Asymptotic Parameterization

of the

Curvature Matrix

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Introduction. Let $\boldsymbol{r}(u,v)$ describe an arbitrarily surface Σ in 3-space. From the 2^{nd} Fundamental Form one obtains the real symmetric 2×2 matrix

$$\mathbb{H}(u,v) = \begin{pmatrix} \boldsymbol{r}_{uu}\cdot\boldsymbol{N} & \boldsymbol{r}_{uv}\cdot\boldsymbol{N} \\ \boldsymbol{r}_{vu}\cdot\boldsymbol{N} & \boldsymbol{r}_{vv}\cdot\boldsymbol{N} \end{pmatrix}$$

that encodes the local curvature structure of Σ. One has

$$\det\mathbb{H} = \lambda_1\lambda_2 < 0 \qquad \begin{cases} \text{locally at points } P \text{ of negative curvature} \\ \text{everywhere on hyperbolic surfaces} \end{cases}$$

At such points and only at such points is it possible to find pairs $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ of real vectors that satisfy the condition

$$(\boldsymbol{a}|\mathbb{H}|\boldsymbol{a}) = 0$$

Such vectors $\boldsymbol{a}_i$ are said to be "self-conjugate," and to indicate "asymptotic directions" at P. My objectives here will be (i) to describe the transformations $\mathbb{H}(u,v) \to \mathbb{H}(x,y)$ induced by parameter adjustments $\{u,v\} \to \{x,y\}$, and (ii) to show that when $\{x,y\}$ refers to an "asymptotic parameterization of Σ"[1]

$$\mathbb{H}(x,y) = \begin{pmatrix} \boldsymbol{r}_{xx}\cdot\boldsymbol{N} & \boldsymbol{r}_{xy}\cdot\boldsymbol{N} \\ \boldsymbol{r}_{yx}\cdot\boldsymbol{N} & \boldsymbol{r}_{yy}\cdot\boldsymbol{N} \end{pmatrix} \quad \text{has invariably the form} \quad \begin{pmatrix} 0 & f \\ f & 0 \end{pmatrix}$$

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[1] Isolated points and regions of negative curvature are of relatively little interest, so I adopt here and henceforth a language that is strictly appropriate only to cases in which Σ is—like the pseudosphere—hyperbolic; *i.e.*, in which the curvature is *everywhere* negative. "Asymptotic parameterizations" are parameterizations with the property that the tangent vectors $\{\boldsymbol{r}_x(x,y), \boldsymbol{r}_y(x,y)\}$ are, for all $\{x,y\}$, asymptotic. The equations $x = \text{constant}$ and $y = \text{constant}$ then inscribe "asymptotic curves" $\{\mathcal{C}_x, \mathcal{C}_y\}$ on Σ.

Response of $\mathbb{H}$ to parameter transformations. We look by way of preparation to the relatively simpler problem posed by the reparameterization of

$$\mathbb{G}(u,v) = \begin{pmatrix} \boldsymbol{r}_u\boldsymbol{\cdot}\boldsymbol{r}_u & \boldsymbol{r}_u\boldsymbol{\cdot}\boldsymbol{r}_v \\ \boldsymbol{r}_v\boldsymbol{\cdot}\boldsymbol{r}_u & \boldsymbol{r}_v\boldsymbol{\cdot}\boldsymbol{r}_v \end{pmatrix}$$

which derives from the 1$^{\text{st}}$ Fundamental Form and encodes the local metric structure of Σ. From

$$\begin{aligned} \boldsymbol{r}_x &= \boldsymbol{r}_u u_x + \boldsymbol{r}_v v_x \\ \boldsymbol{r}_y &= \boldsymbol{r}_u u_y + \boldsymbol{r}_v v_y \end{aligned}$$

it follows by *Mathematica*-assisted quick calculation that

$$\begin{pmatrix} \boldsymbol{r}_x\boldsymbol{\cdot}\boldsymbol{r}_x & \boldsymbol{r}_x\boldsymbol{\cdot}\boldsymbol{r}_y \\ \boldsymbol{r}_y\boldsymbol{\cdot}\boldsymbol{r}_x & \boldsymbol{r}_y\boldsymbol{\cdot}\boldsymbol{r}_y \end{pmatrix} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}\begin{pmatrix} \boldsymbol{r}_u\boldsymbol{\cdot}\boldsymbol{r}_u & \boldsymbol{r}_u\boldsymbol{\cdot}\boldsymbol{r}_v \\ \boldsymbol{r}_v\boldsymbol{\cdot}\boldsymbol{r}_u & \boldsymbol{r}_v\boldsymbol{\cdot}\boldsymbol{r}_v \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

which is to say:

$$\mathbb{G}(x,y) = \mathbb{J}^{\mathsf{T}}\mathbb{G}(u,v)\,\mathbb{J}\Big|_{u\to u(x,y),v\to v(x,y)} \tag{1.1}$$

where $\mathbb{J}$ is the transformation matrix of the non-singular transformation

$$\{u,v\}\longrightarrow\{x,y\} \quad : \quad \det\mathbb{J} = u_x v_y - u_y v_x \neq 0$$

of which the determinant is the Jacobian. Note that (1) preserves symmetry, but preserves trace/determinant/spectrum only when $\mathbb{J}$ is rotational: $\mathbb{J}^{\mathsf{T}} = \mathbb{J}^{-1}$.

Look now to the reparameterization of $\mathbb{H}$. From

$$\begin{aligned} \boldsymbol{r}_{xx} &= (\boldsymbol{r}_{uu}u_x + \boldsymbol{r}_{uv}v_x)u_x + (\boldsymbol{r}_{vu}u_x + \boldsymbol{r}_{vv}v_x)v_x + \boldsymbol{r}_u u_{xx} + \boldsymbol{r}_v v_{xx} \\ \boldsymbol{r}_{xy} &= (\boldsymbol{r}_{uu}u_y + \boldsymbol{r}_{uv}v_y)u_x + (\boldsymbol{r}_{vu}u_y + \boldsymbol{r}_{vv}v_y)\,v_x + \boldsymbol{r}_u u_{xy} + \boldsymbol{r}_v v_{xy} \\ \boldsymbol{r}_{yy} &= (\boldsymbol{r}_{uu}u_y + \boldsymbol{r}_{uv}v_y)u_y + (\boldsymbol{r}_{vu}u_y + \boldsymbol{r}_{vv}v_y)\,v_y + \boldsymbol{r}_u u_{yy} + \boldsymbol{r}_v v_{yy} \end{aligned}$$

we have

$$\begin{pmatrix} \boldsymbol{r}_{xx} & \boldsymbol{r}_{xy} \\ \boldsymbol{r}_{xy} & \boldsymbol{r}_{yy} \end{pmatrix} = \mathbb{J}^{\mathsf{T}}\begin{pmatrix} \boldsymbol{r}_{uu} & \boldsymbol{r}_{uv} \\ \boldsymbol{r}_{uv} & \boldsymbol{r}_{vv} \end{pmatrix}\mathbb{J} + \boldsymbol{r}_u\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} + \boldsymbol{r}_v\begin{pmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{pmatrix}$$

The tangent vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are orthogonal to the normal vector $\boldsymbol{N}$, so the second and third terms in the preceding equation vanish when dotted into $\boldsymbol{N}$. We are left with

$$\begin{pmatrix} \boldsymbol{r}_{xx}\boldsymbol{\cdot N} & \boldsymbol{r}_{xy}\boldsymbol{\cdot N} \\ \boldsymbol{r}_{xy}\boldsymbol{\cdot N} & \boldsymbol{r}_{yy}\boldsymbol{\cdot N} \end{pmatrix} = \mathbb{J}^{\mathsf{T}}\begin{pmatrix} \boldsymbol{r}_{uu}\boldsymbol{\cdot N} & \boldsymbol{r}_{uv}\boldsymbol{\cdot N} \\ \boldsymbol{r}_{uv}\boldsymbol{\cdot N} & \boldsymbol{r}_{vv}\boldsymbol{\cdot N} \end{pmatrix}\mathbb{J}$$

or

$$\mathbb{H}(x,y) = \mathbb{J}^{\mathsf{T}}\mathbb{H}(u,v)\,\mathbb{J}\Big|_{u\to u(x,y),v\to v(x,y)} \tag{1.2}$$

Equation (1.1) is no surprise; it states simply that the elements of the metric matrix $\mathbb{G}$ transform as a covariant tensor of second rank. Equation (1.2)—which says that so also do the elements of the $\mathbb{H}$-matrix—is, on the other hand, a bit of a surprise, since it hinges on the fortuitous vanishing of terms that involve the second derivatives of $u(x,y)$ and $v(x,y)$.[2] From (1.1) and (1.2) follows the coordinate-independence of the Gaussian curvature:

$$K = \frac{\det \mathbb{H}(x,y)}{\det \mathbb{G}(x,y)} = \frac{\det \mathbb{H}(u,v)}{\det \mathbb{G}(u,v)}$$

Inverse problem: the matrices with respect to which a given pair of vectors are asymptotic. It is in service of clarity that I approach the issue before us backwards. Let

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbb{W} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

Clearly, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are asymptotic to $\mathbb{W}$[3]

$$(\boldsymbol{e}_1|\mathbb{W}|\boldsymbol{e}_1) = (\boldsymbol{e}_2|\mathbb{W}|\boldsymbol{e}_2) = 0 \quad \text{if and only if } p = s = 0$$

Restricting our attention henceforth of *symmetric* matrices, we have therefore the trivial statements

$$\begin{matrix} (\boldsymbol{e}_1|\mathbb{Q}|\boldsymbol{e}_1) = 0 \\ (\boldsymbol{e}_2|\mathbb{Q}|\boldsymbol{e}_2) = 0 \end{matrix} \quad : \quad \mathbb{Q} = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} \tag{2}$$

Now let

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

be an arbitrary pair of linearly independent vectors (neither orthogonality nor normalization assumed). From them, construct

$$\mathbb{C} \equiv \big\| |\boldsymbol{a}), |\boldsymbol{b}) \big\| = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \tag{3.1}$$

Immediately

$$\begin{matrix} \mathbb{C}|\boldsymbol{e}_1) = |\boldsymbol{a}) & \qquad (\boldsymbol{e}_1|\mathbb{C}^{\mathsf{T}} = (\boldsymbol{a}| \\ \mathbb{C}|\boldsymbol{e}_2) = |\boldsymbol{b}) & \qquad (\boldsymbol{e}_2|\mathbb{C}^{\mathsf{T}} = (\boldsymbol{b}| \end{matrix}$$

and inversely

$$\begin{matrix} |\boldsymbol{e}_1) = \mathbb{D}|\boldsymbol{a}) & \qquad (\boldsymbol{e}_1| = (\boldsymbol{a}|\mathbb{D}^{\mathsf{T}} \\ |\boldsymbol{e}_2) = \mathbb{D}|\boldsymbol{b}) & \qquad (\boldsymbol{e}_2| = (\boldsymbol{b}|\mathbb{D}^{\mathsf{T}} \end{matrix}$$

where

$$\mathbb{D} = \mathbb{C}^{-1} = \frac{1}{a_1 b_2 - a_2 b_1} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix} \equiv \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} = \left\| \begin{matrix} (\boldsymbol{A}| \\ (\boldsymbol{B}| \end{matrix} \right\| \tag{3.2}$$

So (2) becomes

$$(\boldsymbol{a}|\mathbb{M}|\boldsymbol{a}) = (\boldsymbol{b}|\mathbb{M}|\boldsymbol{b}) = 0 \tag{4.1}$$

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[2] It is the management of such terms that in other contexts motivates the definition of the covariant derivative.

[3] I adopt here and henceforth a variant of Dirac notation, writing $\boldsymbol{x}$ else $|\boldsymbol{x})$ to signify column vectors, $\boldsymbol{x}^{\mathsf{T}}$ else $(\boldsymbol{x}|$ to signify their row-vector transposes.

with[4]

$$\mathbb{M} \equiv \mathbb{D}^{\mathsf{T}}\mathbb{Q}\mathbb{D} = \frac{q}{(a_1b_2 - a_2b_1)^2}\begin{pmatrix} -2a_2b_2 & a_1b_2 + a_2b_2 \\ a_1b_2 + a_2b_2 & -2a_1b_1 \end{pmatrix} \tag{4.2}$$

The essential structure of this result is revealed if one assumes (without loss of generality) that the asymptotic vectors are unit vectors, which can be accomplished by writing

$$|\boldsymbol{a}) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \qquad |\boldsymbol{b}) = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} \tag{5}$$

This reduces the number of adjustable parameters from five to three (α, β, q) and brings (4.2) to the form

$$\mathbb{M} = \frac{q}{\sin^2(\alpha-\beta)}\begin{pmatrix} -2\sin\alpha\sin\beta & \sin(\alpha+\beta) \\ \sin(\alpha+\beta) & -2\cos\alpha\cos\beta \end{pmatrix} \tag{6.1}$$

giving

$$\det\mathbb{M} = -\frac{q^2}{\sin^2(\alpha-\beta)} < 0 \tag{6.2}$$

$$\mathrm{tr}\,\mathbb{M} = -2q\,\frac{\cos(\alpha-\beta)}{\sin^2(\alpha-\beta)} \tag{6.3}$$

$$\lambda_\pm = \tfrac{1}{2}\Big[\mathrm{tr}\,\mathbb{M} \pm \sqrt{(\mathrm{tr}\,\mathbb{M})^2 - 4\det\mathbb{M}}\,\Big] = -q\,\frac{\cos(\alpha-\beta)\pm 1}{\sin^2(\alpha-\beta)} \tag{6.4}$$

Direct problem: from the vectors asymptotic to a given matrix to the asymptotic representation of that matrix. One has in principle only to trace in reverse the procedure described in the preceding section:

$\boxed{\text{STEP ONE}}$ Check that $\det\mathbb{M} < 0$. Construct arbitrarily normalized solutions $|\boldsymbol{a})$ and $|\boldsymbol{b})$ of $(\boldsymbol{x}|\mathbb{M}|\boldsymbol{x}) = 0$.

$\boxed{\text{STEP TWO}}$ Construct

$$\mathbb{C} = \big\||\boldsymbol{a})\;|\boldsymbol{b})\big\| \quad\text{and}\quad \mathbb{D} = \mathbb{C}^{-1} = \left\|\begin{matrix} (\boldsymbol{A}| \\ (\boldsymbol{B}| \end{matrix}\right\|$$

Notice, by the way, that the $\boldsymbol{a}$-basis (with elements $\{|\boldsymbol{a}), |\boldsymbol{b})\}$) and the $\boldsymbol{A}$-basis (with elements $\{|\boldsymbol{A}), |\boldsymbol{B})\}$) are "dual" (the pair are "biorthogonal") in the sense that $(\boldsymbol{A}^i|\boldsymbol{a}_j) = \delta^i{}_j$.[5]

One is placed thus in position to write

$$\begin{aligned} (\boldsymbol{a}|\mathbb{M}|\boldsymbol{a}) &= (\boldsymbol{a}|\mathbb{D}^{\mathsf{T}}\!\cdot\mathbb{C}^{\mathsf{T}}\mathbb{M}\mathbb{C}\cdot\mathbb{D}|\boldsymbol{a}) \\ &= (\boldsymbol{e}_1|\mathbb{Q}|\boldsymbol{e}_1) \\ (\boldsymbol{b}|\mathbb{M}|\boldsymbol{b}) &= (\boldsymbol{b}|\mathbb{D}^{\mathsf{T}}\!\cdot\mathbb{C}^{\mathsf{T}}\mathbb{M}\mathbb{C}\cdot\mathbb{D}|\boldsymbol{b}) \\ &= (\boldsymbol{e}_2|\mathbb{Q}|\boldsymbol{e}_2) \end{aligned}$$

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[4] Accuracy check: ask *Mathematica* to solve $(\boldsymbol{x}|\mathbb{M}|\boldsymbol{x}) = 0$ with $(\boldsymbol{x}| = (x_1, x_2)$, get $x_1 = a_1x_2/a_2$, $x_1 = b_1x_2/b_2$; *i.e.*, $\boldsymbol{x} \sim \boldsymbol{a}$ else $\boldsymbol{x} \sim \boldsymbol{b}$.

[5] See "Biorthogonality–revisited" (June, 2016).

where $\mathbb{Q}$—the asymptotic representation of $\mathbb{M}$—has invariably the form

$$\mathbb{Q} = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} \quad \text{with} \quad \begin{aligned} q^2 &= -(\det \mathbb{C})^2 \det \mathbb{M} \\ &= -(a_1 b_2 - a_2 b_1)^2 \det \mathbb{M} > 0 \end{aligned} \tag{7}$$

Numerical experimentation confirms the swift effectiveness of the procedure.

Again, the procedure is illuminated if one imposes the assumption (5) that the asymptotic vectors $|\boldsymbol{a})$ and $|\boldsymbol{b})$ are unit vectors. Writing

$$\mathbb{M} = \begin{pmatrix} p & q \\ q & s \end{pmatrix}, \qquad |\boldsymbol{x}) = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}$$

one from $(\boldsymbol{x}|\mathbb{M}|\boldsymbol{x}) = 0$ obtains

$$\tan\psi = \frac{-q \pm \sqrt{q^2 - ps}}{s} \quad : \quad \text{real by } q^2 - rs = -\det\mathbb{M} > 0$$

giving

$$\begin{aligned} \alpha &= \arctan\left[\frac{-q + \sqrt{q^2 - ps}}{s}\right] \\ \beta &= \arctan\left[\frac{-q - \sqrt{q^2 - ps}}{s}\right] \end{aligned} \tag{8}$$

From

$$\mathbb{C} = \begin{pmatrix} \cos\alpha & \cos\beta \\ \sin\alpha & \sin\beta \end{pmatrix}$$

we obtain

$$\mathbb{D} = \mathbb{C}^{-1} = \frac{1}{\sin(\alpha - \beta)} \begin{pmatrix} -\sin\beta & \cos\beta \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

and

$$\mathbb{Q} = \mathbb{C}^{\mathsf{T}}\mathbb{M}\,\mathbb{C} = \begin{pmatrix} Q_1 & q \\ q & Q_2 \end{pmatrix}$$

with[6]

$$\begin{aligned} Q_1 &= p\cos^2\alpha + s\sin^2\alpha + q\sin 2\alpha = 0 \\ Q_2 &= p\cos^2\beta + s\sin^2\beta + q\sin 2\beta = 0 \\ q &= \cos\alpha(p\cos\beta + q\sin\beta) + \sin\alpha(q\cos\beta + s\sin\beta) \\ &= -2\frac{q^2 - ps}{\sqrt{(p-s)^2 + 4q^2}} < 0 \end{aligned} \tag{9.1}$$

We know from the symmetry of $\mathbb{M}$ that its eigenvalues are real, and from $\det\mathbb{M} = \lambda_1\lambda_2 < 0$ we know that they are of opposite signs. From (9) and

$$\lambda_\pm = \tfrac{1}{2}\big[(p+s) \pm \sqrt{(p-s)^2 + 4q^2}\,\big]$$

[6] Use (8), `TrigToExp` and `Simplify`.

We therefore have (writing $\{\lambda_1, \lambda_2\}$ for $\{\lambda_+, \lambda_-\}$)

$$q = 2\,\frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \tag{9.2}$$

In normalized asymptotic coordinates (5) the metric matrix becomes

$$\mathbb{G} = \begin{pmatrix} \boldsymbol{a}\cdot\boldsymbol{a} & \boldsymbol{a}\cdot\boldsymbol{b} \\ \boldsymbol{b}\cdot\boldsymbol{a} & \boldsymbol{b}\cdot\boldsymbol{b} \end{pmatrix} = \begin{pmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{pmatrix} \tag{10.1}$$

where

$$\omega = \boldsymbol{a}\angle\boldsymbol{b} = \alpha - \beta$$

One has the identity

$$\cos[\arctan x - \arctan y] = \frac{1 + xy}{\sqrt{(1+x^2)(1+y^2)}}$$

so by (8)

$$\cos\omega = \frac{p + s}{\sqrt{(p-s)^2 + 4q^2}} = \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \tag{10.2}$$

Illustrative application to differential geometry: the unit pseudosphere. We look now to the simplest instance of a hyperbolic surface—the pseudosphere, on which negative curvature is in fact *constant*: $K = -1$ Working from the Beltrami parameterization (1868)

$$\boldsymbol{r}(u,v) = \begin{pmatrix} \operatorname{sech} u\, \cos v \\ \operatorname{sech} u\, \sin v \\ u - \tanh u \end{pmatrix} \tag{11}$$

we find[7]

$$\mathbb{M} = \|h_{ij}(u,v)\| = \begin{pmatrix} -\operatorname{sech} u \tanh u & 0 \\ 0 & \operatorname{sech} u \tanh u \end{pmatrix}$$

and from

$$(d\boldsymbol{u}|\mathbb{M}|d\boldsymbol{u}) = 0 \quad \text{with} \quad |d\boldsymbol{u}) = \begin{pmatrix} du \\ dv \end{pmatrix}$$

we have $du^2 - dv^2 = (du + dv)(du - dv) = 0$ so are led to introduce asymptotic variables

$$\begin{aligned} x &= \tfrac{1}{2}(u + v) \\ y &= \tfrac{1}{2}(u - v) \end{aligned} \quad : \quad \text{the factors } \tfrac{1}{2} \text{ are cosmetic conveniences}$$

whence

$$\begin{aligned} u &= x + y \\ v &= x - y \end{aligned} \tag{12}$$

[7] See "Alternative formulations of the consistency argument that leads from pseudosphere to the sine-Gordon equation," (April, 2016), pages 3–4.

In these variables (11) becomes

$$\boldsymbol{r}(x,y) = \begin{pmatrix} \mathrm{sech}(x+y)\cos(x-y) \\ \mathrm{sech}(x+y)\sin(x-y) \\ (x+y) - \tanh(x+y) \end{pmatrix} \tag{13}$$

and tedious direct calculation gives

$$\|h_{ij}(x,y)\| = \begin{pmatrix} 0 & -2\mathrm{sech}(x+y)\tanh(x+y) \\ -2\mathrm{sech}(x+y)\tanh(x+y) & 0 \end{pmatrix}$$

We are, however, in position now to avoid such tedium: by (12)[8]

$$\mathbb{J} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and we recover the preceding result as an immediate instance of (1.2):

$$\|h_{ij}(x,y)\| = \mathbb{J}^{\mathsf{T}}\|h_{ij}(u,y)\|\,\mathbb{J}\Big|_{u\to u(x,y),v\to v(x,y)} \tag{14.1}$$

In the linear algebraic language of page 4 we have

$$|\boldsymbol{a}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\boldsymbol{b}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbb{C} = \||\boldsymbol{a})\,|\boldsymbol{b})\| = \mathbb{J}$$

in which notation (14.1) reads

$$\mathbb{Q} = \mathbb{C}^{\mathsf{T}}\mathbb{M}\,\mathbb{C}\Big|_{u\to x+y,\, v\to x-y} \tag{14.2}$$

Computational efficiency, as demonstrated above, is, however, secondary fruit of the preceding discussion; my primary objective has been to establish that the asymptotic representations of negative-definite symmetric matrices $\mathbb{M}$ have invariably the distinctive structure of $\mathbb{Q}$ (symmetric, with zeros on the diagonal); in the language of differential geometry

$$\mathbb{H}(u,v) \xrightarrow[\text{asymptotic reparameterization}]{} \mathbb{H}(x,y) = \begin{pmatrix} 0 & f \\ f & 0 \end{pmatrix}$$

Working from (11), one has the metric matrix

$$\mathbb{G}(u,v) = \|g_{ij}(u,v)\| = \begin{pmatrix} \boldsymbol{r}_u\cdot\boldsymbol{r}_u & \boldsymbol{r}_u\cdot\boldsymbol{r}_v \\ \boldsymbol{r}_v\cdot\boldsymbol{r}_u & \boldsymbol{r}_v\cdot\boldsymbol{r}_v \end{pmatrix} = \begin{pmatrix} \tanh^2 u & 0 \\ 0 & \mathrm{sech}^2 u \end{pmatrix}$$

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[8] Notice that $\mathbb{J}$, though not rotational, satisfies $\mathbb{J}^{\mathsf{T}}\mathbb{J} = 2\mathbb{I}$, so is *proportional* to a rotation matrix. Rescaled asymptotic parameters remain asymptotic. Suitable rescaling would in the present instance send $\mathbb{J} \to \tilde{\mathbb{J}} = \frac{1}{\sqrt{2}}\mathbb{J}$, which *is* rotational, so would preserve not only symmetry but also trace, determinant and spectrum.

which in asymptotic parameters (work from (13) or—more efficiently—use (1.1)) becomes (compare (10))

$$\mathbb{G}(x,y) = \|g_{ij}(x,y)\| = \begin{pmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{pmatrix}$$

with

$$\begin{aligned}\omega(x,y) &= \arccos\big[\tanh^2(x+y) - \operatorname{sech}^2(x+y)\big] \\ &= \arccos\big[1 - 2\operatorname{sech}^2(x+y)\big] \\ &= \arccos\left[1 - \frac{4}{1+\cosh(2x+2y)}\right]\end{aligned}$$

This unpromising result acquired historic significance from the circumstance that, as we are informed by *Mathematica* and as Edmond Bour was the first to notice (1862), we have on one hand

$$\omega_{xy}(x,y) = 2\operatorname{sech}(x+y)\tanh(x+y)$$

and on the other hand

$$\sin\omega = 2\operatorname{sech}(x+y)\tanh(x+y)$$

Thus did the SINE-GORDON EQUATION

$$\partial_x\partial_y\,\omega = \sin\omega$$

enter the literature of mathematics, fully half a century before it became central to the physical theory of solitons, the phenomenological basis of which can be traced to an event observed and remarked upon (1834) by John Scott Russell nearly thirty years before Bour' mathematical remark.

Pseudospheric surfaces exist in infinite variety. In a previous essay[9] I provide brief discussion of a class of surfaces attributed to Alfred Enneper (1830–1885):

$$\begin{aligned}\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} &+ \frac{2d}{c}\frac{\sin(dv)\cosh(cu)}{d^2\cosh^2(cu) + c^2\sin^2(dv)}\begin{pmatrix} \sin v \\ -\cos v \\ 0 \end{pmatrix} \\ &+ \frac{2d^2}{c}\frac{\cosh(cu)}{d^2\cosh^2(cu) + c^2\sin^2(dv)}\begin{pmatrix} \cos v\cos(dv) \\ \sin v\cos(dv) \\ -\sinh(cu) \end{pmatrix}\end{aligned}$$

where $c = \sqrt{1-d^2}$ and d is any proper fraction contained within the unit interval: $0 < d \equiv p/q < 1$. There I report calculations (details too intricate to transcribe, so allowed to remain in *Mathematica*'s memory) that establish

$$\mathbb{G}(u,v;d) = \|g_{ij}(u,v;d)\| = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix} \quad : \quad \text{positive definite}$$

$$\mathbb{H}(u,v;d) = \|h_{ij}(u,v;d)\| = \begin{pmatrix} e & 0 \\ 0 & g \end{pmatrix} \quad : \quad \text{negative definite, traceless}$$

[9] "Origin of the sine-Gordon equation," (April, 2016).

and, moreover, that in every instance

$$K = \frac{\det \mathbb{H}(u, v; d)}{\det \mathbb{G}(u, v; d)} = -1$$

The Enneper surfaces comprise, therefore, an *infinite set* of pseudospheres (one for every d). One expects the simplicity of $\mathbb{H}(u, v; d)$ to make it relatively easy to develop the details of the transformation $\{u, v\} \to \{x, y\}$ to asymptotic coordinates, therefore to construct $\mathbb{J}$, therefore—by the methods illustrated above—to produce $\mathbb{G}(x, y; d)$ and $\mathbb{H}(x, y; d)$ without heavy *ab initio* calculation. Graphic representations of Enneper surfaces are, by the way, marvelously intricate, glorious to behold. Such surfaces are of interest to Rogers & Schief[10] because they give rise to stationary multi-soliton "breather" solutions of the sine-Gordon equation.

Remark concerning the origin of this essay. Given the arbitrarily parameterized description $\boldsymbol{r}(u, v)$ of a surface Σ, the 2$^{\text{nd}}$ Fundamental Form leads to a matrix

$$\mathbb{H}(u, v) = \begin{pmatrix} e(u, v) & f(u, v) \\ f(u, v) & g(u, v) \end{pmatrix}$$

The sign of $\det \mathbb{H}(u, v)$ determines whether the Gaussian curvature is positive or negative at the point $\{u, v\}$. At points of negative curvature real-valued self-conjugate tangent vectors (solutions of $(\boldsymbol{x}|\mathbb{H}|\boldsymbol{x}) = 0$) exist, and occur in pairs. Asymptotic parameterizations $\boldsymbol{r}(x, y)$ of hyperbolic surfaces are defined by the condition that the tangent vectors $\{\boldsymbol{r}_x(x, y), \boldsymbol{r}_y(x, y)\}$ are self-conjugate (or "asymptotic") at all points $\{x, y\}$, and serve to inscribe asymptotic curves $\{\mathcal{C}_x, \mathcal{C}_y\}$ on Σ.

Rogers and Schief[11] begin their discussion of Tzitzeica surfaces with the unsupported remark that "in such cases [meaning what?]

$$\mathbb{H}(x, y) \text{ takes the form } \begin{pmatrix} 0 & f(x, y) \\ f(x, y) & 0 \end{pmatrix} \text{"}$$

My problem has been to understand the significance of that remark. Does it pertain to a specific class of cases—if so, which?—or does it refer (as was my hunch, and as emerged) to a general property of the asymptotic parameterizations? Noting that $\{\boldsymbol{r}_u(u, v), \boldsymbol{r}_v(u, v)\}$ and $\{\boldsymbol{r}_x(x, y), \boldsymbol{r}_y(x, y)\}$ provide bases on the tangent plane that are typically non-orthogonal, I look in a preparatory essay[5] to the use of non-orthogonal bases to construct matrix representations of linear operators and to the relationship between such representations, first in $\mathcal{V}_n$, then in $\mathcal{V}_2$ (the case of interest). That discussion did expose an interesting "generalized spectral decomposition theorem," but proved

[10] *Bäcklund and Darboux Transformations: Geometry & Modern Applications in Soliton Theory* (2002), §1.4.4, pages 38–41.

[11] *Ibid.*, page 89.

to be fundamentally misdirected. For such transformations turn out to be of the form

$$\mathbb{M} \to \tilde{\mathbb{M}} = \mathbb{S}\,\mathbb{M}\,\mathbb{S}^{-1}$$

which (except when $\mathbb{S}^{-1} = k\mathbb{S}^{\mathsf{T}}$) do not preserve symmetry, but do preserve trace/determinant/spectrum, and in all those respects are inconsistent with the basic facts of the matter at hand: $\mathbb{H}(u,v)$ and $\mathbb{H}(x,y)$ are invariably both symmetric, and their traces—at least in the cases contemplated by Rogers and Schief—are distinct.

I was brought thus to the belated realization that it is a mistake to conflate basis transformations and transformations of the parameters that support the constructions of $\mathbb{H}(u,v)$ and $\mathbb{H}(x,y)$. I look to transformations of the latter type in a second preparatory essay[12] and am led (as at (1.2) above) to a result of the form

$$\mathbb{M} \to \tilde{\mathbb{M}} = \mathbb{J}^{\mathsf{T}}\mathbb{M}\,\mathbb{J}$$

which *does* preserve symmetry and *does not* preserve trace. I show there that in the specific cases of the hexenhut and pseudosphere (both of which are hyperbolic) one does indeed find that

$$\mathbb{H}(x,y) \text{ has the form } \begin{pmatrix} 0 & f \\ f & 0 \end{pmatrix} \quad : \quad \{x,y\} \text{ asymptotic}$$

Building upon that foundation, I show in the present essay that the preceding statement pertains to *all* hyperbolic surfaces.

I have retained those preparatory essays because both provide supplemental material of some independent interest.

[12] "Parameter transformations *vs.* basis transformations," (June, 2016).