A MATHEMATICAL NOTE

Interrelations among representations of the Dirac delta function

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May 2002

Introduction. I record here a small idea that came to my attention as a result of my effort to understand why a certain computational program proved barren.[1] The child seems to me to be too cute, and potentially too useful, to be allowed to die with his mother, so I take this opportunity to set him adrift amongst the bulrushes.

All proceeds from the elementary integral

$$\int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi}$$

of which

$$e^{b^2/4a} = \sqrt{a/\pi} \int_{-\infty}^{+\infty} e^{-at^2-bt} dt \quad : \quad \Re[a] > 0$$

is (complete the square, change variables) a corollary. By notational adjustment we have

$$e^{-\beta x^2} = \tfrac{1}{2\sqrt{\beta\pi}} \int_{-\infty}^{+\infty} e^{-t^2/4\beta} e^{ixt} dt$$

From

$$\int_{-\infty}^{+\infty} e^{-\beta x^2} dx = \sqrt{\pi/\beta}$$

we are led to construct the β-parameterized family of normalized functions

$$\begin{aligned} g(x,\beta) \equiv \sqrt{\beta/\pi}e^{-\beta x^2} &= \tfrac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-t^2/4\beta} e^{ixt} dt \\ &= \tfrac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-t^2/4\beta} \cos xt \, dt \end{aligned} \tag{1}$$

[1] "Toward an exact theory of lightbeams" (2002), page 32.

1. The basic idea. It is a familiar fact that (Figure 1) the Gaussian functions $g(x,\beta)$ become narrower/taller as β becomes larger, and thus contrive to provide

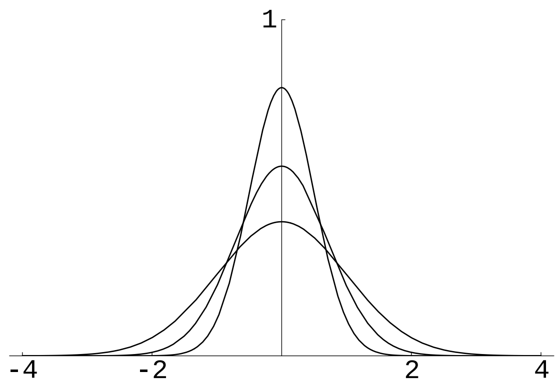

FIGURE 1: *The normalized Gaussians $g(x,\beta)$ become taller and narrower as β increases, and approach $\delta(x)$ in the asymptotic limit. Here β has been assigned the values* 0.5, 1.0 *and* 2.0.

a "representation of the δ-function":

$$\lim_{\beta\uparrow\infty} g(x,\beta) = \delta(x)$$

The idea is to let the $\lim\limits_{\beta\uparrow\infty}$ process be applied to the *right* side of (1): writing

$$\tfrac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-t^2/4\beta}\cos xt\,dt = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty}\big\{1-(t^2/4\beta)+\tfrac{1}{2!}(t^2/4\beta)^2-\cdots\big\}\cos xt\,dt$$

we adopt the interpretation

$$\int_{-\infty}^{+\infty} = \lim_{k\uparrow\infty}\int_{-k}^{+k}$$

and integrate termwise, obtaining

$$g(x,\beta) = \lim_{k\uparrow\infty}\Big\{G_0(x,k) - \tfrac{1}{4}\beta^{-1}G_1(x,k) + \tfrac{1}{32}\beta^{-2}G_2(x,k) - \cdots\Big\} \tag{2}$$

with

$$\begin{aligned} G_0(x,k) &\equiv \tfrac{1}{2\pi}\int_{-k}^{+k}\cos xt\,dt \\ &= \frac{\sin kx}{\pi x} \\ G_1(x,k) &\equiv \tfrac{1}{2\pi}\int_{-k}^{+k} t^2\cos xt\,dt \\ &= \frac{2kx\cos kx + (k^2x^2-2)\sin kx}{\pi x^3} \end{aligned}$$

$$G_2(x,k) \equiv \tfrac{1}{2\pi}\int_{-k}^{+k} t^4 \cos xt\, dt$$
$$= \frac{4kx(k^2x^2-6)\cos kx + (k^4x^4 - 12k^2x^2 + 24)\sin kx}{\pi x^5} \Bigg\}$$
$$\vdots$$

I digress to describe properties of those G-functions, though the main point of this discussion is staring us in the face already at (2):

The functions $G_0(x,k)$, $G_1(x,k)$ and $G_2(x,k)$ are plotted in Figure 2 and all share the same general design: in each case, the central peak becomes higher and the oscillations tighter as the value of k increases. Though it is not obvious

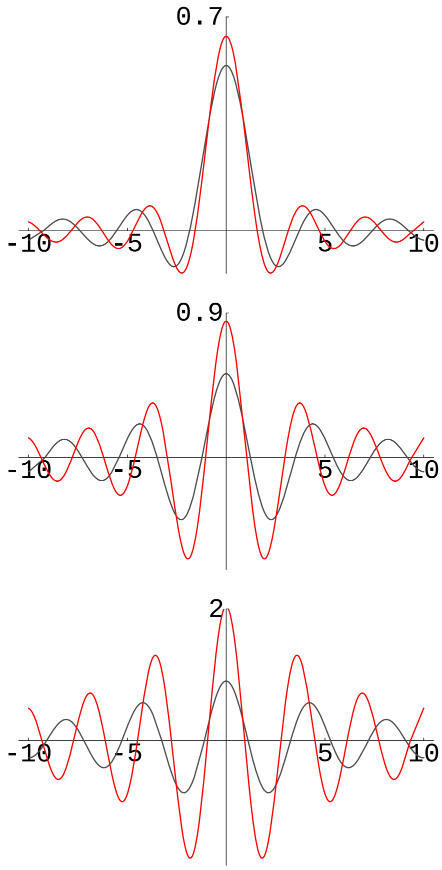

FIGURE 2: *Graphs of (reading top to bottom) $G_0(x,k)$, $G_1(x,k)$ and $G_2(x,k)$ with $k = 1.7$ and $k = 2.0$. The area under each of the top curves is unity, under each of the other curves is zero. Note that the functions $G_n(x,k)$ take progressively longer to die as n increases.*

to the eye, we are informed by *Mathematica* that

$$\begin{aligned}\int_{-\infty}^{+\infty} G_0(x,k)\,dx &= 1\\ \int_{-\infty}^{+\infty} G_1(x,k)\,dx &= 0\\ \int_{-\infty}^{+\infty} G_2(x,k)\,dx &= 0\\ &\vdots\end{aligned}$$

Return now to (2) and notice that the terms G_1, G_2, ... are turned off in the limit $\beta\uparrow\infty$. We are left with

$$\lim_{\beta\uparrow\infty}\Big[g(x,\beta)\equiv\sqrt{\beta/\pi}\,e^{-\beta x^2}\Big]=\delta(x)=\lim_{k\uparrow\infty}\Big[G_0(x,k)\equiv\frac{\sin kx}{\pi x}\Big] \tag{3}$$

One famous representation of $\delta(x)$ has here been transmuted into another.

2. The sech representation. Gaussians are by no means the only functions that Fourier transform into rescaled replicas of themselves: a second example is provided by the (normalized) sech distribution

$$s(x,\beta)\equiv(\beta/\pi)\,\text{sech}\beta x \tag{4}$$

The following identity[2] tells the story:

$$(\beta/\pi)\,\text{sech}\beta x=\tfrac{1}{2\pi}\int_{-\infty}^{+\infty}\text{sech}(\pi t/2\beta)\cos xt\,dt$$

Arguing as before, we have

$$\begin{aligned}&=\tfrac{1}{2\pi}\int_{-\infty}^{+\infty}\Big\{1-\tfrac{1}{2}(\pi t/2\beta)^2+\tfrac{5}{24}(\pi t/2\beta)^4-\cdots\Big\}\cos xt\,dt\\ &=\lim_{k\uparrow\infty}\Big\{G_0(x,k)-\tfrac{\pi^2}{8}\beta^{-2}G_1(x,k)+\tfrac{5\pi^2}{384}\beta^{-4}G_2(x,k)-\cdots\Big\}\\ &\downarrow\\ &=\lim_{k\uparrow\infty}\Big[\frac{\sin kx}{\pi x}\Big]=\delta(x)\qquad\text{as }\beta\uparrow\infty\end{aligned}$$

The interesting point is that $g(x,\beta)$ and $s(x,\beta)$ give rise by this line of argument (for evident reasons) to the *same* alternative representation of $\delta(x)$. Figure 3 provides a comparison of the two distributions here in question.

[2] A. Erdélyi *et al*, *Tables of Integral Transforms* (1954), Volume 1, **1.9.1**, page 30. The topic here touched upon is developed in detail in Chapter IX, "Self-reciprocal Functions" of E. C. Titchmarsh' s *Introduction to the Theory of Fourier Integrals* (2nd edition 1948).

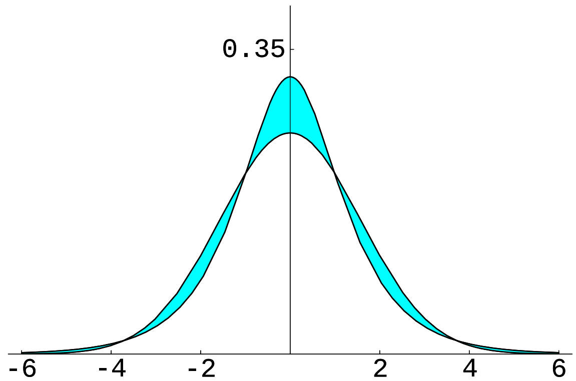

FIGURE 3: *Graphical comparison of* $s(x,\alpha) \equiv (\alpha/\pi)\text{sech}\,\alpha x$ *with* $g(x,\beta) \equiv \sqrt{\beta/\pi}e^{-\beta x^2}$. *In constructing the figure I have set* $\alpha = 1$ *and tuned the value of* β *so as to achieve equal variance*

$$\int_{-\infty}^{+\infty} x^2 s(x,1)\,dx = \int_{-\infty}^{+\infty} x^2 g(x,\beta)\,dx$$

which entails $\beta = 2/\pi^2$. *At the origin the sech-distribution stands here* 125% *taller than the Gaussian distribution. As it happens, a much closer approximation to the normal distribution is provided by a properly tuned "sechsquared-distribution"* $(a/2)\text{sech}^2 ax$*: see in this connection my Mathematica* Lab Manual (2000), Lab 1A.

Fourier self-reciprocity is an interesting property when it occurs, and a feature of both of the examples considered thus far, but it is inessential to the essence of the story. I turn now to an example that demonstrates the point:

3. The box representation. The normalized "box function"

$$b(x,\beta) \equiv \begin{cases} 0 & : \quad x < -a \\ \beta & : \quad -a < x < +a \\ 0 & : \quad +a < x \end{cases} \quad : \quad a \equiv 1/2\beta$$

can, in the language of *Mathematica*, be described

$$\begin{aligned} b(x,\beta) &= \frac{\texttt{Sign[a+x]+Sign[a-x]}}{\texttt{4a}} \\ &= \tfrac{1}{2}\beta\Big\{\texttt{Sign[}\tfrac{1}{2\beta}+x\texttt{]} + \texttt{Sign[}\tfrac{1}{2\beta}-x\texttt{]}\Big\} \end{aligned} \tag{5}$$

Some box functions are shown on the next page. A little exploratory tinkering (I took Erdélyi[2] **1.2.1**, page 7 as my point of departure) supplies the identity

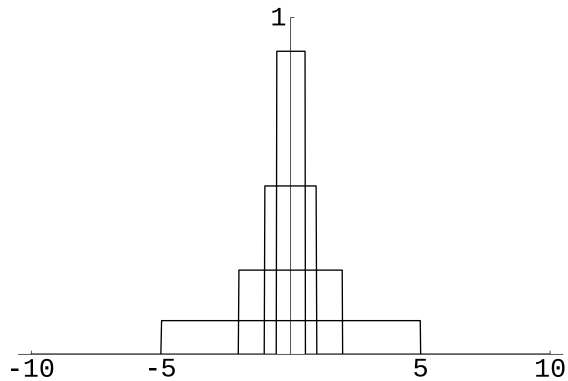

FIGURE 4: *Box functions* $b(x,\beta)$*, drawn by Mathematica on the basis of* (5) *with* $\beta = 0.1, 0.25, 0.50, 0.90$.

$$b(x,\beta) = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\sin(t/2\beta)}{(t/2\beta)} \cos xt\, dt$$

which leads us directly back again to a well-trod trail:

$$\begin{aligned}
&= \tfrac{1}{2\pi}\int_{-\infty}^{+\infty}\Big\{1 - \tfrac{1}{3!}(t/2\beta)^2 + \tfrac{1}{5!}(t/2\beta)^4 - \cdots\Big\}\cos xt\, dt \\
&= \lim_{k\uparrow\infty}\Big\{G_0(x,k) - \tfrac{1}{24}\beta^{-2}G_1(x,k) + \tfrac{1}{1920}\beta^{-4}G_2(x,k) - \cdots\Big\} \qquad (6)\\
&\downarrow \\
&= \lim_{k\uparrow\infty}\Big[\frac{\sin kx}{\pi x}\Big] = \delta(x) \qquad \text{as } \beta\uparrow\infty
\end{aligned}$$

4. Quick look at the landscape. Our short and easy hike has taken us already to a viewpoint, and we stop to look around.

One gets the impression that the preceding equation describes a *universal* representation of $\delta(x)$, in this sense: it will arise as a natural companion to *every* statement of the form

$$\delta(x) = \lim_{\beta\uparrow\infty} u(x,\beta) \quad : \quad u(x,\bullet)\ \underline{\text{even}} \text{ and } \underline{\text{Fourier transformable}}$$

In his short list[3] of the formal properties of $\delta(x)$ Dirac's first entry reads

$$\delta(-x) = \delta(x) \quad : \quad \delta(x) \text{ is to be thought of as an } \textit{even} \text{ function}$$

It becomes in this light natural to look to even representations, though it is certainly possible (but is it ever useful?) to look to representations with odd

[3] *Principles of Quantum Mechanics* (revised 4th edition 1967), page 60.

parts—representations of the form

$$u(x,\beta) + \beta^{-1}\cdot(\text{any odd function of } x)$$

But the asymptotic evaporation of the odd part would serve ultimately to bring us right back to where we already are.

It is my impression that the idea developed above is not quite so trivial as it might at first appear. Or—if trivial—that it may be of some value as a "cartoon" of a this more momentous circumstance: the quantum mechanical propagator can be developed in two quite different ways

$$K(x,t;y,0) = \begin{cases} \sum_n e^{-\frac{i}{\hbar}E_n t}\psi_n(x)\psi_n^*(y) & : \quad \text{spectral representation} \\ A(t)\sum_{\text{paths}} e^{\frac{i}{\hbar}S[\text{path}:(y,0)\rightarrow(x,t)]} & : \quad \text{Feynman representation} \end{cases}$$

The former supplies

$$\lim_{t\downarrow 0} K(x,t;y,0) = \delta(x-y)$$

by power series expansion in t, the latter by asymptotic expansion in t^{-1}.

5. Contact with theory relating to the asymptotic evaluation of integrals. We put our packs back on and head now farther up the trail, deeper into the woods . . .

Quantum mechanics has interesting (because classical!) things to say in the limit $\hbar^{-1}\uparrow\infty$, but its most characteristic statements arise when the limit process is *suspended*. Classical analysis provides a number of techniques[4] for developing asymptotic expansions of the form

$$\int_a^b f(t)e^{\beta g(t)}dt \approx I_0 + \beta^{-1}I_1 + \beta^{-2}I_2 + \cdots$$

and it is to such statements that our methods latently apply . . . for, while we have thus far allowed our δ-functions to prance about nakedly, it is in the decorous shade of $\int$-signs that they properly reside, and do their work. *Do* they work? I must be content here to approach the question anecdotally.

Suppose $f(x) = f_0 + f_1x + \frac{1}{2!}f_2x^2 + \frac{1}{3!}f_3x^3 + \frac{1}{4!}f_4x^4$. Then by direct integration

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)g(x,\beta)\,dx &= \int_{-\infty}^{+\infty} f(x)\sqrt{\beta/\pi}e^{-\beta x^2}dx \\ &= f_0 + \tfrac{1}{4}\beta^{-1}f_2 + \tfrac{1}{32}\beta^{-2}f_4\end{aligned}$$

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[4] See, for example, A. Erdélyi, *Asymptotic Expansions* (1956), Chapter 2 or Frank W. J. Olver, *Asymptotics & Special Functions* (1974 & 1997), Chapters 3, 4 & 9.

From (2) it would follow on the other hand that

$$\begin{aligned}&\int_{-\infty}^{+\infty} f(x)g(x,\beta)\,dx\\&\quad= \lim_{k\uparrow\infty}\int_{-\infty}^{+\infty} f(x)\Big\{G_0(x,k) - \tfrac{1}{4}\beta^{-1}G_1(x,k) + \tfrac{1}{32}\beta^{-2}G_2(x,k) - \cdots\Big\}dx\end{aligned}$$

which, according to *Mathematica*, supplies

$$\begin{aligned}&= \lim_{k\uparrow\infty}\Big\{\frac{\pi\texttt{Sign[k]}}{\pi}f_0 - \tfrac{1}{4}\beta^{-1}\frac{-\pi\texttt{Sign[k]}}{\pi}f_2 + \tfrac{1}{32}\beta^{-2}\frac{+\pi\texttt{Sign[k]}}{\pi}f_4 - \cdots\Big\}\\&= f_0 + \tfrac{1}{4}\beta^{-1}f_2 + \tfrac{1}{32}\beta^{-2}f_4\end{aligned}$$

—exactly as before. The remarkable fact operative here is that

$$\int_{-\infty}^{+\infty} G_n(x,k)\tfrac{1}{m!}x^m\,dx = \begin{cases}(-1)^n & : \quad m = 2n\\ 0 & : \quad \text{otherwise}\end{cases} \tag{7}$$

(or so I gather on the basis of some low-order experimentation): the function sets $\big\{G_0(x,k), G_1(x,k), G_2(x,k),\ldots\big\}$ and $\big\{1, x, \frac{1}{2}x^2,\ldots\big\}$ are, in other words, *biorthogonal*.

Look to a second example: by direct integration

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)b(x,\beta)\,dx &= \int_{-\frac{1}{2\beta}}^{+\frac{1}{2\beta}} f(x)\beta\,dx\\&= f_0 + \tfrac{1}{24}\beta^{-2}f_2 + \tfrac{1}{1920}\beta^{-4}f_4\end{aligned}$$

But this is precisely the asymptotic expansion that follows from (6) by (7).