + (z^2-2)\sin z}{z^3}\\
&= -\tfrac{1}{3} + \tfrac{z^2}{10} - \tfrac{z^4}{168} + \tfrac{z^6}{6480} + \cdots\\
\big(\tfrac{d}{dz}\big)^3\tfrac{\sin z}{z} &= +\tfrac{-z(z^2-6)\cos z + 3(z^2-2)\sin z}{z^4}\\
&= \tfrac{z}{5} - \tfrac{z^3}{42} + \tfrac{z^5}{1080} - \cdots\\
&\ \ \vdots
\end{aligned}$$

[42] I appeal here to a computationally more useful variant of (64.1).

are still too complicated to lend any plausible meaning to $\left(\frac{d}{dz}\right)^{\nu}\frac{\sin z}{z}$. When we turn to the higher-dimensional generalization of (98) things get worse, for we are led to normalization integrals

$$Z \sim \int_0^\infty \tfrac{\sin r}{r}\, r^{N-1}\, dr \quad \text{which are } \textit{undefined} \text{ for } N \neq 1$$

Look finally to the "forward-looking exponential" representation

$$\delta(y-x) = \lim_{\epsilon\downarrow 0}\begin{cases} 0 & : \quad y<x \\ \frac{1}{\epsilon}\exp\left\{-\frac{y-x}{\epsilon}\right\} & : \quad x<y \end{cases} \tag{99}$$

$$= \tfrac{d}{dy}\theta(y-x) \quad \text{with} \quad \theta(y-x) = \lim_{\epsilon\downarrow 0}\begin{cases} 0 & : \quad y<x \\ 1-\exp\left\{-\frac{y-x}{\epsilon}\right\} & : \quad x<y \end{cases}$$

Writing

$$\overrightarrow{E}(z) \equiv \begin{cases} 0 & : \quad z>0 \\ e^{-z} & : \quad 0<z \end{cases}$$

$$= \theta(z)\cdot e^{-z}$$

we have

$$\left(\tfrac{d}{dz}\right)^n \overrightarrow{E}(z) = e^{-z}\cdot \sum_{k=0}^{n}(-)^k \binom{n}{k}\theta^{(n-k)}(z)$$

The $\theta(z)$ factor reflects the fact that $\overrightarrow{E}(z)$ is *discontinuous* at the origin. The occurance of $\theta(z)$ together with its derivatives $\delta(z), \delta'(z), \delta''(z), \ldots$ in a theory intended to accomplish (among other things) the representation of such objects is, of course, intolerable.[43] Normalization of the associated N-dimensional theory is made particularly easy by the circumstance that

$$\int_0^\infty e^{-r} r^{N-1}\, dr = \Gamma(N)$$

but when—drawing inspiration from (97)—one looks to expressions of the form

$$\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{N-1}{r}\tfrac{d}{dr}\right]^m e^{-r}$$

one is led to results which, while interesting in their way (see the Appendix), lead to absurdities, for they fail at the origin, where the tentlike function e^{-r} becomes undifferentiable. Attempts to "manage" the implications of that fact (I don't know precisely how that would be accomplished) would open the theory to the same criticism as we just encountered in the case $N=1$.

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[43] A variant of the same problem arises from the tentlike "Janus exponential"

$$e^{-|r|} = \theta(-z)e^{z} + \theta(z)e^{-z}$$

which, though continuous, has a discontinuous first derivative at the origin.

I conclude on evidence of the preceding examples that the Gaussian representation owes its success to these circumstances:

- it is everywhere continuous, and so are its derivatives of all orders;
- its higher-dimensional analog is normalizable for every N;
- it does not lead into realms unknown to the established theory of higher functions.

It is the last of those circumstances which would have to be rendered more precise if one were to contemplate the possibility of a provable "Gaussian uniqueness theorem." It would be nice, in fact, to possess an alternative to the Gaussian representation, so that one could test experimentally whether "all representation-based formulations of the fractional calculus become identical in the limit $\epsilon \downarrow 0$." I have been led by this thought to consider the

$$\text{"super-Gaussian representation"} \quad \delta(x) = \lim_{\epsilon\downarrow 0} \tfrac{1}{\epsilon\Gamma(\frac{5}{4})}\, e^{-(x/2\epsilon)^4}$$

which is manifestly C^∞, and normalizable in all dimensions because

$$\int_0^\infty e^{-r^4} r^{n-1} = \tfrac{1}{4}\Gamma(\tfrac{n}{4})$$

But it fails to meet the third of the above criteria.

Standing somewhat apart from "representations of the δ-function" of the sort discussed in preceding pages are the representations

$$\delta(\boldsymbol{x}-\boldsymbol{y}) = \sum \varphi_n(\boldsymbol{x})\varphi_n(\boldsymbol{y})$$

supplied by the theory of self-adjoint operators (Sturm-Liouville theory), to which (in particular) quantum mechanics assigns fundamental importance. Such representations will presumably play a basic role in the "fractional multivariate calculus in the presence of boundaries" which I hope to explore on another occasion.

9. Fractional calculus by "interpolation in function space". Construe $f(x)$ and its successive ordinary integrals/derivatives

$$\cdots,\ D^{-2}f(x),\ D^{-1}f(x),\ f(x),\ D^{1}f(x),\ D^{2}f(x),\ \cdots$$

to represent an ennumerated sequence of "points in the function space" $\mathcal{F}$. The fractional calculus serves in effect to thread a ν-parameterized "interpolating curve" $D^\nu f(x)$ through those points. It does its work, however, without appeal to any "best fit" criterion, and makes no claim in that regard. Distinct variants of the fractional calculus give rise to (alternatively: arise from) distinct interpolating curves in $\mathcal{F}$.

It is with that imagery in mind that I undertake now to try to render more concretely explicit some of the central accomplishments of preceding pages, accomplishments which I fear have been obscured by the blizzard of exploratory detail. I draw exclusively upon the Gaussian representation, and work initially in one dimension.

By Taylor's theorem[44]

$$f(x+r) = \sum_{n=0}^{\infty} \tfrac{1}{n!} F_n\, r^n \tag{100}$$
$$F_n \equiv D^n f(x)$$

which we may usefully interpret to be the *generating function* for the data F_n. Standardly, one recovers data from such an "exponential"[20] generating function by performing operations of the design $\lim_{r\downarrow 0}(\frac{d}{dr})^n$, but I have reason now to advocate an alternative procedure, which I illustrate by examples before discussing the generalities of the method: write

$$G_0(r;\epsilon) \equiv \tfrac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{1}{2\epsilon}r^2} \tag{101}$$

and use

$$\int_{-\infty}^{+\infty} r^n\, G_0(r;\epsilon)\, dr = \begin{cases} (2\epsilon)^{\frac{1}{2}n} \frac{1}{\sqrt{\pi}} \Gamma(\frac{n+1}{2}) & : \quad n \text{ even} \\ 0 & : \quad n \text{ odd} \end{cases}$$

to obtain

$$\begin{aligned} \int \Big\{ \sum \tfrac{1}{n!} F_n\, r^n \Big\} G_0(r;\epsilon)\, dr &= F_0 + 0 + \tfrac{1}{2!}\epsilon F_2 + 0 + \tfrac{1}{4!} 3\epsilon^2 F_4 + \cdots \\ &\downarrow \\ &= F_0 \quad \text{in the limit } \epsilon \downarrow 0 \end{aligned}$$

Introduce

$$G_1(r;\epsilon) \equiv -\tfrac{d}{dr} G_0(r;\epsilon) = \tfrac{1}{\epsilon}\, r\, G_0(r;\epsilon)$$

and use

$$\int_{-\infty}^{+\infty} r^n\, G_1(r;\epsilon)\, dr = \begin{cases} 0 & : \quad n \text{ even} \\ 2^{\frac{1}{2}(n+1)} \epsilon^{\frac{1}{2}(n-1)} \frac{1}{\sqrt{\pi}} \Gamma(\frac{n+2}{2}) & : \quad n \text{ odd} \end{cases}$$

to obtain

$$\begin{aligned} \int \Big\{ \sum \tfrac{1}{n!} F_n\, r^n \Big\} G_1(r;\epsilon)\, dr &= 0 + F_1 + 0 + \tfrac{1}{3!} 3\epsilon F_3 + 0 + \tfrac{1}{5!} 15\epsilon^2 F_5 + \cdots \\ &\downarrow \\ &= F_1 \end{aligned}$$

Proceeding similarly to the next higher order, introduce

$$G_2(r;\epsilon) \equiv -\tfrac{d}{dr} G_1(r;\epsilon) = \big(-\tfrac{d}{dr}\big)^2 G_0(r;\epsilon) = \tfrac{r^2-\epsilon}{\epsilon^2}\, G_0(r;\epsilon)$$

[44] I proceed formally, therefore do not stipulate that x be a regular point of a nice function, and that r lie within the circle of convergence.

and use[45]

$$\int_{-\infty}^{+\infty} r^n\, G_2(r;\epsilon)\, dr = \begin{cases} 0 & : \quad n=0 \\ 0 & : \quad n=1 \\ 2 & : \quad n=2 \\ 0 & : \quad n=3 \\ 12\epsilon & : \quad n=4 \\ 0 & : \quad n=5 \\ 90\epsilon^2 & : \quad n=6 \\ & \vdots \end{cases}$$

to obtain

$$\int \Big\{\sum \tfrac{1}{n!} F_n\, r^n\Big\} G_2(r;\epsilon)\, dr = 0 + 0 + \tfrac{1}{2!}2\epsilon F_2 + 0 + \tfrac{1}{4!}12\epsilon F_4 + 0 + \cdots$$
$$\downarrow$$
$$= F_2$$

To describe the situation in general—i.e., to describe

$$\int \Big\{\sum \tfrac{1}{n!} F_n\, r^n\Big\} G_m(r;\epsilon)\, dr \quad \text{with} \quad G_m(r;\epsilon) \equiv \tfrac{1}{\sqrt{2\pi\epsilon}}\big(-\tfrac{d}{dr}\big)^m e^{-\frac{1}{2\epsilon}r^2}$$

—we observe that great simplification can be achieved by slight notational adjustment: writing $r = \sqrt{\epsilon}z$, we find ourselves looking at

$$\int \Big\{\sum \tfrac{1}{n!} F_n\, z^n\Big\} \epsilon^{\frac{1}{2}(n-m)} G_m(z)\, dz \quad \text{with} \quad G_m(z) \equiv \tfrac{1}{\sqrt{2\pi}}\big(-\tfrac{d}{dz}\big)^m e^{-\frac{1}{2}z^2}$$
$$= \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} He_m(z)$$

By reorganization of some elementary data presented on p. 3 we have

$$\left.\begin{aligned} z^0 &= He_0(z) \\ z^1 &= He_1(z) \\ z^2 &= He_2(z) + He_0(z) \\ z^3 &= He_3(z) + 3He_1(z) \\ z^4 &= He_4(z) + 6He_2(z) + 3He_0(z) \\ z^5 &= He_5(z) + 10He_3(z) + 15He_1(z) \\ z^6 &= He_6(z) + 30He_4(z) + 135He_2(z) + 105He_0(z) \\ &\vdots \\ z^{2p} &= He_{2p}(z) + \text{terms of lower even order} \\ z^{2p+1} &= He_{2p+1}(z) + \text{terms of lower odd order} \end{aligned}\right\} \tag{102}$$

[45] Notice the onset of an introductory string of *consecutive* zeros.

The Hermite polynomials are orthogonal in the Gaussian-weighted sense

$$\int_{-\infty}^{+\infty} He_m(z)He_n(z)\, e^{-\frac{1}{2}z^2}dz = \begin{cases} n!\sqrt{2\pi} & : \quad m = n \\ 0 & : \quad m \neq n \end{cases} \tag{103}$$

and they are complete.[46] So one has the Hermite representation formulæ

$$\begin{aligned} J(x) &= \sum_{n=0}^{\infty} J_n \cdot He_n(x) \\ J_n &= \tfrac{1}{n!\sqrt{2\pi}} \int_{-\infty}^{+\infty} J(y)He_n(y)\, e^{-\frac{1}{2}y^2}\, dy \end{aligned} \tag{104}$$

One could use this information to reproduce and extend the list of Hermite expansions (102) but I won't, since we have no pressing need of such data. We are in position now to write

$$\begin{aligned} \int \Big\{ \sum F_n\, r^n \Big\} G_m(r;\epsilon)\, dr &= \sum_{n=0}^{\infty} F_n\, I(n,m)\epsilon^{\frac{1}{2}(n-m)} \\ I(n,m) &\equiv \tfrac{1}{n!\sqrt{2\pi}} \int_{-\infty}^{+\infty} z^n He_m(z) e^{-\frac{1}{2}z^2}\, dz \end{aligned} \tag{105}$$

The integral $I(n,m)$ senses only the He_m-term which may or may not be present in the Hermite expansion of z^n. No such term is present if $n < m$ (this accounts for the "introductory string of consecutive zeros" noticed earlier), nor is any present if $n - m$ is odd (i.e., if n and m are of opposite parity; this accounts for the alternating zeros evident in the examples). If $n > m$ and $n - m$ is even, then $I(n,m)$ assumes a non-zero value, but that value is (owing to presence in (105) of the factor $\epsilon^{\frac{1}{2}(n-m)}$) rendered irrelevant in the limit $\epsilon \downarrow 0$. And if $n = m$ then we have $z^n = He_m(z) + \big\{\text{weighted sum of terms orthogonal to } He_m(z)\big\}$, giving[47] $I(m,m) = 1$. We are brought thus to the conclusion that

$$\lim_{\epsilon \downarrow 0} \int_{-\infty}^{+\infty} \Big\{ \sum_{n=0}^{\infty} \tfrac{1}{n!} F_n\, r^n \Big\} G_m(r;\epsilon)\, dr = \sum F_n \delta_{nm} = F_m \tag{106}$$

The pattern of the preceding argument is, however, susceptible to this serious criticism: Taylor expansion, as invoked at (100), is available only within a circle of convergence, which my integrals $\int_{-\infty}^{+\infty}$ presume to be infinite. The latter presumption will often be in conflict with the facts of the matter, and in such cases the method can be expected to fail. The example $f(x) = e^{ix}$ leads naturally to a variant of the argument which seems to me to be of some independent interest. As a preliminary step, we play again the game first played at (11): we notice that the Weber functions

$$D_\mu(z) \mapsto He_m(z)\, e^{-\frac{1}{4}z^2}$$

[46] For the proof, see QUANTUM MECHANICS (1967), Chapter 2, pp. 64–65.

[47] It is interesting that exactly the right normalization factor has been left behind by constructions/computations prior to this point.

when $\mu \mapsto m = 0, 1, 2, \ldots$ We inform *Mathematica* of our interest in

$$D_\mu(z) \equiv \texttt{Weber[}\mu\texttt{\_,z\_]}$$
$$\texttt{:=}\, 2^{\mu/2}\texttt{Exp[-z}^2\texttt{/4]}\Big(\frac{\texttt{Gamma[1/2]}}{\texttt{Gamma[(1-}\mu\texttt{)/2]}}\texttt{Hypergeometric1F1[-}\mu\texttt{/2,1/2,z}^2\texttt{/2]}$$
$$\texttt{+}\frac{\texttt{z}}{\sqrt{2}}\frac{\texttt{Gamma[-1/2]}}{\texttt{Gamma[-}\mu\texttt{/2]}}\texttt{Hypergeometric1F1[(1-}\mu\texttt{)/2,3/2,z}^2\texttt{/2]}\Big)$$

and proceed from (13), writing

$$\begin{aligned} D^n e^{ix} &= \lim_{\epsilon\downarrow 0} \tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\epsilon}\big)^{\frac{n+1}{2}} \int_{-\infty}^{+\infty} e^{iy}\, e^{-\frac{1}{4}[\frac{y-x}{\sqrt{\epsilon}}]^2} D_n\big(\tfrac{y-x}{\sqrt{\epsilon}}\big)\, dy \\ &= e^{ix}\cdot \lim_{\epsilon\downarrow 0} \tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\epsilon}\big)^{\frac{n+1}{2}} \Big\{ \int_{-\infty}^{+\infty} e^{iu}\, e^{-\frac{1}{4}[\frac{u}{\sqrt{\epsilon}}]^2} D_n\big(\tfrac{u}{\sqrt{\epsilon}}\big)\, du\Big\} \end{aligned}$$

Mathematica finds the integrals difficult,[48] but at length supplies

$$= e^{ix}\cdot \lim_{\epsilon\downarrow 0}\begin{cases} e^{-\frac{1}{2}\epsilon}\cdot e^{0i\frac{\pi}{2}} & : \text{ case } n=0 \\ e^{-\frac{1}{2}\epsilon}\cdot e^{1i\frac{\pi}{2}} & : \text{ case } n=1 \\ e^{-\frac{1}{2}\epsilon}\cdot e^{2i\frac{\pi}{2}} & : \text{ case } n=2 \\ e^{-\frac{1}{2}\epsilon}\cdot e^{3i\frac{\pi}{2}} & : \text{ case } n=3 \\ e^{-\frac{1}{2}\epsilon}\cdot e^{4i\frac{\pi}{2}} & : \text{ case } n=4 \\ & \vdots \end{cases}$$

We are led thus to this uncommon formulation of a very familiar fact:

$$D^n e^{ix} = e^{i(x+n\frac{\pi}{2})} \quad : \quad n = 0, 1, 2, \ldots \tag{107}$$

It becomes natural to speculate[49] that the preceding equation may hold even when n is not an integer, though in §11 of "Construction and physical application of the fractional calculus" (1997) I use standard methods to obtain quite a different result. The integral $\{\text{etc.}\}$ above (after notational adjustment $n \mapsto \nu$ intended to emphasize that we are now relaxing of the presumption of integral order) can be written

$$\begin{aligned} \{\text{etc.}\} &= \sqrt{\epsilon}\int_{-\infty}^{+\infty}\big\{\cos(\sqrt{\epsilon}v) + i\sin(\sqrt{\epsilon}v)\big\}\, e^{-\frac{1}{4}v^2} D_\nu(v)\, du \\ &= \sqrt{\epsilon}\int_0^{\infty}\cos(\sqrt{\epsilon}v)\, e^{-\frac{1}{4}v^2}\big[D_\nu(v) + D_\nu(-v)\big]\, du \\ &\quad + i\sqrt{\epsilon}\int_0^{\infty}\sin(\sqrt{\epsilon}v)\, e^{-\frac{1}{4}v^2}\big[D_\nu(v) - D_\nu(-v)\big]\, du \end{aligned}$$

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[48] Things go well enough if one asks for `Simplify[`$\int_{-\infty}^{0} + \int_0^{+\infty}$`]`.

[49] See in this connection §3 of "A child's garden of fractional derivatives" (July 1998), distributed privately by T. J. Osler & M. Kleinz, who can be contacted at <osler@rowan.edu>.

Mathematica finds both of the latter integrals intractable when ν is not a positive integer, but the second of those integrals is tabulated at **7.741.4** in Gradshteyn & Ryzhik (whose source is **2.14.2** in Erdélyi's *Tables of Integral Transforms*); we are told that

$$\sqrt{\epsilon}\int_0^\infty \sin(\sqrt{\epsilon}v)\, e^{-\frac{1}{4}v^2}\big[D_\nu(v)-D_\nu(-v)\big]\,du = \sqrt{2\pi}\,\epsilon^{\frac{\nu+1}{2}}e^{-\frac{1}{2}\epsilon}\sin\big(\nu\tfrac{\pi}{2}\big)$$

The first of the preceding integrals is *not* tabulated (though quite a similar integral—taken from **1.14.4** in Erdélyi—appears at **7.741.5** in Gradshteyn & Ryzhik), but *if it were the case* that

$$\sqrt{\epsilon}\int_0^\infty \cos(\sqrt{\epsilon}v)\, e^{-\frac{1}{4}v^2}\big[D_\nu(v)+D_\nu(-v)\big]\,du = \sqrt{2\pi}\,\epsilon^{\frac{\nu+1}{2}}e^{-\frac{1}{2}\epsilon}\cos\big(\nu\tfrac{\pi}{2}\big)$$

then we would have

$$D^\nu e^{ix} = e^{i(x+\nu\frac{\pi}{2})}\cdot\lim_{\epsilon\downarrow 0} e^{-\frac{1}{2}\epsilon} \tag{108.1}$$

as previously conjectured. Consistency with the "different result [obtained by standard] methods" would be established by appeal to a point emphasized by Osler & Kleinz: "standard methods" place the fiducial point at the origin, whereas the Gaussian representation places that point at $-\infty$. Slight adjustment[50] of the argument which gave (108.1) gives

$$D^\nu e^{ikx} = e^{ikx}(ke^{i\frac{\pi}{2}})^\nu\cdot\lim_{\epsilon\downarrow 0} e^{-\frac{1}{2}\epsilon k^2} \tag{108.2}$$

No more straightforward fractional generalization of the statement that

$\big(\frac{d}{dx}\big)^n$ becomes "multiply by $(ik)^n$" in the Fourier transform domain

is imaginable; in the latter domain, by this account, one achieves a fractional calculus by direct *interpolation in the exponent*.

In higher-dimensional theory we acquired reason at (52) to have interest in a construction of which (compare (100)) the one-dimensional analog assumes a form

$$\langle f(x;r)\rangle \equiv \tfrac{1}{2}\big[f(x+r)+f(x-r)\big] = \sum_{n=0}^\infty \tfrac{1}{(2n)!}F_{2n}r^{2n} \tag{109}$$

in which only the even-order derivatives $F_{2n}\equiv\big(\frac{d^2}{dx^2}\big)^n f(x)$ are present; odd derivatives are absent owing to the manifest evenness of $\langle f(\bullet;r)\rangle$; the resulting fractional calculus is "supported" by $\frac{d^2}{dx^2}$ rather than by $\frac{d}{dx}$, and was found to be empty at the points one might have expected to be occupied by operators of the form $\big(\frac{d}{dx}\big)^{\text{odd}}$. The question now before us is this: Can we, on the pattern

[50] Make the replacement $\sqrt{\epsilon}\mapsto k\sqrt{\epsilon}$

of the preceding argument, sharpen our understanding of the sense in which that factional theory can be said to be "interpolative"?

In "fractional Laplacian theory" our objective is to give useful meaning to operators of the general type $\nabla^{\frac{2}{3}}$. Within that context, our interest in $\frac{d^2}{dx^2}$ springs in case $N=1$ from an interest in the "radial Laplacian" $\frac{d^2}{dr^2}+\frac{N-1}{r}\frac{d}{dr}$ which we acquired at (64.1). Which is to say, it shifts somewhat when we turn our gaze from $N=1$ to $N=2,3,\ldots$

We proceed from an "even-ized" variant

$$\int\Big\{\sum\tfrac{1}{(2n)!}F_{2n}\,z^{2n}\Big\}\epsilon^{(n-m)}G_{2m}(z)\,dz\quad\text{with}\quad G_{2m}(z)\equiv\tfrac{1}{\sqrt{2\pi}}\big(-\tfrac{d}{dz}\big)^{2m}e^{-\frac{1}{2}z^2}$$
$$=\tfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}He_{2m}(z)$$

of our former starting point. Here as before $z=r/\sqrt{\epsilon}$, but in view of the fact that z^{2n} is always positive it becomes natural to introduce $s\equiv\frac{1}{2}z^2$ and to recall[25] that $He_{2m}(z)=(-2)^m m!\,L_m^{-1/2}(\frac{z^2}{2})$. We then have

$$2\int_0^\infty\Big\{\sum\tfrac{1}{(2n)!}F_{2n}\cdot(2s)^n\Big\}\epsilon^{(n-m)}\mathcal{G}_{2m}(s)\,\tfrac{1}{\sqrt{2s}}ds$$
$$\mathcal{G}_{2m}(s)\equiv\tfrac{1}{\sqrt{2\pi}}e^{-s}He_{2m}(\sqrt{2s}\,)$$
$$=\tfrac{1}{\sqrt{2\pi}}(-2)^m m!\,e^{-s}L_m^{-\frac{1}{2}}(s)$$
$$=\sum_{n=0}^\infty F_{2n}\Big\{\tfrac{1}{(2n)!\sqrt{\pi}}2^{n+m}(-)^m m!\int_0^\infty s^{n-\frac{1}{2}}e^{-s}L_m^{-\frac{1}{2}}(s)\,ds\Big\}\epsilon^{n-m}\tag{110}$$

where the 2 which appears as a prefactor arises from the circumstance that when z ranges on $[-\infty,+\infty]$ the variable $s\equiv\frac{1}{2}z^2$ ranges *twice* on $[0,\infty]$.[51] Thus are we are led, with the assistance of *Mathematica*, to write

$$\begin{aligned}
&= F_0+F_2\tfrac{1}{2}\epsilon+F_4\tfrac{1}{2\cdot4}\epsilon^2+F_6\tfrac{1}{2\cdot4\cdot6}\epsilon^3+F_8\tfrac{1}{2\cdot4\cdot6\cdot8}\epsilon^4+\cdots\quad:\quad\text{case } m=0\\
&\quad\downarrow\\
&\quad F_0\quad\text{in the limit }\epsilon\downarrow 0\\
&= 0+F_2+F_4\tfrac{1}{2}\epsilon+F_6\tfrac{1}{2\cdot4}\epsilon^2+F_8\tfrac{1}{2\cdot4\cdot6}\epsilon^3+\cdots\quad:\quad\text{case } m=1\\
&\qquad\downarrow\\
&\qquad F_2\quad\text{in the limit }\epsilon\downarrow 0\\
&= 0+0+F_4+F_6\tfrac{1}{2}\epsilon+F_8\tfrac{1}{2\cdot4}\epsilon^2+\cdots\quad:\quad\text{case } m=2\\
&\qquad\quad\downarrow\\
&\qquad\quad F_4\quad\text{in the limit }\epsilon\downarrow 0
\end{aligned}$$

[51] Look to this example:

$$\int_{-\infty}^{+\infty}e^{-x^2}dx=2\int_0^\infty e^{-y}\tfrac{1}{2\sqrt{y}}\,dy=\sqrt{\pi}$$

To establish the point more generally, one draws (compare (102)) upon

$$\left.\begin{aligned}
s^0 &= L_0^{-\frac{1}{2}(s)}\\
s^1 &= \tfrac{1}{2}\,L_0^{-\frac{1}{2}}(s) - L_1^{-\frac{1}{2}}(s)\\
s^2 &= \tfrac{3}{4}\,L_0^{-\frac{1}{2}}(s) - 3\,L_1^{-\frac{1}{2}}(s) + 2L_2^{-\frac{1}{2}}(s)\\
s^3 &= \tfrac{15}{8}\,L_0^{-\frac{1}{2}}(s) - \tfrac{45}{4}\,L_1^{-\frac{1}{2}}(s) + 15L_2^{-\frac{1}{2}}(s) - 6L_3^{-\frac{1}{2}}(s)\\
s^4 &= \tfrac{105}{16}L_0^{-\frac{1}{2}}(s) - \tfrac{105}{2}L_1^{-\frac{1}{2}}(s) + 105L_2^{-\frac{1}{2}}(s) - 84L_3^{-\frac{1}{2}}(s) + 24L_4^{-\frac{1}{2}}(s)\\
&\;\vdots\\
s^n &= \text{terms of lower order} + (-)^n n!\,L_n^{-\frac{1}{2}}(s)
\end{aligned}\right\}\qquad(111)$$

and the circumstance that the associated Laguerre polynomials $L_m^\alpha(s)$ are (see Abramowitz & Stegun, **22.2**) orthogonal in the sense that

$$\int_0^\infty L_m^\alpha(s)L_n^\alpha(s)\,e^{-s}s^\alpha\,ds = \delta_{mn}\tfrac{\Gamma(n+\alpha+1)}{n!}$$

which (if completeness can be assumed) entails[52]

$$\begin{aligned}
K(s) &= \sum_{n=0}^\infty K_n\cdot L_n^\alpha(s)\\
&\qquad K_n = \tfrac{n!}{\Gamma(n+\alpha+1)}\int_0^\infty K(t)L_n^\alpha(t)e^{-t}\,t^\alpha\,dt
\end{aligned}$$

Returning with this information to (110) we obtain

$$0+\cdots+0+F_{2m}\underbrace{\Big\{\tfrac{1}{(2m)!\sqrt{\pi}}2^{m+m}(-)^m m!\cdot(-)^m m!\Gamma(m+\tfrac{1}{2})/m!\Big\}}_{1\quad:\quad m=0,1,2,\ldots}\epsilon^0+O(\epsilon)$$

which is a straggle-toothed variant of (106), and susceptible to criticism on the same grounds: we have tacitly assumed the convergence of Taylor's series to be unrestricted. It is in an effort to escape the force of that criticism that...

We look now to the case $f(x) = e^{ikx}$, which entails[53]

$$\langle f(x;r)\rangle = \tfrac{1}{2}\big[f(x+r)+f(x-r)\big] = e^{ikx}\cos kr \qquad (112)$$

[52] The following construction can be (has been) used to reconstruct (111), which I originally obtained by matrix inversion of the equations that describe $L_m^{-1/2}(s)$ as linear combinations of powers s^n $(m, n = 0, 1, 2, \ldots)$.

[53] The following function, when expanded in r, actually *does* converge unrestrictedly, but we will agree to make no use of that fact.

Picking up our former train of thought at a very early stage (p. 73), we seek to recover

$$\nabla^{2m}e^{ikx} = (ik)^{2m}e^{ikx} \tag{113}$$

(here I write ∇^2 for $\frac{d^2}{dx^2}$ to draw attention to the enveloping context of this work) from

$$\begin{aligned}
\nabla^{2m}e^{ikx} &= \lim_{\epsilon\downarrow 0} e^{ikx}\cdot\int_{-\infty}^{+\infty}\cos kr\, G_{2m}(r;\epsilon)\,dr \\
&\qquad G_{2m}(r;\epsilon) = \tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\sqrt{\epsilon}}\big)^{2m+1}e^{-\frac{1}{2\epsilon}r^2}He_{2m}\big(\tfrac{r}{\sqrt{\epsilon}}\big) \\
&\qquad\qquad = \tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\sqrt{\epsilon}}\big)^{2m+1}(-2)^m m!\, e^{-\frac{1}{2\epsilon}r^2}L_m^{-\frac{1}{2}}\big(\tfrac{1}{2\epsilon}r^2\big) \\
&= \lim_{\epsilon\downarrow 0} e^{ikx}\cdot\int_0^\infty \cos k\sqrt{2\epsilon s}\Big\{\tfrac{1}{\sqrt{2\pi}}\big(\tfrac{1}{\sqrt{\epsilon}}\big)^{2m+1}(-2)^m m!\, e^{-s}L_m^{-\frac{1}{2}}(s)\sqrt{\tfrac{2\epsilon}{s}}\,\Big\}ds
\end{aligned}$$

But

$$\Big\{\text{etc.}\Big\} = (-)^m\tfrac{1}{\sqrt{2\pi}}2^{m+\frac{1}{2}}\epsilon^{-m}\,m!\cdot e^{-s}s^{-\frac{1}{2}}L_m^{-\frac{1}{2}}(s) \tag{114}$$

and *Mathematica* reports that

$$\int_0^\infty \cos k\sqrt{2\epsilon s}\,\Big\{\text{etc.}\Big\}\,ds = \begin{cases} e^{-\frac{1}{2}k^2\epsilon} & : \ \text{case m=0} \\ -k^2e^{-\frac{1}{2}k^2\epsilon} & : \ \text{case m=1} \\ +k^4e^{-\frac{1}{2}k^2\epsilon} & : \ \text{case m=2} \\ -k^6e^{-\frac{1}{2}k^2\epsilon} & : \ \text{case m=3} \\ & \ \vdots \end{cases} \tag{115}$$

This is (in the limit $\epsilon\downarrow 0$) precisely the result we sought—an elementary result, obtain here with great labor. The point of the labor—i.e., of the round-about procedure—is that it puts us in position to relax the presumption that m is an integer. At (114) we make the replacements

$$(-)^m \mapsto \cos m\pi \quad \text{and (see again (70))} \quad L_m^{-\frac{1}{2}}(s) \mapsto \tfrac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)\Gamma(\frac{1}{2})}M(-m;\tfrac{1}{2};s)$$

For $m = 0, 1, 2, \ldots$ the adjustment is merely notational/cosmetic, and one again recovers (114).

10. Does there exist a fractional exterior calculus? I have remarked already in connection with (49/50) that the high road to a theory of Laplace-Beltrami operators active upon (antisymmetric) tensor fields of arbitrary dimension N and rank $n \leq N$ is provided by the exterior calculus, and have made reference to an essay in which details supportive of that remark can be found. The exterior calculus provides such a theory, but provides also much more; it leads in particular to vast generalization of Stokes' theorem. How exciting would be the prospect if there existed a "*fractional* exterior calculus." Alas! I am persuaded that the answer to the question posed above is "No." My objective here will be to sketch the argument that leads me to that conclusion. I admit to a measure of disappointment, though it is absurd to draw disappointment from a mathematical fact, and one or two of the ideas which we will encounter along the way do seem to me to be of some independent interest.

I intend to be sketchy—to work typically in only two or three dimensions, to omit factors and references to limiting processes when they either do not contribute directly to the point at issue or can be expected to "take care of themselves." When borrowing ideas from representation theory I will appeal exclusively to *Gaussian* representation theory. And though several variables will be in play, I will have need only of (a slight enlargement of) the fractional calculus of a *single* variable.

The notion of a "fractional partial derivative" would appear to pose no difficulty

APPENDIX. I alluded in §8 to the curious results obtained when one examines hyperspherically symmetric constructions of the form

$$\nabla^{2m}e^{-r} = \left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{N-1}{r}\tfrac{d}{dr}\right]^m e^{-r}$$

The results in question were useless for the purposes of that discussion because they are *invalid at the origin*,[54] but they are in their odd way so pretty that I could not consign them to the trash can. I look serially to the cases $N = 1, 2, 3, \ldots$ to gain a preliminary sense of the matter. In the case $\boldsymbol{N=1}$ we have

$$\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{0}{r}\tfrac{d}{dr}\right]^m e^{-r} = e^{-r} \quad : \quad \text{all } m$$

In the case $\boldsymbol{N=2}$

$$\begin{aligned}
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{1}{r}\tfrac{d}{dr}\right]^0 e^{-r} &= e^{-r}\\
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{1}{r}\tfrac{d}{dr}\right]^1 e^{-r} &= e^{-r}\cdot\tfrac{r-1}{r}\\
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{1}{r}\tfrac{d}{dr}\right]^2 e^{-r} &= e^{-r}\cdot\tfrac{r^3-2r^2-r-1}{r^3}\\
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{1}{r}\tfrac{d}{dr}\right]^3 e^{-r} &= e^{-r}\cdot\tfrac{r^5-3r^4-3r^3-6r^2-9r-9}{r^5}\\
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{1}{r}\tfrac{d}{dr}\right]^1 e^{-r} &= e^{-r}\cdot\tfrac{r^7-4r^6-6r^5-18r^4-51r^3-126r^2-225r-225}{r^7}
\end{aligned}$$

In the case $\boldsymbol{N=3}$

$$\begin{aligned}
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{2}{r}\tfrac{d}{dr}\right]^0 e^{-r} &= e^{-r}\\
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{2}{r}\tfrac{d}{dr}\right]^1 e^{-r} &= e^{-r}\cdot\tfrac{r-2}{r}\\
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{2}{r}\tfrac{d}{dr}\right]^2 e^{-r} &= e^{-r}\cdot\tfrac{r-4}{r}\\
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{2}{r}\tfrac{d}{dr}\right]^3 e^{-r} &= e^{-r}\cdot\tfrac{r-6}{r}\\
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{2}{r}\tfrac{d}{dr}\right]^4 e^{-r} &= e^{-r}\cdot\tfrac{r-8}{r}
\end{aligned}$$

In the case $\boldsymbol{N=4}$

$$\begin{aligned}
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{3}{r}\tfrac{d}{dr}\right]^0 e^{-r} &= e^{-r}\\
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{3}{r}\tfrac{d}{dr}\right]^1 e^{-r} &= e^{-r}\cdot\tfrac{r-3}{r}\\
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{3}{r}\tfrac{d}{dr}\right]^2 e^{-r} &= e^{-r}\cdot\tfrac{r^3-6r^2+3r+3}{r^3}\\
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{3}{r}\tfrac{d}{dr}\right]^3 e^{-r} &= e^{-r}\cdot\tfrac{r^5-9r^4+9r^3+12r^2+9r+9}{r^5}\\
\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{3}{r}\tfrac{d}{dr}\right]^4 e^{-r} &= e^{-r}\cdot\tfrac{r^7-12r^6+18r^5+30r^4+45r^3+90r^2+135r+135}{r^7}
\end{aligned}$$

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[54] A similar remark pertains to statements of the form

$$\left[\left(\tfrac{d}{dr}\right)^2 + \tfrac{N-1}{r}\tfrac{d}{dr}\right]^m \tfrac{1}{r^{N-2}} = 0$$

but (on evidence of a robust potential theory) certainly does not mean that they are useless in *all* contexts

The case $\boldsymbol{N=5}$ displays a relative simplicity reminiscent of that encountered at $N=1$ and $N=3$ [55]

$$\begin{aligned}
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{4}{r}\tfrac{d}{dr}\big]^0e^{-r}&=e^{-r}\\
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{4}{r}\tfrac{d}{dr}\big]^1e^{-r}&=e^{-r}\cdot\tfrac{r-4}{r}\\
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{4}{r}\tfrac{d}{dr}\big]^2e^{-r}&=e^{-r}\cdot\tfrac{r^3-8r^2+8r+8}{r^3}\\
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{4}{r}\tfrac{d}{dr}\big]^3e^{-r}&=e^{-r}\cdot\tfrac{r^3-12r^2+24r+24}{r^3}\\
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{4}{r}\tfrac{d}{dr}\big]^4e^{-r}&=e^{-r}\cdot\tfrac{r^3-16r^2+48r+48}{r^3}
\end{aligned}$$

and encountered again at $\boldsymbol{N=7}$:

$$\begin{aligned}
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{6}{r}\tfrac{d}{dr}\big]^0e^{-r}&=e^{-r}\\
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{6}{r}\tfrac{d}{dr}\big]^1e^{-r}&=e^{-r}\cdot\tfrac{r-6}{r}\\
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{6}{r}\tfrac{d}{dr}\big]^2e^{-r}&=e^{-r}\cdot\tfrac{r^3-12r^2+24r+24}{r^3}\\
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{6}{r}\tfrac{d}{dr}\big]^3e^{-r}&=e^{-r}\cdot\tfrac{r^5-18r^4+72r^3+24r^2-144r-144}{r^5}\\
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{6}{r}\tfrac{d}{dr}\big]^4e^{-r}&=e^{-r}\cdot\tfrac{r^5-24r^4+144r^3-48r^2-576r-576}{r^5}
\end{aligned}$$

The relative complexity encountered at $N=2$ and $N=4$ is encountered also at $\boldsymbol{N=6}$, and is typical of all even-dimensional cases:

$$\begin{aligned}
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{5}{r}\tfrac{d}{dr}\big]^0e^{-r}&=e^{-r}\\
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{5}{r}\tfrac{d}{dr}\big]^1e^{-r}&=e^{-r}\cdot\tfrac{r-5}{r}\\
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{5}{r}\tfrac{d}{dr}\big]^2e^{-r}&=e^{-r}\cdot\tfrac{r^3-10r^2+15r^2+15}{r^3}\\
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{5}{r}\tfrac{d}{dr}\big]^3e^{-r}&=e^{-u}\cdot\tfrac{u^5-15u^4+45u^3+30u^2-45u-45}{u^5}\\
\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{5}{r}\tfrac{d}{dr}\big]^4e^{-r}&=e^{-u}\cdot\tfrac{u^7-20u^6+90u^5+30u^4-195u^3-270u^2-225u-225}{u^7}
\end{aligned}$$

The patterns semi-evident in preceding formulæ are somewhat clarified when one looks to the general case; one has

$$\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{N-1}{r}\tfrac{d}{dr}\big]^m e^{-r}=e^{-r}\cdot\frac{P_m(r;N)}{r^{2m-1}}$$

where

$$P_m(r;N)=\sum_{k=0}^{2m-1}P_{mk}(N)\,r^k$$

[55] Observe that when $N=3$ the exponent in the denominator stabilizes at $1=3-2$, when $N=5$ it stabilizes at $3=5-2$, when $N=7$ it stabilizes at $5=7-2$ (all this owing to the disappearance of low-order coefficients in the numerator), but when N is even no such stabilization takes place.

and computation shows that the coefficients $P_{mk}(N)$ can be described

$$\begin{aligned}
P_{10} &= -\ (N-1)\\
P_{11} &= +1\\[1em]
P_{20} &= +\ (N-1)(N-3)\\
P_{21} &= +\ (N-1)(N-3)\\
P_{22} &= -2(N-1)\\
P_{23} &= +1\\[1em]
P_{30} &= -3(N-1)(N-3)(N-5)\\
P_{31} &= -3(N-1)(N-3)(N-5)\\
P_{32} &= -\ (N-1)(N-3)(N-8)\\
P_{33} &= +3(N-1)(N-3)\\
P_{34} &= -3(N-1)\\
P_{35} &= +1\\[1em]
P_{40} &= +15(N-1)(N-3)(N-5)(N-7)\\
P_{41} &= +15(N-1)(N-3)(N-5)(N-7)\\
P_{42} &= +\ 6(N-1)(N-3)(N-5)(N-9)\\
P_{43} &= +\ (N-1)(N-3)(N-5)(N-19)\\
P_{44} &= -\ 2(N-1)(N-3)\boldsymbol{(2N-13)}\\
P_{45} &= +\ 6(N-1)(N-3)\\
P_{46} &= -\ 4(N-1)\\
P_{47} &= +\ 1\\
P_{50} &= -105(N-1)(N-3)(N-5)(N-7)(N-9)\\
P_{51} &= -105(N-1)(N-3)(N-5)(N-7)(N-9)\\
P_{52} &= -\ 15(N-1)(N-3)(N-5)(N-7)\boldsymbol{(3N-32)}\\
P_{53} &= -\ 5(N-1)(N-3)(N-5)(N-7)\boldsymbol{(2N-33)}\\
P_{54} &= -\ (N-1)(N-3)(N-5)\boldsymbol{(N^2-46N+303)}\\
P_{55} &= +\ 5(N-1)(N-3)(N-5)(N-13)\\
P_{56} &= -\ 10(N-1)(N-3)(N-6)\\
P_{57} &= +\ 10(N-1)(N-3)\\
P_{58} &= -\ 5(N-1)\\
P_{59} &= +\ 1
\end{aligned}$$

Here I owe everything to *Mathematica*, who produced the equations in far less time than it took me to write them out. I have used heavy type to draw attention to the occurrance of certain **goofy factors**. One does not expect polynomials to factor so nicely "for no reason," but neither does one expect such a pattern of nice-factorization to be "occasionally disrupted."

It is, on preceding evidence, clear how the striking simplifications at $N=1$ and $N=3$ come about, and why—after a bumpy start (look to the anomalies which sometimes appear near the bottoms of the "stacks of factors" on the preceding page)—similar simplifications occur whenever N is odd. One can easily say various things about the polynomials $P_m(u;N)$,[56] but I have been unable to bring them within the compass of any established theory of named polynomials.[57] I am therefore unable to relax the presumption that m be an integer, unable to mimic the interpolative and backward extrapolative steps which are central to the fractional theory developed in the text. Unable, that is to say, except in the cases $N=1$ and $N=3$. Looking to the latter...

We have

$$\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{2}{r}\tfrac{d}{dr}\big]^m e^{-r}=e^{-r}\cdot\tfrac{r-2m}{r}\quad:\quad m=1,2,3,\ldots$$

It is in this case by exceptionally elementary calculation that one obtains

$$\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{2}{r}\tfrac{d}{dr}\big]^m e^{-r}\cdot\tfrac{r-\nu}{r}=e^{-r}\cdot\tfrac{r-(2m+\nu)}{r}\quad:\quad m=1,2,3,\ldots$$

and it becomes natural to write

$$\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{2}{r}\tfrac{d}{dr}\big]^{\frac{\mu}{2}} e^{-r}\equiv e^{-r}\cdot\tfrac{r-\mu}{r}\quad:\quad \mu>0\text{ not necessarily an even integer}$$

by way of "fractional interpolation," and

$$\big[\big(\tfrac{d}{dr}\big)^2+\tfrac{2}{r}\tfrac{d}{dr}\big]^{-\frac{\mu}{2}} e^{-r}\equiv e^{-r}\cdot\tfrac{r+\mu}{r}$$

by way of "backward extrapolation." The "law of exponents" is, in this case, a virtual triviality (and in the case $N=1$ it is a blatant triviality). But a lot of good that does us: our formulae lead demonstrably to absurdities unless implications of the singularity at the origin is correctly managed, and in the absence of such management it would be foolhardy to invest confidence in any "theorems" based upon them.

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[56] It is, for example, almost immediate that

$$\begin{aligned}P_{m+1}=\big[(4m^2-2mN+N-1)-(N-4m+1)r+r^2\big]P_m&\\ +\,r\big[(N-4m+1)-2r\big]P_m'+r^2P_m''&\end{aligned}$$

which simplifies markedly in the cases $N=1$ and $N=3$.

[57] I find this development surprising, since the polynomials spring from soil which has been heavily cultivated by many farmers over several centuries, and has in other connections yielded so many valuable crops.