Introduction to the

Theory of Tzitzeica Surfaces

Part 2: Initial Details

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Introduction. As is remarked in the Introduction to Part 1,[1] my interest in the theory of Tzitzeica surfaces was sparked by correspondence with Ahmed Sebbar, reinforced by correspondence with Nicoleta Bîlă[2] and has been informed mainly by Rogers & Schief.[3] In Part 1 I provide a record of my meandering, highly digressive approach to my intended subject; I look more specifically to

- Mathematics associated generally and specifically with the asymptotic parameterization of hyperbolic surfaces.
- Application of that material to the unit pseudosphere.
- An elaborate account of the general theory of surfaces of revolution.
- The theory of geodesics on surfaces of revolution.
- The construction of conformal (or "isothermal") coordinates on such surfaces.
- Basic elements of the theory of Liouville surfaces.

That obliquely preparatory work came to a halt when I encountered in Rogers & Schief the unsupported claim—taken by them to be common knowledge/ obvious—that the 2nd Fundamental Form for asymptotically parameterized hyperbolic surfaces gives rise to a symmetric matrices $\mathbb{H}(x, y)$ that invariably

[1] "Introduction to the theory of Tzitzeica surfaces. Part 1: Preparatory remarks," (April, 2016).

[2] See her "Symmetry groups and Lagrangians associated with Tzitzeica surfaces," Balkan Journal of Geometry and its Applications **10**, 73–91 (2005). Bîlă was a student of the Romanian differential geometer Consantin Udrişte (born in 1940, the year after Txitzeica died) who—now retired—contributed prolifically to the continuation of the Txitzeica school of differential geometry, so is herself a "great granddaughter," so to speak, of the founder of this subject.

[3] Chapter 3 in their *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory* (2002) is entitled "Tzitzeica Surfaces, Conjugate Nets and the Toda Lattice Scheme."

have 0s on the diagonal; *i.e.*, which are invariably of the form

$$\mathbb{H}(x,y) = \begin{pmatrix} 0 & f(x,y) \\ f(x,y) & 0 \end{pmatrix} \quad : \quad \{x,y\} \text{ asymptotic}$$

It is three essays later,[4] with the validity of that claim now established, that I return here to the theory of Tzitzeica surfaces.

General properties of asymptotically parameterized hyperbolic surfaces. Let $\boldsymbol{r}(u,v)$ refer to an arbitrarily parameterized surface Σ. The Gauss equations[5] read

$$\left.\begin{aligned} \boldsymbol{r}_{uu} &= \Gamma^1_{11}\boldsymbol{r}_u + \Gamma^2_{11}\boldsymbol{r}_v + e\boldsymbol{N} \\ \boldsymbol{r}_{uv} &= \Gamma^1_{12}\boldsymbol{r}_u + \Gamma^2_{12}\boldsymbol{r}_v + f\boldsymbol{N} \\ \boldsymbol{r}_{vv} &= \Gamma^1_{22}\boldsymbol{r}_u + \Gamma^2_{22}\boldsymbol{r}_v + g\boldsymbol{N} \end{aligned}\right\} \tag{1}$$

the Weingarten equations read

$$\left.\begin{aligned} \boldsymbol{N}_u &= \frac{fF - eG}{EG - F^2}\boldsymbol{r}_u + \frac{eF - fE}{EG - F^2}\boldsymbol{r}_v \\ \boldsymbol{N}_v &= \frac{gF - fG}{EG - F^2}\boldsymbol{r}_u + \frac{fF - gE}{EG - F^2}\boldsymbol{r}_v \end{aligned}\right\} \tag{2}$$

and the Mainardi-Codazzi (consistency/integrability) equations read

$$\left.\begin{aligned} e_v - f_u &= e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11} \\ f_v - g_u &= e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12} \end{aligned}\right\} \tag{3}$$

By way of specialization, assume Σ to be hyperbolic, and the parameterization to be asymptotic. Then[6] $e = g = 0$ and we have

$$\left.\begin{aligned} \boldsymbol{r}_{xx} &= \Gamma^1_{11}\boldsymbol{r}_x + \Gamma^2_{11}\boldsymbol{r}_y \\ \boldsymbol{r}_{xy} &= \Gamma^1_{12}\boldsymbol{r}_x + \Gamma^2_{12}\boldsymbol{r}_y + f\boldsymbol{N} \\ \boldsymbol{r}_{yy} &= \Gamma^1_{22}\boldsymbol{r}_x + \Gamma^2_{22}\boldsymbol{r}_y \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} \boldsymbol{N}_x &= \frac{fF}{EG - F^2}\boldsymbol{r}_x + \frac{-fE}{EG - F^2}\boldsymbol{r}_y \\ \boldsymbol{N}_y &= \frac{-fG}{EG - F^2}\boldsymbol{r}_x + \frac{fF}{EG - F^2}\boldsymbol{r}_y \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} -f_x &= f(\Gamma^2_{12} - \Gamma^1_{11}) \\ f_y &= f(\Gamma^2_{22} - \Gamma^1_{12}) \end{aligned}\right\} \tag{6}$$

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[4] "Biorthogonality—revisited, and a generalized spectral decomposition theorem," "Parameter transformations *vs.* basis transformations," "Asymptotic parameterization of the curvature matrix," all dated June, 2016.

[5] See "Differential geometry of some surfaces in 3-space," (December, 2015), pages 4–5.

[6] See the last of the essays[4] cited above.

The Christoffel symbols $\Gamma^i{}_{jk} = g^{im}\Gamma_{mjk} = \frac{1}{2}g^{im}\big(g_{mj,k} + g_{mk,j} - g_{jk,m}\big) = \Gamma^i{}_{kj}$ are assembled from the first derivatives and inverse of

$$\|g_{ij}\| = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

and can be described[7]

$$\left.\begin{aligned}
\Gamma^1{}_{11} &= g^{-1}\big\{\ \ \tfrac{1}{2}GE_x - FF_x + \tfrac{1}{2}FE_y\big\}\\
\Gamma^2{}_{11} &= g^{-1}\big\{-\tfrac{1}{2}FE_x + EF_x - \tfrac{1}{2}EE_y\big\}\\
\Gamma^1{}_{12} = \Gamma^1{}_{21} &= g^{-1}\big\{\ \ \tfrac{1}{2}GE_y - \tfrac{1}{2}FG_x\big\}\\
\Gamma^2{}_{12} = \Gamma^2{}_{21} &= g^{-1}\big\{-\tfrac{1}{2}FE_y + \tfrac{1}{2}EG_x\big\}\\
\Gamma^1{}_{22} &= g^{-1}\big\{-\tfrac{1}{2}FG_y + GF_y - \tfrac{1}{2}GG_x\big\}\\
\Gamma^2{}_{22} &= g^{-1}\big\{\ \ \tfrac{1}{2}EG_y - FF_y + \tfrac{1}{2}FG_x\big\}
\end{aligned}\right\} \tag{7}$$

It proves advantageous in the present context to work from a variant[8] of the Mainardi-Codazzi equations (6) that results from the following argument: We have the identity

$$f_x + f(\Gamma^2{}_{12} - \Gamma^1{}_{11}) = \sqrt{g}\Big(\frac{f}{\sqrt{g}}\Big)_x + \tfrac{1}{2}fg^{-1}g_x + f(\Gamma^2{}_{12} - \Gamma^1{}_{11})$$

But

$$\begin{aligned}
&\tfrac{1}{2}g^{-1}g_x + (\Gamma^2{}_{12} - \Gamma^1{}_{11})\\
&\qquad = g^{-1}\big\{\tfrac{1}{2}\big(EG_x + GE_x - 2FF_x\big) + \big(\tfrac{1}{2}EG_x - \tfrac{1}{2}GE_x + FF_x - FE_x\big)\big\}\\
&\qquad = g^{-1}\big\{-FE_y + EG_x\big\}\\
&\qquad = 2\Gamma^2{}_{12}
\end{aligned}$$

and similarly $\frac{1}{2}g^{-1}g_y - (\Gamma^2{}_{22} - \Gamma^1{}_{12}) = 2\Gamma^1{}_{12}$. So equations (6) have become

$$\left.\begin{aligned}
\Big(\frac{f}{\sqrt{g}}\Big)_x + 2\Gamma^2{}_{12}\frac{f}{\sqrt{g}} &= 0\\
\Big(\frac{f}{\sqrt{g}}\Big)_y + 2\Gamma^1{}_{12}\frac{f}{\sqrt{g}} &= 0
\end{aligned}\right\} \tag{8}$$

But the Gaussian curvature is

$$K = \frac{\det\mathbb{H}}{\det\mathbb{G}} = \frac{-f^2}{g} = -\Big(\frac{f}{\sqrt{g}}\Big)^2 \equiv -\frac{1}{\rho^2}$$

so (8) can be written

$$\begin{aligned}
\big[\log\big(\sqrt{-K}\,\big)\big]_x + 2\Gamma^2{}_{12} &= 0\\
\big[\log\big(\sqrt{-K}\,\big)\big]_y + 2\Gamma^1{}_{12} &= 0
\end{aligned}$$

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7 See page 12 in the first of the essays cited on the preceding page.
8 See equations (1.14) and (3.3) in Rogers & Schief.

or finally

$$\left.\begin{aligned}\Gamma^1{}_{12} &= -\tfrac{1}{4}\big[\log\big(-K\big)\big]_y\\ \Gamma^2{}_{12} &= -\tfrac{1}{4}\big[\log\big(-K\big)\big]_x\end{aligned}\right\} \tag{9}$$

These equations are quite general, in the sense that they pertain to *every* asymptotically parameterized hyperbolic surface.

The unit vector $\boldsymbol{N}$ that enters into the construction of the 2^{nd} Fundamental Form stands normal to Σ at the point $\boldsymbol{r}$, and normal therefore to the plane Π (spanned by $\{\boldsymbol{r}_u, \boldsymbol{r}_v\}$) that is tangent to Σ at $\boldsymbol{r}$. The dot product

$$d = \boldsymbol{N}\cdot\boldsymbol{r} \tag{10.1}$$

admits therefore of interpretation as the normal (or shortest) distance from the origin to Π. Generally, d is $\{u, v\}$-dependent, because $\boldsymbol{r}$ and $\boldsymbol{N}$ are.[9] One has

$$d_u = \boldsymbol{N}_u\cdot\boldsymbol{r} + \boldsymbol{N}\cdot\boldsymbol{r}_u = \boldsymbol{N}_u\cdot\boldsymbol{r} \tag{10.2}$$

$$d_v = \boldsymbol{N}_v\cdot\boldsymbol{r} + \boldsymbol{N}\cdot\boldsymbol{r}_v = \boldsymbol{N}_v\cdot\boldsymbol{r} \tag{10.3}$$

by $\boldsymbol{N} \perp \boldsymbol{r}_{u,v}$. Equations (10) quantify the projections of $\boldsymbol{r}$ onto the linearly independent (but generally non-orthogonal) vectors ($\boldsymbol{N}$, $\boldsymbol{N}_u$, $\boldsymbol{N}_v$).

At this point, Rogers & Schief assert (their page 90) that "use of the Weingarten equations yields

$$\boldsymbol{r} = -(d_y/f)\,\boldsymbol{r}_x - (d_x/f)\,\boldsymbol{r}_y + d\boldsymbol{N}\text{ ''} \tag{11}$$

I indicate now how that result can be obtained. Since $\{\boldsymbol{N}_x, \boldsymbol{N}_y\}$ (ditto $\{\boldsymbol{r}_x, \boldsymbol{r}_y\}$) are generally non-orthogonal we are obliged to make use of devices borrowed from the theory of biorthogonality,[10] which supplies

$$\boldsymbol{r} = \boldsymbol{N}_x(\boldsymbol{X}\cdot\boldsymbol{r}) + \boldsymbol{N}_y(\boldsymbol{Y}\cdot\boldsymbol{r}) + \boldsymbol{n}(\boldsymbol{Z}\cdot\boldsymbol{r})$$

where $\{\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}\}$ is the non-orthonormal "dual" of the non-orthonormal basis $\{\boldsymbol{N}_x, \boldsymbol{N}_y,\ \boldsymbol{n} = \boldsymbol{r}_x \times \boldsymbol{r}_y\}$. In constructing the elements of the dual basis we will make heavy use of the following elementary vector identities:

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a} &= (\boldsymbol{a}\cdot\boldsymbol{a})\,\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{a}\\ \boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{b}) &= (\boldsymbol{b}\cdot\boldsymbol{b})\,\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{b}\\ [(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a}]\cdot\boldsymbol{b} = [\boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{b})]\cdot\boldsymbol{a} &= (\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b}) - (\boldsymbol{a}\cdot\boldsymbol{b})^2 \geqslant 0\\ (\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{a}\times\boldsymbol{b}) &= (\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b}) - (\boldsymbol{a}\cdot\boldsymbol{b})^2\end{aligned}$$

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[9] My notation is intended to indicate that the parameterization is arbitrary; it need not be asymptotic. It is, at the moment, not even required that Σ be hyperbolic. But in the next paragraph we reinstate the critical assumption that the parameterization is asymptotic.

[10] See "Biorthonality—Revisited," (June, 2016).

Look first to the construction of $\boldsymbol{Z}$. We know from the Weingarten equations (5) that $\boldsymbol{N}_x$ and $\boldsymbol{N}_y$ lie in the tangent plane, so have $\boldsymbol{n}\perp\{\boldsymbol{N}_x,\boldsymbol{N}_y\}$: $\boldsymbol{n}$ differs from its dual only by a factor. From

$$\boldsymbol{n}\cdot\boldsymbol{n}=(\boldsymbol{r}_x\cdot\boldsymbol{r}_x)(\boldsymbol{r}_y\cdot\boldsymbol{r}_y)-(\boldsymbol{r}_x\cdot\boldsymbol{r}_y)^2=EG-F^2=\det\mathbb{G}\equiv g$$

we achieve $(\boldsymbol{Z}\cdot\boldsymbol{n})=1$ by setting

$$\boldsymbol{Z}=g^{-1}\boldsymbol{n} \tag{12.1}$$

What we have here established is that the unit vector $\boldsymbol{N}=g^{-\frac{1}{2}}\boldsymbol{n}$ is its own dual. Look next to the construction of $\boldsymbol{X}$. We have $\boldsymbol{x}\equiv\boldsymbol{n}\times\boldsymbol{N}_y\perp\{\boldsymbol{n},\boldsymbol{N}_y\}$, and drawing on (5) obtain

$$\begin{aligned}\boldsymbol{x}&=fg^{-1}(\boldsymbol{r}_x\times\boldsymbol{r}_y)\times(-G\boldsymbol{r}_x+F\boldsymbol{r}_y)\\&=-fg^{-1}G\big[(\boldsymbol{r}_x\cdot\boldsymbol{r}_x)\boldsymbol{r}_y-(\boldsymbol{r}_x\cdot\boldsymbol{r}_y)\boldsymbol{r}_x\big]-fg^{-1}F\big[(\boldsymbol{r}_y\cdot\boldsymbol{r}_y)\boldsymbol{r}_x-(\boldsymbol{r}_x\cdot\boldsymbol{r}_y)\boldsymbol{r}_y\big]\\&=-fg^{-1}G\big[E\boldsymbol{r}_y-F\boldsymbol{r}_x\big]-fg^{-1}F\big[G\boldsymbol{r}_x-F\boldsymbol{r}_y\big]\\&=-fg^{-1}(EG-F^2)\boldsymbol{r}_y\\&=-f\boldsymbol{r}_y\end{aligned}$$

So

$$(\boldsymbol{x}\cdot\boldsymbol{N}_x)=(-f\boldsymbol{r}_y)\cdot\big[fg^{-1}(F\boldsymbol{r}_x-E\boldsymbol{r}_y)\big]=-f^2g^{-1}(FF-EG)=f^2$$

and to achieve $(\boldsymbol{X}\cdot\boldsymbol{N}_x)=1$ must set

$$\boldsymbol{X}=-f^{-1}\boldsymbol{r}_y \tag{12.2}$$

A similar argument gives

$$\boldsymbol{Y}=-f^{-1}\boldsymbol{r}_x \tag{12.3}$$

So we have

$$\boldsymbol{r}=\boldsymbol{N}_x(\boldsymbol{X}\cdot\boldsymbol{r})+\boldsymbol{N}_y(\boldsymbol{Y}\cdot\boldsymbol{r})+\boldsymbol{N}(\boldsymbol{N}\cdot\boldsymbol{r})$$

and its dual

$$\begin{aligned}\boldsymbol{r}&=\boldsymbol{X}(\boldsymbol{N}_x\cdot\boldsymbol{r})+\boldsymbol{Y}(\boldsymbol{N}_y\cdot\boldsymbol{r})+\boldsymbol{N}(\boldsymbol{N}\cdot\boldsymbol{r})\\&=-d_xf^{-1}\boldsymbol{r}_y-d_yf^{-1}\boldsymbol{r}_x+d\boldsymbol{N}\end{aligned}$$

which is precisely the result (11) asserted by Rogers & Schief.

Bringing to the Gauss equation

$$\boldsymbol{r}_{xy}=\Gamma^1{}_{12}\boldsymbol{r}_x+\Gamma^2{}_{12}\boldsymbol{r}_y+f\boldsymbol{N} \tag{4$_2$}$$

the descriptions of $\Gamma^1{}_{12}$ and $\Gamma^2{}_{12}$ provided at (9), we have

$$\boldsymbol{r}_{xy}=-\tfrac{1}{4}\big[\log(-K)\big]_y\boldsymbol{r}_x-\tfrac{1}{4}\big[\log(-K)\big]_x\boldsymbol{r}_y+f\boldsymbol{N}$$

But by (11)

$$\begin{aligned}f\boldsymbol{N}&=(f/d)\boldsymbol{r}+(d_x/d)\boldsymbol{r}_y+(d_y/d)\boldsymbol{r}_x\\&=(f/d)\boldsymbol{r}+\tfrac{1}{4}\big[\log d^4\big]_y\boldsymbol{r}_x+\tfrac{1}{4}\big[\log d^4\big]_x\boldsymbol{r}_y\end{aligned}$$

so

$$\boldsymbol{r}_{xy} - (f/d)\boldsymbol{r} = -\tfrac{1}{4}\big[\log(-K/d^4)\big]_y \boldsymbol{r}_x - \tfrac{1}{4}\big[\log(-K/d^4)\big]_x \boldsymbol{r}_y \tag{13}$$

This result pertains to *every* asymptotically parameterized hyperbolic surface.

Tzitzeica's fundamental assumption, and some consequences. Tzitzeica surfaces arise from setting

$$\boldsymbol{r}_{xy} - (f/d)\boldsymbol{r} = 0 \tag{14.1}$$

which by (13) entails

$$-K/d^4 = \text{constant} = c^2 > 0 \tag{14.2}$$

Let (14.1) be abbreviated

$$\boldsymbol{r}_{xy} = h\boldsymbol{r} \quad \text{with} \quad h = f/d \tag{15}$$

It remains to expoit the information supplied by the remaining Gauss equations $(4)_{1,3}$, which by (12) can be written

$$\begin{aligned}\boldsymbol{r}_{xx} &= -f\Gamma^1{}_{11}\boldsymbol{Y} - f\Gamma^2{}_{11}\boldsymbol{X}\\ \boldsymbol{r}_{yy} &= -f\Gamma^1{}_{22}\boldsymbol{Y} - f\Gamma^2{}_{22}\boldsymbol{X}\end{aligned}$$

and by the biorthogonality of $\{\boldsymbol{X},\boldsymbol{Y}\}$ and $\{\boldsymbol{N}_x,\boldsymbol{N}_y\}$ give

$$\left.\begin{aligned}\Gamma^1{}_{11} &= -f^{-1}\boldsymbol{r}_{xx}\cdot\boldsymbol{N}_y \qquad & \Gamma^2{}_{11} &= -f^{-1}\boldsymbol{r}_{xx}\cdot\boldsymbol{N}_x\\ \Gamma^1{}_{22} &= -f^{-1}\boldsymbol{r}_{yy}\cdot\boldsymbol{N}_y \qquad & \Gamma^2{}_{22} &= -f^{-1}\boldsymbol{r}_{yy}\cdot\boldsymbol{N}_x\end{aligned}\right\} \tag{16}$$

from which we must press the juice. Differentiation of $\boldsymbol{r}_{xx}\cdot\boldsymbol{N} = e = 0$ gives $\boldsymbol{r}_{xx}\cdot\boldsymbol{N}_y + \boldsymbol{r}_{xxy}\cdot\boldsymbol{N} = \boldsymbol{r}_{xx}\cdot\boldsymbol{N}_y + (h\boldsymbol{r})_x\cdot\boldsymbol{N} = 0$ whence

$$\begin{aligned}\Gamma^1{}_{11} &= f^{-1}(h\boldsymbol{r})_x\cdot\boldsymbol{N}\\ &= f^{-1}(h_x d + h\boldsymbol{r}_x\cdot\boldsymbol{N}) \quad : \quad f^{-1} = 1/hd, \quad \boldsymbol{r}_x \perp \boldsymbol{N}\\ &= h_x/h\\ &= (\log h)_x\end{aligned} \tag{17.1}$$

Arguing similarly from $\boldsymbol{r}_{yy}\cdot\boldsymbol{N} = g = 0$ we have

$$\begin{aligned}\Gamma^2{}_{22} &= h_y/h\\ &= (\log h)_y\end{aligned} \tag{17.2}$$

Rogers & Schief assert also (without proof) that

$$\Gamma^2{}_{11} = \frac{a(x)}{h} \qquad\qquad \Gamma^1{}_{22} = \frac{b(y)}{h} \tag{17.3}$$

which entail $(h\Gamma^2{}_{11})_y = (h\Gamma^1{}_{22})_x$ and by (16) become

$$\left(\frac{\boldsymbol{r}_{yy}\cdot\boldsymbol{N}_y}{d}\right)_y = 0 \qquad\qquad \left(\frac{\boldsymbol{r}_{xx}\cdot\boldsymbol{N}_x}{d}\right)_x = 0 \tag{17$\star$}$$

But these are equations that I have been thus far unable to verify, unable to recover from the available material.[11] I proceed on the essential assumption that equations (17.3) are correct... to see where it takes me and in the hope that the light may someday dawn.

Returning with the fundamental assumption (15) and equations (17)—which descend from (15) and the Weingarten equations—to the (4), we find that the Gauss equations for asymptotically parameterized Tzitzeica surfaces can be written

$$\boldsymbol{r}_{xx} = (h_x/h)\boldsymbol{r}_x + (a(x)/h)\boldsymbol{r}_y \tag{18.1}$$
$$\boldsymbol{r}_{xy} = h\boldsymbol{r} \tag{18.2}$$
$$\boldsymbol{r}_{yy} = (h_y/h)\boldsymbol{r}_y + (b(y)/h)\boldsymbol{r}_x \tag{18.3}$$

The first pair of equations supplies

$$\begin{aligned}\boldsymbol{r}_{xxy} &= (h_x/h)_y\boldsymbol{r}_x + (h_x/h)h\boldsymbol{r} + (a/h)_y\boldsymbol{r}_y + (a/h)\boldsymbol{r}_{yy}\\ &= (h_x/h)_y\boldsymbol{r}_x + (h_x/h)h\boldsymbol{r} + (a/h)_y\boldsymbol{r}_y + (a/h)[(h_y/h)\boldsymbol{r}_y + (b/h)\boldsymbol{r}_x]\\ &= [(h_x/h)_y + ab/h^2]\boldsymbol{r}_x + h_x\boldsymbol{r} + a[(1/h)_y + h_y/h^2]\boldsymbol{r}_y\\ \boldsymbol{r}_{xyx} &= h_x\boldsymbol{r} + h\boldsymbol{r}_x\end{aligned}$$

which are consistent if and only if

$$[(h_x/h)_y + ab/h^2] = h \tag{19.11}$$
$$[(1/h)_y + h_y/h^2] = 0 \tag{19.21}$$

A similar argument shows the second pair of equations to be consistent if and only if

$$[(h_y/h)_x + ab/h^2] = h \tag{19.12}$$
$$[(1/h)_x + h_x/h^2] = 0 \tag{19.22}$$

Equations (19.2) are valid as elementary identities, so impose no constraint upon h. Equations (19.1) are seen by

$$\left.\begin{aligned}(h_x/h)_y &= \frac{hh_{xy} - h_xh_y}{h^2}\\ (h_y/h)_x &= \frac{hh_{yx} - h_yh_x}{h^2}\end{aligned}\right\} = (\log h)_{xy}$$

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[11] Which I take to be

$$\boldsymbol{r}_x\cdot\boldsymbol{N} = \boldsymbol{r}_y\cdot\boldsymbol{N} = 0 \quad : \quad \text{normality of } \boldsymbol{N}$$
$$e = \boldsymbol{r}_{xx}\cdot\boldsymbol{N} = 0,\ g = \boldsymbol{r}_{yy}\cdot\boldsymbol{N} = 0 \quad : \quad \text{structure of } \mathbb{H}$$
$$\left.\begin{aligned}\boldsymbol{r}_x\cdot\boldsymbol{N}_x = \boldsymbol{r}_y\cdot\boldsymbol{N}_y &= 0\\ \boldsymbol{r}_x\cdot\boldsymbol{N}_y = \boldsymbol{r}_y\cdot\boldsymbol{N}_x &= -f\end{aligned}\right\} \quad : \quad \text{biorthogonalty}$$

together with the assumed relation (15).

to be identical, and lead to the conclusion that the Gauss equations (18) are integrable if and only if $h(x, y)$ is a solution of the non-linear partial differential equation

$$(\log h)_{xy} = h - a(x)b(y)/h^2 \tag{20}$$

Optimally regauged asymptotic parameters. Given the $\{x, y\}$-parameterized description $\boldsymbol{r}(x, y)$ of a surface Σ, the equations

$$x = \text{constant}, \quad y = \text{constant}$$

inscribe curves $\mathcal{C}_x$ and $\mathcal{C}_y$ on Σ. When the parameters are rescaled

$$x = x(u), \quad y = y(v)$$

the members of the populations $\{\mathcal{C}_x\}$, $\{\mathcal{C}_y\}$ acquire new names $\{\mathcal{C}_u\}$, $\{\mathcal{C}_v\}$ but the populations themselves remain unchanged. All of which pertains to arbitrary parameterizations, and to asymptotic parameterizations in particular. It has been established elsewhere[12] that

$$\begin{aligned}
\mathbb{G}(u, v) &= \mathbb{J}^{\mathsf{T}}\mathbb{G}(x, y)\,\mathbb{J}\Big|_{x\to x(u), y\to y(v)}\\
\mathbb{H}(u, v) &= \mathbb{J}^{\mathsf{T}}\mathbb{H}(x, y)\,\mathbb{J}\Big|_{x\to x(u), y\to y(v)}\\
\mathbb{J} &= \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} x_u & 0 \\ 0 & y_v \end{pmatrix}
\end{aligned}$$

In the asymptotic case we therefore have

$$\begin{pmatrix} 0 & \bar{f}(u, v) \\ \bar{f}(u, v) & 0 \end{pmatrix} = \begin{pmatrix} 0 & f(x, y)x_u y_v \\ f(x, y)x_u y_v & 0 \end{pmatrix}_{x\to x(u), y\to y(v)}$$

The parameter $d = \boldsymbol{r}\cdot\boldsymbol{N}$ is intrinsic (parameterization-independent); $h \equiv f/d$ therefore transforms like f:

$$\bar{h}(u, v) = h(x, y)x_u y_v\Big|_{x\to x(u), y\to y(v)}$$

Thus prepared, we look to the response of the Gauss equations (18) to such rescaling. Immediately

$$\boldsymbol{r}_{uv} = \boldsymbol{r}_{xy}x_u y_v = h x_u y_v\,\boldsymbol{r} = \bar{h}\boldsymbol{r}$$

A more elaborate argument gives

$$\begin{aligned}
\boldsymbol{r}_{uu} &= \boldsymbol{r}_{xx}x_u x_u\\
&= \{(h_x/h)\boldsymbol{r}_x + (a(x)/h)\boldsymbol{r}_y\}x_u x_u\\
&= \frac{(\bar{h}u_x v_y)_u u_x}{\bar{h}u_x v_y}(\boldsymbol{r}_u u_x)x_u x_u + \frac{a(x)}{\bar{h}u_x v_y}(\boldsymbol{r}_v v_y)x_u x_u
\end{aligned}$$

[12] "Parameter transformations *vs.* basis transformations," (June, 2016).

But

$$\frac{(\bar{h}u_x v_y)_u u_x}{\bar{h}u_x v_y}(\boldsymbol{r}_u u_x)x_u x_u = (\bar{h}_u/\bar{h})\boldsymbol{r}_u + \frac{(u_x v_y)_u}{u_x v_y}\boldsymbol{r}_u$$
$$= (\bar{h}_u/\bar{h})\boldsymbol{r}_u \quad \text{by} \quad (u_x v_y)_u = v_y(u_u)_x = 0$$

and

$$\frac{a(x)}{\bar{h}u_x v_y}(\boldsymbol{r}_v v_y)x_u x_u = (\bar{a}(u)/\bar{h})\boldsymbol{r}_v$$
$$\bar{a}(u) \equiv a(x(u)) \cdot x_u(u)^3$$

So we have the rescaled equations

$$\begin{aligned} \boldsymbol{r}_{uu} &= (\bar{h}_x/\bar{h})\boldsymbol{r}_u + (\bar{a}(u)/\bar{h})\boldsymbol{r}_v \\ \boldsymbol{r}_{uv} &= \bar{h}\boldsymbol{r} \\ \boldsymbol{r}_{vv} &= (\bar{h}_y/\bar{h})\boldsymbol{r}_v + (\bar{b}(v)/\bar{h})\boldsymbol{r}_u \end{aligned}$$

and are free to adopt a scaling $x \to u(x), y \to v(y)$ such that

$$\bar{a}(u) = \lambda, \quad \bar{b}(v) = \lambda^{-1} \quad : \quad \lambda \text{ any non-zero constant}$$

This done, the asymptotically parameterized Gauss equations (18) assume the "canonical" form[13]

$$\left.\begin{aligned} \boldsymbol{r}_{xx} &= (h_x/h)\boldsymbol{r}_x + (\lambda/h)\boldsymbol{r}_y \\ \boldsymbol{r}_{xy} &= h\boldsymbol{r} \\ \boldsymbol{r}_{yy} &= (h_y/h)\boldsymbol{r}_y + (\lambda^{-1}/h)\boldsymbol{r}_x \end{aligned}\right\} \tag{20}$$

The associated integrability condition reads (for all λ)

$$(\log h)_{xy} = h - \frac{1}{h^2} \tag{21}$$

This nonlinear partial differential equation plays within the theory of Tzitzeica surfaces precisely the role played by the nonlinear sine-Gordon equation within the theory of pseudospheric surfaces.

The dual of a Tzitzeica surface. Let

$$\boldsymbol{s} = -\frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{h}, \text{ which is parallel to } \boldsymbol{N} \tag{22}$$

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[13] Here I reinstate $\{x, y\}$ as the names of the asymptotic parameters. Rogers & Schief remark that equations (20) provide an instance of a "Lax triad."

Rogers & Schief assert that if $\boldsymbol{r}(x,y)$ satisfies the triad (20) then $\boldsymbol{s}(x,y)$ satisfies the "dual" or "adjoint" triad that results from reversing the sign of λ. We look to the demonstration. We by (22) have

$$\begin{aligned}\boldsymbol{s}_x &= \frac{-h(\boldsymbol{r}_x\times\boldsymbol{r}_y)_x + h_x(\boldsymbol{r}_x\times\boldsymbol{r}_y)}{h^2}\\ &= \frac{-h[h_x h^{-1}\boldsymbol{r}_x + \lambda h^{-1}\boldsymbol{r}_y]\times\boldsymbol{r}_y - h\boldsymbol{r}_x\times[h\boldsymbol{r}] + h_x(\boldsymbol{r}_x\times\boldsymbol{r}_y)}{h^2}\\ &= \frac{hh_x\boldsymbol{s} - \lambda\boldsymbol{0} - h^2\boldsymbol{r}_x\times\boldsymbol{r} + h_x[-h\boldsymbol{s}]}{h^2}\\ &= -\boldsymbol{r}_x\times\boldsymbol{r} \end{aligned}\tag{23.1}$$

and by a similar argument

$$\boldsymbol{s}_y = -\boldsymbol{r}\times\boldsymbol{r}_y \tag{23.2}$$

Therefore

$$\begin{aligned}\boldsymbol{s}_{xx} &= -\boldsymbol{r}_{xx}\times\boldsymbol{r} - \boldsymbol{r}_x\times\boldsymbol{r}_x\\ &= -[h_x h^{-1}\boldsymbol{r}_x + \lambda h^{-1}\boldsymbol{r}_y]\times\boldsymbol{r} - \boldsymbol{0}\\ &= h_x h^{-1}\boldsymbol{s}_x - \lambda h^{-1}\boldsymbol{s}_y \qquad (24.1)\\ \boldsymbol{s}_{xy} &= -\boldsymbol{r}_{xy}\times\boldsymbol{r} - \boldsymbol{r}_x\times\boldsymbol{r}_y\\ &= -h\boldsymbol{r}\times\boldsymbol{r} + h\boldsymbol{s}\\ &= h\boldsymbol{s} \qquad (24.2)\\ \boldsymbol{s}_{yy} &= -\boldsymbol{r}_y\times\boldsymbol{r}_y - \boldsymbol{r}\times\boldsymbol{r}_{yy}\\ &= -\boldsymbol{0} - \boldsymbol{r}\times[h_y h^{-1}\boldsymbol{r}_y + \lambda^{-1}h^{-1}\boldsymbol{r}_x]\\ &= h_y h^{-1}\boldsymbol{s}_y - \lambda^{-1}h^{-1}\boldsymbol{s}_x \qquad (24.3)\end{aligned}$$

The argument is elementary, but provides no insight into why it is "obvious/ inevitable" that the substitutions

$$\begin{aligned}\boldsymbol{r} &\rightarrow \boldsymbol{s} = (\boldsymbol{r}_y\times\boldsymbol{r}_x)/h\\ \boldsymbol{r}_x &\rightarrow \boldsymbol{s}_x = \boldsymbol{r}\times\boldsymbol{r}_x\\ \boldsymbol{r}_y &\rightarrow \boldsymbol{s}_y = \boldsymbol{r}_y\times\boldsymbol{r}\end{aligned}$$

send the description $\boldsymbol{r}(x,y)$ of one Tzitzeica surface Σ over into the description $\boldsymbol{s}(x,y)$ of another, Σ^{dual}. Rogers & Schief remark that equations (23) are called "Lelieuvre formulae,"[14] and were noted already by Jonas. We observe finally that

$$\begin{aligned}(\boldsymbol{r}\cdot\boldsymbol{s})_x &= \boldsymbol{r}_x\cdot\boldsymbol{s} + \boldsymbol{r}\cdot\boldsymbol{s}_x\\ &= -h^{-1}\boldsymbol{r}_x\cdot(\boldsymbol{r}_x\times\boldsymbol{r}_y) + \boldsymbol{r}\cdot(\boldsymbol{r}\times\boldsymbol{r}_x)\\ &= 0\\ (\boldsymbol{r}\cdot\boldsymbol{s})_y &= 0\end{aligned}$$

[14] See L. P. Eisenhart, *Differential Geometry of Curves & Surfaces* (1909), §79, pages 193–195; §172, pages 417–420.

by a familiar property of the triple scalar product:

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \text{ unless } \{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\} \text{ are linearly independent}$$

So we have

$$\boldsymbol{r}\cdot\boldsymbol{s} = \text{constant} \tag{25}$$

Example: the unit hexenhut. It is one thing to establish the integrability of the Gauss equations (20) and quite another—given that $h(x,y)$ is a solution of (21)—actually to integrate them: that is an assignment that we are not yet in position to undertake. We do, however, have in hand one certified Tzitzeica surface: Jonas' *hexenhut*, which I have seen referred to (on grounds that I have yet to establish, but of which the plausibility will soon emerge) as the "simplest" Tzitzeica surface. My object here will be to indicate what the general results developed above have to say in this particular case.

Sebbar's cubic surface $x^3+y^3+z^3-3xyz=1$ is seen when plotted to be coaxial with the vector $(1,1,1)$. The rotation that brings that vector to the vector $(0,0,\sqrt{3}\,)$ brings Sebbar's cubic to the form[15]

$$z(x^2+y^2)=\alpha^2 \quad : \quad \alpha^2 = \tfrac{2}{3\sqrt{3}}$$

which clearly describes a surface of revolution, of which

$$\boldsymbol{r}(u,v) = \begin{pmatrix} r(u)\cos v \\ r(u)\sin v \\ u \end{pmatrix} \quad : \quad r(u)=\alpha/\sqrt{u}$$

provides a natural parameterization. The differential geometry latent in $\boldsymbol{r}(u,v)$, particularly as it relates to the Tzitzeica property of that surface, has been discussed in several previous essays.[16] Here I look to the "unit hexenhut," that results from setting $\alpha=1$:

$$\boldsymbol{r}(u,v) = \begin{pmatrix} u^{-\frac{1}{2}}\cos v \\ u^{-\frac{1}{2}}\sin v \\ u \end{pmatrix} \tag{26}$$

For the most part, the theory of Tzitzeica surfaces presumes asymptotic parameterization, but the *intrinsic* properties of surfaces Σ (such, for example, as the *numerical values* assumed by $\{K,d\}$ at a specified point P, as opposed to the functions that describe them) must be parameterization-independent. We are free, therefore, to use the $\{u,v\}$-parameterization (26) to demonstrate

[15] See "Differential geometry of some surfaces in 3-space," (December 2015), page 21.

[16] See, for example, pages 2–4 in "Geodesics on the hexenhut," (January, 2016).

that the unit hexenhut Σ possesses the intrinsic property (14.2), the defining feature of Tzitzeica surfaces:

$$-K/d^4 = \text{constant, the same for all points } P \text{ on } \Sigma$$

Quick (*Mathematica*-assisted) calculation supplies

$$\begin{aligned}
g_{11}(u,v) &= E = 1 + \frac{1}{4u^3}\\
g_{12}(u,v) = g_{21}(u,v) &= F = 0\\
g_{21}(u,v) &= G = \frac{1}{u}
\end{aligned}$$

$$\boldsymbol{N}(u,v) = -\frac{1}{\sqrt{1+4u^3}}\begin{pmatrix} 2u^{\frac{2}{3}}\cos v\\ 2u^{\frac{2}{3}}\sin v\\ 1\end{pmatrix}$$

$$\begin{aligned}
h_{11}(u,v) &= e = -\frac{3}{2u\sqrt{1+4u^3}}\\
h_{12}(u,v) = h_{21}(u,v) &= f = 0\\
h_{22}(u,v) &= g = +\frac{2u}{\sqrt{1+4u^3}}
\end{aligned}$$

$$\det\mathbb{G}(u,v) = \frac{1+4u^3}{4u^4}$$

$$\det\mathbb{H}(u,v) = -\frac{3}{1+4u^2}$$

$$K(u,v) = \frac{\det\mathbb{H}(u,v)}{\det\mathbb{G}(u,v)} = -\frac{12u^4}{(1+4u^3)^2}$$

$$d(u,v) = \boldsymbol{r}(u,v)\cdot\boldsymbol{N}(u,v) = -\frac{3u}{\sqrt{1+4u^3}}$$

whence finally

$$-K/d^4 = \frac{12}{3^4} = \frac{4}{27} \equiv c^2 \quad : \quad \text{all } u, v$$

Curiously, $c^2 = \frac{4}{27} = \big(\frac{2}{3\sqrt{3}}\big)^2 = \alpha^4$. That $\mathbb{G}(u,v)$ and $\mathbb{H}(u,v)$ are both diagonal can come as no surprise, for it was shown at (14) in Part 1 that $F = f = 0$ is a property of *every* naturally parameterized surface of revolution. The point acquires interest from the established fact[17] that for every *asymptotically* parameterized hyperbolic surface —whether or not it be a surface of revolution —$\mathbb{H}(x,y)$ is *anti*diagonal.

We look now to the asymptotic parameterization of the unit hexenhut.[18] From

$$\boldsymbol{du}\cdot\mathbb{H}(u,v)\boldsymbol{du} = e(du)^2 + g(dv)^2 = 0$$

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[17] See "Asymptotic parameterization of the curvature matrix," (June, 2016).
[18] See Part 1, pages 6 & 9; "Geodesics on the hexenhut," (January 2016), page 3 and the essay cited there.

we find that the functions $v(u)$ that describe $\{u,v\}$-parameterized asymptotic curves on the unit hexenhut satisfy

$$\tfrac{dv}{du} = \pm\sqrt{-e/g} = \pm\beta\, u^{-1} \quad : \quad \beta = \tfrac{\sqrt{3}}{2}$$

which gives

$$v(u) = \pm\beta\log(u/u_0)$$

Which is to say: we have

$$v + \beta\log u = x \tag{27.1}$$

$$v - \beta\log u = y \tag{27.2}$$

where (27.1) describes asymptotic curves on one helicity, (27.2) describes curves of the opposite helicity, and where $\{x,y\}$ are the "asymptotic coordinates" that serve to give names to such curves; *i.e.*, to distinguish each from all others. Inversely,

$$\left.\begin{aligned} u &= \exp\Big(\frac{x-y}{2\beta}\Big) \\ v &= \frac{x+y}{2} \end{aligned}\right. \tag{28}$$

which when introduced into (26) give this asymptotic parameterization of the unit hexenhut:

$$\boldsymbol{r} = \begin{pmatrix} \exp\frac{y-x}{4\beta}\cos\frac{x+y}{2} \\ \exp\frac{y-x}{4\beta}\sin\frac{x+y}{2} \\ \exp\frac{x-y}{2\beta} \end{pmatrix} \tag{29}$$

Working from (29), we (with the substantial assistance of *Mathematica*) find

$$\left.\begin{aligned} g_{11} = g_{22} &= \frac{(4\beta^2+1)\exp\frac{y-x}{2\beta} + 4\exp\frac{x-y}{\beta}}{16\beta^2} \\ g_{12} = g_{21} &= \frac{(4\beta^2-1)\exp\frac{y-x}{2\beta} - 4\exp\frac{x-y}{\beta}}{16\beta^2} \end{aligned}\right. \tag{30.1}$$

so $\mathbb{G}(x,y)$—which could alternatively have been obtained from

$$\mathbb{G}(x,y) = \mathbb{J}^{\mathsf{T}}\mathbb{G}(u,v)\mathbb{J}\Big|_{u\to u(x,y),\,v\to v(x,y)} \tag{30.2}$$

$$\mathbb{J} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \frac{1}{2\beta}\exp(\frac{x-y}{2\beta}) & -\frac{1}{2\beta}\exp(\frac{x-y}{2\beta}) \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

—does not share the diagonal structure of $\mathbb{G}(u,v)$. From (30) we get

$$\begin{aligned} \det\mathbb{G}(x,y) &= (g_{11})^2 - (g_{12})^2 \\ &= \det\mathbb{G}(u,v)\cdot(\det\mathbb{J})^2 \\ &= \frac{4\exp(\frac{x-y}{2\beta}) + \exp(\frac{y-x}{\beta})}{16\beta^2} \\ &= \tfrac{1}{12}\big[4\exp(\tfrac{x-y}{2\beta}) + \exp(\tfrac{y-x}{\beta})\big] \end{aligned} \tag{31}$$

Working similarly from

$$\mathbb{H}(x,y) = \mathbb{J}^{\intercal}\mathbb{H}(u,v)\,\mathbb{J}\Big|_{u\to u(x,y),v\to v(x,y)}$$

we find

$$\begin{aligned} h_{11} = h_{22} &= \Big[8\beta^2\sqrt{\exp(\tfrac{y-x}{\beta}) + 4\exp(\tfrac{x-y}{2\beta})}\,\Big]^{-1}(4\beta^2 - 3) \\ &= 0 \quad \text{by } \beta = \tfrac{1}{2}\sqrt{3} \\ h_{12} = h_{21} &= \Big[8\beta^2\sqrt{\exp(\tfrac{y-x}{\beta}) + 4\exp(\tfrac{x-y}{2\beta})}\,\Big]^{-1}(4\beta^2 + 3) \end{aligned} \tag{32}$$

so $\mathbb{H}(x,y)$, which does not share the diagonal structure of $\mathbb{H}(u,v)$, has in fact the *anti*diagonal structure that we know[17] to be characteristic of *all* asymptotically parameteried hyperbolic surfaces. From (32) we have

$$\begin{aligned} \det\mathbb{H}(x,y) &= -(h_{12})^2 \\ &= \det\mathbb{H}(u,v)\cdot(\det\mathbb{J})^2 \end{aligned}$$

and so are led respectively to results

$$\det\mathbb{H}(x,y) = -\big[4\exp(\tfrac{x-y}{2\beta}) + \exp(\tfrac{y-x}{\beta})\big]^{-1}\cdot\begin{cases}(4\beta^2+3)/64\beta^4\\ 3/4\beta^2\end{cases}$$

that are identical by $\beta = \frac{1}{2}\sqrt{3}$, and give simply

$$\det\mathbb{H}(x,y) = -\big[4\exp(\tfrac{x-y}{2\beta}) + \exp(\tfrac{y-x}{\beta})\big]^{-1} \tag{33}$$

From (31) and (33) we obtain

$$K(x,y) = \frac{\det\mathbb{H}(x,y)}{\det\mathbb{G}(x,y)} = \frac{12}{\big[4\exp(\tfrac{x-y}{2\beta}) + \exp(\tfrac{y-x}{\beta})\big]^2} \tag{34}$$

The numerical value of the Gaussian curvature at any point P on Σ is intrinsic, the same whatever parameters are used to address that point, which is to say: curvature transforms as a scalar, in which connection we are gratified to observe that

$$= K(u,v)\Big|_{u\to u(x,y),\; v\to v(x,y)}$$

A similar remark pertains to Tzitzeica's parameter d, so we have

$$\begin{aligned} d(x,y) &= d(u,v)\Big|_{u\to u(x,y),\; v\to v(x,y)} \\ &= -\frac{3}{\big[4\exp(\tfrac{x-y}{2\beta}) + \exp(\tfrac{y-x}{\beta})\big]^{\frac{1}{2}}} \end{aligned} \tag{35}$$

which gives back

$$-K/d^4 = \tfrac{12}{3^4} = \tfrac{4}{27} \quad : \quad \text{constant on the unit hexenhut}$$

and places us in position to construct

$$h(x,y) = f/d = h_{12}/d = \frac{[4\beta^2+3)/8\beta^2]/\sqrt{\text{etc.}}}{-3/\sqrt{\text{etc.}}} = -\tfrac{1}{3} \tag{36}$$

Working from (29) and (36), we verify[19] that

$$\boldsymbol{r}_{xy} = h\boldsymbol{r} \quad : \quad h = -\tfrac{1}{3}$$

which is precisely the characteristic relation (15) stipulated by Tzitzeica. The remaining Gauss equations (18) are brought by (36) to the form

$$\left.\begin{aligned} \boldsymbol{r}_{xx} &= -3a(x)\boldsymbol{r}_y \\ \boldsymbol{r}_{yy} &= -3b(y)\boldsymbol{r}_x \end{aligned}\right. \tag{37}$$

of which the first is found (again use $\beta = \frac{1}{2}\sqrt{3}$) to be satisfied if and only if we set $a(x) = 1/3\sqrt{3}$. The integrability condition (20) then reads

$$0 = h - a(x)b(y)/h^2 = (-\tfrac{1}{3}) - \big(b(y)/3\sqrt{3}\,\big)/(-\tfrac{1}{3})^2 = -\tfrac{1}{3} - \sqrt{3}\,b(y)$$

which supplies $b(y) = -1/3\sqrt{3}$, whereupon the second of the Gauss equations (37) is found also to be satisfied.

At this point the Gauss equations

$$\left.\begin{aligned} \boldsymbol{r}_{xx} &= \Gamma^1_{11}\boldsymbol{r}_x + \Gamma^2_{11}\boldsymbol{r}_y \\ \boldsymbol{r}_{xy} &= \Gamma^1_{12}\boldsymbol{r}_x + \Gamma^2_{12}\boldsymbol{r}_y + f\boldsymbol{N} \\ \boldsymbol{r}_{yy} &= \Gamma^1_{22}\boldsymbol{r}_x + \Gamma^2_{22}\boldsymbol{r}_y \end{aligned}\right\} \tag{4}$$

of the asymptotically parameterized unit hexenhut have assumed the form

$$\begin{aligned} \boldsymbol{r}_{xx} &= \ \ 0\,\boldsymbol{r}_x - \tfrac{1}{\sqrt{3}}\boldsymbol{r}_y \\ \boldsymbol{r}_{xy} &= \Gamma^1_{12}\boldsymbol{r}_x + \Gamma^2_{12}\boldsymbol{r}_y + f\boldsymbol{N} \\ &= -\tfrac{1}{3}\boldsymbol{r} \quad : \quad \text{stipulated at (15) by Tzitzeica} \\ \boldsymbol{r}_{yy} &= \tfrac{1}{\sqrt{3}}\boldsymbol{r}_x + \ 0\,\boldsymbol{r}_y \end{aligned}$$

from which we infer $\Gamma^1_{11} = \Gamma^2_{22} = 0, \Gamma^1_{22} = -\Gamma^2_{11} = \frac{1}{\sqrt{3}}$, as might have been obtained directly from (17) , or alternatively: use (30.1) to write

$$\begin{aligned} E = G &= \frac{(4\beta^2+1)\exp\frac{y-x}{2\beta} + 4\exp\frac{x-y}{\beta}}{16\beta^2} \\ F &= \frac{(4\beta^2-1)\exp\frac{y-x}{2\beta} - 4\exp\frac{x-y}{\beta}}{16\beta^2} \end{aligned}$$

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[19] Use

$$\frac{1+4\beta^2}{16\beta^2} = \frac{1}{4\beta^2} = \frac{1}{3} \quad \text{by} \quad \beta = \tfrac{1}{2}\sqrt{3}$$

and introduce those expressions into (7), which by $E = G$ read

$$\begin{aligned}
\Gamma^1{}_{11} &= g^{-1}\big\{ \quad \tfrac{1}{2}EE_x - FF_x + \tfrac{1}{2}FE_y\big\}\\
\Gamma^2{}_{11} &= g^{-1}\big\{ -\tfrac{1}{2}FE_x + EF_x - \tfrac{1}{2}EE_y\big\}\\
\Gamma^1{}_{22} &= g^{-1}\big\{ -\tfrac{1}{2}FE_y + EF_y - \tfrac{1}{2}EE_x\big\}\\
\Gamma^2{}_{22} &= g^{-1}\big\{ \quad \tfrac{1}{2}EE_y - FF_y + \tfrac{1}{2}FE_x\big\}
\end{aligned}$$

with $g = E^2 - F^2$. *Mathematica* promptly supplies the results in question.

We look now to the parameter rescaling $\{x \to u(x), y \to v(y)\}$ that brings the hexenhut equations to "canonical form." We established on page 10 that

$$\begin{aligned}
\boldsymbol{r}_{xx} &= (a/h)\boldsymbol{r}_y & & \boldsymbol{r}_{uu} = (ax_u^3/hx_u y_v)\boldsymbol{r}_v\\
\boldsymbol{r}_{xy} &= h\boldsymbol{r} & \xrightarrow[\text{transforms to}]{\hspace{3em}} \quad & \boldsymbol{r}_{uv} = (hx_u y_v)\boldsymbol{r}\\
\boldsymbol{r}_{yy} &= (b/h)\boldsymbol{r}_x & & \boldsymbol{r}_{vv} = (by_v^3/hx_u y_v)\boldsymbol{r}_u
\end{aligned}$$

and want to achieve $ax_u^3 = \lambda$, $by_v^3 = \lambda^{-1}$. Since $\{a, b, \lambda\}$ are in this instance constants the functions $x(u)$ and $y(v)$ must be linear. Write

$$x(u) = ku, \quad y(v) = \ell v$$

Then $ak^3 = \lambda, b\ell^3 = \lambda^{-1}$ give

$$\begin{aligned}
k &= \ (\lambda/a)^{\frac{1}{3}} = \quad (\lambda 3^{\frac{3}{2}})^{\frac{1}{3}} = \quad \sqrt{3}\,\lambda^{+\frac{1}{3}}\\
\ell &= (1/\lambda b)^{\frac{1}{3}} = (-3^{\frac{3}{2}}/\lambda)^{\frac{1}{3}} = -\sqrt{3}\,\lambda^{-\frac{1}{3}}
\end{aligned}$$

whence $\bar{h} = hx_u y_v = -\frac{1}{3}k\ell$. So

$$\bar{h} = 1 \quad : \quad \text{simplest possible solution of } (\log \bar{h})_{uv} = \bar{h} - \bar{h}^{-2} \tag{38.1}$$

and the canonical hexenhut equations read

$$\begin{aligned}
\boldsymbol{r}_{uu} &= \lambda\ \ \boldsymbol{r}_v\\
\boldsymbol{r}_{uv} &= \boldsymbol{r}\\
\boldsymbol{r}_{vv} &= \lambda^{-1}\boldsymbol{r}_u
\end{aligned} \tag{38.2}$$

—than which nothing could be sweeter! Many of the preceding results—but particularly this one—lend plausibility to the claim that "Jonas' hexenhut is the simplest of all Tzitzeica surfaces. . . though we encountered serious whitewater in the computational stream, and complications (of which *Mathematica* takes no notice) intrude on introduction of the canonical asymptotic coordinates:

$$\boldsymbol{r}(x, y)\Big|_{x \to ku,\ y \to \ell v}$$

Define $\boldsymbol{s} = -\boldsymbol{r}_u \times \boldsymbol{r}_v$. A simplified reprise of the arguments that gave (23) and (24) gives

$$\begin{aligned}
\boldsymbol{s}_u &= -\boldsymbol{r}_u \times \boldsymbol{r}\\
\boldsymbol{s}_v &= \quad \boldsymbol{r}_v \times \boldsymbol{r}
\end{aligned} \tag{39.1}$$

$$\begin{aligned} \boldsymbol{s}_{uu} &= -\lambda \boldsymbol{s}_v \\ \boldsymbol{s}_{uv} &= \boldsymbol{s} \\ \boldsymbol{s}_{vv} &= -\lambda^{-1} \boldsymbol{s}_u \end{aligned} \tag{39.2}$$

Were we to define $\boldsymbol{t} = -\boldsymbol{s}_u \times \boldsymbol{s}_v$ we would be led to $\boldsymbol{t}$-equations that mimic (38.2). But[20]

$$\begin{aligned} \boldsymbol{t} = -\boldsymbol{s}_u \times \boldsymbol{s}_v &= (\boldsymbol{r}_u \times \boldsymbol{r}) \times (\boldsymbol{r}_v \times \boldsymbol{r}) \\ &= (\boldsymbol{r}_u \boldsymbol{r}\, \boldsymbol{r})\boldsymbol{r}_v - (\boldsymbol{r}_u \boldsymbol{r}\, \boldsymbol{r}_v)\boldsymbol{r} \\ &= [(\boldsymbol{r}_u \times \boldsymbol{r}_v)\cdot \boldsymbol{r}]\boldsymbol{r} \end{aligned}$$

We draw now on a fact upon which we have not previouisly had occasion to remark; from the description (29) of $\boldsymbol{r}(x,y)$ if follows by computation that

$$(\boldsymbol{r}_x \times \boldsymbol{r}_y)\cdot \boldsymbol{r} = -4/3\beta = -\tfrac{1}{2}\sqrt{3} = -\beta \quad : \quad \text{a constant!}$$

Therefore

$$(\boldsymbol{r}_u \times \boldsymbol{r}_v)\cdot \boldsymbol{r} = [(\boldsymbol{r}_x \times \boldsymbol{r}_y)\cdot \boldsymbol{r}]x_u y_v = -\beta k\ell = 3\beta$$

So $\boldsymbol{t} = 3\beta \boldsymbol{r}$. The linear system of $\boldsymbol{t}$-equations differs from the $\boldsymbol{r}$-system (which was our point of departure) only by an irrelevant multiplicative constant. It is for this reason that the $\boldsymbol{r}$-system and the $\boldsymbol{s}$-system can be said to be "dual:" $\boldsymbol{r}(u,v)$ provides the canonical description of the unit hexenhut Σ, and $\boldsymbol{s}(u,v)$ the canonical description of Σ^{dual}. Reverting (for merely typographic reasons) to our original asymptotic parameters $\{x,y\}$, we see from

$$\boldsymbol{r}(x,y) = \begin{pmatrix} \exp\frac{y-x}{4\beta}\cos\frac{x+y}{2} \\ \exp\frac{y-x}{4\beta}\sin\frac{x+y}{2} \\ \exp\frac{x-y}{2\beta} \end{pmatrix} \qquad \boldsymbol{s}(x,y) = \begin{pmatrix} \frac{1}{2\beta}\exp\frac{x-y}{4\beta}\cos\frac{x+y}{2} \\ \frac{1}{2\beta}\exp\frac{x-y}{4\beta}\sin\frac{x+y}{2} \\ \frac{1}{4\beta}\exp\frac{y-x}{2\beta} \end{pmatrix}$$

that $\boldsymbol{r}(x,y)$ and its dual are structurally quite similar. It is therefore not surprising that the associated surfaces are also similar; Σ^{dual} looks (because $\frac{1}{2\beta} = 0.57735 < 1$, $\frac{1}{4\beta} = 0.28867 < 1$) like an emaciated hexenhut.

Where does that leave us? As was remarked already on page 11, it is one thing to establish the integrability of the Gauss equations (20) and quite another—given that $h(x,y)$ is a solution of (21)—actually to integrate them. That is an issue that I reserve for a separate essay (Part 3 in this series).

[20] Use the identity $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{abd})\boldsymbol{c} - (\boldsymbol{abc})\boldsymbol{d}$ and properties of the triple scalar product to which reference was made on page 12.