EXERCISES

Mathematica 6 ~ Lab Number 5

NOTE: Your ability to make effective use of *Mathematica* in the course of your own future work will hinge critically on your ability to look things up, as needed. You will need to cultivate a technique for discovering the techniques of which you have momentary need. To encourage you in that effort I will, in this last pair of problem sets, allow myself to move farther and farther from material specifically covered in the lab manuals. But not to worry: I will provide lots of hints.

Problem 1: Invaded Planetary System. First, use Graphics Primitives to draw a figure consisting of

- a black point of **AbsolutePointSize[10]** at $\{0,0\}$
- a red point of **AbsolutePointSize[4]** at $\{1,0\}$
- a blue point of **AbsolutePointSize[8]** at $\{10,4\}$
- a blue circle of unit radius, centered at $\{0,0\}$
- a blue line from $\{-10,4\}$ to $\{+10,4\}$.

Suppose for the moment that the gravitational field of the blue body has been turned off. Imagine the black body to be pinned at the origin, and allow the red body (assumed to have unit mass) to move dynamically; if we work in units where $G=1$ and assume the black body to have mass 9, then the motion of the red body is described

$$\ddot{x}(t) = -\frac{9x(t)}{[x^2(t)+y^2(t)]^{\frac{3}{2}}}$$
$$\ddot{y}(t) = -\frac{9y(t)}{[x^2(t)+y^2(t)]^{\frac{3}{2}}}$$

We will assume that initially

$$x(0)=1, \quad y(0)=0, \quad \dot{x}(0)=0, \quad \dot{y}(0)=3$$

Give the name **UnperturbedOrbit** to the result of using **NDSolve** to solve the equations of motion for times $0<t<3$, then plot **UnperturbedOrbit** and color it red; this exercise serves to demonstrate that we have successfully rigged the initial data to yield a *circular* orbit of unit radius. Be sure to install the options

```
AspectRatio->Automatic
PlotRange->All
```

Now turn on the gravitational field of the blue body (assumed to have mass $\frac{1}{2}$, and to be pinned at $\{10, 4\}$); the equations of motion become

$$\ddot{x} = -\frac{9x}{[x^2+y^2]^{\frac{3}{2}}} + \frac{\frac{1}{2}(10-x)}{[(10-x)^2+(4-y)^2]^{\frac{3}{2}}}$$
$$\ddot{y} = -\frac{9y}{[x^2+y^2]^{\frac{3}{2}}} + \frac{\frac{1}{2}(4-y)}{[(10-x)^2+(4-y)^2]^{\frac{3}{2}}}$$

Preserving the former initial conditions, and working now on $0 < t < 80$, again solve the equations (call the solution **PerturbedOrbit**) and plot the orbit.

Now assume the blue body to move uniformly along the blue line, from right to left. Assume more particularly that it departs the point $(10, 4)$ at $t = 0$ and arrives at the point $(-10, 4)$ at time $t = 80$. Again solve and plot the solution (called now **ImpactedOrbit**), working again on $0 < t < 80$.

Finally, repeat that exercise on the assumption that the blue body starts at the left end of the blue line and moves (with its former uniform speed) to the right. When trying to make carry-home intuitive sense of your result, don't forget to take into account the fact that in both cases the red particle is moving counterclockwise.

Problem 2: Nonlinear Oscillations. In the following "scrubbed" instance of the unforced damped oscillator equation

$$\ddot{x} + b\dot{x} + x = 0$$

we will allow b to become x-dependent (which will destroy the linearity of the equation: solution + solution will no longer be a solution!). In particular, we set

$$b \longrightarrow b(x) \equiv \mu(x^2 - 1) \quad : \quad \mu \geqslant 0$$

and obtain the "unforced Van der Pol equation," which is fundamental to the theory of nonlinear oscillations.[1] **Plot** $b(x)$ and notice that the $\dot{x}$-term describes damping (energy loss) if $x < -1$ or $1 < x$, but "antidamping" (energy injection) if x is small: $-1 < x < +1$. The Van der Pol oscillator is "self-exciting," with consequences that will soon become evident.

Create a link to the website to the scholarpedia article about the Van de Pol oscillator.

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[1] Balthasar van der Pol (1889–1959) was an engineer working for the N. V. Philips' Glowlamp Works in Eindhover, Holland when his study of the operating characteristics of triodes (early radio tubes with cathode, anode & grid) led him to publish "Forced oscillations in a system with non-linear resistance," Phil. Mag. **3**, 65 (1927), of which the Dutch version had appeared already in 1924.

Set $\mu = \frac{1}{8}$, $x(0) = 1$ and $\dot{x}(0) = 0$, then (for $0 < t < 30$) **NDSolve** and **Plot** the solution of

$$\ddot{x}(t) + \mu[x^2(t) - 1]\dot{x}(t) + x(t) = 0$$

Notice that the oscillation grows to an apparently stable value.

Do the same with $x(0) = 4$. The oscillation damps to an apparently stable value.

Now set $\mu = 8$, $x(0) = 1$ and $\dot{x}(0) = 0$ and proceed as before. You have entered the regime most characteristic of Van der Pol oscillators.

The idea now is to promote $\dot{x}$ to the status of an independent variable, display the Van der Pol equation as a coupled pair of first order equations

$$\dot{x} = y$$
$$\dot{y} + \mu(x^2 - 1)y + x = 0$$

and to look at the phase plot. To that end, give names **phase1**, **phase2** and **phase3** to the results of **NDSolve**'ing the preceding system—work as before on $0 < t < 30$—with

$$\mu = 8, \quad x(0) = 0.0, \quad y(0) = 01.0$$
$$\mu = 8, \quad x(0) = 0.0, \quad y(0) = 12.0$$
$$\mu = 8, \quad x(0) = 2.1, \quad y(0) = 17.2$$

Plot those results, calling the results **phasemap1**, **phasemap2** and **phasemap3**. Adopt the options

```
PlotRange->All
AspectRatio->Automatic
Ticks->None
```

and make the figures respectively red, black and blue.

Finally, use **Show** to superimpose those figures. Notice that after initial transcients have died down they *precisely coincide*! This is but one of many wonderful properties—properties of high practical importance—exhibited by nonlinear oscillators.[2]

Problem 3: Oscillators, Coupled in Various Ways. The equations

$$\ddot{x} + 2^2 x = 0$$
$$\ddot{y} + 5^2 y = 0$$

describe the motion of an uncoupled pair of oscillators, with natural frequencies that stand in the ratio 2:5. Assume that initially

$$x(0) = 1, \quad y(0) = 0, \quad \dot{x}(0) = 0, \quad \dot{y}(0) = 2$$

Use **NDSolve** to obtain the solution (for $0 \leqslant t \leqslant 100$) of those equations, and call it **Uncoupled**. Command

```
ParametricPlot[{x[t],y[t]}/.Uncoupled,{t,0,100},
AspectRatio→Automatic, PlotRange→{{-1,1},{-0.4,0.4}}
```

[2] For a nice account of this subject—a famously "difficult" subject which *Mathematica* makes much more accessible—see A. H. Nayfeh & D. T. Mook, *Nonlinear Oscillations* (1979).

to generate a representation of the motion. You recognize this to be a simple Lissajous figure.

Now introduce weak damping:

$$\ddot{x} + 2^2 x = -0.1\dot{x}$$
$$\ddot{y} + 5^2 y = -0.1\dot{y}$$

Obtain the solution (same initial conditions as before, and as henceforth), call it **DampedUncoupled** and plot it.

Now turn off the damping, turn on ordinary coupling terms

$$\ddot{x} + 2^2 x = 5y$$
$$\ddot{y} + 5^2 y = 5x$$

Obtain the solution, call it **UndampedCoupled**, plot it (with the **PlotRange** option deleted).

Do the same with the coupling constant increased from 5 to 9. Do the same with the coupling constant increased to 9.999.

Look next to the very weekly **DampedCoupled** system

$$\ddot{x} + 2^2 x = 9y - 0.005\dot{x}$$
$$\ddot{y} + 5^2 y = 9x - 0.005\dot{y}$$

You will need to install the option **PlotRange→{{-1.3,1.3},{-0.5,0.5}}**.

Look similarly to the "**GyroCoupled**" system

$$\ddot{x} + 2^2 x = -0.6\dot{y}$$
$$\ddot{y} + 5^2 y = +0.6\dot{x}$$

The **PlotRange** option can again be deleted.

Look finally to the **HybridCoupled** system

$$\ddot{x} + 2^2 x = -0.6\dot{y}$$
$$\ddot{y} + 5^2 y = +0.6x \quad \text{(no dot on the } x\text{)}$$