Geodesics

on the

Pseudosphere & Hexenhut

Nicholas Wheeler
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Introduction. In "Differential geometry of some surfaces in 3-space" (December 2015) I consider two classes of curves inscribed on simple instances of such surfaces: rectilinear "rules" inscribed on hyperboloids of a single sheet, and "asymptotic curves" inscribed on hyperboloids, the hexenhut and (because they mark the birthplace of the sine-Gordon equation[1]) the pseudosphere. Conspicuously absent from that discussion is any reference to the inscribed curves that (particularly for a physicist) come first to mind, the curves which Liouville was the first to call "geodesics."

Ahmed Sebbar[2] has reported that he and Chapman University colleagues Mihaela & Adrian Vajiac have established that the hexenhut

$$x^3 + y^3 + z^3 - 3xyz = 1 \tag{1}$$

is a Tzitzeica surface. And that, drawing inspiration from Manfredo do Carmo's demonstration[3] that "any geodesic on a paraboloid of revolution $z = x^2 + y^2$ which is not a meridian intersects itself an infinite number of times," he as acquired interest in geodesics inscribed on the hexenhut (which possess perhaps that same property?). Before undertaking any computation relating to the latter question, Sebbar would like to possess graphic indication of what hexenhut geodesics look like. And that it is my ultimate intention here to provide.

Demonstration that the hexenhut is a Tzitzeica surface. It was in lecture notes supplied to me by Sebbar[4] that I first encountered the description (1) of a surface that he preferred to call the "Appell sphere," but remarked is "by some

[1] See "Some remarks concerning the sine-Gordon equation," (November 2015).
[2] Private communication, 3 January 2016.
[3] *Differential Geometry of Curves and Surfaces* (1976). This text is available as a free download on the web. The demonstration appears as Example 6 at the end (page 258) of §4-4: "Parallel transport; geodesics" in Chapter 4: "The intrinsic geometry of surfaces."
[4] Private communication, 16 September 2015.

physicists called a 'hexenhut'." On consulting Google, I was referred to page 105 in C. Rogers & W. K. Schief, *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory* (2002), a beautiful monograph which I acquired, and which has been central to my subsequent work. But only today have I noticed that the Rogers-Schief reference to "Jonas' hexenhut" occurs in a chapter devoted to Tzitzeica surfaces.[5] Little wonder, therefore, that—as Sebbar and colleagues have discovered, and as I show below—the hexenhut exhibits properties characteristic of Tzitzeica surfaces.

The points that satisfy (1) are seen when plotted to be symmetrically distributed about the axis indicated by the vector $(1,1,1)$. Rotate the coordinate frame so as to bring the 3-axis into alignment with that vector and find[6] that (1) has assumed the form

$$z(x^2+y^2)=\alpha^2 \quad \text{where} \quad \alpha^2=\tfrac{2}{3\sqrt{3}} \tag{2}$$

In (2) we have a surface of revolution Σ, of which

$$\boldsymbol{r}=\begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ u \end{pmatrix} \quad \text{with} \quad f(u)=\alpha/\sqrt{u} \tag{3}$$

provides a natural parameterization. The 1st and 2nd fundamental forms that follow from (3) supply

$$\begin{aligned}
g_{11}\equiv E &= 1+\frac{\alpha^2}{4u^3}\\
g_{12}=g_{21}\equiv F &= 0\\
g_{22}\equiv G &= \frac{\alpha^2}{u}\\
h_{11}\equiv e &= -\frac{3\alpha^2}{2u\sqrt{4u^3\alpha^2+\alpha^4}}\\
h_{12}=h_{21}\equiv f &= 0\\
h_{22}\equiv g &= \frac{2u\alpha^2}{\sqrt{4u^3\alpha^2+\alpha^4}}
\end{aligned}$$

Writing

$$\mathbb{G}=\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}, \quad \mathbb{H}=\begin{pmatrix} h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix}$$

[5] Chapter 3: Tzitzeica Surfaces. Conjugate Nets and the Toda Lattice Scheme. The authors write at some length about the occurance of Tzitzeica surfaces in the theory of an integrable gasdynamics system, and mention in a footnote that "the hexenhut arises naturally in hydrodynamics as a free surface bounding axi-symmetric jets impinging normally on a plate and [in the theory of] whirlpools."

[6] The details are developed on page 21 of "Surfaces in 3-space" (December 2015). I will adopt the abbreviation SURFACES in future references to that essay.

we find that the Gaussian curvature at the point $\{u, v\}$ is given by[7]

$$K = \frac{\det \mathbb{H}}{\det \mathbb{G}} = -\frac{12u^4}{(4u^3+\alpha^2)^2} \tag{4}$$

Asymptotic curves are inscribed on Σ (see SURFACES, page 22) by equations of the form

$$v(u) = \pm\beta \log(u/u_0) \quad : \quad \beta = \tfrac{\sqrt{3}}{2}$$

Which is to say: we have

$$v + \beta \log u = p \tag{5.1}$$

$$v - \beta \log u = q \tag{5.2}$$

where (5.1) describes asymptotic curves of one helicity, (5.2) describes curves of the opposite helicity, and where $\{p, q\}$ are the "asymptotic coordinates" that serve to give names to such curves; *i.e.*, to distinguish each from all others. Inversely

$$u = \exp\Big(\frac{p-q}{2\beta}\Big) \tag{6.1}$$

$$v = \frac{p+q}{2} \tag{6.2}$$

by virtue of which (3) acquires the $\{p, q\}$-parameterization

$$\boldsymbol{r} = \begin{pmatrix} \alpha \exp \frac{q-p}{4\beta} \cos \frac{p+q}{2} \\ \alpha \exp \frac{q-p}{4\beta} \sin \frac{p+q}{2} \\ \exp \frac{p-q}{2\beta} \end{pmatrix} \tag{7}$$

Working from (7), we (with the substantial assistance of *Mathematica*) find

$$\begin{aligned} g_{11} = g_{22} &= \frac{\alpha^2(4\beta^2+1)\exp\frac{q-p}{2\beta} + 4\exp\frac{p-q}{\beta}}{16\beta^2} \\ g_{12} = g_{21} &= \frac{\alpha^2(4\beta^2-1)\exp\frac{q-p}{2\beta} - 4\exp\frac{p-q}{\beta}}{16\beta^2} \end{aligned} \tag{8}$$

To construct the 2nd fundamental form—and for another purpose peculiar to the theory of Tzitzeica surfaces—we need a description of the unit normal $\boldsymbol{N}$ at the point $\{p, q\}$. The unnormalized normal is

$$\boldsymbol{n} = \boldsymbol{r}_p \times \boldsymbol{r}_q = \begin{pmatrix} -\dfrac{\alpha \exp \frac{p-q}{4\beta} \cos \frac{p+q}{2}}{2\beta} \\ -\dfrac{\alpha \exp \frac{p-q}{4\beta} \sin \frac{p+q}{2}}{2\beta} \\ -\dfrac{\alpha^2 \exp \frac{q-p}{2\beta}}{4\beta} \end{pmatrix} \tag{9.1}$$

which gives

$$\boldsymbol{n}\cdot\boldsymbol{n} = \frac{4\alpha^2\exp\frac{p-q}{2\beta} + \alpha^4\exp\frac{q-p}{\beta}}{16\beta^2} \tag{9.2}$$

[7] In a draft version of SURFACES the denominator is written $(4u^2+\alpha^2)^2$, which is incorrect.

and from (9) we obtain

$$\boldsymbol{N} = \frac{\boldsymbol{n}}{\sqrt{\boldsymbol{n}\cdot\boldsymbol{n}}}$$

which it would be pointless to spell out in detail, is best allowed to remain in the memory of *Mathematica*. With this result in hand, we are in position to construct

$$\begin{aligned} h_{11} = h_{22} &= \alpha^2\frac{4\beta^2-3}{32\beta^3}\cdot\frac{1}{\sqrt{\boldsymbol{n}\cdot\boldsymbol{n}}} \\ h_{12} = h_{21} &= \alpha^2\frac{4\beta^2+3}{32\beta^3}\cdot\frac{1}{\sqrt{\boldsymbol{n}\cdot\boldsymbol{n}}} \end{aligned} \tag{10}$$

From (8) we obtain

$$\det\mathbb{G} = \frac{4\alpha^2\exp\frac{p-q}{2\beta}+\alpha^4\exp\frac{q-p}{\beta}}{16\beta^2} \tag{11.1}$$

while (10) gives

$$\det\mathbb{H} = -\frac{3\alpha^4\exp\frac{2p+q}{2\beta}}{4\beta^2\big(4\alpha^2\exp\frac{3p}{2\beta}+\alpha^4\exp\frac{3q}{2\beta}\big)} \tag{11.2}$$

From (11) we obtain this description in asymptotic variables of the Gaussian curvature:

$$K = \frac{\det\mathbb{H}}{\det\mathbb{G}} = -\frac{12\exp\frac{2p+q}{2\beta}}{\big(4\exp\frac{3p}{2\beta}+\alpha^2\exp\frac{3q}{2\beta}\big)^2} \tag{12}$$

It is gratifying to discover that when we use (5) to pass from $\{p,q\}$ to $\{u,v\}$ variables we recover precisely (4):

$$= -\frac{12\,u^4}{(4u^3+\alpha^2)^2}$$

The vector $\boldsymbol{n}$, since it stands normal to Σ at the point $\{p,q\}$, is normal to the plane Π that is tangent to Σ at $\{p,q\}$. The dot product

$$d = \boldsymbol{N}\cdot\boldsymbol{r}$$

admits therefore of geometric interpretation as the normal (or shortest) distance between the origin and Π. By computation we find

$$\begin{aligned} d^4 &= \frac{81\alpha^4\exp\frac{2p+q}{2\beta}}{\big(4\exp\frac{3p}{2\beta}+\alpha^2\exp\frac{3q}{2\beta}\big)^2} \\ &= -\frac{81\,\alpha^4}{12}K \end{aligned}$$

We have established therefore that

$$K = -c^2d^4 \tag{13}$$

with $c^2 = \frac{4}{27\alpha^4} = \frac{27}{4}$. But (13) is precisely the condition (see Rogers & Schief, page 90) that distinguishes Tzitzeica surfaces from other surfaces in 3-space.