Some Properties of a Population of

Chordally Partitioned Disks

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Introduction. This story begins with David Borwein's chance observation that if

$$s(x; a_0, a_2, \dots, a_n) \equiv \prod_{k=0}^{n} \text{sinc}(a_k x) \quad : \quad \text{all } a_k > 0$$

then

$$S_0 \equiv \int_{-\infty}^{+\infty} s(x; 1)dx = \pi$$

$$S_1 \equiv \int_{-\infty}^{+\infty} s(x; 1, \tfrac{1}{3})dx = \pi$$

$$S_2 \equiv \int_{-\infty}^{+\infty} s(x; 1, \tfrac{1}{3}, \tfrac{1}{5})dx = \pi$$

$$S_3 \equiv \int_{-\infty}^{+\infty} s(x; 1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7})dx = \pi$$

$$S_4 \equiv \int_{-\infty}^{+\infty} s(x; 1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \tfrac{1}{9})dx = \pi$$

$$S_5 \equiv \int_{-\infty}^{+\infty} s(x; 1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \tfrac{1}{9}, \tfrac{1}{11})dx = \pi$$

$$S_6 \equiv \int_{-\infty}^{+\infty} s(x; 1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \tfrac{1}{9}, \tfrac{1}{11}, \tfrac{1}{13})dx = \pi$$

but

$$\begin{aligned} S_7 &= \pi \cdot \tfrac{467807924713440738696537864460}{467807924720320453655260875000} \\ &= \pi \cdot 9999999999852937 < S_6 \\ S_8 &= \pi \cdot 9999999880796184 < S_7 \end{aligned}$$

For this surprising development—initially attributed to computer error—David and Jonathan Borwein (father and son) managed finally to provide an intricate theoretical explanation.[1] Hanspeter Schmid[2] traced the "Borwein phenomenon" to the circumstance that the sequence

$$\begin{aligned}
\sigma_0 &= 1 = 1\\
\sigma_1 &= 1 - \tfrac{1}{3} = \tfrac{2}{3}\\
\sigma_2 &= 1 - \tfrac{1}{3} - \tfrac{1}{5} = \tfrac{7}{15}\\
\sigma_3 &= 1 - \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} = \tfrac{34}{105}\\
\sigma_4 &= 1 - \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{9} = \tfrac{67}{315}\\
\sigma_5 &= 1 - \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{9} - \tfrac{1}{11} = \tfrac{422}{3465}\\
\sigma_6 &= 1 - \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{9} - \tfrac{1}{11} - \tfrac{1}{13} = \tfrac{2021}{45045}\\
\sigma_7 &= 1 - \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{9} - \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{15} = -\tfrac{982}{45045}\\
\sigma_8 &= 1 - \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{9} - \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{15} - \tfrac{1}{17} = -\tfrac{61739}{765765}
\end{aligned}$$

changes sign at σ_7.

I was inspired by the pattern of Schmid's argument to look again to the theory of random walks with diminishing steps,[3] first to *harmonic* walks—in which the k^{th} step has length

$$a_k = \frac{1}{pk+q} \quad : \quad pq \neq 0$$

(these become "Borwein walks" in the case $p = 2, q = -1$)—and then to *geometric* walks, in which the k^{th} step has length

$$a_k = \lambda^k$$

Geometric walks acquire special interest from the fact that they are *bounded* (by $\pm(1-\lambda)^{-1}$) if $\lambda < 1$. When one looks to the endpoints achieved by N simulated n-step geometric walks (n large, $N \gg n$ but for practical reasons much less than the total number 2^n of such walks) patterns emerge—patterns that change often radically in response to small adjustments of λ, but for some λ-values are quite distinctive. Most frequently encountered in the literature are the "Golden Walks" generated by

$$\lambda = \frac{1}{\text{Golden Ratio } \varphi} = 0.618034$$

for which the endpoint distribution possesses conspicuous self-similar/fractal

[1] "Some remarkable properties of sinc and related integrals," The Ramanujan Journal **6**, 73–89 (2001).

[2] "Two curious integrals and a graphic proof," Elemente der Mathematik **69**, 11–17 (2014).

[3] "On some Borwein-inspired properties of random walks with shrinking steps" (2016).

properties that P. L. Krapivsky & S. Redner[4] have discussed in useful detail. $\varphi = \frac{1}{2}(1+\sqrt{5})$ is a real root of the irreducible monic polynomial $x^2 - x - 1 = 0$ of which the other root $\frac{1}{2}(1-\sqrt{5})$ has modulus < 1, so is a "Pissot number." Endpoint distributions with properties analogous to the Golden distribution are generated by

$$\lambda = \frac{1}{\text{Pissot number}}$$

where the Pissot numbers obtainable from *quadratic* monic polynomials proceed $\varphi = 1.6180, 2.4142, 2.6180, 2.7421, 3.3028, 3.4121\ldots$ The *smallest* Pissot number (C. L. Segel, 1944) is the $\varphi_0 = 1.3247$ produced by $x^3 - x - 1 = 0$, and numbers $\varphi_0 <$ Pissot $< \varphi$ are produced by higher-order polynomials (see the Wikipedia article "Pissot numbers"). But I digress.

I was led to walks with shrinking steps from an initial interest in Borwein integrals. Krapivsky & Redner, on the other hand, took their interest in such walks from their applications (to physico-chemical problems, molecular spectroscopy in disordered media and such like), and in the course of their argument were led *back again* to sequences of Borwein-like integrals (though they appear to have been unaware of Borwein's work).

Recent work by S. N. Majundar & E. Trizac[5] has, in effect, closed the circle. By clever elaboration of Schmid's argument they manage not only to account for the Borwein phenomenon

$$S_n \equiv \int_{-\infty}^{+\infty} \prod_{k=0}^{n} \text{sinc}\big(\tfrac{x}{2k+1}\big)dx = \begin{cases} \pi & : \quad n = 0, 1, 2, \ldots, 6 \\ < S_{n-1} & : \quad n = 7, 8, 9, \ldots \end{cases} \tag{1}$$

but to establish (for example) the more vivid result

$$T_n \equiv \int_{-\infty}^{+\infty} \cos x \prod_{k=0}^{n} \text{sinc}\big(\tfrac{x}{2k+1}\big)dx = \begin{cases} \frac{1}{2}\pi & : \quad n = 0, 1, 2, \ldots, 55 \\ < T_{n-1} & : \quad n = 56, 57, 58\ldots \end{cases} \tag{2}$$

This is no mean accomplishment: *Mathematica* v11 running on MacOS 10.14.4 takes oddly staggered amounts of time to evaluate the S_n integrals: $(S_4, \ldots, S_8)$ took (0.54, 1.35, 21.66, 22.27, 1.92) seconds, respectively. Ditto the T_n integrals: $(T_4, \ldots, T_8)$, which took (0.52, 1.02, 21.65, 22.36, 1.93) seconds. But T_{55}, T_{56} appear to be quite out of the reach of anything less than a supercomputer,

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[4] "Random walk with shrinking steps," AJP **72**, 591–598 (2004). See in this connection also references cited in Wheeler[3].

[5] See "When random walkers help solving intriguing integrals," PR Letters **123**, 02021 (2019) and "When random walkers help solving intriguing integrals: supplemental material," (unpublished). I became aware of this work when David Griffiths called to my attention the fact that a synopsis "Random walkers illuminate a math problem: a family of tricky integrals can now be solved without explicit calculation" by Heather Hall was the featured article in the September issue of PHYSICS TODAY (pages 18–19).

though *Mathematica* is in this regard pretty super; it took only 1.92 seconds to produce

$$S_8 = \pi \cdot \frac{1770869518305619064249731553062842229556986511 9}{17708695394150597647449176493763755467520000000}$$

In their introductory remarks, Majundar & Trizac—to illustrate that sequences that do not adhere to the pattern suggested by their leading terms are a fairly commonplace mathematical phenomenon—borrow from John Conway and Richard Guy an example discussed on pages 76–79 of their *The Book of Numbers* (1996). It is that example that comprises my principal subject matter.

TOPOLOGICAL INVARIANTS OF PARTITIONED DISKS

The basic construction. Position n points ("nodes") on a circle in such a way that

- none of the $\binom{n}{2}$ chords are parallel;
- none of their points of intersection ("interior vertices") are coincident.

Call the resulting construction $\mathbb{D}_n$. Erasure of the counding circle produces a complete connected graph, $\mathbb{G}_n$. Figures 1–5 illustrate the cases $n = \{2, 3, 4, 5, 6\}$.[6]

Let $\{v(n), e(n), f(n)\}$ and $\{V(n), E(n), F(n)\}$ denote the number of vertices/edges/faces evident in $\mathbb{G}_n$ and $\mathbb{D}_n$, respectively. Those numbers are "topological invariants" in the sense that they are invariant under nodal displacements that preserve the stipulated conditions. Our ultimate objective is to describe $F(n)$. Inspection of the $\mathbb{G}$-figures ($n = 2, 3, 4, 5, 6$) supplies the following data:

n	$v(n)$	$e(n)$	$f(n)$	$v(n) - e(n) + f(n)$
2	2	1	0	1
3	3	3	1	1
4	5	8	4	1
5	10	20	11	1
6	21	45	25	1

The final column demonstrates compliance with the relevant instance of Euler's Formula.

Vertex counting. The n nodal points define a population of $\binom{n}{2}$ non-parallel lines, called "chords" where they fall inside the bounding circle. Those intersect at $\binom{\binom{n}{2}}{2}$ points, which may be coincident at nodes, but are otherwise distict. The nodal points mark the corners of $\binom{n}{4}$ distinct quadrilaterals with non-parallel sides. The sides of any given one of those quadrilaterals intersect at a pair of points that (see Figure 6) fall *outside* the circle, while the diagonals intersect

[6] Regular placement of the nodes produces figures that (for $n > 3$) may violate the stipulated conditions, so in constructing the figures of order n I assign to the nodes the angular addresses $\theta_k = k(2\pi/n) + \alpha_k : k = 0, 1, 2, \ldots, n-1$, where the α_k are drawn randomly from the interval $[0, \frac{1}{3}(2\pi/n)]$.

at a solitary *interior* point. So all together we have

$$\begin{aligned} 2\binom{n}{4} &\text{ exterior vertices}\\ \binom{n}{4} &\text{ interior vertices}\end{aligned}$$

At eachof the n nodes $\binom{n-1}{2}$ vertices become coincident, and so far as $\mathbb{D}_n$ and $\mathbb{G}_n$ are concerned count as a single vertex.[7] We are brought thus to the conclusion that

$$\begin{aligned} v(n) = V(n) &= \binom{n}{4} + n\\ &= \binom{n}{4} + \binom{n}{1} \end{aligned} \tag{3}$$

which conforms to the tabulated v-data. When we attempt to fit a function of the form

$$\varphi(n; a, b, c, d, e) = a\binom{n}{0} + b\binom{n}{1} + c\binom{n}{2} + d\binom{n}{3} + e\binom{n}{4}$$

to the tabulated v-data by setting

$$\begin{aligned} \varphi(2; a, b, c, d, e) &= 2\\ \varphi(3; a, b, c, d, e) &= 3\\ \varphi(4; a, b, c, d, e) &= 5\\ \varphi(5; a, b, c, d, e) &= 10\\ \varphi(6; a, b, c, d, e) &= 21 \end{aligned}$$

Mathematica supplies

$$v(n) = V(n) = \varphi(n; 0, 1, 0, 0, 1) = \binom{n}{1} + \binom{n}{4}$$

and so gives back (3).

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[7] Since, on the other hand, $\binom{n}{2}$ non-parallel lines intersect at $\binom{\binom{n}{2}}{2}$ points, we have the curious identity

$$\begin{aligned} \binom{\binom{n}{2}}{2} &= 3\binom{n}{4} + n\binom{n-1}{2}\\ &= \tfrac{1}{8}(2n - n^2 - 2n^3 + n^4) \end{aligned}$$

which checks out numerically.

Edge & face counting. Proceeding similarly from the tabulated e-data we obtain

$$e(n) = E(n) = \varphi(n; 0, 0, 1, 0, 2) = \binom{n}{2} + 2\binom{n}{4} \tag{4}$$

while the tabulated f-data gives

$$f(n) = \varphi(n; 1, -1, 1, 0, 1) = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \binom{n}{4} \tag{5}$$

We are satisfied that the accuracy of (3) extends beyond the tabulated data to all n, and are encouraged by

$$\begin{aligned} v(n) - e(n) + f(n) &= \left\{\binom{n}{1} + \binom{n}{4}\right\} - \left\{\binom{n}{2} + 2\binom{n}{4}\right\} \\ &\qquad + \left\{\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \binom{n}{4}\right\} \\ &= \binom{n}{0} = 1 \end{aligned} \tag{6}$$

to think that the accuracy of (4) and (5) do too. But the Euler relation (6) is stable under adjustments of the form

$$\begin{aligned} e(n) &\longrightarrow e(n) + k(n) \\ f(n) &\longrightarrow f(n) + k(n) \end{aligned}$$

To close the argument we would have to establish that no such $k(n)$ can exist. This I will not linger to do.

The point of it all. $\mathbb{D}_n$ possesses all the faces of $\mathbb{G}_n$ plus an additional $n = \binom{n}{1}$ crescent faces, so

$$F(n) = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} \tag{7}$$

according to which the numbers $F(n)$ advance in the sequence

$$1, 2, 4, 8, 16, \mathbf{31}, 57, 99, 163, 256, \ldots$$

The leading terms suggest the progression 2^{n-1}, which would give

$$1, 2, 4, 8, 16, \mathbf{32}, 64, 128, 256, 512, \ldots$$

and so fails for $n \geqslant 6$. The Pascal identity $\binom{n}{m} = \binom{n-1}{m-1} + \binom{m-1}{m}$ can be used to bring (7) to the form

$$F(n) = \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4}$$

adopted by Conway & Guy.

ADDENDUM

Reading further into Conway & Guy (with random walks/Markov processes still alive in the back of my brain) I was interested to encounter (page 167) passing reference to things called "Markov numbers," which turn out to be any of the integers m encountered in the highly-structured infinite family of triples (x, y, z) that satisfy the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz$$

That equation rang bells, because—as I realized at length—it resembles the equation

$$x^3 + y^3 + z^3 - 3xyz = 1$$

that defines the "hexenhut," a pseudosphere-like surface the differential geometry of which I was in 2016 stimulated by correspondence with Ahmed Sebar to study in extravagant detail.[8] But those, obviously, are horses of quite different colors.

[8] "Geodesics on the pseudosphere & hexenhut" (January 2016); "Geodesics on surfaces of revolution: general theory applied to paraboloid & hexenhut" (February 2016).