

EXERCISES

Mathematica 6 ~ Lab Number 6

Problem 1: River Meanders. It was apparently Einstein (1926) who first discussed why rivers, in their progress across gently sloped alluvial plains, do not pursue straight courses (such motion is evidently unstable), but meander. In recent reading I was alerted¹ to the existence of a recent contribution to that pastoral subject: Hans-Henrik Støllem, in “River meandering as a self-organization process,” *Science* **271**, 1710 (1996), brings modern computer resources, the theory of fractals and field data to bear on the problem, and is led to the striking conclusion that typically—on average—one can expect to find

$$\frac{\text{wet distance from source to mouth}}{\text{distance as a crow flies}} \sim \pi$$

The question arise: what would such a river look like in an idealized case? We begin by drawing a picture of idealized “River Alpha.” Enter these commands:

$$\alpha = \pi/5$$

```
a=Sin[α]
b=Cos[α]
```

```
x1[θ_]:=a+Cos[θ]
x2[θ_]:=3a+Cos[θ]
x3[θ_]:=5a+Cos[θ]
x4[θ_]:=7a+Cos[θ]
x5[θ_]:=9a+Cos[θ]
y[θ_]:=+b+Sin[θ]
z[θ_]:= -b+Sin[θ]
```

```
upperloops
```

```
=ParametricPlot[{{x1[θ],y[θ]}, {x3[θ],y[θ]}, {x5[θ],y[θ]}},
{θ,-π/2+α,3π/2-α},
AspectRatio->Automatic, PlotStyle->{
```

¹ See Simon Singh, *Fermat's Enigma* (1997), page 17. See also page 329 in A. Pais, *Subtle is the Lord...:The Science and the Life of Albert Einstein* (1982).

2

```
{Blue, Thickness[0.02]},
{Blue, Thickness[0.02]},
{Blue, Thickness[0.02]}}], Axes->False,
PlotRange->{{-1,8},{-2,2}}]

lowerloops
=ParametricPlot[{{x2[θ],z[θ]}, {x4[θ],z[θ]}},
{θ,-3π/2+α,π/2-α},
AspectRatio->Automatic, PlotStyle->{
{Blue, Thickness[0.02]},
{Blue, Thickness[0.02]}}], Axes->False,
PlotRange->{{-1,8},{-2,2}}]

Show[upperloops, lowerloops, Graphics[
{Thickness[0.01], Red, Line[{{0,0}, {10a,0}}]}]]]
```

You have with these commands constructed an illustration of the meandering river (five semi-circular loops) and the flight of a crow. The figure can be considered to be have been assembled from “primitive cells”:

```
bluearc=ParametricPlot[{Cos[θ], Sin[θ]}, {θ,α,π},
AspectRatio->Automatic, Ticks->False, Axes->False,
PlotStyle->{Blue, Thickness[0.02]}]

lines=Graphics[{
Line[{{-1,0}, {Cos[α],0}}],
Line[{{0,0}, {Cos[α],Sin[α]}]},
{Red, Thickness[0.01],
Line[{{Cos[α],0}, {Cos[α],Sin[α]}]}],
Text[Styleform["α", FontSize->14], {0.15, 0.04}]}]

Show[bluearc, lines]
```

Take a moment to recognize how this figure—after a clockwise 90° rotation—relates to our image of a meandering river. Use **Plot** to estimate the value of α implied by Støllem’s relation, then use **FindRoot** to obtain the more precise value

$$\alpha = 0.827859 \text{ radians} = 47.4328^\circ$$

Now back up to the beginning, assign the computed value (in radian measure) to α , and reconstruct the picture of the river... which is now a “Støllem meander.”

One can think of ways to make the model more realistic, to make it dynamical. Note the characteristic events that result in the creation of oxbow lakes and temporary straightening of the channel.

Problem 2: Solitons. An equation central to the theory of nonlinear waves (continuous analog of the theory of nonlinear oscillators) is the “KdV equation”²

$$u_t + 6uu_x + u_{xxx} = 0$$

which D. J. Korteweg & G. de Vries (1882) devised to account for certain solitary hydraulic waves first noted by John Scott Russell in 1834. Notice that it is again the middle term that contributes the nonlinearity, and causes it to be the case that [solutions added to solutions are not again solutions](#).

Remind yourself what *Mathematica* has to say about **?D**, then use that information to ask *Mathematica* for a demonstration that every function of the form

$$u(x, t; b) \equiv 2b^2 \operatorname{sech}^2(bx - 4b^3t) \quad : \quad b \text{ arbitrary} \quad (1)$$

is a particular solution of the KdV equation.

Animate (with $0 \leq t \leq 4$) the superimposed graphs of

$$u(x, t, 1) \text{ and } u(x, t, 1/\sqrt{2}) \quad : \quad -4 \leq x \leq 20$$

Notice that (i) each wave *preserves its shape* as it moves (that’s why such waves are called “solitons”); (ii) amplitude is coupled to wavespeed (tall/narrow waves move faster than shorter/broader ones).

The occurrence/preservation of stable forms (“fixed points” in some broad sense) is reminiscent of our experience with the with the Van der Pol equation, and also with iterated maps. For helpful discussion of nonlinear wave systems (and a valuable bibliography) see the recent thesis of Adam Halverson.³

Problem 3: Fourier Analysis of a KdV Soliton. It is our habit to resolve complex waves into their elementary (Fourier) components. In nonlinear wave theory that exercise is still possible but less useful: the Fourier components will not themselves be primitive *solutions* of the wave equation! But let’s pursue the idea anyway, to see where it leads.

Ask what *Mathematica* has to say in response to

?FourierTransform

?InverseFourierTransform

Let’s see how this works in some simple cases. Assign the name **normedgauss** to the function

$$g(x; a) \equiv \sqrt{a/2\pi} e^{-\frac{1}{2}ax^2}$$

and demonstrate that

$$\int_{-\infty}^{+\infty} g(x; a) dx = 1 \quad : \quad \text{all positive real } a$$

Now use **FourierTransform[expr, x, y]** to construct

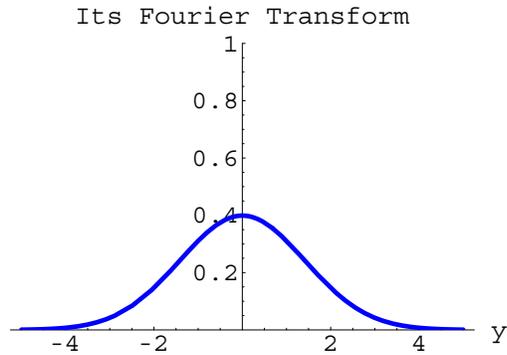
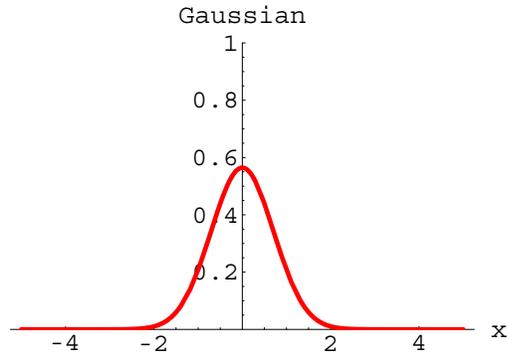
$$G(y; a) \equiv \text{Fourier transform of } g(x; a)$$

² Subscripts are used here to denote partial derivatives.

³ “The inverse scattering transform and the Korteweg-de Vries equation,” (Reed College 2000).

4

Now use **Manipulate** to construct a display of the following design, (use **GraphicsGrid**) where the upper figure plots $g(x, a)$, the lower figure plots $G(y, a)$ with $-5 \leq x, y \leq 5, \frac{1}{5} \leq a \leq 5$.



Use **InverseFourierTransform** to recover $g(x, a)$ from $G(x, a)$.

Do for the functions

$$h(x; a) \equiv \frac{a}{\pi} \operatorname{sech}(ax) \quad \text{and its transform} \quad H(y; a)$$

all the things just done with $g(x; a)$ and $G(y; a)$.

Problem 4: Gibbs' Phenomenon. As we have just seen, a powerful and easy-to-use Fourier analytic capability is built right into *Mathematica 6*. But that power is greatly expanded when one installs the “Fourier Series” Standard Add-on Package, which is done by commanding **Needs["Fourier Series`"]**. Command **?FourierSeries** to gain some sense of the capabilities of this resource.

Define

```
squarewave[x_]:= -UnitStep[x+3]+2UnitStep[x+2]-2UnitStep[x+1]
+2UnitStep[x+0]-2UnitStep[x-1]+2UnitStep[x-2]
```

and give the name **Exact** to the figure produced when you **Plot** that function on $-3 < x < 3$, subject to the options **AspectRatio->Automatic**, **PlotStyle->{Red, Thick}**.

Evidently, $\text{squarewave}(x)$ is an odd periodic function, with period 2. We expect it to be describable as a Fourier sine series.

Command (this will take a minute)

```
FourierTrigSeries[squarewave[x],x,15,FourierParameters->{0,1/2}]
```

and execute the command **Expand[%]**. We are motivated by this result to introduce the function

$$f(x, n) = \sum_{k=0}^n \frac{4 \sin[(2k+1)\pi x]}{(2k+1)\pi}$$

(which in the case $n = 7$ gives back the function we just constructed). Use **Manipulate** to demonstrate what happens when $f(x, n)$ is plotted on $-3 \leq x \leq 3$ and n is made to range from 5 to 25 in steps of 5. You see how the approximating function (which is continuous) struggles to achieve the discontinuity which we built into the design of the exact squarewave, and how in particular it acquires “ears.” This is illustrative of a general circumstance, known as “Gibbs’ phenomenon” (consult the discussion provided by Wikipedia).

It is to expose the situation more clearly that we define

$$\text{Gibbs}(x, n) = 1 - f(x, n)$$

and (do it) use **Manipulate** to demonstrate what happens when the absolute value of $\text{Gibbs}(x, n)$ is plotted on $0 \leq x \leq 1$ and n is made to range as before.

Problem 5: Griffiths’ “Needle Problem”. For reasons that will soon emerge, we undertake to place a red dot at each of the curve produced by the command **Plot[Abs[Gibbs[x,7]],{x,0,1}, PlotRange->{0,1}]**. To that end, we define

$$\text{GibbsPrime}(x) = \frac{\partial \text{Gibbs}(x, 7)}{\partial x}$$

and command **NSolve[{GibbsPrime[x]==0, Abs[x]==x},x]** and define **peaks={x,Abs[Gibbs[x,7]]}/.%**. Which places us in position to command **Plot[Abs[Gibbs[x,7]],{x,0,1}, PlotRange->{0,1}, Epilog->{PointSize[0.02],Red,Map[Point,peaks]}**

It seems to be a principle of scholarship that if you know two things—*any* two things—you are destined to see a relationship between them. As it happens, I have seen such point distributions before. . . in the thesis of Ye Li.⁴ An effort was made there to fit the points to a curve of the form

$$w_p(x; a, b) = a + b \frac{1}{[x(1-x)]^p}$$

Griffiths had theoretical reasons for setting $p = \frac{1}{3}$, but Li (motivated only by a desire to achieve goodness of fit) set $p = \frac{1}{2}$. Let us follow Li's lead. Enter the command **?Fit**, and on the basis of the information thus provided, command

```
Fit[peaks, {1,  $\frac{1}{(x(1-x))^{\frac{1}{2}}}$ }, x]
```

and assign the name **GriffithsAndLi[x_]** to the function thus produced. Finally create

```
Plot[GriffithsAndLi[x], {x,0,1}, PlotRange->{0,1},  
Epilog->{PointSize[0.03], Hue[1], Map[Point, peaks]}]
```

Looks pretty good (and $p = \frac{1}{3}$ is found similarly to look a bit less good).

As the Fourier series is (prior to truncation) carried to higher and higher, we—on this evidence

```
Plot[{(4x(1-x))- $\frac{1}{2}$ , (4x(1-x))- $\frac{1}{4}$ , (4x(1-x))- $\frac{1}{8}$ }, {x,0,1},  
PlotRange->{0,4},  
PlotStyle->{Red, Black, Blue}]
```

—expect to have to make p smaller and smaller. But Gibbs' phenomenon is persistent to very high order; we expect the process $p \downarrow 0$ to be quite slow.

By a Fourier-analytic argument rooted in the fractional calculus⁵ I have shown—I think to Griffiths' satisfaction—that the charge density on a needle is actually uniform, and that the seeming “horns” are artifacts of his numerical approximation procedure (but artifacts that persist to very high order).

Many questions now arise, among them this one: Is it possible that Gibbs' phenomenon has only incidentally to do with Fourier series, and is really an aspect of the general theory of approximation? I will not pursue the matter; it seems fitting that this introduction to *Mathematica* should end with a question—a question which we would scarcely have been in position even to pose had we been deprived of the assistance of this powerful new tool.

⁴ “Charge density on a needle,” (Reed College 1994). An expanded version of that work was published as David Griffiths & Ye Li, “Charge density on a conducting needle,” *AJP* **64**, 706 (1996).

⁵ See §10 in “Construction & physical application of the fractional calculus” (1997).

Problem 6: Link Cration & Imported Graphic. Create a link to the Wikipedia site that discusses Caravaggio's "Narcissus."

Import the painting. As a preliminary step you may need to drag the figure to a blank TextEdit page, and to give that page a filename.