A MATHEMATICAL NOTE

Solution of a transcendental equation encountered in the theory of single slit diffraction

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Introduction. Monochromatic light (wave number $k = 2\pi/\lambda$), when normally incident upon a single slit of width b, gives rise[1] in the far zone (i.e., in the Fraunhofer approximation) to a diffraction pattern which can be described

$$I(x) = I_0\Big(\frac{\sin x}{x}\Big)^2 \tag{1}$$

where x is the dimensionless variable defined $x \equiv \frac{1}{2}kb\sin\theta$.[2] The function

$$\text{sinc}\,(x) \equiv \frac{\sin x}{x}$$

—sometimes called[3] the "sampling function"— is encountered also in many other physical contexts, and participates in a rich population of relations with the other "special functions" of higher analysis. From[4]

$$\frac{\sin ax}{x} = \int_0^\infty f(y)\cos xy\,dy$$

$$f(y) \equiv \begin{cases} 1 & 0 < y < a \\ 0 & a < y \end{cases}$$

[1] For discussion of the physical details see, for example, §10.2.1 of E. Hecht & A. Zajac, *Optics* (1979).

[2] Here θ is the angular address of the pattern point in question, defined in the natural way; see Hecht & Zajac's Fig. 10.10. For some purposes it is useful to re-scale the variable x, writing $x = \pi\xi$, which ranges on $\{0, \pi, 2\pi, \ldots\}$ as ξ ranges on $\{0, 1, 2, \ldots\}$.

[3] See p. 306 of Spanier & Oldham, who adopt the alternative definition

$$\text{Sinc}\,(\xi) \equiv \frac{\sin \pi\xi}{\pi\xi}$$

[4] See A. Erdélyi *et al*, *Tables of Integral Transforms I*, p. 7.

we learn, for example, that

$$\text{sinc}\,(x) = \text{Fourier cosine transform of the } \textit{step function of unit width}$$

while

$$\text{Si}(x) \equiv \int_0^x \text{sinc(t)}\,\text{dt}$$

serves to define the so-called "sine-integral."[5] The form of the function $\text{Sinc}(\xi)$ is indicated in the first of the following figures:

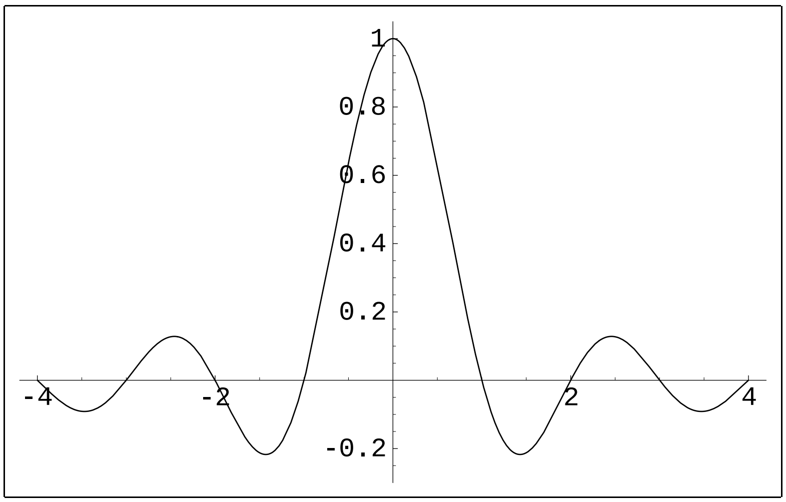

FIGURE 1: *The function* $\text{Sinc}(\xi)$, *as plotted by Mathematica.*

The central peak of $s(x;a) \equiv a\,\text{sinc}\,(ax)$ becomes simultaneously taller and narrower as a increases, but

$$\int_{-\infty}^{+\infty} s(x;a)\,dx = \int_{-\infty}^{+\infty} \frac{\sin(ax)}{x}\,dx = \pi \quad : \quad \text{all positive } a$$

One is led thus to the often useful[6] "sinc representation of the delta function"

$$\delta(x - x_0) = \lim_{a\to\infty} \frac{\sin[a(x - x_0)]}{(x - x_0)}$$

[5] See W. Magnus & F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics* (1954), p. 97; Chapter 5 of M. Abramowitz & I. Stegun, *Handbook of Mathematical Functions* (1964). It is interesting to note that $\text{Si}(x)$ and its immediate cognates are the functions which H. Jahnke & F. Emde, in *Tables of Functions* (1945), take as their point of departure.

[6] See, for example, p. 215 of D. Bohm's *Quantum Theory* (1951) or §3 in Chapter I of S. Chandrasekhar, "Stochastic Problems in Physics & Astronomy," Rev. Mod. Phys. **15**, 1 (1943).

From (1) we acquire physical interest in the function $f(x) \equiv \text{sinc}^2(x)$, which is plotted in Figure 2:

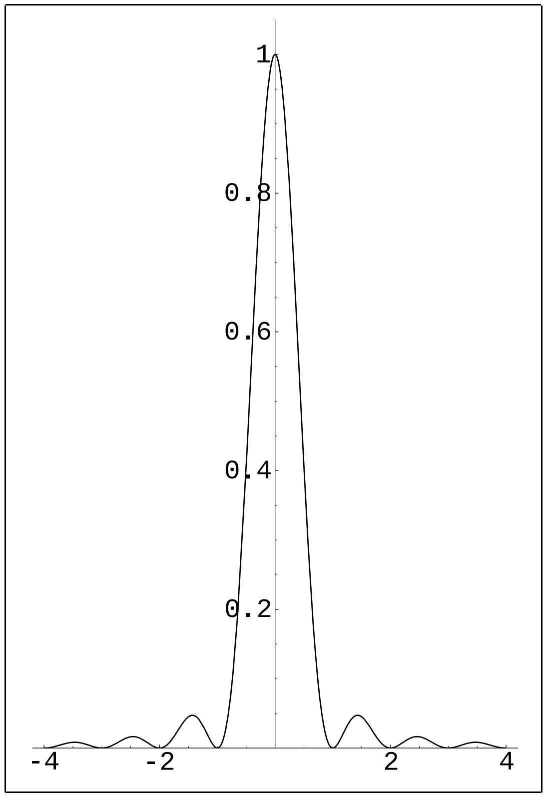

FIGURE 2: *The function* $\text{Sinc}^2(\xi)$*, which is familiar to physicists as the (normalized)* single slit diffraction pattern.

We have particular interest in the points at which $\text{sinc}^2(x)$ assumes its extremal values. From

$$\begin{aligned} f'(x) = \frac{d}{dx}\text{sinc}^2(x) &= 2\,\text{sinc}(x)\cdot\frac{x\cos x - \sin x}{x^2} \\ &= 2\cdot\frac{1}{x^3}\cdot\sin x\cdot(x\cos x - \sin x) \end{aligned}$$

we infer that

- $f(x)$ becomes flat as $x \longrightarrow \pm\infty$
- $f(x)$ is minimal at the zeros $\{\pm\pi, \pm 2\pi, \pm 3\pi, \ldots\}$ of $\sin x$
- $f(x)$ is maximal at $x = 0$ and at the roots $\{\pm x_1, \pm x_2, \pm x_3, \ldots\}$ of the transcendental equation

$$x = \tan x \tag{2}$$

and it is with the description of those roots that we are mainly concerned.

1. Numerical location of the roots and Bonfim's construction. Let the location of the n^{th} root of (2) be notated

$$x_n = \pi\xi_n$$

Mathematica supplies the following information:

$$\begin{array}{rcrcll}
\xi_1 &=& 1.4302967 &=& (1+\frac{1}{2})-q_1 & \text{with}\quad q_1 = 0.0697033\\
\xi_2 &=& 2.4590240 &=& (2+\frac{1}{2})-q_2 & \text{with}\quad q_2 = 0.0409760\\
\xi_3 &=& 3.4708897 &=& (3+\frac{1}{2})-q_3 & \text{with}\quad q_3 = 0.0291103\\
\xi_4 &=& 4.4774086 &=& (4+\frac{1}{2})-q_4 & \text{with}\quad q_4 = 0.0225914\\
\xi_5 &=& 5.4815367 &=& (5+\frac{1}{2})-q_5 & \text{with}\quad q_5 = 0.0184633\\
\xi_6 &=& 6.4843871 &=& (6+\frac{1}{2})-q_6 & \text{with}\quad q_6 = 0.0156129\\
\xi_7 &=& 7.4864742 &=& (7+\frac{1}{2})-q_7 & \text{with}\quad q_7 = 0.0135258\\
\xi_8 &=& 8.4880687 &=& (8+\frac{1}{2})-q_8 & \text{with}\quad q_8 = 0.0119313\\
\xi_9 &=& 9.4893266 &=& (9+\frac{1}{2})-q_9 & \text{with}\quad q_9 = 0.0106734\\
\xi_{10} &=& 10.4903444 &=& (10+\frac{1}{2})-q_{10} & \text{with}\quad q_{10} = 0.0096556\\
\xi_{20} &=& 20.4950567 &=& (20+\frac{1}{2})-q_{20} & \text{with}\quad q_{20} = 0.0049433\\
\xi_{30} &=& 30.4966778 &=& (30+\frac{1}{2})-q_{30} & \text{with}\quad q_{30} = 0.0033222\\
\xi_{40} &=& 40.4974981 &=& (40+\frac{1}{2})-q_{40} & \text{with}\quad q_{40} = 0.0025019\\
\xi_{50} &=& 50.4979936 &=& (50+\frac{1}{2})-q_{50} & \text{with}\quad q_{50} = 0.0020064\\
\xi_{100} &=& 100.4989918 &=& (100+\frac{1}{2})-q_{100} & \text{with}\quad q_{100} = 0.0010082\\
\xi_{200} &=& 200.4994947 &=& (200+\frac{1}{2})-q_{200} & \text{with}\quad q_{200} = 0.0005053
\end{array}$$

The representation

$$x_n = \pi\xi_n \quad \text{with} \quad \xi_n = (n+\tfrac{1}{2})-q_n$$
$$q_n \downarrow 0 \quad \text{as} \quad n \uparrow \infty$$

was inspired by (see Figure 3) the graphical solution of (2).

Oz Bonfim has noticed[7] that the points $(n, 1/q_n)$ fall very nearly on a straight line (see Figure 4), which (if we take the first ten of those points as our data points) can in least squares approximation be described

$$y \equiv \frac{1}{q} = 4.585419 + 9.904152n$$

The implication is that

$$\begin{aligned}
\xi_n &\approx (n+\tfrac{1}{2}) - \frac{1}{4.585419 + 9.904152n} && (3)\\
&\approx (n+\tfrac{1}{2}) - \frac{1}{\sqrt{21}+10n} && \text{BONFIM'S FORMULA}
\end{aligned}$$

[7] Private communication (April 1997). To avoid expository clutter I will take certain liberties in my account of the details of Bonfim's work.

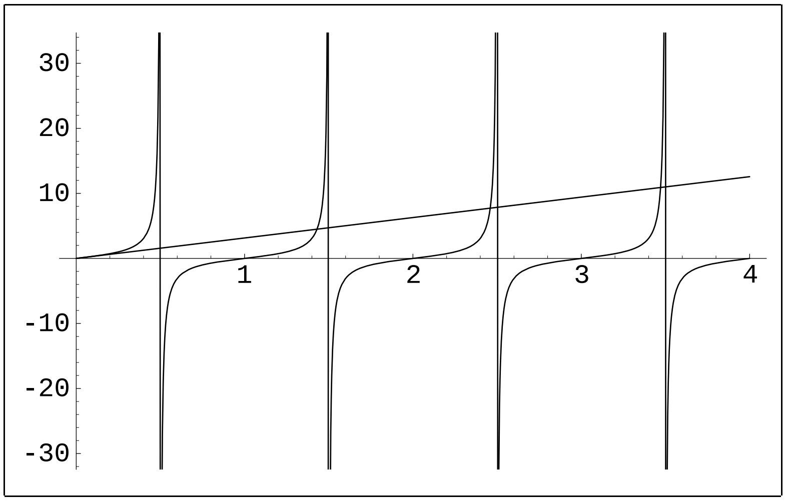

FIGURE 3: *Superimposed graphs of* $y = \pi\xi$ *and* $y = \tan\pi\xi$. *The graphs intersect at points* ξ_n *that stand just to the left of (and ever closer to) the points* $\xi = n + \frac{1}{2}$ *at which* $\tan\pi\xi$ *becomes singular.*

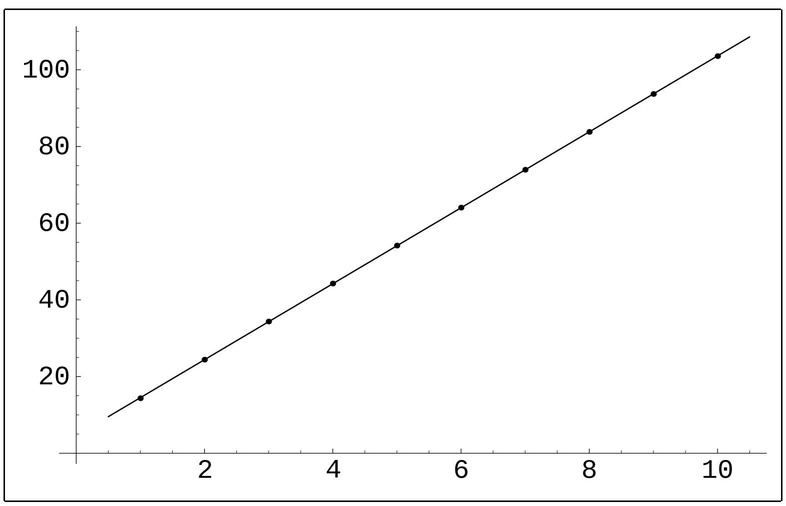

FIGURE 4: *The first ten of the points* $(n, 1/r_n)$ *and the line*

$$y = 4.585419 + 9.904152n$$

to which they give rise in least squares approximation.

Bonfim's formula gives 3-place accuracy (after round-off) already at $n = 1$, and (3) does even better.

2. First steps toward a theoretical account of Bonfim's construction. My main objective in subsequent pages will be to remove some of the mystery which attaches to Bonfim's striking result—to clarify its analytical origins, and to indicate how it might, in principle, be refined. I begin with some elementary observations intended to sharpen our understanding of the analytical problem which Bonfim's formula presents.

With $x_n = \pi[(n+\frac{1}{2}) - q_n]$ in mind, let us suppose for the moment that x has been resolved $x = R - r$. Then (2) reads

$$\begin{aligned} R - r &= \tan(R-r) \\ &= \frac{\tan R - \tan r}{1 + \tan R \tan r} \\ &= \frac{1 - \frac{\tan r}{\tan R}}{\frac{1}{\tan R} + \tan r} \longrightarrow -\frac{1}{\tan r} \quad \text{as} \quad \tan R \uparrow \infty \end{aligned}$$

The implication (if we set $R \mapsto R_n \equiv \pi(n+\frac{1}{2})$ and $r \mapsto r_n \equiv \pi q_n$) is that (2) can be written

$$r = R - \frac{1}{\tan r}$$

which on the assumption that r is small ($\tan r = r + \frac{1}{3}r^3 + \frac{2}{15}r^5 + \cdots \approx r$) becomes

$$r + \frac{1}{r} = R$$

or again

$$r^2 - Rr + 1 = 0$$

This is a quadratic with the property that if r is a root then so also is r^{-1}; if one root is small then the other is large. We have interest in the small root

$$\begin{aligned} r &= \tfrac{1}{2}\Big[R - \sqrt{R^2 - 4}\Big] \\ &= \tfrac{1}{2}R\Big[1 - \sqrt{1 - (2/R)^2}\Big] \\ &= \tfrac{1}{2}R\Big[1 - \Big\{1 - \tfrac{1}{2}(2/R)^2 + \cdots\Big\}\Big] = \frac{1}{R} + \cdots \end{aligned}$$

Thus are we led to write

$$\begin{aligned} \xi_n &= (n+\tfrac{1}{2}) - q_n \quad \text{with} \quad q_n = \tfrac{1}{\pi}r_n \approx 1/(\pi R_n) \\ &\approx (n+\tfrac{1}{2}) - \frac{1}{\pi^2(n+\frac{1}{2})} \\ &\approx (n+\tfrac{1}{2}) - \frac{1}{4.934802 + 9.869604n} \end{aligned} \tag{4}$$

This equation does exhibit the qualitative features of Bonfim's formula, but is quantitatively much less accurate.[8] We confront therefore a new question: *Why* is (4) less precise than (3)?

[8] I will have occasion to amend this remark.

3. What the literature has to say. That our subject has in fact an ancient and honorable history came first to my attention upon perusal of §34.7 in Spanier & Oldham. The equation discussed there reads

$$x = b\tan x \quad : \quad -\infty < b < \infty \tag{5}$$

and gives back (2) in the special case $b = 1$. Spanier & Oldham observe that the positive roots $\{x_1(b), x_2(b), \ldots\}$ of (5) are joined by an additional root $x_0(b)$ if $b > 1$, and assign "especial importance" to the case $b = 1$, in which connection they remark that "the values of $x_n \equiv x_n(1)$ correspond to the zeros of the spherical Bessel functions of the first kind." The functions to which they allude are standardly defined

$$j_n(x) \equiv \sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x) \quad : \quad n = 0, \pm1, \pm2, \ldots$$

Evidently discussion of the zeros of $j_n(x)$ amounts, in effect, to discussion of the zeros of $J_{n+\frac{1}{2}}(x)$, and *Mathematica*, when asked to describe `BesselJ[3/2,x]`, returns the information that

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\Big[\frac{\sin x}{x} - \cos x\Big]$$

The problem posed by (2) is equivalent, therefore, to the problem of exhibiting the zeros of $J_{\frac{3}{2}}(x)$, and this is a particular instance of a problem that has been much studied.[9] We are referred on p. 440 of Abramowitz & Stegun to their equation **9.5.12** p. 371, which under the title "McMahon's expansion[10] for large zeros" asserts that the n^{th} zero of $J_\nu(x)$ can (if $n \gg \nu$ and $\mu \equiv 4\nu^2$) be described

$$x_n \sim R - \frac{\mu-1}{8R} - \frac{4(\mu-1)(7\mu-31)}{3(8R)^3} - \frac{32(\mu-1)(83\mu^2-982\mu+3779)}{15(8R)^5} - \cdots$$

In the case $\nu = \frac{3}{2}$ (which entails $\mu = 9$) we therefore have

$$x_n \sim R - \frac{1}{R} - \frac{2}{3R^3} - \frac{13}{15R^5} - \cdots \quad : \quad R = R_n \equiv (n + \tfrac{1}{2})\pi$$

which removes some of the mystery from the equation

$$x_n = R - \frac{1}{R} - \frac{2}{3R^3} - \frac{13}{15R^5} - \frac{146}{105R^7} - \frac{781}{315R^9}\cdots \tag{6}$$

displayed on p. 325 of Spanier & Oldham. Our own equation (4) can in present notation be written

$$x_n = R - \frac{1}{R} \tag{7}$$

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[9] See, for example, Chapter 4 in C. J. Tranter, *Bessel Functions with Some Physical Applications* (1968) or Chapter XV in G. N. Watson, *Theory of Bessel Functions* (1966).

[10] From Tranter's §4.5 I infer that Major McMahon's expansion is merely a refinement of a result original to Stokes.

of which (6) represents obviously a major refinement. It is interesting to note also that Spanier & Oldham attach no asymptotic proviso $n \gg \nu$ to their equation—for the good and sufficient reason that it works wonderfully well already at $n = 1$:

	Exact	Bonfim	1st	2nd	3rd
x_1	4.4934096	4.4969543	4.5001824	4.4938117	4.4934388
x_2	7.7252517	7.7261841	7.7266577	7.7252816	7.7252526
x_3	10.904122	10.904731	10.904629	10.904127	10.904122
x_4	14.066194	14.066700	14.066431	14.066195	14.066194
x_5	17.220755	17.221203	17.220885	17.220756	17.220755
x_6	20.371303	20.371708	20.371381	20.371303	20.371303
x_7	23.519452	23.519823	23.519504	23.519453	23.519452
x_8	26.666054	26.666395	26.666089	26.666054	26.666054
x_9	29.811599	29.811915	29.811624	29.811599	29.811599
x_{10}	32.956389	32.956684	32.956408	32.956389	32.956389

This data shows—contrary to my initial impression—that (7)—which in the table I call the 1st approximation to (6)—actually surpasses the accuracy of Bonfim's formula for $n \geq 3$.[11] And that, remarkably, the 3rd approximation to (6) achieves 5-place accuracy already at $n = 1$, and 8-place accuracy for $n \geq 3$.

I have acquired an obligation to sketch the argument from which (6) proceeds. Asymptotically (i.e., for large values of x) one has[12]

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left[P(\nu;x)\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) - Q(\nu;x)\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] \tag{8}$$

[11] In my account of Bonfim's method I took the first ten data points and in least squares approximation was led to (3). Bonfim himself actually took the first fifteen data points, and obtained a formula

$$\xi_n \approx (n + \tfrac{1}{2}) - \frac{1}{\sqrt{19} + 10n}$$

which gives improved precision for small values of n:

	Exact	Bonfim
x_1	4.4934096	4.4935983
x_2	7.7252517	7.7250106
x_3	10.904122	10.904140
x_4	14.066194	14.066345
x_5	17.220755	17.220966

but becomes less accurate than my 1st approximation at $n = 5$.

[12] See Spanier & Oldham, p. 526.

where

$$\begin{aligned}P(\nu;x) &\sim 1 - \frac{(\frac{9}{4}-\nu^2)(\frac{1}{4}-\nu^2)}{2!(2x)^2} + \frac{(\frac{49}{4}-\nu^2)(\frac{25}{4}-\nu^2)(\frac{9}{4}-\nu^2)(\frac{1}{4}-\nu^2)}{4!(2x)^4} - \cdots\\ Q(\nu;x) &\sim \qquad\quad -\frac{(\frac{1}{4}-\nu^2)}{1!(2x)} + \frac{(\frac{25}{4}-\nu^2)(\frac{9}{4}-\nu^2)(\frac{1}{4}-\nu^2)}{3!(2x)^3} - \cdots\end{aligned}$$

The $\sim$ notation is intended to emphasize that the preceding statements hold only asymptotically, but when

$$\nu = \frac{\text{odd integer}}{2}$$

the series terminate, and the statements become exact. In the particular case $\nu = \frac{3}{2}$ the resulting simplifications are especially dramatic; we have

$$\begin{aligned}P(\tfrac{3}{2};x) &= 1\\ Q(\tfrac{3}{2};x) &= 1/x\end{aligned}$$

giving

$$\begin{aligned}J_{\frac{3}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\Big[\cos(x-\pi) - \frac{1}{x}\sin(x-\pi)\Big]\\ &= \sqrt{\frac{2}{\pi x}}\Big[\frac{\sin x}{x} - \cos x\Big]\end{aligned}$$

which is precisely and exactly the result quoted previously. Central to Stokes' line of argument is the observation that (8)—which I shall abbreviate

$$\sqrt{\frac{\pi x}{2}}J_\nu(x) = P\cos\xi - Q\sin\xi$$

—admits of "polar representation" in this familiar sense: Write $P = A\cos\theta$ and $-Q = A\sin\theta$, which entail $A = \sqrt{P^2+Q^2}$ and $\tan\theta = -Q/P$. Then

$$\begin{aligned}&= A\cos(\xi-\theta)\\ &= 0 \quad \text{when} \quad \xi - \theta = (n+\tfrac{1}{2})\pi \quad : \quad n = 0, \pm1, \pm2, \ldots\end{aligned}$$

In the case $\nu = \frac{3}{2}$ one has $\xi = x - \pi$ and $\tan\theta = -1/x$; we infer that the positive roots zeros of $J_{\frac{3}{2}}(x)$ satisfy

$$\begin{aligned}x &= (n+\tfrac{1}{2})\pi - \arctan(1/x) \quad : \quad n = 1, 2, \ldots \qquad (9)\\ &= R - \Big\{\frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots\Big\}\end{aligned}$$

Into this statement Stokes feeds the *assumption* that x_n can be described

$$x = R + \frac{a}{R} + \frac{b}{R^3} + \frac{c}{R^5} + \cdots$$

which entails

$$\begin{aligned}
\frac{1}{x} &= \frac{1}{R}\left[1+\left(\frac{a}{R^2}+\frac{b}{R^4}+\cdots\right)\right]^{-1}\\
&= \frac{1}{R}\left[1-\left(\frac{a}{R^2}+\frac{b}{R^4}+\cdots\right)+\left(\frac{a}{R^2}+\cdots\right)^2-\cdots\right]\\
&= \frac{1}{R}-\frac{a}{R^3}+\frac{a^2-b}{R^5}+\cdots\\
\frac{1}{x^3} &= \frac{1}{R^3}\left[1+\left(\frac{a}{R^2}+\frac{b}{R^4}+\cdots\right)\right]^{-3}\\
&= \frac{1}{R^3}\left[1-3\left(\frac{a}{R^2}+\cdots\right)+\cdots\right]\\
&= \frac{1}{R^3}-\frac{3a}{R^5}+\cdots\\
\frac{1}{x^5} &= \frac{1}{R^5}\left[1+\left(\frac{a}{R^2}+\frac{b}{R^4}+\cdots\right)\right]^{-5}\\
&= \frac{1}{R^5}+\cdots\\
&\ \vdots
\end{aligned}$$

So we have

$$\begin{aligned}
R+\frac{a}{R}+\frac{b}{R^3}+\frac{c}{R^5}+\cdots &= R-\left[\frac{1}{R}-\frac{a}{R^3}+\frac{a^2-b}{R^5}+\cdots\right]\\
&\qquad+\frac{1}{3}\left[\frac{1}{R^3}-\frac{3a}{R^5}+\cdots\right]-\frac{1}{5}\left[\frac{1}{R^5}+\cdots\right]+\cdots\\
&= R-\frac{1}{R}+\frac{a+\frac{1}{3}}{R^3}+\frac{-a^2+b-a-\frac{1}{5}}{R^5}+\cdots
\end{aligned}$$

and for consistency are obligated to set $a=-1$, therefore $b=-\frac{2}{3}$, therefore $c=-\frac{13}{15}$, therefore… Thus by elegant refinement of the argument that led us to (4) do we recover precisely (6).

Equation (2) can be written $x=\text{Arctan}(x)=n\pi+\arctan x$ or again

$$x-n\pi=\arctan x \tag{10}$$

which is plotted in Figure 5. Spanier & Oldham observe—and the figure makes clear—that the graphical technique standardly used to locate fixed points of iterative processes

$$x\mapsto f(x)\mapsto f(f(x))\mapsto\cdots$$

can by slight modification be used to construct x_n, and that (because the graph of $\arctan x$ is so flat) convergence is typically quite rapid. Since

$$n\pi<x_n<(n+\tfrac{1}{2})\pi$$

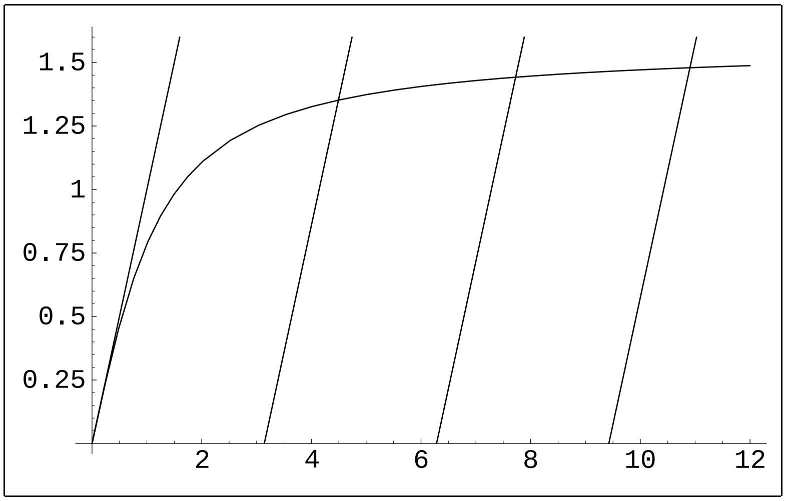

FIGURE 5: *Graphical representation of (10). The n^{th} rising line constitutes a graph of $y = x - n\pi$; it has unit slope, intercepts the x-axis at $n\pi$ and intercepts the curve at the point x_n; i.e., at the n^{th} root of (2).*

it proves convenient in 0^{th} approximation to set $X_0 \equiv x_n^{\text{seed}} = (n + \frac{1}{4})\pi$ and then to proceed

$$X_1 = \arctan X_0 + n\pi \mapsto X_2 = \arctan X_1 + n\pi \mapsto \cdots \mapsto X_N \approx x_n$$

In the illustrative case $n = 2$ we obtain

$$\begin{aligned} X_0 &= 7.0685835 \\ X_1 &= 7.7250569 \\ X_2 &= 7.7252486 \\ X_3 &= 7.7252517 = \text{first 8 digits of } x_2^{\text{exact}} \end{aligned}$$

The efficient computational algorithm just described proceeds from (10), while Stokes' analytical argument proceeded from (9). The equivalence of those equations follows from the observation[13] that

$$\arctan\left(\tfrac{1}{x}\right) = -\arctan x + \tfrac{\pi}{2}$$

4. Connections with other topics. The roots x_n of (2) enter not very mysteriously into the infinite product

$$J_{\frac{3}{2}}(x) = \frac{(x/2)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\left(1 - \frac{x^2}{x_1^2}\right)\left(1 - \frac{x^2}{x_2^2}\right)\left(1 - \frac{x^2}{x_3^2}\right)\cdots$$

[13] See Spanier & Oldham, p. 336.

More mysterious is the claim[14] that

$$\sum_{n=1}^{\infty}\frac{1}{x_n^2}=\frac{1}{10}$$
$$\sum_{n=1}^{\infty}\frac{1}{x_n^4}=\frac{1}{350}$$
$$\sum_{n=1}^{\infty}\frac{1}{x_n^6}=\frac{1}{7875}$$
$$\vdots$$

and that such sums occur in "certain problems." I cannot, at the moment, imagine such a problem, or how to prove such a result (but see below!). Spanier & Oldham remark finally that the numbers x_n occur in connection with the so-called "Langevin function," which arises in connection with the theory of dielectrics[15] and is defined

$$\mathcal{L}(x)\equiv\coth x-\frac{1}{x}$$

The connection they have in mind can be written

$$\frac{1}{\mathcal{L}(x)}=\frac{3}{x}+2x\sum_{n=1}^{\infty}\frac{1}{x^2+x_n^2}$$

or again

$$\sum_{n=1}^{\infty}\frac{1}{x^2+x_n^2}=\frac{1/\mathcal{L}(x)-3/x}{2x}$$

Mathematica, when asked to expand the expression on the right side of the preceding equation, responds

$$=\frac{1}{10}-\frac{1}{350}x^2+\frac{1}{7875}x^4-\frac{37}{6063750}x^6+\frac{59}{197071875}x^8-\cdots$$

This striking result establishes contact with—and at the same time serves to extend—the list of sum formulæ presented at the top of the page.

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[14] Spanier & Oldham, p. 325.
[15] See pp. 25–29 of R. Coelho, *Physics of Dielectrics for the Engineer* (1979).