ELECTRODYNAMICAL APPLICATIONS
OF THE
EXTERIOR CALCULUS

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June 1996

Physical introduction. It is a fact of experience that the objects which comprise the central subject matter of mechanics—the localized entities (planets, pith balls, electrons) which Newton was led to endow with "mass"—are sometimes endowed also (and independently) with "electric charge." Charged bodies exert forces upon one another by virtue of their charge (just as "gravitationally charged" bodies do), and respond to those forces in accordance with the established principles of mechanics.

The electromagnetic interaction of particles brings naturally to mind the "action at a distance" which is a characteristic feature (and was always considered to be a philosophically bothersome feature) of Newton's account of the gravitational interaction of massive bodies, which in many ways it resembles. It was by slow degrees, and reluctantly, that during the course of the 19^{th} Century this mental image of events gave way to another image—an image wherein charged particles act (not directly upon one another but) upon an intermediating "electromagnetic field," and only by mediated indirection upon other charged particles. The new image, however profound its consequences, is not at all alien to simple experience; when I step into my boat, the boat tied up nearby moves (after a brief delay) not because of some imagined "boat-boat interaction" but in consequence of the "boat-water interaction." I could account quantitatively for the motion of the neighboring boat if I possessed sufficiently detailed knowledge of (i) the dynamics of water (a subject which might be pursued quite independently of any reference to boats) and (ii) the forces which boats and contiguous water exert upon one another.

But while the "invention of water" placed no great demand upon the imaginative powers of sailors, the "invention of the electromagnetic field" is universally acknowledged to have been one of the most pregnantly imaginative acts in the history of physics. It was a messy birth, and it took decades (which is to say, until about the first decade of the present century) for physicists to appreciate that the electromagnetic field is a dynamical system like any other (except that, instead of being a spatially localized system like a "particle," it is a distributed system), and that the proper place to gain a view of the stark simplicities of the subject is not in a universe filled with the "stuff" dear to Victorian experimentalists (sealing wax, transformer oil, mica, etc.) but in a vacuum. The hardest lesson of all had to do with the fact that the vacuum in question is devoid even of an "electromagnetic medium" (or "æther")—that in electrodynamics one confronts something very like a theory of "water waves in the absence of water."

1. Maxwell's equations. Electrodynamics rests upon a phenomenological base which can be summarized as follows:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \rho \tag{1.1}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \tfrac{1}{c}\mathbf{j} \tag{1.2}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{1.3}$$

$$\boldsymbol{\nabla}\times\mathbf{E} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\mathbf{B} = \mathbf{0} \tag{1.4}$$

Here ρ, $\mathbf{j}$, $\mathbf{E}$ and $\mathbf{B}$ are fields—functions of the independent variables (x^1, x^2, x^3, t)—with these physical meanings:

ρ	describes charge density, assumed to have been prescribed
$\mathbf{j}$	describes current density, assumed to have been prescribed
$\mathbf{E}$	describes the electric field
$\mathbf{B}$	describes the magnetic field

and c is a fundamental "constant of Nature" with the physical dimensionality of velocity.[1] Equation (1.1) derives from the observation that electrostatic forces fall off "geometrically" (i.e., as $1/r^2$), which is the upshot of COULOMB'S LAW. Equation (1.2) provides a description of the magnetic fields which arise from steady currents (AMPERE'S LAW). Equation (1.3) asserts that magnetic fields never arise from "magnetic charges"—the conceivable (but assertedly non-existent) magnetic analogs of electric charges. Equation (1.4) describes the electric fields called into being by time-dependent magnetic fields (FARADAY'S LAW).

Such was the state of electromagnetic knowledge at the time of Maxwell's birth,[2] and such did it remain when, in 1854, Maxwell began his own

[1] It is a curious historical fact that c first entered the theory not as a "constant of Nature" but as a specifically electromagnetic construct

$$c \equiv \sqrt{1/\epsilon_0\mu_0}$$

where ϵ_0 and μ_0 no *not* individually enjoy "fundamental" status.

[2] Maxwell was born on 13 July 1831, and Faraday took up the work which quickly led to (1.4) in August of that same year. Maxwell died on 5 November 1879, not quite seven months after the birth of Einstein.

electrodynamical research. Maxwell was aware that the functions ρ and $\mathbf{j}$ which appear on the righthand sides of the "sourcey field equations" (1.1) & (1.2) cannot be independently specified, but are subject by charge conservation to the "continuity equation"

$$\tfrac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\!\cdot\!\mathbf{j} = 0 \tag{2}$$

and that equations (1) are, as they stand (since (1.2) entails $\boldsymbol{\nabla}\!\cdot\!\mathbf{j} = 0$), inconsistent with (2). In 1859 he observed that consistency could, however, be achieved (and that consistency with the then-available experimental evidence would not be damaged) if in place of (1) one wrote

$$\boldsymbol{\nabla}\!\cdot\!\mathbf{E} = \rho \tag{3.1}$$

$$\boldsymbol{\nabla}\times\mathbf{B} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\mathbf{E} = \tfrac{1}{c}\mathbf{j} \tag{3.2}$$

$$\boldsymbol{\nabla}\!\cdot\!\mathbf{B} = 0 \tag{3.3}$$

$$\boldsymbol{\nabla}\times\mathbf{E} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\mathbf{B} = \mathbf{0} \tag{3.4}$$

i.e., if in (1.2) one made the replacement $\mathbf{j} \longrightarrow \mathbf{j} + \frac{\partial}{\partial t}\mathbf{E}$. By this slight formal adjustment (i.e., by the invention of the so-called "displacement current") did the full-blown Maxwell-Lorentz equations (3) come into being; all subsequent physics (the invention of relativity, of quantum mechanics, of much else) and much subsequent technology (therefore much subsequent political/ cultural history) was shaped by a mathematical event which went largely unnoticed at the time, and the importance of which has remained largely unnoticed to this day.

2. Notational innovations. It was a formal property of the Maxwell-Lorentz equations (3) which in 1905 led Einstein (writing under the title "On the electrodynamics of moving bodies") to invention of special relativity. And it was Einstein's accomplishment which led Minkowski in 1907 to the formal developments to which I now turn.

Take advantage of the electrodynamic availability (which special relativity serves to universalize) of the constant c to write $x^0 \equiv ct$; x^0 has the status of a (rescaled) "time coordinate," but has the dimensionality of a length. Write

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \quad \text{with } \mu = 0, 1, 2, 3$$

and notice that $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$. Notice that the first pair of Maxwell-Lorentz equations (3) can in this notation be written

$$\begin{pmatrix} \partial_0 & \partial_1 & \partial_2 & \partial_3 \end{pmatrix} \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ +E_1 & 0 & -B_3 & +B_2 \\ +E_2 & +B_3 & 0 & -B_1 \\ +E_3 & -B_2 & +B_1 & 0 \end{pmatrix} = \tfrac{1}{c}\begin{pmatrix} c\rho & j_1 & j_2 & j_3 \end{pmatrix} \tag{4.1}$$

while their homogeneous companions become

$$\begin{pmatrix} \partial_0 & \partial_1 & \partial_2 & \partial_3 \end{pmatrix} \begin{pmatrix} 0 & +B_1 & +B_2 & +B_3 \\ -B_1 & 0 & -E_3 & +E_2 \\ -B_2 & +E_3 & 0 & -E_1 \\ -B_3 & -E_2 & +E_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2}$$

More compactly

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{5.1}$$
$$\partial_\mu G^{\mu\nu} = 0 \tag{5.2}$$

where $\sum_\mu : \mu = 0, 1, 2, 3$ is (by the so-called "Einstein summation convention") understood, where $J^0 \equiv \rho$ and $\mathbf{J} \equiv \frac{1}{c}\mathbf{j}$, and where $||F^{\mu\nu}||$ and $||G^{\mu\nu}||$ are the 4×4 matrices which equations (4) serve explicitly to describe in terms of the components of the $\mathbf{E}$ and $\mathbf{B}$ vectors.

Note the manifest antisymmetry of the matrices $||F^{\mu\nu}||$ and $||G^{\mu\nu}||$, and that they stand obviously in some very close relationship. We have, by simple notational play, been led to associate the $3 + 3 = 6$ degrees of freedom of a pair of 3-vectors with the $1 + 2 + 3 = 6$ degrees of freedom of an antisymmetric 4×4 matrix…in, actually, two distinct ways; substitutional adjustment $\{\mathbf{E}, \mathbf{B}\} \longrightarrow \{-\mathbf{B}, +\mathbf{E}\}$ of the $||F^{\mu\nu}||$ defined by (4.1) sends $||F^{\mu\nu}|| \longrightarrow ||G^{\mu\nu}||$. But there is a deeper and more natural way to describe how $||G^{\mu\nu}||$ relates to $||F^{\mu\nu}||$. It is natural in view of the 4-dimensional antisymmetry

$$F^{\mu\nu} = -F^{\nu\mu} \tag{6}$$

of $||F^{\mu\nu}||$ to construct

$$||\tilde{F}_{\mu\nu}|| \equiv ||\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}|| = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ +B_1 & 0 & -E_3 & +E_2 \\ +B_2 & +E_3 & 0 & -E_1 \\ +B_3 & -E_2 & +E_1 & 0 \end{pmatrix}$$

which resembles $||G^{\mu\nu}||$ but from which, however, it differs in two respects: the B's all wear the wrong signs, and $\tilde{F}_{\mu\nu}$ wears subscripts instead of superscripts (which will become consequential when we look to the *transformational* properties of the developing theory). To solve the latter problem we write

$$\tilde{F}^{\mu\nu} \equiv g^{\mu\alpha}\tilde{F}_{\alpha\beta}g^{\beta\nu} \tag{7}$$

and to solve the former problem we assign to $||g^{\mu\nu}||$ the structure (enforced to within an overall sign)

$$||g^{\mu\nu}|| \equiv \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{8}$$

Thus do we obtain

$$G^{\mu\nu} = \tilde{F}^{\mu\nu} \tag{9}$$

In words: $G^{\mu\nu}$ is none other than the contravariant companion $\tilde{F}^{\mu\nu}$ (indices lifted by means of the "Lorentz metric" $g^{\mu\nu}$) of the Levi-Civita dual $\tilde{F}_{\mu\nu}$ of the electromagnetic field tensor $F^{\mu\nu}$. Returning with this information to (5.2) we have $\partial_\mu(\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}) = 0$ giving this ν-indexed quartet of equations:

$$\epsilon^{\nu\mu\alpha\beta}\partial_\mu F_{\alpha\beta} = 0 \tag{10.1}$$

Explicitly, we have the "windmill sum conditions" $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$ (all μ, ν, λ) which are $4^3 = 64$ in number but have only four things to say:

$$\left.\begin{aligned} \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0 \\ \partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} = 0 \\ \partial_0 F_{31} + \partial_3 F_{10} + \partial_1 F_{03} = 0 \\ \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = 0 \end{aligned}\right\} \tag{10.2}$$

From

$$F_{\mu\nu} = g_{\mu\alpha}F^{\alpha\beta}g_{\beta\nu} \quad \text{and} \quad ||g_{\mu\nu}|| \equiv ||g^{\mu\nu}||^{-1} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{11}$$

we have

$$||F_{\mu\nu}|| = \begin{pmatrix} 0 & +E_1 & +E_2 & +E_3 \\ -E_1 & 0 & -B_3 & +B_2 \\ -E_2 & +B_3 & 0 & -B_1 \\ -E_3 & -B_2 & +B_1 & 0 \end{pmatrix} \tag{12}$$

and on this basis recognize (10.2) to be precisely equivalent to (i.e., a merely notational variant of) the sourceless field equations (4.2).

Several comments are now in order. The use of a "metric connection" $g_{\mu\nu}$ to manipulate (i.e., to raise and lower) indices is entirely standard to tensor algebra; the present situation is special only in that at (8) we were motivated by electromagnetic circumstance to assign to $g_{\mu\nu}$ a particular—and especially simple—structure. "Tensors" are, of course, defined by their transformation properties, not by their indicial decorations; we have deployed indices in anticipation of a discussion of the transformation properties of the Maxwell-Lorentz equations, but until such a discussion has actually been undertaken it is only informally that we allow ourselves to speak (for example) of a "field tensor." At (8) we called into being "spacetime," the (3+1)-dimensional metric space where

$$(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{13}$$

and upon which, according to special relativity, not only electrodynamics but *all* physics is to be inscribed; special relativity and all that flows from it is, in other words, implicit in (8).

3. Essentials of the exterior calculus. "Exterior algebra/calculus" are the names given to one orchestration—one mode of writing out—the *algebra/calculus of fully antisymmetric tensors*. As such, it is at once less and more than the algebra/calculus of tensors in general; it is "less" in that it has nothing to say about tensors which lack the property of complete antisymmetry, but it is "more" in that it brings into sharp focus the special circumstances which permit the theory of antisymmetric tensors to support not only a differential calculus but also an integral calculus. The exterior calculus is relatively sharp because it is relatively specialized, its focus relatively narrow. But the frequently encountered claim that it is in some sense an intrinsically "superior" construct does not impress me, does not seem to me any less spurious than do other manifestations of the claim that "French culture is superior to German/Italian/ English culture." Nor do I accept an obligation to honor the "coordinate free" methods which are traditional within the field. I adopt the view that "exterior algebra/calculus" is a sub-field within the broader field called "tensor analysis"—a sub-field within which "antisymmetry" is the hallmark.

I turn now to some remarks intended to fix my notation and to assemble some tensor-theoretic facts essential to the work at hand. The dimension n of the manifold is taken for the moment to be arbitrary. A change of coordinates

$$x^i \longrightarrow y^i = y^i(x^1, x^2, \ldots, x^n)$$

induces

$$dx^i \longrightarrow dy^i = T^i{}_a\, dx^a \quad \text{with} \quad T^i{}_j \equiv \partial y^i/\partial x^j \tag{14}$$

while the partial derivatives of a scalar field transform

$$\frac{\partial\varphi}{\partial x^j} \longrightarrow \frac{\partial\varphi}{\partial y^j} = \frac{\partial\varphi}{\partial x^b}\, t^b{}_j \quad \text{with} \quad t^i{}_j \equiv \partial x^i/\partial y^j \tag{15}$$

The "transformation matrices" $||T^i{}_j||$ and $||t^i{}_j||$ are inverse to each other:

$$||T^i{}_a|| \cdot ||t^a{}_j|| = ||T^i{}_a t^a{}_j|| = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{16}$$

Objects X^i which transform in imitation of (14)

$$X^i \longrightarrow Y^i = T^i{}_a X^a \tag{17.1}$$

are said to transform as "the components of a contravariant tensor of rank $r = 1$" while objects which transform in imitation of (15)

$$X_j \longrightarrow Y_j = t^b{}_j X_b \tag{17.2}$$

are said to transform as "the components of a covariant tensor of rank $s = 1$." More generally, multiply-indexed objects $X^{i_1 i_2 \cdots i_r}{}_{j_1 j_2 \cdots j_s}$ which transform by the multilinear rule

$$\begin{aligned} &Y^{i_1 i_2 \cdots i_r}{}_{j_1 j_2 \cdots j_s} \\ &\quad = t^{\mathrm{W}} \cdot T^{i_1}{}_{a_1} T^{i_2}{}_{a_2} \cdots T^{i_r}{}_{a_r} t^{b_1}{}_{j_1} t^{b_2}{}_{j_2} \cdots t^{b_s}{}_{j_s} X^{a_1 a_2 \cdots a_r}{}_{b_1 b_2 \cdots b_s} \\ &\quad | \\ &\quad t \equiv \det ||t^i{}_j|| = 1/\det ||T^i{}_j|| \end{aligned} \tag{18}$$

are said to transform as "the components of a mixed tensor density of contravariant rank r, covariant rank s and weight W." The transformation rule (18) gives back (17.1) in the special case $\{r, s; W\} = \{1, 0; 0\}$, and it gives back (17.2) in the case $\{0, 1; 0\}$.

Latent in (18) are certain universally available tensorial objects of which—insofar as the exterior calculus is our objective—we have immediate need, objects which are "universally available" in that their definitions and properties entail the introduction of no additional apparatus (no metric connection, no affine connection). The simplest of these objects comes into being as follows: Let $X^i{}_j$ be number-valued, defined as follows

$$X^i{}_j \equiv \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

and let $X^i{}_j$ transform as a tensor density of weight $W = 0$. It follows then from (18) by (16) that the righthand side of (19) serves also to describe $Y^i{}_j$; i.e., that the second-rank object thus defined transforms *by numerical invariance*. It is "Kronecker's tensor," standardly denoted $\delta^i{}_j$. Consider next the object (of contravariant rank equal to its dimensionality: $r = n$)

$$X^{i_1 i_2 \cdots i_n} \equiv \begin{cases} \text{sgn}\begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ 1 & 2 & \cdots & n \end{pmatrix} & \text{if } i_1 i_2 \cdots i_n \text{ are distinct} \\ 0 & \text{otherwise} \end{cases} \tag{20.1}$$

It follows then from (18) that the object thus defined transforms by numerical invariance provided we set $W = +1$. It is the "contravariant Levi-Civita tensor density," denoted $\varepsilon^{i_1 i_2 \cdots i_n}$ and notable for its total *antisymmetry*. By the same line of argument

$$X_{j_1 j_2 \cdots j_n} \equiv \begin{cases} \text{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix} & \text{if } j_1 j_2 \cdots j_n \text{ are distinct} \\ 0 & \text{otherwise} \end{cases} \tag{20.2}$$

transforms by numerical invariance provided we set $W = -1$; it is the "covariant Levi-Civita tensor density," standardly denoted $\epsilon_{j_1 j_2 \cdots j_n}$.[3] It becomes apparent upon a moment's reflection that one can write

$$\epsilon_{j_1 j_2 \cdots j_n} = \begin{vmatrix} \delta^1{}_{j_1} & \delta^1{}_{j_2} & \cdots & \delta^1{}_{j_n} \\ \delta^2{}_{j_1} & \delta^2{}_{j_2} & \cdots & \delta^2{}_{j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^n{}_{j_1} & \delta^n{}_{j_2} & \cdots & \delta^n{}_{j_n} \end{vmatrix}$$

[3] Note carefully the distinction between ε and ϵ. Though typographically slight, it signals a mathematical distinction which is anything but slight. The covariant tensor (density) $\epsilon_{j_1 j_2 \cdots j_n}$ was, by the way, encountered already (case $n = 4$) in connection with the construction of $\tilde{F}_{\mu\nu}$.

and (more generally)

$$\varepsilon^{i_1 i_2 \cdots i_n}\epsilon_{j_1 j_2 \cdots j_n} = \begin{vmatrix} \delta^{i_1}{}_{j_1} & \delta^{i_1}{}_{j_2} & \cdots & \delta^{i_1}{}_{j_n} \\ \delta^{i_2}{}_{j_1} & \delta^{i_2}{}_{j_2} & \cdots & \delta^{i_2}{}_{j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{i_n}{}_{j_1} & \delta^{i_n}{}_{j_2} & \cdots & \delta^{i_n}{}_{j_n} \end{vmatrix}$$

$$\equiv \delta^{i_1 i_2 \cdots i_n}{}_{j_1 j_2 \cdots j_n} \tag{21}$$

Equation (21) sets up an interrelationship among all the numerically invariant objects thus far assembled, and serves to define the (doubly antisymmetric and numerically invariant) "generalized Kronecker tensor" of mixed rank $r = s = n$ and weight $W = 0$. By explicit calculation one easily obtains

$$\begin{aligned} \delta^{i_1 i_2 a}{}_{j_1 j_2 a} &= \begin{vmatrix} \delta^{i_1}{}_{j_1} & \delta^{i_1}{}_{j_2} \\ \delta^{i_2}{}_{j_1} & \delta^{i_2}{}_{j_2} \end{vmatrix} \equiv \delta^{i_1 i_2}{}_{j_1 j_2} \\ \delta^{i_1 a_1 a_2}{}_{j_1 a_1 a_2} &= 2 \cdot \delta^{i_1}{}_{j_1} \\ \delta^{a_1 a_2 a_3}{}_{a_1 a_2 a_3} &= 3 \cdot 2 \end{aligned}$$

and thus gains insight into the origin and meaning of the following general contraction formula:

$$\begin{aligned} \varepsilon^{i_1 \cdots i_{n-m} a_1 \cdots a_m}\epsilon_{j_1 \cdots j_{n-m} a_1 \cdots a_m} &= \delta^{i_1 \cdots i_{n-m} a_1 \cdots a_m}{}_{j_1 \cdots j_{n-m} a_1 \cdots a_m} \\ &= m! \cdot \delta^{i_1 \cdots i_{n-m}}{}_{j_1 \cdots j_{n-m}} \end{aligned} \tag{22}$$

In (22) we encounter for the first time the entire population of (weightless, doubly antisymmetric and numerically invariant) generalized Kronecker tensors of various ranks: $1 \leq r = s = (n-m) \leq n$. It follows easily from (22) that

$$\delta^{i_1 \cdots i_p}{}_{a_1 \cdots a_p}\delta^{a_1 \cdots a_p}{}_{j_1 \cdots j_p} = p! \cdot \delta^{i_1 \cdots i_p}{}_{j_1 \cdots j_p} \tag{23}$$

If $X^{i_1 \cdots i_p}$ are the components of a tensor $\mathbf{X}$ then

$$A^{i_1 \cdots i_p} \equiv \frac{1}{p!}\delta^{i_1 \cdots i_p}{}_{a_1 \cdots a_p}X^{a_1 \cdots a_p}$$

are the components of a tensor $\mathbf{A}$ of the same rank and weight which it is natural to call the "antisymmetric part" of $\mathbf{X}$. From (23) it follows that "extraction of the antisymmetric part is a *projective* process." While $\mathbf{X}$ lives in a vector space

$$\mathcal{V}^p \equiv \overbrace{\mathcal{V} \times \mathcal{V} \times \cdots \mathcal{V}}^{p \text{ times}}$$

of n^p dimensions, its antisymmetric part $\mathbf{A}$ lives in an invariant subspace

$$\mathcal{A}^p \equiv \overbrace{\mathcal{V} \wedge \mathcal{V} \wedge \cdots \mathcal{V}}^{p \text{ times}} \equiv \wedge^p \mathcal{V} \quad \subseteq \quad \mathcal{V}^p$$

of $\binom{n}{p}$ dimensions.[4]

Now—but only now—we assume V to be an inner product space. More particularly, we assume ourselves to be in possession of a (symmetric non-singular) "metric connection" with components $g_{ij}(x^1, x^2, \ldots, x^n)$, and by this means to be empowered to manipulate indices (i.e., to set up an associations of the type $X^i \longleftrightarrow X_i : X_i = g_{ia}X^a$, etc.). In particular, we are empowered to manipulate the indices which decorate the numerically invariant tensors and tensor densities encountered in the preceeding paragraph. By definition

$$g \equiv \det ||g_{ij}|| = g_{1a_1}g_{2a_2}\cdots g_{na_n}\varepsilon^{a_1a_2\cdots a_n}$$

whence

$$g \cdot \epsilon_{i_1i_2\cdots i_n} = g_{i_1a_1}g_{i_2a_2}\cdots g_{i_na_n}\varepsilon^{a_1a_2\cdots a_n} \equiv \varepsilon_{i_1i_2\cdots i_n} \tag{24}$$

from which it follows (since $\epsilon_{i_1i_2\cdots i_n}$ has weight $W = -1$ and $\varepsilon_{i_1i_2\cdots i_n}$ has weight $W = +1$) that

$$g \text{ is a scalar density of weight } W = +2$$

It follows also from (24) that, while the values of $\varepsilon^{i_1i_2\cdots i_n}$ range universally on $\{-1, 0, +1\}$, the values of $\varepsilon_{i_1i_2\cdots i_n}$ range contingently on $\{-g, 0, +g\}$, and—to say the same thing another way—while the values of $\epsilon_{i_1i_2\cdots i_n}$ range universally on $\{-1, 0, +1\}$, the values of $\epsilon^{i_1i_2\cdots i_n}$ range contingently on $\{-g^{-1}, 0, +g^{-1}\}$.

If $\mathbf{A}$ is an antisymmetric tensor density of contravariant rank p and weight W then

$$A^\star{}_{i_1\cdots i_{n-p}} \equiv \frac{1}{p!}\sqrt{g}\,\epsilon_{i_1\cdots i_{n-p}a_1\cdots a_p}A^{a_1\cdots a_p} \tag{25}$$

serves to define the action $\mathbf{A} \longrightarrow \star\mathbf{A}$ of the "Levi-Civita $\star$ operator." $\star\mathbf{A}$ has (owing to the presence of the $\sqrt{g}$-factor) the same weight as $\mathbf{A}$ but is of complementary rank (i.e., of covariant rank $n - p$). Note that (25) draws only weakly (i.e., only *via* the "scalar density connection" $\sqrt{g}$) upon the resources latent in the metric connection g_{ij}. The action $\mathbf{A} \longrightarrow *\mathbf{A}$ of the "Hodge $*$ operator"—defined by

$$A^*{}_{i_1\cdots i_{n-p}} \equiv \frac{1}{p!}\sqrt{g}\,\epsilon_{i_1\cdots i_{n-p}}{}^{a_1\cdots a_p}A_{a_1\cdots a_p} \tag{26}$$

—makes, on the other hand, full use of those resources. It follows from (26)—and even more transparently from (25)—that

$$**\mathbf{A} = (-)^{p(n-p)}\mathbf{A} = \begin{cases} -\mathbf{A} & \text{if } n \text{ is even and } p \text{ is odd} \\ +\mathbf{A} & \text{otherwise} \end{cases} \tag{27}$$

[4] It would be quite natural at this point to define and develop the properties of the so-called "wedge product" $\mathbf{P} \wedge \mathbf{Q}$, where $\mathbf{P} \in A^p$ and $\mathbf{Q} \in A^q$ entail $\mathbf{P} \wedge \mathbf{Q} \in A^{p+q}$ with $p + q \leq n$. I prefer, however, to await the specifically electrodynamical motivation which will arise in §10. See the footnote on p. 50.

In the language standard to exterior algebra—language which at this point becomes useful to us—our "fully antisymmetric covariant tensors of rank p" are called "p-forms" (subject to a weight restriction which derives its motivation not from algebra but from the calculus: see below). The p-form $\mathbf{A}$ lives in the space $\wedge^p\mathcal{V}$ of dimension $\binom{n}{p}$, while its companion $*\mathbf{A}$ lives in a space $\wedge^{n-p}\mathcal{V}$ of that same dimension: $\binom{n}{n-p}=\binom{n}{p}$. But it is (except in some specially-constructed contexts—analogs of the physicists' "Fock space") permissible to *conflate* $\mathbf{A}$ and $*\mathbf{A}$ —to contemplate constructions such as $\mathbf{A}+*\mathbf{A}$, and statements such as $\mathbf{A}=*\mathbf{A}$ —only if those tensors have the *same* rank (and weight). This entails $n=2p$, in which case (27) becomes

$$**\mathbf{A}=(-)^{p^2}\mathbf{A}=\begin{cases}-\mathbf{A} & \text{if } n=2p \text{ and } p \text{ is odd}\\ +\mathbf{A} & \text{if } n=2p \text{ and } p \text{ is even}\end{cases} \tag{28}$$

It is interesting to observe in this connection that the identity

$$\mathbf{A}=\tfrac{1}{2}(\mathbf{A}+*\mathbf{A})+\tfrac{1}{2}(\mathbf{A}-*\mathbf{A})$$

serves to resolve $\mathbf{A}$ into its "self-dual" and "antiself-dual" components only when p is itself even; when p is odd it becomes on these grounds sometimes more natural to speak not of $*$ but of $i*$.

Turning now from algbra to calculus, let us (for illustrative purposes) look to the transform properties of the partial derivatives $X_{ij,k}\equiv\partial X_{ij}/\partial x^k$ of a covariant tensor X_{ij} of arbitrary weight W:

$$X_{ij}\longrightarrow Y_{ij}=\Big|\frac{\partial x}{\partial y}\Big|^{\mathrm{W}}\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}X_{ab}$$

Immediately

$$\begin{aligned}Y_{ij,k}&=\frac{\partial x^c}{\partial y^k}\frac{\partial}{\partial x^c}\left\{\Big|\frac{\partial x}{\partial y}\Big|^{\mathrm{W}}\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}X_{ab}\right\}\\ &=\Big|\frac{\partial x}{\partial y}\Big|^{\mathrm{W}}\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\frac{\partial x^c}{\partial y^k}X_{ab,c}+\underbrace{X_{ab}\frac{\partial}{\partial y^k}\left\{\Big|\frac{\partial x}{\partial y}\Big|^{\mathrm{W}}\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\right\}}_{\text{unwelcome term}}\end{aligned} \tag{29}$$

In the absence of the "unwelcome term" (29) would assert—quite independently of any appeal to the assumed antisymmetry of X_{ij} —that the $X_{ij,k}$ transform tensorially. And the unwelcome term does in fact vanish under some special circumstances; it vanishes spontaneously if the coordinate transformation $x(y)\longleftarrow y$ is linear, and it vanishes by contrivance if in place of $X_{ij,k}$ one writes (relative to some prescribed "affine connection") the "covariant derivative" $X_{ij;k}$. In the exterior calculus one proceeds, however, otherwise; one "antisymmetrizes" to kill the offending term. The argument runs as follows: we have

$$\begin{aligned}\text{unwelcome term}=X_{ab}\Big|\frac{\partial x}{\partial y}\Big|^{\mathrm{W}}\bigg\{&\frac{\partial^2x^a}{\partial y^k\partial y^i}\frac{\partial x^b}{\partial y^j}+\frac{\partial x^a}{\partial y^i}\frac{\partial^2x^b}{\partial y^k\partial y^j}\\ &+W\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\frac{\partial}{\partial y^k}\log\Big|\frac{\partial x}{\partial y}\Big|\bigg\}\end{aligned}$$

Antisymmetricization (i.e., summing over the signed permutations of $\{i, j, k\}$) kills the first two of the three terms in braces by virtue of the symmetry of the second derivative: $\partial^2/\partial y^i \partial y^j = \partial^2/\partial y^j \partial y^i$. We obtain

$$
\begin{aligned}
&(Y_{ij} - Y_{ji})_{,k} + (Y_{jk} - Y_{kj})_{,i} + (Y_{ki} - Y_{ik})_{,j} \\
&\quad = \left|\frac{\partial x}{\partial y}\right|^{\mathrm{W}} \bigg\{ \frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\frac{\partial x^c}{\partial y^k} - \frac{\partial x^a}{\partial y^j}\frac{\partial x^b}{\partial y^i}\frac{\partial x^c}{\partial y^k} \\
&\qquad\qquad + \frac{\partial x^a}{\partial y^j}\frac{\partial x^b}{\partial y^k}\frac{\partial x^c}{\partial y^i} - \frac{\partial x^a}{\partial y^k}\frac{\partial x^b}{\partial y^j}\frac{\partial x^c}{\partial y^i} \\
&\qquad\qquad + \frac{\partial x^a}{\partial y^k}\frac{\partial x^b}{\partial y^i}\frac{\partial x^c}{\partial y^j} - \frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^k}\frac{\partial x^c}{\partial y^j} \bigg\} X_{ab,c} \\
&\qquad\qquad + W \cdot \left|\frac{\partial x}{\partial y}\right|^{\mathrm{W}} X_{ab} \bigg\{ \frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\frac{\partial}{\partial y^k} - \frac{\partial x^a}{\partial y^j}\frac{\partial x^b}{\partial y^i}\frac{\partial}{\partial y^k} \\
&\qquad\qquad\qquad\qquad + \frac{\partial x^a}{\partial y^j}\frac{\partial x^b}{\partial y^k}\frac{\partial}{\partial y^i} - \frac{\partial x^a}{\partial y^k}\frac{\partial x^b}{\partial y^j}\frac{\partial}{\partial y^i} \\
&\qquad\qquad\qquad\qquad + \frac{\partial x^a}{\partial y^k}\frac{\partial x^b}{\partial y^i}\frac{\partial}{\partial y^j} - \frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^k}\frac{\partial}{\partial y^j} \bigg\} \log\left|\frac{\partial x}{\partial y}\right| \\
&= \left|\frac{\partial x}{\partial y}\right|^{\mathrm{W}} \frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\frac{\partial x^c}{\partial y^k} \Big\{ (X_{ab} - X_{ba})_{,c} + (X_{bc} - X_{cb})_{,a} + (X_{ca} - X_{ac})_{,b} \Big\} \\
&\qquad + W \cdot \left|\frac{\partial x}{\partial y}\right|^{\mathrm{W}} (X_{ab} - X_{ab}) \\
&\qquad\qquad \cdot \bigg\{ \frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\frac{\partial}{\partial y^k} + \frac{\partial x^a}{\partial y^j}\frac{\partial x^b}{\partial y^k}\frac{\partial}{\partial y^i} + \frac{\partial x^a}{\partial y^k}\frac{\partial x^b}{\partial y^i}\frac{\partial}{\partial y^j} \bigg\} \log\left|\frac{\partial x}{\partial y}\right| \qquad (30)
\end{aligned}
$$

We note with interest that if one writes

$$X_{ij} = \tfrac{1}{2}(X_{ij} + X_{ji}) + \tfrac{1}{2}(X_{ij} - X_{ji}) \equiv \text{symmetric part} + \text{antisymmetric part}$$

then it is *only the antisymmetric parts* of X_{ij} and Y_{ij} which enter into the construction of (30). And that unrestricted tensoriality is achieved if one assumes $\mathbf{X}$ to be weightless: $W = 0$. Equation (30) then reads

$$
\begin{aligned}
&(Y_{ij} - Y_{ji})_{,k} + (Y_{jk} - Y_{kj})_{,i} + (Y_{ki} - Y_{ik})_{,j} \\
&\quad = \frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\frac{\partial x^c}{\partial y^k} \Big\{ (X_{ab} - X_{ba})_{,c} + (X_{bc} - X_{cb})_{,a} + (X_{ca} - X_{ac})_{,b} \Big\}
\end{aligned}
$$

The implication is that, if X_{jk} is a weightless tensor, then $\partial_i X_{jk}$ in general does not—but its antisymmetrized companion $A_{ijk} \equiv \frac{1}{2}\delta_{ijk}{}^{abc}\partial_a X_{bc}$ invariably does—transform tensorially. Enlarging upon such experience, one defines the "exterior derivative of a p-form" to be the antisymmetrization of the ordinary derivative. More specifically: if $\mathbf{A}$ is an n-dimensional p-form (if, that is to say, its components $A_{i_1 i_2 \cdots i_p}$ are totally antisymmetric, and transform as a tensor density of weight $W = 0$) then

$$\mathbf{dA} \prec \frac{1}{p!}\,\delta_{i_1 i_2 \cdots i_{p+1}}{}^{b a_1 \cdots a_p} \partial_b A_{a_1 \cdots a_p} \qquad (31)$$

where (as henceforth) I use $\prec$ to mean "is the p-form whose components are given by." If $\mathbf{A}$ lives in $\wedge^p\mathcal{V}$ then $\mathbf{dA}$ lives in $\wedge^{p+1}\mathcal{V}$ but if $p=n$ then the construction (31) becomes meaningless; formally, if $\mathbf{A}\in\wedge^n\mathcal{V}$ then $\mathbf{dA}=\mathbf{0}$. By an easy argument[5]

$$\mathbf{d}^2\mathbf{A}\prec\frac{1}{p!}\delta_{i_1\cdots i_{p+2}}{}^{b_1b_2a_1\cdots a_p}\frac{\partial^2}{\partial x^{b_1}\partial x^{b_2}}A_{a_1\cdots a_p}$$

which vanishes by virtue of the b_1b_2-symmetry of the 2nd derivative: $\mathbf{d}^2=\mathbf{0}$.

$*$ and $\mathbf{d}$ collectively exhaust the "operational alphabet" of the exterior calculus, and since

$$** \sim \mathbf{1} \qquad \text{and} \qquad \mathbf{dd}=\mathbf{0} \tag{32}$$

the only "words" constructable from them are snippets from the string

$$\cdots *\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\cdots$$

The 1st-order differential operators available to the theory are four in number:

$$\left.\begin{array}{rcl} \mathbf{d} & : & \text{sends } p\text{-forms} \longrightarrow \quad (p+1)\text{-forms}\\ *\mathbf{d} & : & \text{sends } p\text{-forms} \longrightarrow (n-p-1)\text{-forms}\\ \mathbf{d}* & : & \text{sends } p\text{-forms} \longrightarrow (n-p+1)\text{-forms}\\ *\mathbf{d}* & : & \text{sends } p\text{-forms} \longrightarrow \quad (p-1)\text{-forms} \end{array}\right\} \tag{33.1}$$

The available 2nd-order differential operators are also four in number:

$$\left.\begin{array}{rcl} \mathbf{d}*\mathbf{d} & : & \text{sends } p\text{-forms} \longrightarrow \quad (n-p)\text{-forms}\\ *\mathbf{d}*\mathbf{d} & : & \text{sends } p\text{-forms} \longrightarrow \quad p\text{-forms}\\ \mathbf{d}*\mathbf{d}* & : & \text{sends } p\text{-forms} \longrightarrow \quad p\text{-forms}\\ *\mathbf{d}*\mathbf{d}* & : & \text{sends } p\text{-forms} \longrightarrow \quad (n-p)\text{-forms} \end{array}\right\} \tag{33.2}$$

Operators of type $\cdots *\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}$ annihilate n-forms, but for $p<n$ achieve the cycle

$$p \quad\longrightarrow\quad p+1 \quad\longrightarrow\quad n-p-1 \quad\longrightarrow\quad n-p \quad\longrightarrow\quad p \quad \cdots$$

while operators of the type $\cdots *\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*$ promptly annihilate 0-forms, but for $p>0$ achieve

$$p \quad\longrightarrow\quad n-p \quad\longrightarrow\quad n-p+1 \quad\longrightarrow\quad p-1 \quad\longrightarrow\quad p \quad \cdots$$

---

[5] See ELECTRODYNAMICS 1972 p. 164 for the details. The argument hinges on the identity

$$\delta^{i_1\cdots i_{p+q}}{}_{j_1\cdots j_p a_1\cdots a_q}\delta^{a_1\cdots a_q}{}_{k_i\cdots k_q}=q!\,\delta^{i_1\cdots i_{p+q}}{}_{j_1\cdots j_p k_1\cdots k_q}$$

It has been my experience that argument involving such operator sequences is greatly facilitated by use of diagrams of the following design:

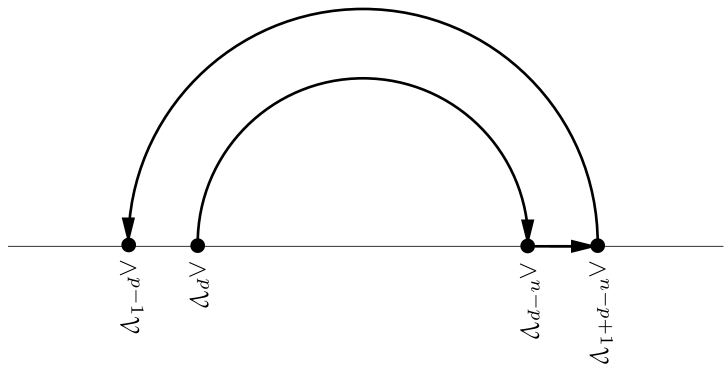

FIGURE 1: *A "Wheeler diagram," illustrative here of the action on a p-form of the operator sequence* $\boldsymbol{*}\mathbf{d}\boldsymbol{*}$.

4. Catalog of "accidentally tensorial" derivative constructions. In CLASSICAL ELECTRODYNAMICS (1980) I was motivated to assemble (at pp. 174–176) a short catalog of "accidentally tensorial derivative constructs." Each of the objects listed in that catalog (see below) is assembled by application of ordinary ∂-processes to a tensor of specified type, and tensoriality is (in all cases but one) achieved by "lucky cancellations;" in no case is appeal made to the imposition of (linearity) restrictions upon $x \longrightarrow y = y(x)$, and in no case is the "covariant derivative" concept invoked. My catalog was in the first instance inspired by the discussion which appears on pp. 22–24 of E. Schrödinger's elegant monograph *Space-time Structure* (Cambridge, 1954). My objective here will be to show that the "accidentally tensorial derivative constructs" here in question spring naturally and in a unified way (in the company of many others) from the exterior calculus—a fact of which Schrödinger himself appears to have been unaware.

> *If φ is a scalar field of weight $W = 0$ then $\varphi_i \equiv \partial_i\varphi$ transforms as a (weightless) vector field.*

In exterior terminology this amounts simply to the statement that if $\boldsymbol{\varphi}$ is a 0-form then $\mathbf{d}\boldsymbol{\varphi}$ is a 1-form, with

$$\mathbf{d}\boldsymbol{\varphi} \prec \delta_i{}^b\partial_b\varphi = \partial_i\varphi$$

Clearly, the catalog speaks here simply of the familiar "gradient" of φ—commonly notated $\boldsymbol{\nabla}\varphi$—and achieves tensoriality independently of any kind of "cancellation" mechanism. The exterior calculus permits/invites us to speak now more generally about the "gradient of a p-form $\mathbf{A}$" ($\mathrm{p} = 0, 1, 2, \ldots, n-1$) when we have in mind the $(p+1)$-form $\mathbf{dA}$.

If V_i is a vector field of weight $W = 0$ then $V_{ij} \equiv \partial_i V_j - \partial_j V_i$ transforms as a (weightless) tensor field.

In exterior terminology this amounts to the statement that if $\mathbf{V}$ is a 1-form then $\mathbf{dV}$ is a 2-form, with

$$\mathbf{dV} \prec \delta_{ij}{}^{ba}\partial_b V_a = \partial_i V_j - \partial_j V_i$$

In Euclidean 3-space one standardly construes $C_1 \equiv V_{23}$, $C_2 \equiv V_{31}$ and $C_3 \equiv V_{12}$ to be the components of the "curl" of the vector in question: $\mathbf{C} = \boldsymbol{\nabla} \times \mathbf{V}$. It becomes natural on the basis of this remark to observe (i) that the presence of a (Euclidean) metric permits us to assign meaning to $*$, and (ii) in 3-space $*$ acts on 2-forms to yield 1-forms. We are led thus to examine

$$*\mathbf{dV} \prec \tfrac{1}{2}\sqrt{g}\,\epsilon_i{}^{ab}(\partial_a V_b - \partial_b V_a) = \begin{cases} \partial_2 V_3 - \partial_2 V_3 & \text{for i=1} \\ \partial_3 V_1 - \partial_1 V_3 & \text{for i=2} \\ \partial_1 V_2 - \partial_2 V_1 & \text{for i=3} \end{cases}$$

where the equality exploits familiar special properties of the Euclidean metric. By extension, we allow ourselves to speak of the "curl of the n-dimensional p-form $\mathbf{A}$" when we have in mind the $(n-p-1)$-form $*\mathbf{dA}$.

If A_{ij} is an antisymmetric tensor field of weight $W = 0$ then so also do the quantities A_{ijk} defined

$$A_{ijk} \equiv \partial_i A_{jk} + \partial_j A_{ik} + \partial_k A_{ij}$$

transform as a (weightless) tensor field.

We have already authenticated this claim in elaborate detail, in the motivational material which led us to (31). What we have here in hand is the lowest-order non-trivial instance of the exterior generalization of "gradient;" if $\mathbf{A}$ is a p-form then $\mathbf{dA}$ is a $(p+1)$-form:

$$\begin{aligned} \mathbf{A} \prec A_{ij} \Rightarrow \mathbf{dA} &\prec \tfrac{1}{2}\delta_{ijk}{}^{abc}\partial_a A_{bc} \\ &= \tfrac{1}{2}\big\{(\partial_i A_{jk} - \partial_i A_{kj}) + (\partial_j A_{ki} - \partial_j A_{ik}) + (\partial_k A_{ij} - \partial_k A_{ji})\big\} \\ &= \partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij} \end{aligned}$$

If V^i is a vector density of weight $W = 1$ then $V \equiv \partial_i V^i$ transforms as a scalar density of weight $W = 1$.

Implicit here is the claim that if $W \neq 1$ then $\partial_i V^i$ does *not* tranform tensorially. The reason physicists are so successful in their general disregard of the weight restriction is that they work typically in situations where $x \longrightarrow y = y(x)$ has such a specialized character as to render invisible the complications typical of the general case; reverting to the notation of (30) we discover that

$$Y^i = \left|\frac{\partial x}{\partial y}\right|^{\mathrm{W}} \frac{\partial y^i}{\partial x^a} X^a \Rightarrow Y^i{}_{,i} = \left|\frac{\partial x}{\partial y}\right|^{\mathrm{W}} \left\{ X^a{}_{,a} + (W-1)X^a \frac{\partial}{\partial x^a} \log\left|\frac{\partial x}{\partial y}\right| \right\}$$

which shows clearly both the origin of the weight restriction and the condition which must be satisfied to preserve tensoriality when that restriction is violated. To construct the "divergence" is by a derivative process $\mathbf{V} \longrightarrow \boldsymbol{\nabla}\cdot\mathbf{V}$ to construct a scalar from a vector. The exterior calculus supplies only one means to achieve such an objective; it is $\mathbf{A} \longrightarrow *\mathbf{d}*\mathbf{A}$, which sends p-forms into $(p-1)$-forms $(0 < p \le n)$. Explicitly we have (by straightforward calculation in the case $p = 1$; for the details see ELECTRODYNAMICS (1972)) p. 168 or specialize the general result achieved at (38)below)

$$*\mathbf{d}*\mathbf{A} \prec (-)^{n-1} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^a} \big(\sqrt{g}\, A^a \big)$$

It is interesting in the light of this result to recall (see p. 145 of the notes just cited) that on a metrically connected manifold (with affine connection taken therefore to be the Christoffel connection) the "covariant divergence" of a vector density of weight W is given by

$$X^a{}_{;a} = \frac{1}{\sqrt{g^{1-W}}} \frac{\partial}{\partial x^a} \big(\sqrt{g^{1-W}}\, X^a \big) = \begin{cases} X^a{}_{,a} & \text{when } W = 1 \\ \dfrac{1}{\sqrt{g}} \dfrac{\partial}{\partial x^a} \big(\sqrt{g}\, X^a \big) & \text{when } W = 0 \end{cases}$$

and that in the exterior calculus A^i is in fact assumed to be weightless (causing $V^i \equiv \sqrt{g}A^i$ to have weight $W = 1$). But what to make of the peculiar sign factor $(-)^{n-1}$? This is a question to which I will soon return.

> *If* A^{ij} *is a an antisymmetric tensor density then* $A^j \equiv \partial_i A^{ij}$ *transforms as a vector density if* $W = 1$*, but not otherwise.*

This provides again an instance of $\mathbf{A} \longrightarrow *\mathbf{d}*\mathbf{A}$, but with $p = 2$; details can be extracted from a general result soon to be developed.

My account of the "exterior origins of Schrödinger's catalog" is now complete. It becomes natural, on the basis of the preceeding discussion, to contemplate associations of the form

$$\text{grad} \quad \longleftrightarrow \quad \mathbf{d} \tag{34.1}$$

$$\text{curl} \quad \longleftrightarrow \quad *\mathbf{d} \tag{34.2}$$

$$\pm\,\text{div} \quad \longleftrightarrow \quad *\mathbf{d}* \tag{34.3}$$

But those tentative associations would provide (vast) *generalizations* of some familiar notions, and generalization/bootstrapping is an inherently ambiguous process. We have encountered no evidence that sign ambiguities attach to the generalized meanings of "grad" and "curl," but do have reason to suppose that a factor of the form $(-)^{f(n,p)}$ may properly enter into the exterior definition of "div." It is to that problem that I will in a moment turn. But even in advance of its resolution we are (by (32) placed)in position to write

$$\text{curl}\cdot\text{grad} \qquad \longleftrightarrow \qquad \boldsymbol{*}\mathbf{d}\cdot\mathbf{d}=\mathbf{0} \tag{35.1}$$

$$\text{div}\cdot\text{curl} \qquad \longleftrightarrow \qquad \boldsymbol{*}\mathbf{d}\boldsymbol{*}\cdot\boldsymbol{*}\mathbf{d}=\mathbf{0} \tag{35.2}$$

and thus to give much-generalized meaning to a pair of familiar identites—identities which are the source of all the "potentials" encountered in physical applications.

I turn now to some detailed calculation the results of which will prove useful in a variety of connections, but which are intended mainly to clarify the sign ambiguity which attaches at (34.3) to the generalized definition of "div." The discussion will serve also to illustrate the elementary tedium characteristic of some exterior-algebraic calculation (though in the main the exterior calculus is notable for the painless swiftness with which it leads to its elegant results). We proceed from prior definitions which I here repeat: if $\mathbf{A}$ is an n-dimensional p-form, then

$$\boldsymbol{*}\mathbf{A} \prec \frac{1}{p!}\sqrt{g}\,\epsilon_{i_1\cdots i_{n-p}a_1\cdots a_p}A^{a_1\cdots a_p} \tag{36}$$

and

$$\mathbf{dA} \prec \frac{1}{p!}\,\delta_{i_1i_2\cdots i_{p+1}}{}^{ba_1\cdots a_p}\partial_b A_{a_1\cdots a_p} \tag{37}$$

So (rolling up our sleeves)

$$\begin{aligned}\mathbf{d}\boldsymbol{*}\mathbf{A} &\prec \frac{1}{(n-p)!}\delta_{j_i\cdots j_{n-p+1}}{}^{ba_1\cdots a_{n-p}}\partial_b\frac{1}{p!}\sqrt{g}\epsilon_{a_1\cdots a_{n-p}k_1\cdots k_p}A^{k_1\cdots k_p}\\ &= \frac{1}{p!(n-p)!}\delta_{j_i\cdots j_{n-p+1}}{}^{ba_1\cdots a_{n-p}}\epsilon_{a_1\cdots a_{n-p}k_1\cdots k_p}\partial_b\big(\sqrt{g}A^{k_1\cdots k_p}\big)\end{aligned}$$

$$\begin{aligned}\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{A} &\prec \frac{1}{p!(n-p)!(n-p+1)!}\sqrt{g}\\ &\qquad\cdot\,\epsilon_{i_1\cdots i_{p-1}}{}^{j_i\cdots j_{n-p+1}}\delta_{j_i\cdots j_{n-p+1}}{}^{ba_1\cdots a_{n-p}}\epsilon_{a_1\cdots a_{n-p}k_1\cdots k_p}\partial_b\big(\sqrt{g}A^{k_1\cdots k_p}\big)\\ &= \frac{1}{p!(n-p)!}\sqrt{g}\epsilon_{i_1\cdots i_{p-1}}{}^{ba_1\cdots a_{n-p}}\epsilon_{a_1\cdots a_{n-p}k_1\cdots k_p}\partial_b\big(\sqrt{g}A^{k_1\cdots k_p}\big)\\ &= \frac{1}{p!(n-p)!}\sqrt{g}(-)^{p-1}\epsilon_{bi_1\cdots i_{p-1}a_1\cdots a_{n-p}}(-)^{p(n-p)}\\ &\qquad\qquad\cdot\,\underbrace{\epsilon^{k_1\cdots k_pa_1\cdots a_{n-p}}}_{=\frac{1}{g}\,\varepsilon^{k_1\cdots k_pa_1\cdots a_{n-p}}}\partial^b\big(\sqrt{g}A_{k_1\cdots k_p}\big)\end{aligned}$$

Drawing now upon (23) we obtain

$$= (-)^{p(n-p)+p-1}\frac{1}{p!}\frac{1}{\sqrt{g}}\delta_{bi_1\cdots i_{p-1}}{}^{k_1\cdots k_p}\partial^b\big(\sqrt{g}A_{k_1\cdots k_p}\big)$$

$$= (-)^{p(n-p)}\frac{1}{p!}\frac{1}{\sqrt{g}}\delta_{i_1\cdots i_{p-1}b}{}^{k_1\cdots k_p}\partial^b\big(\sqrt{g}A_{k_1\cdots k_p}\big) \tag{38}$$

which gives back the result previously quoted in the case $p=1$. The implication appears to be[6] that in place of (34.3) we should adopt the sharper association

$$\text{div} \quad \longleftrightarrow \quad (-)^{f(n,p)}\boldsymbol{*}\mathbf{d}\boldsymbol{*} \quad \text{with } f(n,p)\equiv p(n-p) \tag{39}$$

We have observed that the second order differential operators $\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{d}$ and $\mathbf{d}\boldsymbol{*}\mathbf{d}\boldsymbol{*}$ both yield p-forms when applied to p-forms. We are in position now to sharpen that observation. In n-dimensional theory one has frequent occasion to speak of the "laplacian" of a scalar field, with this intention:

$$\text{lap} \quad \longleftrightarrow \quad \text{div}\cdot\text{grad}$$

and in 3-dimensional theory one has occasional need of the "vector laplacian"

$$\text{lap} \quad \longleftrightarrow \quad \text{grad}\cdot\text{div} - \text{curl}\cdot\text{curl}$$

It is a curious fact that "div · grad" and "curl · curl" are (up to an occasional sign) *identical* constructs in their exterior generalizations:

$$\text{div}\cdot\text{grad} \quad \longleftrightarrow \quad \boldsymbol{*}\mathbf{d}\boldsymbol{*}\cdot\mathbf{d} = \boldsymbol{*}\mathbf{d}\cdot\boldsymbol{*}\mathbf{d} \quad \longleftrightarrow \quad \text{curl}\cdot\text{curl}$$

Both, moreover, are distinct from grad · div $\sim \mathbf{d}\cdot\boldsymbol{*}\mathbf{d}\boldsymbol{*}$ (which is moot when active upon 0-forms). An association of the form

$$\text{lap} \quad \longleftrightarrow \quad \mathbf{d}\boldsymbol{*}\mathbf{d}\boldsymbol{*} + (-)^{p(n-p+1)}\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{d}$$

would conform (when n is arbitrary but $p=0$, and in the case $n=3$, $p=1$) to our expectations in the particular cases discussed above, but to pursue the "lap definition problem" to a definitive conclusion would take me too far afield; having made my essential point, I drop the problem.

5. Exterior integral calculus. The statement $\mathbf{dd}=\mathbf{0}$ is known in the literature as the "Poincaré Lemma," and was seen at (32) to be almost trivial in its origin (though profound in its consequences). Not at all trivial—and certainly no less consequential—is the celebrated

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[6] The reader who is curious about my use of a phrase so soft as "appears to be" may wish to consult ELECTRODYNAMICS (1972) p. 170, where study of the inner products in $\wedge^p\mathcal{V}$ relative to the metric structure induced by that of $\mathcal{V}$ leads to the concept of a differential operator "adjoint" to $\mathbf{d}$. That operator turns out to be quite similar to the generalized "div" defined above, but to involve a somewhat different sign factor.

CONVERSE OF THE POINCARÉ LEMMA: *If* $\mathbf{A}$ *is a* p-form $(p \geq 1)$ *with the property that* $\mathbf{dA} = \mathbf{0}$ *then there exists a* $(p-1)$-form $\mathbf{F}$ *such that* $\mathbf{A} = \mathbf{dF}$.

$\mathbf{F}$ is determined only up to a "gauge transformation"

$$\mathbf{F} \longrightarrow \mathbf{F}' = \mathbf{F} + \mathbf{dG} \qquad \text{where } \mathbf{G} \text{ is an arbitrary } (p-2)\text{-form}$$

and—this is the amazing part!—can (under weak hypotheses, and up to gauge) be described explicitly:

$$F_{i_1 \cdots i_{p-1}}(x) = \int_0^1 A_{\alpha i_1 \cdots i_{p-1}}(\tau x) x^\alpha \tau^{p-1}\, d\tau \tag{40}$$

where $x \equiv (x^1, x^2, \ldots, x^n)$.

In multivariable calculus courses one encounters integral identites of which the following are typical:

GREEN'S THEOREM: $$\int\!\!\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_{\partial R} (P\, dx + Q\, dy)$$

STOKES' THEOREM: $$\int\!\!\int_D \text{curl}\mathbf{A} \cdot \mathbf{dS} = \oint_{\partial D} \mathbf{A} \cdot \mathbf{ds}$$

GAUSS' THEOREM: $$\int\!\!\int\!\!\int_B \text{div}\mathbf{A}\; dV = \int\!\!\int_{\partial B} \mathbf{A} \cdot \mathbf{dS}$$

Here R is a region inscribed on the 2-dimensional plane, and ∂R is the curve which bounds it; D is an orientable "potato chip" suspended in 3-space, and ∂D its bounding curve; B is a "bubble" suspended in 3-space, and ∂B its bounding surface. Those identities and their multitudinous relatives and corollaries—of which

$$\int\!\!\int\!\!\int_B \big\{\phi\nabla^2\psi - \psi\nabla^2\phi\big\}\, dV = \int\!\!\int_{\partial B} \big\{\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi\big\}\cdot\mathbf{dS}$$

is typical—share with the

FUNDAMENTAL THEOREM OF THE CALCULUS: $$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$$

the following architecture:

$$\int_{n\text{-dimensional region } R} \text{differentiated multilinear object}$$
$$= \int_{(n-1)\text{-dimensional boundary } \partial R \text{ of } R} \text{undifferentiated multilinear object}$$

It was applied mathematics which motivated the discovery and development of this population of integral relations (Green, Stokes—and in this connection also

Gauss and Newton—were physicists!), but it was mathematics of a purer stripe (Poincaré, working in 1887) that led to the insight that all the statements here in question are but particular instances of a single statement of vast generality, a statement to which the exterior calculus gives most natural voice:

$$\text{Poincaré's version of Stokes Theorem:} \qquad \int_R \mathbf{dA} = \int_{\partial R} \mathbf{A} \tag{41}$$

Here $\mathbf{A}$ is an n-dimensional $(p-1)$-form, and ∂R is the $(p-1)$-dimensional boundary of an orientable p-dimensional region R in n-space. But $\mathbf{A}$ has no longer quite the meaning it has had heretofore; it has acquired the character of a "differential form" (and it was in anticipation of this development—basic to the theory of integration, but irrelevant in other contexts—that the "p-form" terminology was devised); if, for example, $n = 3$ then "$\mathbf{A}$ is a 1-form" means

$$\mathbf{A} = A_1\,dx^1 + A_2\,dx^2 + A_3\,dx^3$$

"$\mathbf{A}$ is a 2-form" means

$$\begin{aligned} \mathbf{A} &= \tfrac{1}{2}(A_{12} - A_{12})\,dx^1dx^2 + \tfrac{1}{2}(A_{23} - A_{32})\,dx^2dx^3 + \tfrac{1}{2}(A_{31} - A_{13})\,dx^3dx^1 \\ &= \tfrac{1}{2!}A_{ij}\,dx^i \wedge dx^j \\ &\qquad dx^i \wedge dx^j = -dx^j \wedge dx^i \end{aligned}$$

while "$\mathbf{A}$ is a 3-form" means

$$\begin{aligned} \mathbf{A} &= \tfrac{1}{6}(A_{123} + A_{231} + A_{312} - A_{321} - A_{123} - A_{213})\,dx^1dx^2dx^3 \\ &= \tfrac{1}{3!}A_{ijk}\,dx^i \wedge dx^j \wedge dx^k \end{aligned}$$

and more generally

"$\mathbf{A}$ is an n-dimensional p-form" means $\mathbf{A} = \frac{1}{p!}A_{i_1 i_2 \cdots i_p}\,dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$

with indices ranging on $\{1, 2, \ldots, n\}$. Poincare's generalized formulation (41) of Stokes' theorem is established by a line of argument which it best serves my expository purpose to illustrate in a special case. Let $\mathbf{A}$ be an n-dimensional 1-form: $\mathbf{A} = A_i\,dx^i$ where the functions $A_i \equiv A_i(x) \equiv A_i(x^1, x^2, \ldots, x^n)$ are given. And let $\mathbf{B}$ be the 2-form $\mathbf{B} \equiv \mathbf{dA} = \frac{1}{2}B_{ij}(x)\,dx^i \wedge dx^j$ with $B_{ij} = \partial_i A_j - \partial_j A_i$. R is a 2-dimensional "potato chip" in n-space which we will (without essential loss of generality) assume contains the origin as one of its interior points. Think of R as a "distorted disk" upon which we have inscribed a correspondingly "distorted polar coordinate system" $\{\tau, \varphi\}$ in such a way that the origin lies at $\tau = 0$ and ∂R is achieved by setting $\tau = 1$; more particulary, write $x^i = \tau\cdot\xi^i(\varphi)$. Then

$$\int_R \mathbf{dA} \equiv \tfrac{1}{2}\int_R B_{ij}(x)\,dx^i\wedge dx^j$$

$$\begin{aligned}
dx^i\wedge dx^j &= \Big(\frac{\partial x^i}{\partial\tau}\,d\tau + \frac{\partial x^i}{\partial\varphi}\,d\varphi\Big)\wedge\Big(\frac{\partial x^j}{\partial\tau}\,d\tau + \frac{\partial x^j}{\partial\varphi}\,d\varphi\Big)\\
&= \Big(\frac{\partial x^i}{\partial\tau}\frac{\partial x^j}{\partial\varphi} - \frac{\partial x^i}{\partial\varphi}\frac{\partial x^j}{\partial\tau}\Big)d\tau\wedge d\varphi\\
&= \tau\Big(\xi^i\frac{\partial\xi^j}{\partial\varphi} - \xi^j\frac{\partial\xi^i}{\partial\varphi}\Big)d\tau\wedge d\varphi
\end{aligned}$$

$$= \int\underbrace{\Big\{\int_0^1 B_{ij}(\tau\xi(\varphi))\tau\xi^i(\varphi)\,d\tau\Big\}}\frac{\partial\xi^j}{\partial\varphi}\,d\varphi$$

$$= A_j(\xi(\varphi))\quad\text{by (40), which is available by } \mathbf{dB}=\mathbf{0}$$

$$= \int_{\partial R} A_j\,dx^j \equiv \int_{\partial R}\mathbf{A}$$

which is precisely the result we sought to establish. This lovely argument—which by straightforward adjustment serves to establish (41) in the general case—hinges critically on (40), and shows the generality of (41) to be in fact coextensive with that of (40). One is not surprised to learn that it is the handiwork of the leading topologist of his day, and the inventor of the so-called "integral invariants"[7] of classical mechanics. For more elaborate discussion and many illustrative examples I refer the reader to my ELECTRODYNAMICS (1972), and to the publications cited there. Here my objective has been simply to make clear the meaning of (41) and to capture the flavor of the argument upon which it rests. And to support this general observation:

The tensor concept supports an elaborate differential calculus, but only in the presence of a "complete antisymmetry" assumption does it support also an integral calculus. This fact can be traced to the circumstance that mensuration theory is founded upon an essentially determinental (therefore antisymmetric) construct; the volume V of the tetrahedron whose vertices lie (relative to a cartesian frame) at $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ is given, after all, by

$$V_{\text{3-dimensional simplex}} = \frac{1}{3!}\begin{vmatrix} 1 & x_0 & y_0 & z_0\\ 1 & x_1 & y_1 & z_1\\ 1 & x_2 & y_2 & z_2\\ 1 & x_3 & y_3 & z_3\end{vmatrix}$$

$$= \frac{1}{3!}\begin{vmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{vmatrix}\quad\text{when } \boldsymbol{x}_0 \text{ lies at the origin}$$

[7] See H. Goldstein, *Classical Mechanics* (1951) §8.3 and the cautionary footnote which appears on p. 403 of Goldstein's 2nd edition (1980).

so we have

$$dV = \underbrace{\frac{1}{3!}\begin{vmatrix} dx & 0 & 0 \\ 0 & dy & 0 \\ 0 & 0 & dz \end{vmatrix}}_{\equiv\, dx\wedge dy\wedge dz} \qquad \text{in a differential special case}$$

and these statements generalize straightforwardly to the n-dimensional case. Which comes—bottom line—to this: if a tensorial theory admits of integral formulation, then it does so *by virtue of Stokes' Theorem*, and antisymmetric structures cannot be far away; the exterior calculus stands ready to give natural expression to the essentials of the theory in question. And conversely: if a theory is dominated by antisymmetric structures, then the exterior calculus is likely to afford the language of choice, and the theory is likely to admit of integral formulation.

6. Exterior formulations of the Maxwell-Lorentz equations. It is a familiar fact that the Maxwell-Lorentz equations—displayed as a system of coupled partial differential equations at (3)—admit also of integral formulation, which is achieved as follows: (3.1) entails

$$\underbrace{\iiint_B \text{div}\mathbf{E}\, dV}_{\displaystyle = \iint_{\partial B} \mathbf{E}\cdot\mathbf{dS} \quad \text{by Gauss' Theorem}} = \iiint_B \rho\, dV \qquad \text{for all "bubbles" B}$$

while by the same argument (3.3) becomes

$$\iint_{\partial B} \mathbf{B}\cdot\mathbf{dS} = 0$$

And (3.2) entails, for all t-independent "potato chips" C,

$$\begin{aligned}\underbrace{\iint_C \text{curl}\mathbf{B}\cdot\mathbf{dS}}_{\displaystyle = \oint_{\partial C}\mathbf{B}\cdot\mathbf{ds} \quad\text{by Stokes' Theorem}} &= \tfrac{1}{c}\iint_C \Big\{\mathbf{j} + \tfrac{\partial}{\partial t}\mathbf{E}\Big\}\cdot\mathbf{dS} \\ &= \tfrac{1}{c}\iint_C \mathbf{j}\cdot\mathbf{dS} + \tfrac{1}{c}\tfrac{d}{dt}\iint_C \mathbf{E}\cdot\mathbf{dS}\end{aligned}$$

while by the same argument (3.4) becomes $\oint_{\partial C}\mathbf{E}\cdot\mathbf{ds} = -\frac{1}{c}\frac{d}{dt}\iint_C \mathbf{B}\cdot\mathbf{dS}$. The partial differential equations (3) can, in short—by arguments all of which turn upon instances of (41)—be written

$$\iint_{\partial B} \mathbf{E}\cdot\mathbf{dS} = \iiint_B \rho\, dV \tag{42.1}$$

$$\oint_{\partial C} \mathbf{B}\cdot\mathbf{ds} = +\tfrac{1}{c}\tfrac{d}{dt}\iint_C \mathbf{E}\cdot\mathbf{dS} + \tfrac{1}{c}\iint_C \mathbf{j}\cdot\mathbf{dS} \tag{42.2}$$

$$\iint_{\partial B} \mathbf{B}\cdot\mathbf{dS} = 0 \tag{42.3}$$

$$\oint_{\partial C} \mathbf{E}\cdot\mathbf{ds} = -\tfrac{1}{c}\tfrac{d}{dt}\iint_C \mathbf{B}\cdot\mathbf{dS} \tag{42.4}$$

We have in Maxwellian electrodynamics an instance of a theory that "admits of integral formulation," and—already on those grounds alone—we expect its differential formulation to admit of articulation in language afforded by the exterior calculus. To that end:

Let us adopt an explicitly non-relativistic point of view. Let us agree, in other words, to think of $\mathbf{E}(x^1, x^2, x^3; t)$ and $\mathbf{B}(x^1, x^2, x^3; t)$ as 3-component fields defined on 3-space, and to regard t as an autonomous parameter. Since

when $n = 3$ 0-forms have 1 component
1-forms have 3 components
2-forms have 3 components
3-forms have 1 components

we have options. It is, perhaps, most natural to take $\mathbf{E}$—whence (by (3.2)) also $\mathbf{j}$—to be a 1-form. Then (3.1) can be written $\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{E} = \rho$, which forces ρ to be an 0-form. Looking next to (3.4), we obtain $\boldsymbol{*}\mathbf{dE} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = \mathbf{0}$, which forces $\mathbf{B}$ to be also a 1-form. Thus are the Maxwell-Lorentz equations (3) brought to the form

$$\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{E} = \rho \tag{43.1}$$

$$\boldsymbol{*}\mathbf{dB} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\mathbf{E} = \tfrac{1}{c}\mathbf{j} \tag{43.2}$$

$$\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{B} = 0 \tag{43.3}$$

$$\boldsymbol{*}\mathbf{dE} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\mathbf{B} = \mathbf{0} \tag{43.4}$$

Had we, on the other hand, taken $\mathbf{E}$—whence also $\mathbf{j}$—to be a 2-form (which might seem on component-counting grounds to be a possibility) then $\text{div}\mathbf{E} = \boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{E} = \rho$ would have forced ρ to be a 1-form, which is absurd. It is interesting to observe that if we hit (43.1) with $\frac{1}{c}\frac{\partial}{\partial t}$, hit (43.2) with $\boldsymbol{*}\mathbf{d}\boldsymbol{*}$ and add, we obtain

$$\underbrace{\boldsymbol{*}\mathbf{d}\boldsymbol{**}\mathbf{d}}_{=\,\mathbf{0}\quad\text{by (32)}}\mathbf{B} = \tfrac{1}{c}\Big\{\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{j} + \tfrac{\partial}{\partial t}\rho\Big\}$$

and thus recover the charge conservation equation (2). Note also that it is (by the converse of the Poincaré Lemma) an implication of (43.3) that there exists a 1-form $\mathbf{A}$ such that $\boldsymbol{*}\mathbf{B} = \mathbf{dA}$, which (since according to (27) we have $\boldsymbol{**} = \mathbf{1}$ for all p because $n = 3$ is odd) entails

$$\mathbf{B} = \boldsymbol{*}\mathbf{dA} \tag{44.1}$$

Returning with this information (which renders (43.3) automatic) to (43.4) we obtain $\boldsymbol{*}\mathbf{d}\big\{\mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}\big\} = \mathbf{0}$, from which it follows (again by the converse of the Poincaré Lemma) that there exists an 0-form φ such that $\mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} = \mathbf{d}\varphi$, giving

$$\mathbf{E} = -\mathbf{d}\phi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\mathbf{A} \tag{44.2}$$

where in unforced deference to convention we have reversed the sign of ϕ. Equation (44.1) is invariant under $\mathbf{A} \longrightarrow \mathbf{A} + \mathbf{d}\chi$ where χ is an arbitrary 0-form. The invariance of (44.2) then entails $\phi \longrightarrow \phi - \frac{1}{c}\frac{\partial}{\partial t}\chi$. Introducing (44) into (43.1) and (43.2) we obtain

$$\left.\begin{aligned} -\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{d}\varphi - \boldsymbol{*}\mathbf{d}\boldsymbol{*}\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)\mathbf{A} &= \rho \\ \boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{dA} + \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\mathbf{A} + \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)\mathbf{d}\varphi &= \tfrac{1}{c}\mathbf{j} \end{aligned}\right\} \tag{45}$$

If—as without loss of generality we may do[8] —we assume the potentials φ and $\mathbf{A}$ to satisfy the so-called "Lorentz gauge condition"

$$\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{A} + \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)\varphi = 0 \tag{46}$$

then we can use $\boldsymbol{*}\mathbf{d}\boldsymbol{*}\big(\frac{1}{c}\frac{\partial}{\partial t}\big)\mathbf{A} = -\big(\frac{1}{c}\frac{\partial}{\partial t}\big)^2\varphi$ and $\big(\frac{1}{c}\frac{\partial}{\partial t}\big)\mathbf{d}\varphi = -\mathbf{d}\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{A}$ in (45) to obtain

$$\left.\begin{aligned} \Big\{\big[\qquad\quad \boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{d}\big] - \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\Big\}\varphi &= \rho \\ \Big\{\big[\mathbf{d}\boldsymbol{*}\mathbf{d}\boldsymbol{*} - \boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{d}\big] - \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\Big\}\mathbf{A} &= \tfrac{1}{c}\mathbf{j} \end{aligned}\right\} \tag{47}$$

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[8] The argument runs as follows: Let $\mathbf{A}$ and φ be potentials which, while they do give rise to the physical $\mathbf{E}$ and $\mathbf{B}$ fields, do *not* satisify the Lorentz gauge condition (46), and let $\mathbf{A}'$ and φ' be gauge-equivalent potentials:

$$\left.\begin{aligned} \mathbf{A}' &= \mathbf{A} + \mathbf{d}\chi \\ \varphi' &= \varphi - \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)\chi \end{aligned}\right\} \tag{48}$$

Then

$$\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{A}' = \boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{A} + \boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{d}\chi$$

$$\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)\varphi' = \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)\varphi - \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\chi$$

and, writing $f \equiv -\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{A} - \big(\frac{1}{c}\frac{\partial}{\partial t}\big)\varphi$, we have only to take χ to be a solution (any solution) of

$$\Big\{\big[\qquad\quad \boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{d}\big] - \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\Big\}\chi = f$$

to insure that $\mathbf{A}'$ and φ' *do* satisfy the Lorentz condition.

We recall an argument to the effect that

$$\text{lap} \longleftrightarrow \mathbf{d{*}d{*}} + (-)^{p(n-p+1)}\mathbf{{*}d{*}d} = \begin{cases} \big[\qquad\qquad \mathbf{{*}d{*}d}\big] & \text{when } n=3, p=0 \\ \big[\mathbf{d{*}d{*}} - \mathbf{{*}d{*}d}\big] & \text{when } n=3, p=1 \end{cases}$$

so equations (47) can, if we wish, be cast in this more attractively symmetric form

$$\begin{aligned} \Big\{\text{lap} - \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\Big\}\,\varphi &= \rho \\ \Big\{\text{lap} - \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\Big\}\mathbf{A} &= \tfrac{1}{c}\mathbf{j} \end{aligned}$$

We have here, by a clever exercise of our gauge freedom, achieved a quartet of *uncoupled* second-order partial differential equations (inhomogeneous wave equations).

So what conclusions can we draw from the preceding few lines of text? Minimally, that it is *possible* to construct an exterior formulation of the Maxwell-Lorentz equations (3), to pour old wine into a new bottle (though in the present instance the bottle is in fact nearly as old as the wine!). Possible, and actually fairly easy: numerology combined with the sharp logic of the exterior calculus to make clear the path we should follow. We saw something also of the power latent in the simple identities $\mathbf{**} \sim \mathbf{1}$ and $\mathbf{dd} = \mathbf{0}$, which are in themselves equivalent to what would in standard notation be a fairly lengthly list of identities, and which give rise spontaneously to such constructions as the "vector laplacian." But it does seem somewhat forced to think of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{j}$ and ρ as "antisymmetric structures." and it would be hard to argue on present evidence that the exterior calculus affords tools so uniquely sharp and efficient as to merit inclusion in every physicist's toolbox. However...

Let us now—in a kind of "proto-relativistic" spirit—agree to write $x^0 \equiv ct$ and to construe our fields to be defined on a 4-dimensional spacetime. To do so is—since

when $n = 4$	0-forms have 1 component
	1-forms have 4 components
	2-forms have 6 components
	3-forms have 4 components
	4-forms have 1 components

—to confront quite a different set of options. Simple numerology suggests that the $3+3=6$ collective components of $\mathbf{E}$ and $\mathbf{B}$ should be deployed (how?) as a 2-form, and that the $1+3=4$ collective components of ρ and $\mathbf{j}$ should be deployed (how?) either as a 1-form or as a 3-form (which?). Let $\mathbf{F}$ denote the conjectured field-form, let $\mathbf{J}$ denote the source-form, and let us take note of the fact that according to (27)

$$\mathbf{**} = (-)^p\mathbf{1} \quad \text{in the case } n=4$$

while

$$\text{lap} = \boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{d} + \mathbf{d}\boldsymbol{*}\mathbf{d}\boldsymbol{*} \quad \text{for all } p \text{ in the case } n = 4$$

To extract $\mathbf{J}$ from the 2-form $\mathbf{F}$ by an exterior differentiation process (as two of the Maxwell-Lorentz equations suggest we want to do) is most naturally to write $\mathbf{dF} = \mathbf{J}$, which is in effect to assign to $\mathbf{J}$ the status of a 3-form. Notational play has, on the other hand, suggested (at (5.1)) the naturalness of assigning to $\mathbf{J}$ the status of a 1-form, which we could accomplish by writing either $\boldsymbol{*}\mathbf{dF} = \mathbf{J}$ or $\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{F} = \mathbf{J}$. Since the $\sqrt{g}$ factors which enter into the definition of $\boldsymbol{*}$ introduce factors of i when $g < 0$, and since we can in such cases nevertheless avoid the "intrusive complexification" of our real physics if we can arrange to have the $\boldsymbol{*}$ operators enter in pairs,[9] we tentatively adopt the latter option, writing

$$\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{F} = \mathbf{J} \tag{49.1}$$

To obtain exterior equations corresponding to (5.2) we write $\mathbf{G} = k_1 \cdot \mathbf{F} + k_2 \cdot \boldsymbol{*}\mathbf{F}$ (which would appear to exhaust the possibilities) and require $\mathbf{dG} = \mathbf{0}$. This—by (49.1); i.e., by $\mathbf{d}\boldsymbol{*}\mathbf{F} = \boldsymbol{*}\mathbf{J}$—enforces $k_2 = 0$, and we obtain $\mathbf{dF} = \mathbf{0}$, which I prefer to formulate as a relation between (not 3-forms but) 1-forms:

$$\boldsymbol{*}\mathbf{d}\ \mathbf{F} = \mathbf{0} \tag{49.2}$$

(Here an i, if present, could be painlessly discarded). It is (by the converse of the Poincaré Lemma) an implication of (49.2) that there exists a 1-form $\mathbf{A}$—unique to within a gauge transformation

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \mathbf{d}\chi \qquad \text{with } \chi \text{ an arbitrary 0-form} \tag{50}$$

—such that

$$\mathbf{F} = \mathbf{dA} \tag{51}$$

Returning with (51)—which renders (49.2) automatic—to (49.1) we obtain

$$\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{dA} = \mathbf{J} \tag{52}$$

which can be written

$$\Box\mathbf{A} = \mathbf{J} \qquad \text{with} \qquad \Box \equiv \boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{d} + \mathbf{d}\boldsymbol{*}\mathbf{d}\boldsymbol{*} \tag{53}$$

provided $\mathbf{A}$ is annihilated by $\mathbf{d}\boldsymbol{*}\mathbf{d}\boldsymbol{*}$. But look to (50): to achieve (the yet stronger condition) $\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{A}' = \mathbf{0}$ we have only to require that χ satisfies $\Box\chi = f$, where $f \equiv -\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{A}$ and where $\Box \equiv \boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{d}$ is the laplacian operator appropriate

[9] Alternatively, one might make the replacement $g \longrightarrow |g|$. It is, however, my experience that algebraic infelicities then pop up elsewhere in the formalism, and it is not clear on balance that one has come out ahead.

to 0-forms.[10] We conclude that it is without essential loss of generality that at (52) one invokes the "Lorentz gauge condition"

$$\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{A} = \mathbf{0} \tag{54}$$

to achieve (53). We observe that "charge conservation"

$$\boldsymbol{*}\mathbf{d}\boldsymbol{*}\,\mathbf{J} = \mathbf{0} \tag{55}$$

is (by (32)) an immediate implication of (49.1). Finally, we note that equations (49) admit of integral formulation:

$$\iint_{\partial R} \boldsymbol{*}\mathbf{F} = \iiint_R \boldsymbol{*}\mathbf{J} \tag{56.1}$$

$$\iint_{\partial R} \mathbf{F} = 0 \tag{56.2}$$

Thus in just a few lines—lines notable for their stark simplicity and utter naturalness—have we managed to construct a theory[11] which embodies the architectural *design* of Maxwellian electrodynamics. But by what constellation of specific associations (if any) does it become an *expression in fact* of the physics latent in (5)? To answer such a question we must look to the finer details, which I now therefore do. If

$$\mathbf{F} \prec F_{\mu\nu}$$

is 4-dimensional (and antisymmetric) then according to (36) and (37) (see also (38))

$$\begin{aligned}
\boldsymbol{*}\mathbf{d}\ \mathbf{F} &\prec \frac{1}{3!2!}\sqrt{g}\epsilon_\mu{}^{k_1k_2k_3}\delta_{k_1k_2k_3}{}^{\beta\alpha_1\alpha_2}\partial_\beta F_{\alpha_1\alpha_2}\\
&= \tfrac{1}{2}\sqrt{g}g_{\mu\nu}\underbrace{\epsilon^{\nu\beta\alpha_1\alpha_2}}_{=\frac{1}{g}\cdot\varepsilon^{j\beta\alpha_1\alpha_2}\quad\text{by (24)}}\partial_\beta F_{\alpha_1\alpha_2}\\
&= \frac{1}{\sqrt{g}}\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03}\\ g_{10} & g_{11} & g_{12} & g_{13}\\ g_{20} & g_{21} & g_{22} & g_{23}\\ g_{30} & g_{31} & g_{32} & g_{33}\end{pmatrix}\begin{pmatrix} \partial_1F_{23}+\partial_2F_{31}+\partial_3F_{12}\\ \partial_0F_{32}+\partial_2F_{03}-\partial_3F_{02}\\ \partial_0F_{13}+\partial_3F_{01}-\partial_1F_{03}\\ \partial_0F_{21}+\partial_1F_{02}-\partial_2F_{01}\end{pmatrix}
\end{aligned} \tag{57.1}$$

[10] Compare this with the equivalent—but much clumsier—argument sketched in the preceeding footnote.

[11] Actually a *broad class* of theories, for no restriction has been placed (except perhaps implicitly) either upon the value of n or upon the construction of the metric g_{ij}. We catch here our first glimpse of the power of the exterior calculus to expand—and at the same time to guide—the creative imaginations of physicists and applied mathematicians.

$$\ast\mathbf{d}\ast\mathbf{F} \prec +\frac{1}{2}\frac{1}{\sqrt{g}}\delta_{\mu\beta}{}^{k_1k_2}\partial^\beta\big(\sqrt{g}F_{k_1k_2}\big) = \frac{1}{\sqrt{g}}\begin{pmatrix}\partial^\alpha\big(\sqrt{g}F_{0\alpha}\big)\\ \partial^\alpha\big(\sqrt{g}F_{1\alpha}\big)\\ \partial^\alpha\big(\sqrt{g}F_{2\alpha}\big)\\ \partial^\alpha\big(\sqrt{g}F_{3\alpha}\big)\end{pmatrix} \tag{57.2}$$

with $\partial^\alpha \equiv g^{\alpha\beta}\partial_\beta$ while

$$\mathbf{J} \prec J_\mu = \begin{pmatrix}J_0\\ J_1\\ J_2\\ J_3\end{pmatrix} \tag{57.3}$$

We want, according to (4), to achieve

$$\left.\begin{aligned}\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0\\ \partial_0 B_1 + \partial_2 E_3 - \partial_3 E_2 = 0\\ \partial_0 B_2 + \partial_3 E_1 - \partial_1 E_3 = 0\\ \partial_0 B_3 + \partial_1 E_2 - \partial_2 E_1 = 0\end{aligned}\right\} \tag{58.1}$$

as an instance of (49.2), and to achieve

$$\left.\begin{aligned}\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = \rho\\ -\partial_0 E_1 + \partial_2 B_3 - \partial_3 B_2 = \tfrac{1}{c}j_1\\ -\partial_0 E_2 + \partial_3 B_1 - \partial_1 B_3 = \tfrac{1}{c}j_2\\ -\partial_0 E_3 + \partial_1 B_2 - \partial_2 B_1 = \tfrac{1}{c}j_3\end{aligned}\right\} \tag{58.2}$$

as an instance of (49.1). Comparing of the last three of equations (58.1) with their counterparts according to (57.1), we obtain

$$\begin{pmatrix}F_{00} & F_{01} & F_{02} & F_{03}\\ F_{10} & F_{11} & F_{12} & F_{13}\\ F_{20} & F_{21} & F_{22} & F_{23}\\ F_{30} & F_{31} & F_{32} & F_{33}\end{pmatrix} \sim \begin{pmatrix}\ast & E_1 & E_2 & E_3\\ \ast & \ast & \ast & B_2\\ \ast & B_3 & \ast & \ast\\ \ast & \ast & B_1 & \ast\end{pmatrix}$$

which, if we set the constant of proportionality equal to unity and invoke antisymmetry, gives

$$= \begin{pmatrix}0 & +E_1 & +E_2 & +E_3\\ -E_1 & 0 & -B_3 & +B_2\\ -E_2 & +B_3 & 0 & -B_1\\ -E_3 & -B_2 & +B_1 & 0\end{pmatrix} \qquad (12) \leftarrow (59)$$

and achieves conformity also with the first of the equations (58.1). We note that *no enforced constraint has been has been imposed thus far upon the structure of the metric* $g_{\mu\nu}$. Our assignment now is (by (57.2)) to display (57.2) as an instance of

$$\partial^\alpha F_{\mu\alpha} + F_{\mu\alpha}\cdot\partial^\alpha \log\sqrt{g} = J_\mu$$

To kill the $(\partial^\alpha \log\sqrt{g})$-term I assume g to be (which is much weaker than assuming $g_{\mu\nu}$ to be) x-independent, and obtain

$$\begin{aligned}\partial^1 F_{01} + \partial^2 F_{02} + \partial^3 F_{03} &= J_0\\ \partial^0 F_{10} + \partial^2 F_{12} + \partial^3 F_{13} &= J_1\\ \partial^0 F_{20} + \partial^1 F_{21} + \partial^3 F_{23} &= J_2\\ \partial^0 F_{30} + \partial^1 F_{31} + \partial^2 F_{32} &= J_3\end{aligned}$$

which by (59) can after slight rearrangement be written

$$\left.\begin{aligned}\partial^1 E_1 + \partial^2 E_2 + \partial^3 E_3 &= J_0\\ -\partial^0 E_1 - \partial^2 B_3 + \partial^3 B_2 &= J_1\\ -\partial^0 E_2 - \partial^3 B_1 + \partial^1 B_3 &= J_2\\ -\partial^0 E_3 - \partial^1 B_2 + \partial^2 B_1 &= J_3\end{aligned}\right\} \tag{60}$$

The last three of the preceeding equations become identical with their counterparts in (58.2) provided we write

$$\begin{pmatrix}\partial^0\\ \partial^1\\ \partial^2\\ \partial^3\end{pmatrix} = \underbrace{\begin{pmatrix}+1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}}_{\equiv\,||g^{\mu\nu}||}\begin{pmatrix}\partial_0\\ \partial_1\\ \partial_2\\ \partial_3\end{pmatrix} \qquad (8) \leftarrow (61)$$

and set

$$\begin{pmatrix}J_0\\ J_1\\ J_2\\ J_3\end{pmatrix} = \tfrac{1}{c}\begin{pmatrix}*\\ j_1\\ j_2\\ j_3\end{pmatrix}$$

The remaining equation then reads $-\partial_1 E_1 - \partial_2 E_2 - \partial_3 E_3 = J_0$ and entails that we set $J_0 = -c\rho$, giving

$$\begin{pmatrix}J_0\\ J_1\\ J_2\\ J_3\end{pmatrix} = \tfrac{1}{c}\begin{pmatrix}-c\rho\\ j_1\\ j_2\\ j_3\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}J^0\\ J^1\\ J^2\\ J^3\end{pmatrix} = \begin{pmatrix}+J_0\\ -J_1\\ -J_2\\ -J_3\end{pmatrix}$$

Equations (60) now read

$$\begin{aligned}-\partial_1 E_1 - \partial_2 E_2 - \partial_3 E_3 &= J_0\\ -\partial_0 E_1 + \partial_2 B_3 - \partial_3 B_2 &= J_1\\ -\partial_0 E_2 + \partial_3 B_1 - \partial_1 B_3 &= J_2\\ -\partial_0 E_3 + \partial_1 B_2 - \partial_2 B_1 &= J_3\end{aligned}$$

which (since (59) and (61) give back precisely the $||F^{\mu\nu}||$ of (4.1)) can be written (compare (5.1))

$$\begin{aligned}
-\partial_1 F^{10} - \partial_1 F^{10} - \partial_1 F^{10} &= +J^0\\
+\partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} &= -J^1\\
+\partial_0 F^{02} + \partial_1 F^{12} + \partial_3 F^{32} &= -J^2\\
+\partial_0 F^{03} + \partial_1 F^{13} + \partial_2 F^{23} &= -J^3
\end{aligned}$$

We come thus to this conclusion: the exterior differential equations

$$\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{F} = \mathbf{J} \qquad \text{and} \qquad \boldsymbol{*}\mathbf{d}\ \mathbf{F} = \mathbf{0} \tag{49}$$

do provide a formulation of the Maxwell-Lorentz equations (3)=(5) *provided* one takes the metric $g_{\mu\nu}$ to be given by (11), the components $F_{\mu\nu}$ of the 2-form $\mathbf{F}$ to be given by (12), and the components J_μ of the 1-form $\mathbf{J}$ to be the *negatives* of those associated with (5.1).[12] Alternatively and equivalently, one can write

$$\iint_{\partial R} \boldsymbol{*}\mathbf{F} = \iiint_R \boldsymbol{*}\mathbf{J} \qquad \text{and} \qquad \iint_{\partial R} \mathbf{F} = 0 \tag{56}$$

where R is any orientable "3-bubble" in 4-dimensional spacetime. Equations (56) are not "decorated with implicit indices;" they comprise a pair of number-valued statements which are (as it happens) fundamental to the theory of electromagnetic Green's functions.

There is, I think, a valuable general lesson to be drawn from the fact that Maxwellian electrodynamics admits of exterior differential formulation *in two distinct ways*—one 3-dimensional, the other 4-dimensional. And from the fact that it is the latter—the less obvious—formulation which turned out to be both the simpler and (insofar as it gave us special relativity) physically the richer.

7. Exterior elaborations of the Maxwellian electrodynamics. From the converse of the Poincaré Lemma it follows that if the 4-potential $\mathbf{A}$ satisfies the Lorentz gauge condition $\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{A} = \mathbf{0}$ then there exists an imaginary 2-form $\mathbf{H}$ such that $\boldsymbol{*}\mathbf{A} = \mathbf{dH}$, giving

$$\mathbf{A} = \boldsymbol{*}\mathbf{dH}: \quad \text{Lorentz gauge condition now automatic} \tag{62}$$

It follows similarly from (55) that there exists an imaginary 2-form $\mathbf{C}$ such that

$$\mathbf{J} = \boldsymbol{*}\mathbf{dC}: \quad \text{charge conservation now automatic} \tag{63}$$

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[12] It was to expose the origins of that minus sign that I have recently been working in such fussy detail. Note, by the way, that the "Lorentz covariance of Maxwell's equations" was established—and special relativity brought implicitly into being—when at (61) the Lorentzian structure of $g_{\mu\nu}$ was forced. The question arises: by what exterior line of argument does the less well-known *conformal* covariance of electrodynamics come naturally to light? This is a question which I propose to pursue on another occasion; in the meantime, see ELECTRODYNAMICS (1980), pp. 221-243.

The "superpotential" $\mathbf{H}$ and the "supercurrent" $\mathbf{C}$ are unique only up to gauge transformations

$$\left.\begin{aligned}\mathbf{H}&\longrightarrow\mathbf{H}'=\mathbf{H}+\mathbf{dh}\\ \mathbf{C}&\longrightarrow\mathbf{C}'=\mathbf{C}+\mathbf{dc}\end{aligned}\right\}\tag{64}$$

where $\mathbf{h}$ and $\mathbf{c}$ are arbitrary (imaginary) 1-forms.[13] Returning with (62) and (63) to field equation (53), we obtain

$$*\mathbf{d}*\mathbf{d}*\mathbf{dH}=*\mathbf{dC}\tag{65}$$

This quartet of conditions on six variables would follow from (but do not strictly entail) the following sextet of conditions:

$$*\mathbf{d}*\mathbf{dH}=\mathbf{C}\tag{66}$$

Inequivalent to (66)—but equally capable of giving back (65), and in some respects more attractive than (66)—is the sextet

$$\Box\mathbf{H}=\mathbf{C}\tag{67.1}$$

where, as previously remarked, $\Box=*\mathbf{d}*\mathbf{d}+\mathbf{d}*\mathbf{d}*$ for all p in the case $n=4$. Equivalently

$$\Box*\mathbf{H}=*\mathbf{C}\tag{67.2}$$

by the following line of argument: bring $**=(-)^p\mathbf{1}$ (which, as was remarked at (27), holds whenever n is even) to the special construction of $\Box$ and obtain $*\Box*=(-)^p\Box$. If (as in (67.1)) p is even we on these grounds have $**=\mathbf{1}$ and $*\Box*=\Box$, from which (67.1)$\Longleftrightarrow$(67.2) follows at once. Equations (67) both exhibit the structure

$$\Box(\text{2-form related to field structure})=\text{2-form related to source structure}$$

They are, moreover, mathematically—and hence physically—equivalent. It becomes therefore natural to consider them in linear combination, writing

$$\Box\mathbf{H}(u,v)=\mathbf{C}(u,v)\qquad\text{where}\qquad\left\{\begin{aligned}\mathbf{H}(u,v)&\equiv u\cdot\mathbf{H}+iv\cdot*\mathbf{H}\\ \mathbf{C}(u,v)&\equiv u\cdot\mathbf{C}+iv\cdot*\mathbf{C}\end{aligned}\right.\tag{68}$$

from which (67.1) and (67.2) can be recovered as special cases. In (68) we have a (u,v)-paramaterized family of equations, each of which is in itself equivalent to the full set of Maxwell-Lorentz equations (49). Introducing polar coordinates onto the (u,v)-plane

$$\begin{aligned}u&=\lambda\cos\theta\\ v&=\lambda\sin\theta\end{aligned}$$

[13] The "superpotential" generalizes an idea introduced into pre-relativistic electrodynamics by H. Hertz, and is sometimes called the "Hertz potential." The pre-relativistic precursor of the "supercurrent" was called the "stream potential" by Hertz, who borrowed his terminology from fluid mechanics.

we obtain $\lambda\cdot\Box\mathbf{H}(\theta)=\lambda\cdot\mathbf{C}(\theta)$ with

$$\mathbf{H}(\theta)\equiv\cos\theta\cdot\mathbf{H}+i\sin\theta\cdot{*\mathbf{H}} \tag{69.1}$$

$$\mathbf{C}(\theta)\equiv\cos\theta\cdot\mathbf{C}+i\sin\theta\cdot{*\mathbf{C}} \tag{69.2}$$

The λ-factors drop away (in consequence essentially of the *scale invariance* of electromagnetic theory) and we are left with a one-parameter family of structurally identical field equations

$$\Box\mathbf{H}(\theta)=\mathbf{C}(\theta) \tag{70}$$

which give back (67.1) at $\theta=0$ and (67.2) at $\theta=\frac{1}{2}\pi$. Enlarging similarly upon (62) and (63), we introduce

$$\mathbf{A}(\theta)\equiv{*\mathbf{d}}\mathbf{H}(\theta) \tag{71}$$

(which, for all values of θ, conforms automatically to the Lorentz gauge condition) and

$$\mathbf{J}(\theta)\equiv{*\mathbf{d}}\mathbf{C}(\theta) \tag{72}$$

(which yields automatic "charge conservation"). In this notation (70)—multiply by $*\mathbf{d}$ and make use of $\mathbf{dd}=\mathbf{0}$—assumes the form

$$\Box\mathbf{A}(\theta)=\mathbf{J}(\theta) \tag{73}$$

But

$$\begin{aligned}\Box\mathbf{A}(\theta)&=\big[{*\mathbf{d}*\mathbf{d}}+\mathbf{d}\,\underbrace{{*\mathbf{d}*}\big]\mathbf{A}(\theta)}_{=\,\mathbf{0}\text{ by the Lorentz gauge condition; i.e., by (71)}}\\&={*\mathbf{d}*}\mathbf{F}(\theta)\\&\qquad\big|\\&\qquad\mathbf{F}(\theta)\equiv\mathbf{d}\mathbf{A}(\theta)\end{aligned} \tag{74}$$

so we have

$$*\mathbf{d}*\mathbf{F}(\theta)=\mathbf{J}(\theta) \tag{75.1}$$

and, in automatic consequence of (74),

$$*\mathbf{d}\;\;\mathbf{F}(\theta)=\mathbf{0} \tag{75.2}$$

These equations are structurally identical to the Maxwell-Lorentz equations (49), to which they reduce when $\theta=0 \bmod 2\pi$.

Looking now beyond the fine details to the underlying pattern of what we have accomplished: It would, in view of the structure of equations (69), be natural to call the transformations

$$\mathbf{H}\longrightarrow\mathbf{H}(\theta)\quad\text{and}\quad\mathbf{C}\longrightarrow\mathbf{C}(\theta) \tag{76}$$

"duality rotations." According to (71) and (72) they induce

$$\mathbf{A} \longrightarrow \mathbf{A}(\theta) \quad \text{and} \quad \mathbf{J} \longrightarrow \mathbf{J}(\theta) \tag{77}$$

where

$$\mathbf{A}(\theta) = \cos\theta \cdot \mathbf{A} + i\sin\theta \cdot \mathbf{A}^h \qquad \text{with} \qquad \begin{cases} \mathbf{A} \;\equiv {*}\mathbf{d}\;\;\mathbf{H} \\ \mathbf{A}^h \equiv {*}\mathbf{d}{*}\mathbf{H} \neq {*}\mathbf{A} \end{cases} \tag{78.1}$$

$$\mathbf{J}(\theta) = \cos\theta \cdot \mathbf{J} + i\sin\theta \cdot \mathbf{J}^h \qquad \text{with} \qquad \begin{cases} \mathbf{J} \;\equiv {*}\mathbf{d}\;\;\mathbf{C} \\ \mathbf{J}^h \equiv {*}\mathbf{d}{*}\mathbf{C} \neq {*}\mathbf{J} \end{cases} \tag{78.2}$$

The transformations (77) do emphatically *not* have the character of "duality rotations"; to write (say) "$\mathbf{A}(\theta) = \cos\theta \cdot \mathbf{A} + i\sin\theta \cdot {*}\mathbf{A}$" would, in fact, be nonsensical, since it would entail the addition of a 1-form to a 3-form. Finally we have

$$\mathbf{F} \longrightarrow \mathbf{F}(\theta) \tag{79}$$

where

$$\mathbf{F}(\theta) = \cos\theta \cdot \mathbf{F} + i\sin\theta \cdot \mathbf{F}^h \qquad \text{with} \qquad \begin{cases} \mathbf{F} \;\equiv \mathbf{dA} \;\;= \mathbf{d}{*}\mathbf{d}\;\;\mathbf{H} \\ \mathbf{F}^h \equiv \mathbf{dA}^h = \mathbf{d}{*}\mathbf{d}{*}\mathbf{H} \end{cases} \tag{80}$$

Here (since $\mathbf{F}$ and ${*}\mathbf{F}$ are both 2-forms) it would not be patently "nonsensical" to write "$\mathbf{F}(\theta) = \cos\theta \cdot \mathbf{F} + i\sin\theta \cdot {*}\mathbf{F}$," but the meaning of (80) lies actually elsewhere; we have

$$\begin{aligned} \mathbf{F}^h &= \mathbf{d}{*}\mathbf{d}{*}\mathbf{H} \\ &= (\Box - {*}\mathbf{d}{*}\mathbf{d})\mathbf{H} \\ &= \mathbf{C} - {*}\mathbf{F} \end{aligned} \tag{81}$$

giving

$$\mathbf{F}(\theta) = \cos\theta \cdot \mathbf{F} - i\sin\theta \cdot {*}\mathbf{F} + i\sin\theta \cdot \mathbf{C} \tag{82}$$

Returning with (82) and (78.2) to (75) we obtain

$$\begin{aligned} &{*}\mathbf{d}{*}(\cos\theta \cdot \mathbf{F} - i\sin\theta \cdot {*}\mathbf{F}) \\ &\qquad\qquad = (\cos\theta \cdot \mathbf{J} + \underbrace{i\sin\theta \cdot \mathbf{J}^h) - {*}\mathbf{d}{*}(i\sin\theta \cdot \mathbf{C})}_{=\,\mathbf{0} \text{ by definition (78.2) of } \mathbf{J}^h} \\ &{*}\mathbf{d}\;\;(\cos\theta \cdot \mathbf{F} - i\sin\theta \cdot {*}\mathbf{F}) = -\underbrace{{*}\mathbf{d}\;\;(i\sin\theta \cdot \mathbf{C})}_{=\,\mathbf{J}} \end{aligned}$$

whence

$$\left.\begin{aligned} \cos\theta \cdot {*}\mathbf{d}{*}\mathbf{F} - i\sin\theta \cdot {*}\mathbf{d}\;\;\mathbf{F} &= \cos\theta \cdot \mathbf{J} \\ i\cos\theta \cdot {*}\mathbf{d}\;\;\mathbf{F} + \;\sin\theta \cdot {*}\mathbf{d}{*}\mathbf{F} &= \sin\theta \cdot \mathbf{J} \end{aligned}\right\} \tag{83}$$

To see what's going on, set $\theta = 0$ and recover (49), then set $\theta = \frac{1}{2}\pi$ and recover (a signed variant of) the same equations *in reversed sequence*.

It is universally assumed that if the world contained *magnetic* charges and currents, then in place of

$$\ast\mathbf{d}\ast\mathbf{F} = \mathbf{J} \qquad \text{and} \qquad \ast\mathbf{d}\;\;\mathbf{F} = \mathbf{0} \tag{49}$$

one would write

$$\ast\mathbf{d}\ast\mathbf{F} = \mathbf{J}_e \qquad \text{and} \qquad \ast\mathbf{d}\;\;\mathbf{F} = \mathbf{J}_m \tag{84}$$

where $\mathbf{J}_m$ is an imaginary 1-form descriptive of the magnetic current. And it is "common knowledge" that the adjustment $\mathbf{0} \longrightarrow \mathbf{J}_m$ entails sacrifice of the 4-potential $\mathbf{A}$. The exterior calculus leads however—by force of its own internal logic, almost "automatically"—to an elegant solution of the latter problem. In place of (51) write

$$\mathbf{F} = \mathbf{d}\mathbf{A}_e + \ast\mathbf{d}\mathbf{A}_m \tag{85}$$

where $\mathbf{A}_e$ and $\mathbf{A}_m$ are a *pair* of 1-form potentials (the former real, the latter imaginary). We observe that equation (85) admits of an enlarged group of gauge transformations

$$\begin{aligned} \mathbf{A}_e &\longrightarrow \mathbf{A}'_e = \mathbf{A}_e + \mathbf{a}_e \\ \mathbf{A}_m &\longrightarrow \mathbf{A}'_m = \mathbf{A}_m + \mathbf{a}_m \end{aligned} \qquad \text{with} \quad \mathbf{d}\mathbf{a}_e + \ast\mathbf{d}\mathbf{a}_m = \mathbf{0} \tag{86}$$

where it is sufficient *but not necessary* that $\mathbf{a}_e = \mathbf{d}\chi_e$ and $\mathbf{a}_m = \mathbf{d}\chi_m$. When we return now with (85) to (84) we obtain

$$\ast\mathbf{d}\ast\mathbf{d}\mathbf{A}_e = \mathbf{J}_e \qquad \text{and} \qquad \ast\mathbf{d}\ast\mathbf{d}\mathbf{A}_m = \mathbf{J}_m \tag{87}$$

which entail "extended charge conservation"

$$\ast\mathbf{d}\ast\mathbf{J}_e = \ast\mathbf{d}\ast\mathbf{J}_m = 0 \tag{88}$$

and can in the "extended Lorentz gauge"

$$\ast\mathbf{d}\ast\mathbf{A}_e = \ast\mathbf{d}\ast\mathbf{A}_m = 0 \tag{89}$$

be written

$$\Box\mathbf{A}_e = \mathbf{J}_e \qquad \text{and} \qquad \Box\mathbf{A}_m = \mathbf{J}_m \tag{90}$$

Enlarging similarly upon (62) and (63), we introduce an electric/magnetic *pair* of superpotentials and a corresponding *pair* of supercurrents, writing

$$\mathbf{A}_e = \ast\mathbf{d}\mathbf{H}_e \qquad \text{and} \qquad \mathbf{A}_m = \ast\mathbf{d}\mathbf{H}_m \tag{91}$$

$$\mathbf{J}_e = \ast\mathbf{d}\,\mathbf{C}_e \qquad \text{and} \qquad \mathbf{J}_m = \ast\mathbf{d}\,\mathbf{C}_m \tag{92}$$

The continuity equations (88) and gauge conditions (89) are then rendered automatic, and the argument that gave (67.1) now gives

$$\Box\mathbf{H}_e = \mathbf{C}_e \qquad \text{and} \qquad \Box\mathbf{H}_m = \mathbf{C}_m \tag{93.1}$$

which are *not* mathematically/physically equivalent, though evidently they are respectively equivalent to

$$\Box{*}\mathbf{H}_e = {*}\mathbf{C}_e \qquad \text{and} \qquad \Box{*}\mathbf{H}_m = {*}\mathbf{C}_m \tag{93.2}$$

The equations developed in the preceeding paragraph occur in identically-structured pairs, and it is in light of this circumstance natural to introduce

$$\mathbf{F}(\varphi) \equiv \cos\varphi\cdot\mathbf{F}\ + i\sin\varphi\cdot{*}\mathbf{F} \tag{94}$$

and

$$\mathbf{J}_e(\varphi) \equiv \cos\varphi\cdot\mathbf{J}_e + i\,\sin\varphi\cdot\ \mathbf{J}_m \tag{95.1}$$

In this notation the generalized Maxwell-Lorentz equations (84) can be written

$${*}\mathbf{d}{*}\mathbf{F}(\varphi) = \mathbf{J}_e\,(\varphi) \tag{96.1}$$

Equivalently[14]

$${*}\mathbf{d}\ \ \mathbf{F}(\varphi) = \mathbf{J}_m(\varphi) \tag{96.2}$$

where $\mathbf{J}_e(\varphi+\frac{1}{2}\pi)\equiv i\,\mathbf{J}_m(\varphi)$ gives

$$\mathbf{J}_m(\varphi) = \cos\varphi\cdot\mathbf{J}_m + i\,\sin\varphi\cdot\mathbf{J}_e \tag{95.2}$$

One can recover (84) either from (96) by setting $\varphi=0$ or from the stipulation that (96.1) hold for *all* values of φ. Turning now to the associated potential theory, we introduce (85) into (94) and obtain

$$\begin{aligned}\mathbf{F}(\varphi) &= \cos\varphi\cdot\big\{\mathbf{dA}_e + {*}\mathbf{dA}_m\big\} + i\,\sin\varphi\cdot\big\{{*}\mathbf{dA}_e + \mathbf{dA}_m\big\}\\ &= \mathbf{dA}_e(\varphi) + {*}\mathbf{dA}_m(\varphi)\end{aligned} \tag{97}$$

with

$$\left.\begin{aligned}\mathbf{A}_e\,(\varphi) &\equiv \cos\varphi\cdot\mathbf{A}_e\ + i\,\sin\varphi\cdot\mathbf{A}_m\\ \mathbf{A}_m(\varphi) &\equiv \cos\varphi\cdot\mathbf{A}_m + i\,\sin\varphi\cdot\mathbf{A}_e\end{aligned}\right\} \tag{98}$$

Returning with (97) and (98) to (96) we obtain equations

$${*}\mathbf{d}{*}\mathbf{dA}_e(\varphi) = \mathbf{J}_e(\varphi) \qquad \text{and} \qquad {*}\mathbf{d}{*}\mathbf{dA}_m(\varphi) = \mathbf{J}_m(\varphi) \tag{99}$$

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[14] The elementary argument runs as follows:

$$\begin{aligned}{*}\mathbf{d}{*}\mathbf{F}(\varphi) &= {*}\mathbf{d}\cdot{*}\big\{\ \cos\varphi\cdot\mathbf{F} + i\,\sin\varphi\cdot{*}\mathbf{F}\big\}\\ &= {*}\mathbf{d}\cdot\ \ \big\{i\,\sin\varphi\cdot\mathbf{F} +\ \ \cos\varphi\cdot{*}\mathbf{F}\big\}\\ &= i\,{*}\mathbf{d}\cdot\underbrace{\big\{\cos(\varphi-\tfrac{1}{2}\pi)\cdot\mathbf{F} + i\,\sin(\varphi-\tfrac{1}{2}\pi)\cdot{*}\mathbf{F}\big\}}_{=\,\mathbf{F}(\varphi-\frac{1}{2}\pi)} = \mathbf{J}(\varphi)\end{aligned}$$

which entail

$$\ast\mathbf{d}\ast\mathbf{J}_e(\varphi) = \ast\mathbf{d}\ast\mathbf{J}_m(\varphi) = 0 \tag{100}$$

and can in the "extended Lorentz gauge"

$$\ast\mathbf{d}\ast\mathbf{A}_e(\varphi) = \ast\mathbf{d}\ast\mathbf{A}_m(\varphi) = 0 \tag{101}$$

be written

$$\Box\,\mathbf{A}_e(\varphi) = \mathbf{J}_e(\varphi) \qquad \text{and} \qquad \Box\,\mathbf{A}_m(\varphi) = \mathbf{J}_m(\varphi) \tag{102}$$

Equations (99)–(102) are structurally identical to (87)–(90), which they return at $\varphi = 0$. Standing in a similar relationship to (91)–(93) are

$$\mathbf{A}_e(\varphi) = \ast\mathbf{d}\mathbf{H}_e(\varphi) \qquad \text{and} \qquad \mathbf{A}_m(\varphi) = \ast\mathbf{d}\mathbf{H}_m(\varphi) \tag{103}$$

$$\mathbf{J}_e(\varphi) = \ast\mathbf{d}\,\mathbf{C}_e(\varphi) \qquad \text{and} \qquad \mathbf{J}_m(\varphi) = \ast\mathbf{d}\,\mathbf{C}_m(\varphi) \tag{104}$$

which render automatic both the continuity equations (100) and the gauge conditions (101), and entail

$$\Box\;\;\mathbf{H}_e(\varphi) = \;\;\mathbf{C}_e(\varphi) \qquad \text{and} \qquad \Box\;\;\mathbf{H}_m(\varphi) = \;\;\mathbf{C}_m(\varphi) \tag{105.1}$$

—equivalently

$$\Box\ast\mathbf{H}_e(\varphi) = \ast\mathbf{C}_e(\varphi) \qquad \text{and} \qquad \Box\ast\mathbf{H}_m(\varphi) = \ast\mathbf{C}_m(\varphi) \tag{105.2}$$

—where the presumption is that one has defined

$$\left.\begin{aligned} \mathbf{H}_e\,(\varphi) &\equiv \cos\varphi\cdot\mathbf{H}_e\; + i\,\sin\varphi\cdot\mathbf{H}_m \\ \mathbf{H}_m(\varphi) &\equiv \cos\varphi\cdot\mathbf{H}_m + i\,\sin\varphi\cdot\mathbf{H}_e \end{aligned}\right\} \tag{106}$$

$$\left.\begin{aligned} \mathbf{C}_e\,(\varphi) &\equiv \cos\varphi\cdot\mathbf{C}_e\; + i\,\sin\varphi\cdot\mathbf{C}_m \\ \mathbf{C}_m(\varphi) &\equiv \cos\varphi\cdot\mathbf{C}_m + i\,\sin\varphi\cdot\mathbf{C}_e \end{aligned}\right\} \tag{107}$$

There are, within this enlarged conception of Maxwellian electrodynamics, two distinct and independent ways—one electrical, the other magnetic—in which to realize the sequence (70)–(83) of statements which were seen to radiate from (69); one proceeds from

$$\mathbf{H}_e\,(\theta_e\,) \equiv \cos\theta_e\;\cdot\mathbf{H}_e\; + i\sin\theta_e\;\cdot\ast\mathbf{H}_e \tag{108.1}$$

$$\mathbf{C}_e\,(\theta_e\,) \equiv \cos\theta_e\;\cdot\mathbf{C}_e\; + i\sin\theta_e\;\cdot\ast\mathbf{C}_e \tag{108.2}$$

and the other from

$$\mathbf{H}_m(\theta_m) \equiv \cos\theta_m\cdot\mathbf{H}_m + i\sin\theta_m\cdot\ast\mathbf{H}_m \tag{109.1}$$

$$\mathbf{C}_m(\theta_m) \equiv \cos\theta_m\cdot\mathbf{C}_m + i\sin\theta_m\cdot\ast\mathbf{C}_m \tag{109.2}$$

but since the statements latent in (108/109) are in no respect surprising I will not take the trouble to write them out.

We come thus to the striking conclusion that the equations which (in the hypothetical presence even of magnetic sources) serve to describe the physics of the electromagnetic field admit—collectively, and with stark vividness in (105)—of a 3-parameter group of symmetries. The symmetry group in question has nothing (at least nothing directly) to do with such spacetime symmetries as may be associated with the solutions of the field equations in special cases; it pertains universally and globally to *all* electromagnetic source/field systems and the equations that describe them; it has the character (like isotopic spin) of an "intrinsic" symmetry, an "internal" symmetry, a structured "folding amongst themselves" of the several components of the diverse objects that serve to describe the physical field. To expose most clearly the symmetry in question, we will find it convenient to adopt the language of a model; specifically, we discard the electromagnetic specifics of the matter at hand, and look the transformation properties of complex 2-vector

$$\mathbf{z} \equiv \begin{pmatrix} z_e \\ z_m \end{pmatrix} \quad \text{with} \quad \begin{cases} z_e \;= x_e \;+ i\, y_e \\ z_m = x_m + i\, y_m \end{cases}$$

By way of preparation, we observe in connection with (108) and (109) that the following equations

$$\left.\begin{aligned} \mathbf{H}_e \;\longrightarrow\; \mathbf{H}_e\,(\theta_e\,) &\equiv \;\;\cos\theta_e \;\cdot \mathbf{H}_e \;+ \sin\theta_e \;\cdot i{*}\mathbf{H}_e \\ i{*}\mathbf{H}_e \;\longrightarrow\; i{*}\mathbf{H}_e\,(\theta_e\,) &\equiv -\sin\theta_e \;\cdot \mathbf{H}_e \;+ \cos\theta_e \;\cdot i{*}\mathbf{H}_e \end{aligned}\right\} \tag{110}$$

are (by $(i\,{*})(i\,{*}) = -\mathbf{1}$) equivalent each to the other, and that so also (on the same grounds) are

$$\left.\begin{aligned} i\,\mathbf{H}_m \longrightarrow\; i\,\mathbf{H}_m(\theta_m) &\equiv \;\;\cos\theta_m \cdot i\,\mathbf{H}_m + \sin\theta_m \cdot i{*}i\,\mathbf{H}_m \\ i{*}i\,\mathbf{H}_m \longrightarrow i{*}i\,\mathbf{H}_m(\theta_m) &\equiv -\sin\theta_m \cdot i\,\mathbf{H}_m + \cos\theta_m \cdot i{*}i\,\mathbf{H}_m \end{aligned}\right\} \tag{111}$$

The first pair of equations describes a linear relationship between a pair of imaginary 2-forms $\mathbf{H}_e$ and $i{*}\mathbf{H}_e$, while the latter pair of equations describes an identically structured relationship between 2-forms $i\,\mathbf{H}_m$ and $i{*}i\,\mathbf{H}_m$ which are also imaginary.[15] By (106) we are led similarly to write

$$\left.\begin{aligned} \mathbf{H}_e \;\longrightarrow\; \mathbf{H}_e\,(\varphi) &\equiv \;\;\cos\varphi \cdot \mathbf{H}_e + \sin\varphi \cdot i\,\mathbf{H}_m \\ i\,\mathbf{H}_m \longrightarrow i\,\mathbf{H}_m(\varphi) &\equiv -\sin\varphi \cdot \mathbf{H}_e + \cos\varphi \cdot i\,\mathbf{H}_m \\ \\ i{*}\mathbf{H}_e \;\longrightarrow\; i{*}\mathbf{H}_e\,(\varphi) &\equiv \;\;\cos\varphi \cdot i{*}\mathbf{H}_e + \sin\varphi \cdot i{*}i\,\mathbf{H}_m \\ i{*}i\,\mathbf{H}_m \longrightarrow i{*}i\,\mathbf{H}_m(\varphi) &\equiv -\sin\varphi \cdot i{*}\mathbf{H}_e + \cos\varphi \cdot i{*}i\,\mathbf{H}_m \end{aligned}\right\} \tag{112}$$

[15] Since the Lorentz metric is negative definite ($g = -1$) the $*$ operator, in consequence of its definition (26), sends real p-forms into imaginary q-forms ($q = 4 - p$). Since $\mathbf{F}$ is a real 2-form, it follows from (85) that $\mathbf{A}_e$ is real and $\mathbf{A}_m$ imaginary. From this by (91) it follows that $\mathbf{H}_e$ is imaginary and $\mathbf{H}_m$ real.

where the last pair of equations are implications of the first pair. Accepting now the associations

$$\begin{pmatrix} \mathbf{H}_e \\ i{*}\mathbf{H}_e \\ i\,\mathbf{H}_m \\ i{*}i\,\mathbf{H}_m \end{pmatrix} \iff \begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \end{pmatrix}$$

we see that (110) can be represented

$$\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \end{pmatrix} \longrightarrow \begin{pmatrix} +\cos\theta & +\sin\theta & 0 & 0 \\ -\sin\theta & +\cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \end{pmatrix}$$

or—more compactly—

$$\begin{pmatrix} z_e \\ z_m \end{pmatrix} \longrightarrow \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_e \\ z_m \end{pmatrix} = e^{-i\frac{1}{2}\theta} \cdot \begin{pmatrix} e^{-i\frac{1}{2}\theta} & 0 \\ 0 & e^{+i\frac{1}{2}\theta} \end{pmatrix} \begin{pmatrix} z_e \\ z_m \end{pmatrix} \tag{113}$$

while

$$\begin{pmatrix} z_e \\ z_m \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} z_e \\ z_m \end{pmatrix} = e^{-i\frac{1}{2}\theta} \cdot \begin{pmatrix} e^{+i\frac{1}{2}\theta} & 0 \\ 0 & e^{-i\frac{1}{2}\theta} \end{pmatrix} \begin{pmatrix} z_e \\ z_m \end{pmatrix} \tag{114}$$

serves in the same sense to represent (111). Working from (112) we obtain

$$\begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \end{pmatrix} \longrightarrow \begin{pmatrix} +\cos\varphi & 0 & +\sin\varphi & 0 \\ 0 & +\cos\varphi & 0 & +\sin\varphi \\ -\sin\varphi & 0 & +\cos\varphi & 0 \\ 0 & -\sin\varphi & 0 & +\cos\varphi \end{pmatrix} \begin{pmatrix} x_e \\ y_e \\ x_m \\ y_m \end{pmatrix}$$

which can be notated

$$\begin{pmatrix} z_e \\ z_m \end{pmatrix} \longrightarrow \begin{pmatrix} +\cos\varphi & +\sin\varphi \\ -\sin\varphi & +\cos\varphi \end{pmatrix} \begin{pmatrix} z_e \\ z_m \end{pmatrix} \tag{115}$$

The transformations (113), (114) and (115) are all of the type (which is to say, each can be obtained as a special case of)

$$\boldsymbol{z} \longrightarrow \boldsymbol{z}' = e^{i\theta}\cdot \mathbb{U}\,\boldsymbol{z} \tag{116}$$

$$\Big| \quad \mathbb{U} \equiv \exp\big\{i\big[\phi_1\mathbb{S}_1 + \phi_2\mathbb{S}_2 + \phi_3\mathbb{S}_3\big]\big\}$$

where

$$\mathbb{S}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbb{S}_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathbb{S}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the familiar (hermitian) "Pauli matrices." The matrices $\mathbb{U}$ provide, as is well known, a representation of the group $SU(2)$, while the matrices $e^{i\theta}\cdot\mathbb{U}$ provide a representation of $U(2) = U(1)\times SU(2)$. We conclude that electrodynamics—the physical theory latent in equations (105.1)—possesses $U(2)$ as its "global internal symmetry group;" electrodynamics is, in other words, structurally invariant with respect to the linear transformations latent in the stipulation that

$$\langle \boldsymbol{z}|\boldsymbol{z}\rangle \equiv z_e^* z_e + z_m^* z_m \equiv x_e^2 + y_e^2 + x_m^2 + y_m^2 = \text{constant}$$

be invariant under $\boldsymbol{z} \longrightarrow \boldsymbol{z}'$. And when one takes scale-invariance $\boldsymbol{z} \longrightarrow \lambda\boldsymbol{z}$ into account, it becomes invariant with respect to the enveloping "internal conformal group," the group which results at (116) from the complexification of θ; i.e., from the weakened stipulation that

$$\langle \boldsymbol{z}|\boldsymbol{z}\rangle \equiv z_e^* z_e + z_m^* z_m \equiv x_e^2 + y_e^2 + x_m^2 + y_m^2 = 0$$

be invariant under $\boldsymbol{z} \longrightarrow \boldsymbol{z}'$.

Let the internal symmetry group, as described above, be denoted $\mathcal{G}$. It is, I take it, evident that $\mathcal{G}$ affords (multiple) means to (at any given spacetime point) "rotate $\mathbf{J}_m$ to extinction." Maxwell's equations are seen in this light to conceal—in the zeros that appear on the righthand sides of equations (3.3) and (3.4)—a *profound physical claim, wrapped in a convention*. The "profound claim" is that there is within $\mathcal{G}$ an element that "rotates $\mathbf{J}_m$ to extinction *universally—at all spacetime points, for all electromagnetic systems*. It is, however, *by convention* that we have selected the "$\mathbf{J}_m = \mathbf{0}$ representation"—the existence of which is an implausible physical surprise—to be the representation within which to frame our operational definitions, our physical thinking, our calculations. It is, in this regard, interesting to notice that $\mathbf{J}_m = \mathbf{0}$ is implied by *but does not imply* $\mathbf{C}_m = \mathbf{0}$, and that the latter condition does not by itself insure $\mathbf{H}_m = \mathbf{0}$. The figure on the following page is intended to make clear the essential burden of these remarks.

It would be natural at this point to ask whether *conservation laws* can be associated with the $\mathcal{G}$-invariance of the electrodynamical equations. Or to ask of the important body of theory that yields $\mathbf{A}$ as a "gauge field" what natural modifications yield $\mathbf{A}_e$ and $\mathbf{A}_m$ as *joint* gauge fields? Is it possible to obtain $\mathbf{H}_e$ and $\mathbf{H}_m$ as joint "supergauge fields"? I propose, however, to pursue no such speculative issues, but to engage in a little simple "equation counting." Maxwell's equations, in their standard form (3), serve to subject six field variables to a total of eight conditions. Maxwell's equations must, on these grounds, conceal some redundancy. The question is: how redundant are they, and where does the redundancy reside? When formulated in terms of the 4-potential $\mathbf{A}$ the standard theory presents us with a quartet of equations (52) in four field variables; the equations retain nevertheless (at least) the redundancy implicit in gauge condition (50). But when formulated alternatively in terms of the superpotential $\mathbf{H}$ the standard theory imposes six conditions (67.1) on six field variables which are, however, subject to gauge transformations (64)

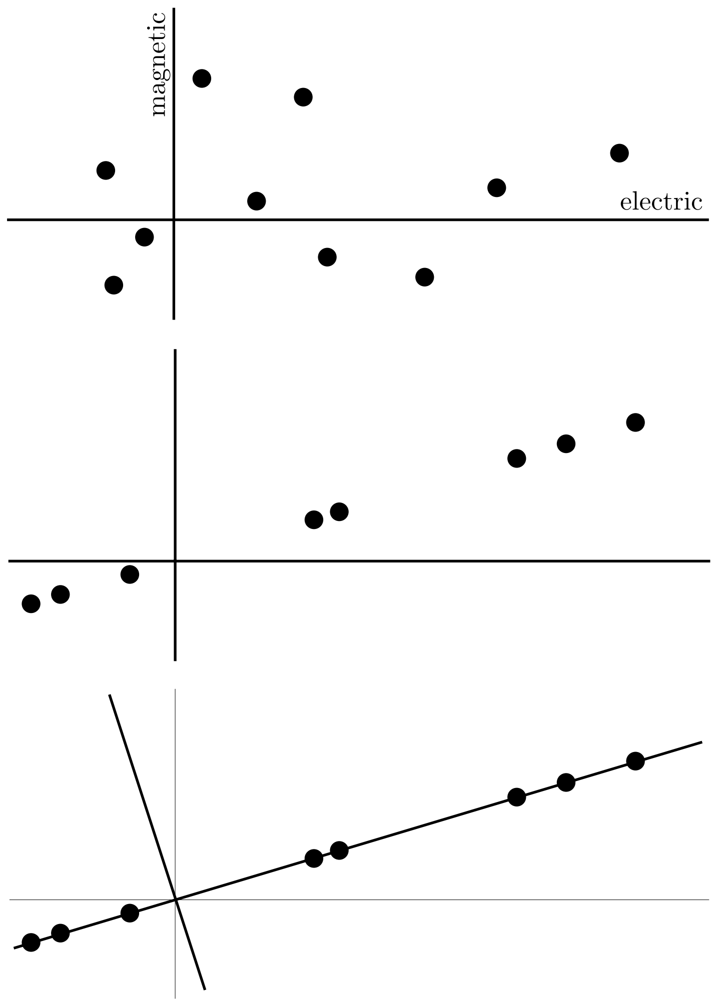

FIGURE 2: *The top figure illustrates how sources might* conceivably *be distributed in electric/magnetic charge space; the middle figure indicates how, according to Maxwellian electrodynamics, they* are in physical fact *distributed; the bottom figure indicates how that fact of nature is* by tacit convention standardly represented. *Additional (non-classical) ideas are required to account for the* quantization *of charge.*

which permit the specification of four arbitrary functions. The implication appears to be that *the electromagnetic field—for all its rich structure—has actually only two degrees of freedom*. This surprising fact was first recognized and exploited by E. T. Whittaker (1903), and was promptly elaborated by

P. W. Debye and (independently) T. J. Bromwich;[16] I know, however, of no account of the "scalar superpotential" idea which conforms naturally/manifestly to the Lorentz covariance and internal $\mathcal{G}$-invariance of the theory in hand. It is, by the way, my impression that in the presence of magnetic sources—real magnetic sources, that can't be transformed to extinction—the number of degrees of freedom increases (not to twice two equals four but, owing to the extra gauge freedom afforded by (86), only) from two to three.

My objective in the preceeding discussion (as also in that which follows) has—let me remind my reader—been not in any sense to "push back the frontiers of electrodynamical knowledge" but simply to demonstrate the power of the exterior calculus to guide our imagination into new channels, to lend the force of "naturalness" to questions that otherwise might not have occurred to us, and to place in our hands some remarkably powerful analytical tools—tools which have turned out to be in fact quite easy to use.

8. Dimensional generalizations of Maxwellian electrodynamics. To ask "What would classical mechanics, what would quantum mechanics look like if the world were $N \neq 3$-dimensional?" is to ask a question the answer to which is on formal grounds so obvious, and so well supported by simplified experience, that the question is in fact seldom asked; the exploratory and expository advantages of "looking first to the lower-dimensional case" are in such fields so familiar that we have recourse to them almost unthinkingly, unapologetically. But to ask the same question of electrodynamics is to ask a question which is seldom asked for the opposite reason. Electrodynamics seems so essentially dependent upon cross products, curls and other "intrinsically 3-dimensional constructs" as to defy dimensional generalization. The formal devices (the "infinite line charges," etc.) which serve effectively to reduce the dimensionality of particular idealized problems are recognized to be precisely that—"formal devices" of quite restricted utility, and those broad subtopics (for example, potential theory) which do admit of dimensional generalization invariably discard more of the theory than they retain. Has the electrodynamicist therefore to abandon any hope of realizing the exploratory/expository advantages that come (in other fields) from dimensional simplification? Is the electrodynamicist forever cut off from the insight that can come from examining a theory in the context afforded by a population of "neighboring" theories?

It is a remarkable fact—if a fact not widely known—that the exterior calculus *does* permit one to assign natural meaning to the notion of an "n-dimensional electrodynamics." The idea is to retain

[16] For details and references see §32 of Melba Phillips' article "Classical Electrodynamics" in Volume IV of the *Handbuch der Physik* (Springer, 1962). Phillips' §§27–32 are given over to a discussion of the Hertz potential and its applications, and are the source of my own introduction to the field. See also pp. 219-229 of W . K. H. Panofsky & M. Phillips, *Classical Electricity & Magnetism* (Addison-Wesley, 1955).

$$\ast\mathbf{d}\ast\mathbf{F} = \mathbf{J} \qquad \text{and} \qquad \ast\mathbf{d}\ \mathbf{F} = \mathbf{0} \qquad\qquad (49) \leftarrow (117)$$

—equivalently

$$\iint_{\partial R} \ast\mathbf{F} = \iiint_R \ast\mathbf{J} \qquad \text{and} \qquad \iint_{\partial R} \mathbf{F} = 0 \qquad\qquad (56) \leftarrow (118)$$

—and to assign to the field-form **F** and the source-form **J** such meanings as may appear most appropriate. In (4-dimensional) Maxwellian electrodynamics **F** is a p-form *and so also is* $\ast\mathbf{F}$; both, that is to say (since $p = 4 - p \Longrightarrow p = 2$) are 2-forms. There would appear, on the basis of this remark, to be two main ways to proceed: one might insist that the field-form be a 2-form irrespective of the dimensionality—then duality rotations

$$\mathbf{F} \longrightarrow \mathbf{F}(\varphi) \equiv \cos\varphi \cdot \mathbf{F} \ + \sqrt{\pm 1}\,\sin\varphi \cdot \ast\mathbf{F} \qquad\qquad (94) \leftarrow (119)$$

are special to the case $n = 4$—or one might (in order to insure the *universal* availability of (119)) insist that $p = n - p$, which entails that n be even and $p = \frac{1}{2}n$. I am, by formal instinct, inclined to favor the later option, even though it cuts me off from any possibility of responding to questions of the form "What would electrodynamics look like if spacetime were 17-dimensional?" And it is to some illustrative implications of the latter option that I confine my remarks.

Looking first to **the case N = 2**, we remind ourselves that

when $n = 2$ 0-forms have 1 component
1-forms have 2 components
2-forms have 0 components

Assuming **F** to be a 1-form

$$\mathbf{F} \prec F_\mu = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix} \equiv \begin{pmatrix} E \\ B \end{pmatrix} \tag{120}$$

we (by (26)) have

$$\ast\mathbf{F} \prec \sqrt{g}\,\epsilon_{\mu\alpha}F^\alpha = \sqrt{g}\begin{pmatrix} +F^1 \\ -F^0 \end{pmatrix} \tag{121}$$

and it follows from $\ast\mathbf{d}\ast\mathbf{F} = \mathbf{J}$ that **J** is an 0-form:

$$\mathbf{J} \prec J \tag{122}$$

From $\ast\mathbf{dF} = \mathbf{0}$ we infer the existence of an 0-form

$$\mathbf{A} \prec A \tag{123}$$

such that

$$\mathbf{F} = \mathbf{dA} \prec \begin{pmatrix} \partial_0 A \\ \partial_1 A \end{pmatrix} \tag{124}$$

Since **A** and **J** are 0-forms, the operator $\mathbf{d}*$ is inapplicable to them (or—formally—might be considered to yield a pair of automatic zeroes); evidently the Lorentz gauge and charge conservation conditions

$$*\mathbf{d}*\mathbf{A} = \mathbf{0} \qquad \text{and} \qquad *\mathbf{d}*\mathbf{J} = \mathbf{0}$$

have no proper role to play within 2-dimensional electrodynamics (alternatively, they may be construed to be statements of *automatic* validity). Though we are therefore deprived of our original motivations (see again (62) and (63)) for introducing **H** and **C** such that

$$\mathbf{A} = *\mathbf{dH} \qquad \text{and} \qquad \mathbf{J} = *\mathbf{dC}$$

we are not on these grounds *prevented* from doing so; note, however, that **H** and **C**, formerly 2-forms, are now 1-forms. Nor are we prevented from introducing magnetic sources by mimicry of

$$*\mathbf{d}*\mathbf{F} = \mathbf{J}_e \qquad \text{and} \qquad *\mathbf{d}\;\mathbf{F} = \mathbf{J}_m \tag{84) \leftarrow (125}$$

and

$$\mathbf{F} = \mathbf{dA}_e + *\mathbf{dA}_m \tag{85) \leftarrow (126}$$

provided we take $\mathbf{A}_m$ and $\mathbf{J}_m$ to be 0-forms. We conclude that 2-dimensional electrodynamics, for all its structural simplicity, is nevertheless rich enough to support in its entirety the internal symmetry group $\mathcal{G}$ of the 4-dimensional theory. No enforced constraint has been imposed thus far upon the structure of the metric $g_{\mu\nu}$. If we yield to habit and write

$$||g_{\mu\nu}|| = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

then (121) becomes

$$*\mathbf{F} \prec \sqrt{g}\epsilon_{\mu\alpha}g^{\alpha\beta}F_\beta = -i\begin{pmatrix} B \\ E \end{pmatrix}$$

and, working from (38), we obtain this explicit rendition of the "2-dimensional Maxwell equations:"

$$\left.\begin{aligned} \partial_1 B - \partial_0 E &= J \\ \partial_1 E - \partial_0 B &= 0 \end{aligned}\right\} \tag{127}$$

Introducing the scalar potential A by $E = -\partial_0 A$ and $B = -\partial_1 A$ (the minus signs are a cosmetic innovation) we find that the last equation is automatically satisfied, while the former becomes

$$\big(\partial_0^2 - \partial_1^2\big)A = J \tag{128}$$

It follows also from (127) that in regions where $J = 0$ we have

$$\left.\begin{aligned} \partial_0\left(\tfrac{1}{2}\left[E^2+B^2\right]\right)-\partial_1(EB)=0\\ \partial_0(EB)-\partial_1\left(\tfrac{1}{2}\left[E^2+B^2\right]\right)=0 \end{aligned}\right\} \tag{129}$$

which suggests how "energy/momentum density" and "energy/momentum flux" should be defined in the 2-dimensional theory. We conclude—rather to our surprise—that the 2-dimensional theory is in fact rich enough to serve as a kind of "laboratory" within which one can undertake to study *many* of the ideas and analytical methods most characteristic of the full Maxwellian theory. It is, in several respects, simply the theory of a driven string, but if one returns to (128) and writes

$$(\partial_0+\partial_1)(\partial_0-\partial_1)A = J$$

one sees that it is by an act of non-obvious cleverness that one would be led to "invent" the Maxwell equations (127).

Looking next to **the case N = 6**, we remind ourselves that

when $n = 6$ 0-forms have 1 component
1-forms have 6 components
2-forms have 15 components
3-forms have 20 components
4-forms have 15 components
5-forms have 6 components
6-forms have 1 components

Assuming the field-form to be a 3-form $\mathbf{F} \prec F_{\lambda\mu\nu}$, it is natural to take the 10 components of the form $F_{0\mu\nu}$ to be "electrical," and the remaining 10 components (which include no zeroes among their subscripts) to be "magnetic." The source-form is in this theory a 2-form $\mathbf{J} \prec J_{\mu\nu}$; it is natural to take the 5 components of the form $J_{0\mu}$ to be "charge-density-like," and the remaining 10 components to be "current-like." The potential $\mathbf{A}$ is also a 2-form (as so, more generally, are $\mathbf{J}_e$, $\mathbf{J}_m$, $\mathbf{A}_e$ and $\mathbf{A}_m$). The superpotential $\mathbf{H}$ and supercurrent $\mathbf{C}$ —designed to automate the Lorentz gauge and charge conservation conditions— are (as so also are their generalizations) 3-forms; they are susceptible to gauge transformations of the form (64), where $\mathbf{h}$ and $\mathbf{c}$ are arbitrary 2-forms; the electromagnetic field in 6-space would, by a previous line of argument, appear on these grounds to possess $20 - 15 = 5$ degrees of freedom. The internal symmetry group is again $\mathcal{G}$. However "natural" it may be to set

$$||g_{\mu\nu}|| = \begin{pmatrix} +1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

no specifically electromagnetic circumstance appears to enforce that assignment of metric structure. Pretty clearly, the pattern of remarks when $N = 8, 10, \ldots$ is similar; such differences as arise will be consequences only of the occasionally strange signs introduced by (27). In all cases the "generalized Lorentz metric" entails $\sqrt{g} = i$.

The question arises: In what respect (if any) is the physical case $N = 4$ "distinguished?" I can think of only one: Lorentz matrices $\mathbb{M}$ are, for all N, 2×2 objects, and proper Lorentz matrices can in all cases be represented

$$\mathbb{M} = e^{\mathbb{A}}$$

where $\mathbb{A} = ||A^\mu{}_\nu||$—the "generator" of $\mathbb{M}$—has the property that

$$A_{\mu\nu} \equiv g_{\mu\alpha}A^\alpha{}_\nu \quad \text{is } \textit{antisymmetric} \quad : \quad A_{\mu\nu} = -A_{\nu\mu}$$

Only in the case $N = 4$ are the generators of Lorentz transformations and electromagnetic field tensors (at least from a matrix-theoretic point of view) "similar kinds of objects;" only in that case is it possible to construe $F^\mu{}_\nu(x)$ to be the "logarithm" of a "field of Lorentz matrices"

$$\mathbb{M}(x) = \exp\left\{\tfrac{1}{\epsilon}\cdot\mathbb{F}(x)\right\} \quad \text{where } \mathbb{F} \text{ satisfies Maxwell's equations}$$

provided we can assemble from (or adjoin to) the "constants of Nature" a constant ϵ with the dimensionality of $\sqrt{\text{energy density}}$. This speculative remark is intended, of course, to call to mind the construction

$$\psi(x,t) = \exp\left\{\tfrac{1}{\hbar}\cdot S(x,t)\right\}$$

where $S(x,t)$ satisfies the Hamilton-Jacobi equation.

9. Remarks concerning the exterior construction of constitutive relations. In phenomenological descriptions of fields in the presence of electromagnetically active matter one expects in place of

$$\boldsymbol{*d*F} = \mathbf{J} \qquad \text{and} \qquad \boldsymbol{*d}\ \mathbf{F} = \mathbf{0} \qquad (49) \leftarrow (130)$$

to write

$$\boldsymbol{*d*G} = \mathbf{J} \qquad \text{and} \qquad \boldsymbol{*d}\ \mathbf{F} = \mathbf{0} \qquad (103) \leftarrow (131)$$

where $\mathbf{G}$ is a 2-form-valued function of $\mathbf{F}$ that reduces to $\mathbf{F}$ in spacetime regions devoid of matter. When I speak of "Maxwell's equations" I in fact have in mind equations (131), since those most nearly resemble—in substance and in spirit—the equations bequeathed to us by Maxwell himself; equations (130), which I have heretofore been careful to call the "Maxwell-Lorentz equations," came clearly into focus only later. We note that equations (131) are in themselves incomplete; they call for specification of the "constitutive relations" which describe, in terms appropriate to the matter in question, the **F**-dependence of **G**. The point I wish now to make is that *the exterior calculus provides a natural mechanism for constructing candidate constitutive relations*. To see how this works, let

$$
\begin{aligned}
&\mathbf{S} \text{ be a generic 0-form}\\
&\mathbf{V} \text{ be a generic 1-form}\\
&\mathbf{T} \text{ be a generic 2-form}\\
&\mathbf{A} \text{ be a generic 3-form}\\
&\mathbf{P} \text{ be a generic 4-form}
\end{aligned}
$$

where my notation has been intended to call to mind scalars, vectors, tensors, axial vectors and psuedoscalars. Then

$$
\begin{aligned}
\mathbf{S}\wedge\cdot\,\mathbf{F} &\quad \text{is a 2-form linear in } \mathbf{F}\\
\mathbf{V}_2\wedge *\mathbf{V}_1\wedge\cdot\,\mathbf{F} &\quad \text{is a 2-form linear in } \mathbf{F}\\
\mathbf{T}_2\wedge *\mathbf{T}_1\wedge\cdot\,\mathbf{F} &\quad \text{is a 2-form linear in } \mathbf{F}\\
&\ \ \vdots\\
\mathbf{V}_2\wedge *\mathbf{F}\wedge *\mathbf{V}_1\wedge\cdot\,\mathbf{F} &\quad \text{is a 2-form quadratic in } \mathbf{F}\\
&\ \ \vdots
\end{aligned}
$$

The idea is to write

$$\mathbf{G} = \mathbf{F} + \text{linear combination of such expressions}$$

If one were to pursue this topic—which threatens to become intricate, both algebraically and physically (particularly in the presence either of differentially moving media or of strong fields)—one would need to know[17] that if **A** is a p-form and **B** a q-form ($p+q\le n$) then

$$\mathbf{d}\,(\mathbf{A}\wedge\mathbf{B}) = \mathbf{dA}\wedge\mathbf{B} + (-)^p\mathbf{A}\wedge\mathbf{dB} \tag{132}$$

Here I must be content to record only a few illustrative results: we have (by (59))

$$\mathbf{F} \prec F_{\mu\nu} = \begin{pmatrix} 0 & +E_1 & +E_2 & +E_3\\ -E_1 & 0 & -B_3 & +B_2\\ -E_2 & +B_3 & 0 & -B_1\\ -E_3 & -B_2 & +B_1 & 0\end{pmatrix}$$

whence

$$i\,*\mathbf{F} \prec \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3\\ +B_1 & 0 & -E_3 & +E_2\\ +B_2 & +E_3 & 0 & -E_1\\ +B_3 & -E_2 & +E_1 & 0\end{pmatrix}$$

[17] See ELECTRODYNAMICS (1972), p. 164.

and agree to write

$$\mathbf{G} \equiv \mathbf{F} + \mathbf{Z} \prec G_{\mu\nu} \equiv \begin{pmatrix} 0 & +D_1 & +D_2 & +D_3 \\ -D_1 & 0 & -H_3 & +H_2 \\ -D_2 & +H_3 & 0 & -H_1 \\ -D_3 & -H_2 & +H_1 & 0 \end{pmatrix} \tag{133}$$

where

$$\mathbf{Z} = \text{some 2-form-valued function of } \mathbf{F} \text{ and/or } i\,{*}\mathbf{F}$$

Suppose, for example, that

$$\mathbf{Z} = \mathbf{U} \wedge \mathbf{W}$$

with

$$\mathbf{W} \equiv i\,{*}(\mathbf{V} \wedge i\,{*}\mathbf{F})$$

where **U** and **V** are arbitrary 1-forms (imagined to be be descriptive of an electromagnetic property of some material). By quick calculation

$$\mathbf{W} = \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix} \quad \text{with} \quad \begin{cases} W_0 = V_1E_1 + V_2E_2 + V_3E_3 \\ W_1 = -V_0E_1 \,+\, (V_2B_3 - V_3B_2) \\ W_2 = -V_0E_2 \,+\, (V_3B_1 - V_1B_3) \\ W_3 = -V_0E_3 \,+\, (V_1B_2 - V_2B_1) \end{cases}$$

so

$$\mathbf{Z} \prec \begin{pmatrix} 0 & Z_{01} & Z_{02} & Z_{03} \\ * & 0 & Z_{12} & Z_{12} \\ * & * & 0 & Z_{23} \\ * & * & * & 0 \end{pmatrix}$$

with (writing out just enough to make clear the pattern of events)

$$\begin{aligned} Z_{01} &= (U_0W_1 - U_1W_0) \\ &= -(U_0V_0)E_1 - U_1(V_1E_1 + V_2E_2 + V_3E_3) + U_0(V_2B_3 - V_3B_2) \\ &\;\vdots \\ Z_{12} &= (U_1W_2 - U_2W_1) \\ &= -(U_1V_1 + U_2V_2 + U_3V_3)B_3 + V_3(U_1B_1 + U_2B_2 + U_3B_3) \\ &\qquad\qquad\qquad\qquad\qquad\qquad - V_0(U_1E_2 - U_2E_1) \\ &\;\vdots \end{aligned}$$

In short,

$$\left.\begin{aligned} \mathbf{D} &= \mathbf{E} + \big\{ - \quad U_0V_0 \cdot \mathbf{E} - (\mathbf{V}\cdot\mathbf{E})\cdot\mathbf{U} - U_0 \cdot \mathbf{V}\times\mathbf{B} \big\} \\ \mathbf{H} &= \mathbf{B} + \big\{ + (\mathbf{U}\cdot\mathbf{V})\cdot\mathbf{B} - (\mathbf{U}\cdot\mathbf{B})\cdot\mathbf{V} + V_0 \cdot \mathbf{U}\times\mathbf{E} \big\} \end{aligned}\right\} \tag{134}$$

where for the moment I have allowed myself to use **boldface** to denote not p-forms but 3-vectors. Equations (134), though not utterly devoid of physical interest in themselves, have, of course, been developed mainly to illustrate a method, to demonstrate once again the power of the exterior calculus to cast physical problems in a constructive new light. It would, in this light, be amusing—and methodologically natural—to consider what the method outlined above would have to say within the context of 2-dimensional electrodynamics. In the presence of "1-dimensional matter" we would, in place of (127), write

$$\left.\begin{aligned} \partial_1 H - \partial_0 D &= J \\ \partial_1 E - \partial_0 B &= 0 \end{aligned}\right\} \tag{135}$$

An interesting first assignment would be to exhibit a 1-form $\mathbf{Z}$ which, by the mechanism latent in (133), yields constitutive relations of the form

$$D = \mu(x)\cdot E \qquad \text{and} \qquad H = \mathfrak{T}(x)\cdot B$$

for then, in place of (128), we would obtain

$$\mu\,\partial_0^2 A - \partial_1(\mathfrak{T}\,\partial_1 A) = J \tag{136}$$

which (ref. SOPHOMORE NOTES (1981), p. 336) is of the form characteristic of a forced string of variable density under variable tension. For such a system one has (when μ and $\mathfrak{T}$ are constant and in the absence of sources)

$$\partial^\mu S_{\mu\nu} = 0$$

with

$$\mathbb{S} = \begin{pmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(ED + BH) & BD \\ EH & \frac{1}{2}(ED+BH) \end{pmatrix} \tag{137}$$

which give back (129) when the matter is "turned off:" $\mu = \mathfrak{T} = 1$.

A world containing magnetic charges/currents is evidently a world in which a correspondingly enriched set of constitutive relations is required to describe the electromagnetic properties of bulk matter; these also can be constructed by application of the principles just outlined. And we expect the presence of matter—here as in field theories generally—to reduce the symmetry inherent in the theory. These, however, are topics not immediately germane to my present interest, so I set them aside.

10. Construction of the stress-energy tensor; stretching the exterior calculus. It has been argued that the electromagnetic field announces is physical presence not directly, but only by way of its effects (specifically: by energy/momentum exchange); that in our descent from abstractions

superpotentials

$\Downarrow$

potentials

$\Downarrow$

electromagnetic field

we have yet one further descent to make

$$\Downarrow$$

stress-energy tensor

before we encounter the stuff of observable reality. While such an assertion may seem odd to those who imagine the electromagnetic field to be in fact *defined* by the forces experienced by "test charges," such persons have tacitly in mind those specialized electromagnetic fields which are at once "strong" and "quasi-static," and will concede that when we speak of quantum fields (photonic electrodynamics) we *invariably* speak in terms of energy/momentum exchange. One has physical reason enough, on these and other grounds, to entertain an interest in the stress-energy tensor. My own immediate interest is, however, more purely mathematical, and has to do with the *symmetry* of the stress-energy tensor:

$$S^{\mu\nu} = S^{\nu\mu} \tag{138}$$

How in general do symmetrical objects manage to find a natural home in a world so devoutly antisymmetrical as that created by the exterior algebra and calculus?

The question acquires sharpness from the observation that it touches on an issue which simply *does not arise* when one adopts the (more conventional) view that it is tensor analysis—not the exterior calculus—which provides the natural language of electrodynamics (as of field theories generally). The tensor theorist standardly takes the field and its sources to be described by (respectively) an antisymmetric tensor $F_{\mu\nu}$ and a vector J_μ, manipulates indices with the aid of a $g_{\mu\nu}$ assumed to transform as weightless tensor, restricts his attention to those (proper) transformations which achieve the numerical invariance of $g_{\mu\nu}$, and posits field equations of the form

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{139.1}$$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \tag{139.2}$$

He then (drawing upon any of several lines of physical/formal motivation) constructs

$$S^{\mu\nu} \equiv F^\mu{}_\alpha F^{\alpha\nu} - \tfrac{1}{4}(F^{\alpha\beta}F_{\beta\alpha})g^{\mu\nu} \tag{140}$$

and observes it to be an implication of the field equations that

$$\begin{aligned} \partial_\mu S^{\mu\nu} &= J_\alpha F^{\alpha\nu} \\ &\Downarrow \\ &= 0 \qquad \text{in source-free regions of spacetime} \end{aligned}$$

I have here passed by in silence many instructive subtleties in order to move undistractedly to my main point, which is this: the stress-energy tensor $S^{\mu\nu}$ defined by (140) is manifestly symmetric (also manifestly traceless: $S^\mu{}_\mu = 0$), and is from a tensor-theoretic point of view quite unproblematically so. The question before us—made urgent by the acknowledged physical importance of the stress-energy tensor—is this: How is $S^{\mu\nu}$ to be described in language afforded by the exterior calculus?[18]

What we want to do, in view of the structure of (140), is to import into exterior algebra some variant of the fundamental tensor-theoretic notion of "contraction;" if $\mathbf{A} \prec A_{i_1\cdots i_p}$ is a p-form, and $\mathbf{B} \prec B_{j_1\cdots j_q}$ a q-form, we want to be able to assign natural exterior meaning to objects of the form $A_{i_1\cdots i_{p-1}a}g^{ab}B_{bj_1\cdots j_{q-1}}$, $A_{i_1\cdots i_{p-2}a_1a_2}g^{a_1b_1}g^{a_2b_2}B_{b_1b_2j_1\cdots j_{q-2}}$, etc. Owing to the total antisymmetry of $A_{i_1\cdots i_p}$ and $B_{j_1\cdots j_q}$ we have

$$
\begin{aligned}
&A_{i_1\cdots i_{p-2}a_1a_2}g^{a_1b_1}g^{a_2b_2}B_{b_1b_2j_1\cdots j_{q-2}}\\
&\quad= \frac{A_{i_1\cdots i_{p-2}a_1a_2} - A_{i_1\cdots i_{p-2}a_2a_1}}{2}g^{a_1b_1}g^{a_2b_2}\frac{B_{b_1b_2j_1\cdots j_{q-2}} - B_{b_2b_1j_1\cdots j_{q-2}}}{2}
\end{aligned}
$$

[18] Moments ago I claimed it to be an implication of the field equations that $\partial_\mu S^{\mu\nu} = 0$ in source-free regions of spacetime. The argument runs as follows:

$$
\partial_\mu S^\mu{}_\nu = \underbrace{F^{\mu\alpha}\partial_\mu F_{\alpha\nu}} + \underbrace{\partial_\mu F^{\mu\alpha}}\cdot F_{\alpha\nu} - \tfrac{1}{4}\partial_\nu(F^{\alpha\beta}F_{\beta\alpha})
$$

$$
\begin{aligned}
\partial_\mu F^{\mu\alpha} &= J^\alpha \qquad \text{by the 1}^{\text{st}}\text{ set of field equations}\\
F^{\mu\alpha}\partial_\mu F_{\alpha\nu} &= \tfrac{1}{2}F^{\mu\alpha}(\partial_\mu F_{\alpha\nu} - \partial_\alpha F_{\mu\nu}) \qquad \text{by } F^{\mu\alpha} = -F^{\alpha\mu}\\
&= \tfrac{1}{2}F^{\mu\alpha}(\partial_\mu F_{\alpha\nu} + \partial_\alpha F_{\nu\mu}) \qquad \text{by } F_{\mu\nu} = -F_{\nu\mu}\\
&= -\tfrac{1}{2}F^{\mu\alpha}\partial_\nu F_{\mu\alpha} \qquad \text{by the 2}^{\text{nd}}\text{ set of field equations}\\
&= +\tfrac{1}{4}\partial_\nu(F^{\alpha\beta}F_{\beta\alpha})
\end{aligned}
$$

For an alternative argument, see below p. 59.

I alluded also to certain "subtleties." The origin of many of those has to do with the fact that—as was remarked in §3—(110.2) exhibits "accidental general covariance" provided $F_{\mu\nu}$ has weight $W = 0$, while (110.1) has the same property provided $F^{\mu\nu}$ has weight $W = +1$. Those statements can be reconciled any of several distinct mechanisms, one of which led E. Cunningham (1909) and H. Bateman (1910) to the important discovery that the Maxwell-Lorentz equations, if written down in the right way, are (not merely Lorentz covariant, they are) *conformally* covariant. Written another way, the equations acquire automatic *general* covariance, as was first emphasized by D. van Dantzig (1934). Here as generally, the transformation properties of physical equations *depend critically upon how they are written down*. Note that we had in the text no formal reason to assign $g_{\mu\nu}$ the Lorentzian structure (11), no formal reason to assign preemptive importance to the Lorentz group. And that weight distinctions—the main driving force behind many of the formal developments to which I allude—become invisible/moot when the transformation group is unimodular. For further discussion, see pp. 179–186 of ELECTRODYNAMICS (1980).

$$= A_{i_1\cdots i_{p-2}a_1a_2}G^{a_1a_2b_1b_2}B_{b_1b_2j_1\cdots j_{q-2}} \tag{141}$$

where

$$\begin{aligned}G^{a_1a_2b_1b_2} &\equiv \tfrac{1}{4}\big\{g^{a_1b_1}g^{a_2b_2} - g^{a_1b_2}g^{a_2b_1} - g^{a_2b_1}g^{a_1b_2} + g^{a_2b_2}g^{a_1b_1}\big\}\\ &= \tfrac{1}{2}\begin{vmatrix} g^{a_1b_1} & g^{a_1b_2}\\ g^{a_2b_1} & g^{a_2b_2}\end{vmatrix}\\ &= \tfrac{1}{2!}\delta^{a_1a_2}{}_{k_1k_2}g^{k_1b_1}g^{k_2b_2}\\ &\equiv \tfrac{1}{2!}g^{(a_1a_2)(b_1b_2)}\end{aligned}$$

The 4$^{\text{th}}$-rank tensor $g^{(a_1a_2)(b_1b_2)}$—more generally $g^{(a_1\cdots a_p)(b_1\cdots b_p)}$—is simply the "all indices up" variant of the "generalized Kronecker delta" introduced at (21); it is totally antisymmetric in $(a_1\cdots a_p)$ as also in $(b_1\cdots b_p)$, but symmetric under $(a_1\cdots a_p)$-$(b_1\cdots b_p)$ interchange. The metric g^{ij} was introduced to lend metric structure to the space $\mathcal{V}$ of 1-forms, where it assigns meaning to the inner product

$$(\mathbf{A},\mathbf{B}) \prec A_a g^{ab}B_b \succ (\mathbf{B},\mathbf{A}) \tag{142.1}$$

In quite a natural sense (see again footnote 5), $g^{(a_1\cdots a_p)(b_1\cdots b_p)}$ assigns "induced metric structure" to the space $\wedge^p\mathcal{V}$ of n-dimensional p-forms; if $\mathbf{A}$ and $\mathbf{B}\in\wedge^p\mathcal{V}$ we agree to write

$$\begin{aligned}(\mathbf{A},\mathbf{B}) &\prec \underbrace{\tfrac{1}{p!p!}A_{a_1\cdots a_p}g^{(a_1\cdots a_p)(b_1\cdots b_p)}B_{b_1\cdots b_p}}\succ(\mathbf{B},\mathbf{A}) \qquad (142.2)\\ &= \tfrac{1}{p!p!}A_{a_1\cdots a_p}\delta^{a_1\cdots a_p}{}_{b_1\cdots b_p}B^{b_1\cdots b_p}\\ &= \tfrac{1}{p!}A_{a_1\cdots a_p}B^{a_1\cdots a_p}\end{aligned}$$

It is interesting in this connection to observe that $*\mathbf{A}\wedge\mathbf{B}$ and $*\mathbf{B}\wedge\mathbf{A}$ are (since both n-forms, and since $\wedge^n\mathcal{V}$ is 1-dimensional) necessarily *proportional* to one another; straightforward calculation[19] gives

$$*\mathbf{A}\wedge\mathbf{B} = *\mathbf{B}\wedge\mathbf{A} \prec (\mathbf{A},\mathbf{B})\cdot\sqrt{g}\,\epsilon_{i_1\cdots i_n} \tag{143}$$

To describe "partial contractions" of the sort encountered at (141) we might, in this spirit, write (for example)

$$(\mathbf{A},\mathbf{B})_{(i_1\cdots i_{p-2})(j_1\cdots j_{q-2})} \prec \tfrac{1}{2!2!}A_{i_1\cdots i_{p-2}a_1a_2}g^{(a_1a_2)(b_1b_2)}B_{b_1b_2j_1\cdots j_{q-2}} \tag{144}$$

[19] We acquire—here for the first time in these pages—explicit need of the definition of the "wedge product" (i.e., of the antisymmetrized tensor product), which follows: if $\mathbf{P}$ is an n-dimensional p-form, and $\mathbf{Q}$ a q-form, and $p+q\le n$ then

$$\mathbf{P}\wedge\mathbf{Q} \prec \tfrac{1}{p!q!}P_{a_1\cdots a_p}Q_{b_1\cdots b_q}\delta^{a_1\cdots a_pb_1\cdots b_q}{}_{i_1\cdots i_{p+q}} \in \wedge^{p+q}\mathcal{V}$$

For a review of the basic properties of $\mathbf{P}\wedge\mathbf{Q}$ (and an elaborate discussion of the properties of induced metrics) see pp. 153 *et seq* of ELECTRODYNAMICS (1972).

Such structures are, so far as I am aware, not standard to the exterior algebraic literature (which the electromagnetic stress-energy tensor has motivated me here to "stretch" a bit); I will, for lack of a better term, call them "fractional inner products." By an easy argument

$$(\mathbf{A},\mathbf{B})_{(i_1\cdots i_{p-2})(j_1\cdots j_{q-2})} = (\mathbf{B},\mathbf{A})_{(j_1\cdots j_{q-2})(i_1\cdots i_{p-2})} \tag{145}$$

from which (142) can be recovered as special cases.

Looking now (with an eye to the assembly of the stress-energy tensor) to some particular implications of the material introduced in the preceeding paragraph, we note that if $\mathbf{F}$ is a 4-dimensional 2-form then

$$\begin{aligned}
(\mathbf{F},\mathbf{F}) &\prec \tfrac{1}{2}F_{ab}F^{ab} = -\tfrac{1}{2}F_{ab}F^{ba}\\
(\mathbf{F},\mathbf{F})_{(i)(j)} &\prec F_{ia}g^{ab}F_{bj}\\
g^{ij}(\mathbf{F},\mathbf{F})_{(i)(j)} &\prec g^{ij}F_{ia}g^{ab}F_{bj} = F^{jb}F_{bj} = F_{ab}F^{ba} \succ -2(\mathbf{F},\mathbf{F})
\end{aligned}$$

permit (140) to be notated

$$S_{\mu\nu} = (\mathbf{F},\mathbf{F})_{(\mu)(\nu)} + \tfrac{1}{2}(\mathbf{F},\mathbf{F})\cdot g_{\mu\nu} \tag{146}$$

In the "$2p$-dimensional electrodynamics" sketched in §8 the "electromagnetic field" is represented by a p-form $\mathbf{F}$, and "electrical current" by a $(p-1)$-form $\mathbf{J}$. The result just achieved suggests that to describe the stress-energy tensor appropriate to such a theory we should write

$$S_{ij} \equiv (\mathbf{F},\mathbf{F})_{(i)(j)} + \lambda\cdot(\mathbf{F},\mathbf{F})\cdot g_{ij} \tag{147}$$

$$\lambda \text{ fixed by the requirement that } S^i{}_i = 0$$

Explicitly

$$(\mathbf{F},\mathbf{F})_{(i)(j)} \prec \tfrac{1}{(p-1)!(p-1)!}F_{ia_1\cdots a_{p-1}}g^{(a_1\cdots a_{p-1})(b_1\cdots b_{p-1})}F_{b_1\cdots b_{p-1}j}$$

so

$$\begin{aligned}
g^{ij}(\mathbf{F},\mathbf{F})_{(i)(j)} &\prec (-)^{p-1}\tfrac{1}{(p-1)!(p-1)!}F_{ia_1\cdots a_{p-1}}g^{ij}g^{(a_1\cdots a_{p-1})(b_1\cdots b_{p-1})}F_{jb_1\cdots b_{p-1}}\\
&= (-)^{p-1}\tfrac{1}{p-1!}F_{ia_1\cdots a_{p-1}}F^{ia_1\cdots a_{p-1}}\\
&\succ (-)^{p-1}p(\mathbf{F},\mathbf{F})
\end{aligned}$$

and the S_{ij} of (147) will be traceless if and only if $(-)^{p-1}p+n\lambda=0$, which (by $n=2p$) entails $\lambda=(-)^p\frac{1}{2}$ We are led thus—tentatively—to write

$$S_{ij} \equiv (\mathbf{F},\mathbf{F})_{(i)(j)} + (-)^p\tfrac{1}{2}\cdot(\mathbf{F},\mathbf{F})\cdot g_{ij} \tag{148}$$

from which we recover (146) in the special case $p=2$. (If we had any actual physics in hyperspace to worry about we would have interest in the sign of the "energy density" S_{00}, and might want to introduce an overall sign factor to insure its non-negativity.[20]) This algebraic accomplishment raises, however, a question: Is it in fact the case that (148) and the field equations (49) jointly entail $\partial^i S_{ij} = 0$ at source-free points in hyperdimensional spacetime? And to approach the question we must—having been obligated to "stretch" the exterior algebra—dig now a bit deeper into the exterior calculus and subject it to a corresponding deformation.

When written out in detail, (148) reads

$$S^i{}_j = \tfrac{1}{(p-1)!}F^{ia_1\cdots a_{p-1}}F_{a_1\cdots a_{p-1}j} + (-)^p\tfrac{1}{2}\cdot\tfrac{1}{p!}F^{a_1\cdots a_p}F_{a_1\cdots a_p}\cdot\delta^i{}_j$$

so

$$\begin{aligned}\partial_i S^i{}_j = \tfrac{1}{(p-1)!}\big(\partial_i F^{ia_1\cdots a_{p-1}}\big)F_{a_1\cdots a_{p-1}j} + \tfrac{1}{(p-1)!}F^{ia_1\cdots a_{p-1}}\big(\partial_i F_{a_1\cdots a_{p-1}j}\big)\\ + (-)^p\tfrac{1}{p!}F^{a_1\cdots a_p}\big(\partial_j F_{a_1\cdots a_p}\big)\end{aligned} \tag{149}$$

Now it follows from (38) that (compare (57.2))

$$\ast\mathbf{d}\ast\mathbf{F} \prec \tfrac{1}{p!}\tfrac{1}{\sqrt{g}}\delta_{i_1\cdots i_{p-1}\beta}{}^{k_1\cdots k_p}\partial^\beta\big(\sqrt{g}F_{k_1\cdots k_p}\big) = (-)^{p-1}\tfrac{1}{\sqrt{g}}\partial^\beta\big(\sqrt{g}F_{\beta i_1\cdots i_{p-1}}\big)$$

so it is an implication of $\ast\mathbf{d}\ast\mathbf{F} = \mathbf{J}$ that (at least in cases where the $\sqrt{g}$-factors are constant, and therefore cancel, as in fact the Lorentz metric entails)

$$\tfrac{1}{(p-1)!}\big(\partial_i F^{ia_1\cdots a_{p-1}}\big)F_{a_1\cdots a_{p-1}j} = \tfrac{1}{(p-1)!}F_{ja_1\cdots a_{p-1}}J^{a_1\cdots a_{p-1}}$$

And it is an implication of $\mathbf{dF} = \mathbf{0}$ that

---

[20] When, in particular, the hypermetric is taken to be Lorentzian one has

$$\begin{aligned}S_{00} &= (-)^{p-1}\tfrac{1}{p-1!}(-)^{p-1}\sum\big(F_{0a_1\cdots a_{p-1}}\big)^2\\ &\quad + (-)^p\tfrac{1}{2}\tfrac{1}{p!}\Big\{(-)^{(p-1)}p\sum\big(F_{0a_1\cdots a_{p-1}}\big)^2 + (-)^p\sum{}'\big(F_{a_1\cdots a_p}\big)^2\Big\}\\ &= \tfrac{1}{2}\tfrac{1}{p-1!}\sum\big(F_{0a_1\cdots a_{p-1}}\big)^2 + \tfrac{1}{2}\tfrac{1}{p!}\sum{}'\big(F_{a_1\cdots a_p}\big)^2 = \text{ sum of squares } \geq 0\end{aligned}$$

No factor of the sort contemplated in the text is in fact required. The reader who takes the trouble to write out the details of the little argument just sketched will, I think, share my feeling that the Lorentz metric is working here almost "purposefully/conspiratorially." But Nature is *not* purposeful/conspiratorial. One is led thus to the vision of *population* of worlds, with all possible metrics, of which all but those with Lorentzian metric structure "die because unstable with respect to electromagnetic energy collapse." This (only semi-serious!) argument casts in new the light the question "Why did God select the Lorentz metric?" but fails to illuminate the question "Why did God set $p = 2$?"

$$\sum \partial_{\underbrace{i F_{a_1 \cdots a_{p-1} j}}} = 0$$

where the underbrace identifies the subscripts which participate in the sum-over-signed-permutations. But

$$= p \cdot \sum \partial_i \underbrace{F_{a_1 \cdots a_{p-1}}}\,{}_j + (-)^p\, \partial_j \sum F\, \underbrace{{}_{ia_1 \cdots a_{p-1}}}$$

so

$$\begin{aligned} \tfrac{1}{(p-1)!} F^{ia_1 \cdots a_{p-1}} \big(\partial_i F_{a_1 \cdots a_{p-1} j}\big) &= -\tfrac{1}{(p-1)!} F^{ia_1 \cdots a_{p-1}} (-)^p \tfrac{1}{p}\, \partial_j F_{ia_1 \cdots a_{p-1}} \\ &= -(-)^p \tfrac{1}{p!} F^{ia_1 \cdots a_{p-1}} \big(\partial_j F_{ia_1 \cdots a_{p-1}}\big) \end{aligned}$$

Returning with these results to (149) we obtain

$$\begin{aligned} \partial^i S_{ij} &= \tfrac{1}{(p-1)!} F_{ja_1 \cdots a_{p-1}} J^{a_1 \cdots a_{p-1}} \\ &\Downarrow \\ &= 0 \qquad \text{at source-free points in hyperspace} \end{aligned} \tag{150}$$

The argument will be recognized to be structurally identical to that encountered already in footnote [18] but to possess hybrid features not standard to **d**-theory. It would be nice to possess the sharpened calculus which would remove the awkwardly *ad hoc* quality from arguments involving the derivative properties of fractional inner products, but I have at the moment no clear sense of how such a goal is to be achieved; future practical needs may provide both the requisite strong motivation and some essential clues. I am content to await the occasion.

The credentials of (148) are by (150) made plausibly secure. It should, however, be observed that (148) entails a construction which becomes meaningless in the case $p = 1$, as so also does the righthand side of (150). In "2-dimensional electrodynamics" we (according to (129), and consistently with (137)) have this special construction

$$\mathbb{S} = \begin{pmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(E^2 + B^2) & EB \\ EB & \frac{1}{2}(B^2 + E^2) \end{pmatrix} \tag{151}$$

which is *not* traceless,[21] but which by (120) and (127) yields a result

$$\partial^i S_{ij} + J F_j = 0 \tag{152}$$

which manages nevertheless to be neatly consonant with (150).

---

[21] The traceless symmetry of the stress-energy tensor lies at the foundation of angular momentum conservation, which in 2-dimensional spacetime is, after all, not a consideration!

In §8 "$2p$-dimensional elecrodynamics" was found to possess a group $\mathcal{G}$ of internal symmetries. The questions arises: How does the stress-energy tensor S_{ij} respond to such transformations? I restrict my explicit remarks to the subgroup of $\mathcal{G}$ which has to do with "duality rotations." We have, as a preparatory step, to resolve the ambiguity present in (119). Since $\mathbf{F}$ is a $2p$-dimensional p-form we have, according to (28), $**\mathbf{F} = (-)^p\mathbf{F}$ so each of the following equations implies the other:

$$\begin{aligned} \mathbf{F}(\varphi) &= \cos\varphi\cdot\mathbf{F} + \sin\varphi\cdot\sqrt{\pm1}*\mathbf{F} \\ \sqrt{\pm1}*\mathbf{F}(\varphi) &= \underbrace{\pm(-)^p}_{=-1\ \text{ entails }\ \begin{cases}\text{take upper sign when } p \text{ is odd}\\ \text{take lower sign when } p \text{ is even}\end{cases}}\sin\varphi\cdot\mathbf{F} + \cos\varphi\cdot\sqrt{\pm1}*\mathbf{F} \end{aligned}$$

We therefore understand the phrase "duality rotation" to mean

$$\mathbf{F} \longrightarrow \begin{cases} \mathbf{F}(\varphi) \equiv \cos\varphi\cdot\mathbf{F} + \sin\varphi\cdot\ *\mathbf{F} & \text{when } p \text{ is odd} \\ \mathbf{F}(\varphi) \equiv \cos\varphi\cdot\mathbf{F} + \sin\varphi\cdot i*\mathbf{F} & \text{when } p \text{ is even} \end{cases} \tag{153}$$

It is a property of the $2p$-dimensional Lorentz metric that $\sqrt{g} = i$ (all p), so if $\mathbf{F}$ is real then $*\mathbf{F}$ is in all cases imaginary; it is therefore an implication of (153) that

$$\mathbf{F}(\varphi) \text{ is } \begin{cases} \text{generally complex when } p \text{ is odd} \\ \text{invariably real when } p \text{ is even} \end{cases}$$

The question now before us is this: How in $2p$-dimensional electrodynamics does the stress-energy tensor respond to transformations of type (153)? Looking first—because it is in some respects especially simple, and in others simply special—to **the case $\mathbf{p = 1}$**, we have (recall (120) and (121))

$$\mathbf{F} \prec \begin{pmatrix} E \\ B \end{pmatrix} \qquad \text{and} \qquad \mathbf{G} \equiv *\mathbf{F} \prec -i\begin{pmatrix} B \\ E \end{pmatrix} \tag{154}$$

and observe that (122) can be written

$$\begin{aligned} \mathbb{S} &= \tfrac{1}{2}\left\{\begin{pmatrix} \bar{F}_0F_0 & \bar{F}_0F_1 \\ \bar{F}_1F_0 & \bar{F}_1F_1 \end{pmatrix} + \begin{pmatrix} \bar{G}_0G_0 & \bar{G}_0G_1 \\ \bar{G}_1G_0 & \bar{G}_1G_1 \end{pmatrix}\right\} \\ &= \tfrac{1}{2}\left\{\bar{\mathbf{F}}\otimes\mathbf{F} + \bar{\mathbf{G}}\otimes\mathbf{G}\right\} \end{aligned} \tag{155}$$

Identical formulae describe $\mathbb{S}(\varphi)$ in terms of

$$\mathbf{F}(\varphi) \prec \begin{pmatrix} F_0(\varphi) \\ F_1(\varphi) \end{pmatrix} = \begin{pmatrix} +\cos\varphi\cdot E - i\sin\varphi\cdot B \\ +\cos\varphi\cdot B - i\sin\varphi\cdot E \end{pmatrix} \tag{156.1}$$

and

$$\mathbf{G}(\varphi) \prec \begin{pmatrix} G_0(\varphi) \\ G_1(\varphi) \end{pmatrix} = \begin{pmatrix} -\sin\varphi\cdot E - i\cos\varphi\cdot B \\ -\sin\varphi\cdot B - i\cos\varphi\cdot E \end{pmatrix} \tag{156.2}$$

Quick calculation now gives (compare (137))

$$\mathbb{S}(\varphi) = \tfrac{1}{2}\begin{pmatrix} E^2+B^2 & 2EB \\ 2EB & E^2+B^2 \end{pmatrix} \qquad \text{(all } \varphi) \tag{157}$$

from which we conclude that in "2-dimensional electrodynamics" the stress-energy tensor is *invariant* under duality rotation. Analysis of **the general case** $\mathbf{p > 1}$ is almost equally simple, provided one is in possession of certain identities which I digress now to acquire. If $\mathbf{A}$ and $\mathbf{B}$ are arbitrary $2p$-dimensional p-forms, then it follows from (22), (27), (142.2) and (144) that

$$\begin{aligned}(\ast\mathbf{A},\ast\mathbf{B}) &= \tfrac{1}{p!}\Big(\tfrac{1}{p!}\sqrt{g}\epsilon_{i_1\cdots i_p a_1\cdots a_p}A^{a_1\cdots a_p}\Big)\Big(\tfrac{1}{p!}\sqrt{g}\epsilon^{i_1\cdots i_p b_1\cdots b_p}B_{b_1\cdots b_p}\Big)\\ &= \tfrac{1}{p!p!}A^{a_1\cdots a_p}\delta_{a_1\cdots a_p}{}^{b_1\cdots b_p}B_{b_1\cdots b_p}\\ &= \tfrac{1}{p!}A_{k_1\cdots k_p}B^{k_1\cdots k_p}\\ &= (\mathbf{A},\mathbf{B})\end{aligned} \tag{158}$$

and

$$\begin{aligned}(\ast\mathbf{A},\ast\mathbf{B})^i{}_j &= \tfrac{1}{(p-1)!}\Big(\tfrac{1}{p!}\sqrt{g}\epsilon^{ik_1\cdots k_{p-1}a_1\cdots a_p}A_{a_1\cdots a_p}\Big)\\ &\qquad\qquad \cdot\Big(\tfrac{1}{p!}\sqrt{g}\epsilon_{k_1\cdots k_{p-1}jb_1\cdots b_p}B^{b_1\cdots b_p}\Big)\\ &= -\tfrac{1}{p!p!}\delta^{ia_1\cdots a_p}{}_{b_1\cdots b_p j}A_{a_1\cdots a_p}B^{b_1\cdots b_p}\end{aligned}$$

which by Laplace expansion of $\delta^{ia_1\cdots a_p}{}_{b_1\cdots b_p j}$ gives[22]

$$\begin{aligned}&= -(-)^p\Big\{\delta^i{}_j\Big(\tfrac{1}{p!}A_{k_1\cdots k_p}B^{k_1\cdots k_p}\Big) + (-)^p\Big(\tfrac{1}{(p-1)!}B^{ik_1\cdots k_{p-1}}A_{k_1\cdots k_{p-1}j}\Big)\Big\}\\ &= -(\mathbf{B},\mathbf{A})^i{}_j - (-)^p(\mathbf{B},\mathbf{A})\delta^i{}_j\end{aligned} \tag{159}$$

[22] One writes $\delta^{ia_1\cdots a_p}{}_{b_1\cdots b_p j} = (-)^p\delta^{ia_1\cdots a_p}{}_{jb_1\cdots b_p}$ and argues from (21):

$$\delta^{ia_1a_2a_3}{}_{jb_1b_2b_3} = \begin{vmatrix} \delta^i{}_j & \delta^i{}_{b_1} & \delta^i{}_{b_2} & \delta^i{}_{b_3}\\ \delta^{a_1}{}_j & \delta^{a_1}{}_{b_1} & \delta^{a_1}{}_{b_2} & \delta^{a_1}{}_{b_3}\\ \delta^{a_2}{}_j & \delta^{a_2}{}_{b_1} & \delta^{a_2}{}_{b_2} & \delta^{a_2}{}_{b_3}\\ \delta^{a_3}{}_j & \delta^{a_3}{}_{b_1} & \delta^{a_3}{}_{b_2} & \delta^{a_3}{}_{b_3}\end{vmatrix}$$

$$= \delta^i{}_j\cdot\delta^{a_1a_2a_3}{}_{b_1b_2b_3} - \delta^i{}_{b_1}\cdot\delta^{a_1a_2a_3}{}_{jb_2b_3} + \delta^i{}_{b_2}\cdot\delta^{a_1a_2a_3}{}_{jb_1b_3} - \delta^i{}_{b_3}\cdot\delta^{a_1a_2a_3}{}_{jb_1b_2}$$

so

$$\begin{aligned}&\delta^{ia_1a_2a_3}{}_{jb_1b_2b_3}A_{a_1a_2a_3}B^{b_1b_2b_3}\\ &\quad= \delta^i{}_j\cdot\big(\delta^{a_1a_2a_3}{}_{b_1b_2b_3}A_{a_1a_2a_3}B^{b_1b_2b_3}\big) - \big(\delta^{a_1a_2a_3}{}_{jb_2b_3}A_{a_1a_2a_3}B^{ib_2b_3}\big)\\ &\qquad\qquad + \big(\delta^{a_1a_2a_3}{}_{jb_1b_3}A_{a_1a_2a_3}B^{b_1ib_3}\big)\\ &\qquad\qquad - \big(\delta^{a_1a_2a_3}{}_{jb_1b_2}A_{a_1a_2a_3}B^{b_1b_2i}\big)\\ &\quad= \delta^i{}_j\cdot\big(\delta^{a_1a_2a_3}{}_{b_1b_2b_3}A_{a_1a_2a_3}B^{b_1b_2b_3}\big) - 3(-)^{3-1}\big(B^{ik_1k_2}A_{a_1a_2a_3}\delta^{a_1a_2a_3}{}_{k_1k_2j}\big)\\ &\quad= \delta^i{}_j\cdot\big(3!\,A_{k_1k_2k_3}B^{k_1k_2k_3}\big) + 3(-)^3\big(3!\,B^{ik_1k_2}A_{k_1k_2j}\big)\end{aligned}$$

Though written out here in the illustrative special case $p = 3$, the pattern of the argument in the general case should be clear.

In the course of the argument that led us from (147) to (148) we showed in effect that

$$(\mathbf{A},\mathbf{B})^k{}_k = -(-)^p\, p(\mathbf{A},\mathbf{B}) \tag{160}$$

When with the aid of (160) we form the trace of (159) we recover (158) which, in view of the tedious complexity of some of our arguments, provides a very gratifying check on the consistency of our results. The preceeding identities provide generalized formulation of some identities which (in the special case $p = 2$) are fundamental to the analysis of the Lorentz group which appears in ELEMENTS OF SPECIAL RELATIVITY (1966). It is a particular implication of (159) that

$$(-)^p \tfrac{1}{2}\cdot(\mathbf{F},\mathbf{F})\cdot g_{ij} = -\tfrac{1}{2}\,\{(\mathbf{F},\mathbf{F}) + (\mathbf{G},\mathbf{G})\}_{(i)(j)}\Big|_{\mathbf{G}\equiv *\mathbf{F}}$$

so (148) can be written

$$\begin{aligned} S_{ij} &= \big\{(\mathbf{F},\mathbf{F}) - (\mathbf{G},\mathbf{G})\big\}_{(i)(j)} \\ &= \big\{(\bar{\mathbf{F}},\mathbf{F}) + (\bar{\mathbf{G}},\mathbf{G})\big\}_{(i)(j)} \end{aligned} \tag{161}$$

which bears a striking formal resemblance to (155). If p is odd then according to (153) we should write

$$\begin{aligned} \mathbf{F} &\longrightarrow \mathbf{F}(\varphi) = \cos\varphi\cdot\mathbf{F} + \sin\varphi\cdot\mathbf{G} \\ \mathbf{G} &\longrightarrow \mathbf{G}(\varphi) = -\sin\varphi\cdot\mathbf{F} + \cos\varphi\cdot\mathbf{G} \end{aligned}$$

By easy calculation we then obtain

$$S_{ij}(\varphi) = S_{ij}(0) \qquad (\text{all } \varphi) \tag{162}$$

If, on the other hand, p is even then

$$\begin{aligned} \mathbf{F} &\longrightarrow \mathbf{F}(\varphi) = \cos\varphi\cdot\mathbf{F} + i\,\sin\varphi\cdot\mathbf{G} \\ \mathbf{G} &\longrightarrow \mathbf{G}(\varphi) = +\,i\,\sin\varphi\cdot\mathbf{F} + \cos\varphi\cdot\mathbf{G} \end{aligned}$$

and by a variant of the preceeding calculation one again recovers (162). We come thus to the conclusion that invariance of the stress-energy tensor with respect to duality rotations is a property which pertains *in general* (i.e., for all values of p) to "$2p$-dimensional electrodynamics."

Physicists are well acquainted with the fact that statements of the form $\partial^i J_i = 0$ admit of integral formulation, along the following familiar lines:

$$\frac{\partial}{\partial t}\iiint \text{density}\; d\,(\text{volume}) + \iint \textbf{current}\;\cdot\mathbf{d}(\textbf{area}) = 0$$

And I have in §6 promoted the view that the J_i can advantageously be construed to be the components of a 1-form $\mathbf{J}$, for then

$$*\mathbf{d}*\mathbf{J} = 0 \qquad \text{entails} \qquad \int_{\partial R} *\mathbf{J} = 0$$

It would in this light seem natural to construe the components S_{ij} of the stress-energy tensor to be the components of a j-indexed set of 1-forms $\mathbf{S}_j$, and in place of $\partial^i S_{ij} = 0$ to write

$$*\mathbf{d}*\mathbf{S}_j = 0 \tag{163}$$

To do so is (by the converse of the Poincaré lemma) to recognize the possibility of writing[23]

$$*\mathbf{S}_j = \mathbf{d}\mathbf{W}_j \tag{164.1}$$

where the $\mathbf{W}_j$ are 2-forms, and are defined only to within gauge transformations of the form

$$\mathbf{W}_j \longrightarrow \mathbf{W}_j + \mathbf{d}\mathbf{w}_j \tag{164.2}$$

where the $\mathbf{w}_j$ are arbitrary 1-forms. More remarkably, the $\mathbf{S}_j$ would—if subject to no physical constraint or desideratum additional to (163)—be themselves susceptible to gauge transformation

$$\mathbf{S}_j \longrightarrow \mathbf{S}_j + *\mathbf{d}\mathbf{T}_j \tag{165}$$

where the $\mathbf{T}_j$ are arbitrary 2-forms. This seldom-remarked fact was brought most recently to my attention when, in January of 1995, I received for review, from the editors of Physical Review, a manuscript bearing the title "Alternate Electromagnetic Energy-Momentum Tensors." I confess that I found the manuscript in question to be in several respects so idiosyncratic—which is to say, written from an experience so different from my own—as to be virtually impenetrable, but buried within it was a pretty observation which (in the relativistic notation which the author unaccountably declines to exploit) can be formulated as follows: it is, as previously remarked, an implication of (140) that

$$\partial^\mu S_{\mu\nu} = J^\alpha F_{\alpha\nu} \tag{166.1}$$

But let

$$\begin{aligned}\tilde{S}_{\mu\nu} \equiv \tfrac{1}{2}\big\{ - J^\alpha A_\alpha g_{\mu\nu} + 2J_\mu A_\nu + A_\alpha(\partial_\mu\partial_\nu A^\alpha) \\ - (\partial_\mu A_\alpha)(\partial_\nu A^\alpha) + (\partial_\nu A_\mu)(\partial_\alpha A^\alpha) - A_\mu(\partial_\nu\partial_\alpha A^\alpha)\big\}\end{aligned}$$

Then

$$\partial^\mu \tilde{S}_{\mu\nu} = J^\alpha F_{\alpha\nu} \tag{166.2}$$

[23] To minimize the element of strangeness in these and subsequent remarks I will, in the absence of statements to the contrary, be assuming spacetime to be 4-dimensional.

Or (to say the same thing another way)

$$\partial^\mu T_{\mu\nu} = 0 \qquad \text{even in the presence of sources} \tag{167}$$

where

$$T_{\mu\nu} \equiv \tilde{S}_{\mu\nu} - S_{\mu\nu} \tag{168}$$

The alternative stress-energy tensors $S_{\mu\nu}$ and $\tilde{S}_{\mu\nu}$ are physically equivalent in the sense of (166), but because they ascribe distinct meanings/values to the energy-momentum densities of an electromagnetic field the author associates himself with the view put forward long ago by M. Mason & W. Weaver, who at p.264 of their text (THE ELECTROMAGNETIC FIELD (1929)) report that

> . . . *"we do not believe that 'Where?' is a fair or sensible question to ask concerning energy. Energy is a function of configuration, just as beauty of a certain black-and-white design is a function of configuration. We see no more reason or excuse for speaking of a spatial energy density than we would for saying, in the case of a design, that its beauty was distributed over it with a certain density. . . "*

The author—who does in fact have a name; he is F. R. Morgenthaler, of the Department of Electrical Engineering and Computer Science at MIT and a self-described disciple of J. A. Stratton (whose influential ELECTROMAGNETIC THEORY appeared in 1941)—is at pains to demonstrate that his idea is robust enough to accommodate the presence of linear media, and that $\tilde{S}_{\mu\nu}$ is in some illustrative cases (selected mainly because of their engineering importance) actually easier to compute than $S_{\mu\nu}$. He seems to hover at the edge of an inclination to attach physical interest to the circumstance that (owing mainly to the J-factors which enter into the construction of $\tilde{S}_{\mu\nu}$) energy-momentum density tends to become *concentrated in the local vicinity of the sources* in the new representation, but stops short of actual advocacy of such a view; adapting to the interpretation of (167) a line taken from Mason & Weaver's discussion (at their p. 326) of the purported properties of the electromagnetic æther

"All statements are true if they are made about nothing"

Morgenthaler suggests that his $T_{\mu\nu}$, as defined at (168), might appropriately be called "electromagnetic beauty." The suggested terminology has about it an unfortunate air of cute self-congratulation, though it is intended only to echo the language—and to honor the insight—of Mason & Weaver. In my view (with which I think Morgenthaler would concur) the physical importance of (167) is of a mainly cautionary nature: it advises physicists not to fall causually into an instance of what philosophers call "the fallacy of misplaced concreteness." That in itself is a valuable accomplishment. Unfortunately, Morganthaler is content simply to pull (166.2) out of a hat; he provides his readers with no indication of the train of thought which led him to his discovery,[24] no hint of

[24] Was he perhaps the victim of a lucky accident, such as led D. M. Lipkin—another electrical engineer—to the discovery of "zilch" in 1964?

the "mechanism" by which (166.2) acquires its validity. We confront, therefore, some questions of a mainly mathematical nature: How does one *establish* the validity of (166.2)? In what sense are $\tilde{S}_{\mu\nu}$ and $T_{\mu\nu}$ "natural" constructs?

By way of preparation for a discussion of the first of those questions, we revisit the proof of (166.1). Taking (140) once again as out point of departure, we write

$$\partial_\mu S^\mu{}_\nu = \partial_\mu F^{\mu\alpha}\cdot F_{\alpha\nu} + \big[F^{\mu\alpha}\partial_\mu F_{\alpha\nu} - \tfrac{1}{4}\partial_\nu(F^{\alpha\beta}F_{\beta\alpha})\big]$$

and use $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ to reformulate the expression interior to the square brackets; we obtain

$$\begin{aligned}
[\text{ etc. }] &= (\partial^\mu A^\alpha - \partial^\alpha A^\mu)\partial_\mu(\partial_\alpha A_\nu - \partial_\nu A_\alpha)\\
&\qquad\qquad - \tfrac{1}{2}(\partial^\alpha A^\beta - \partial^\beta A^\alpha)\partial_\nu(\partial_\beta A_\alpha - \partial_\alpha A_\beta)\\
&= (\partial^\alpha A^\beta)\big\{(\partial_\alpha\partial_\beta A_\nu - \partial_\beta\partial_\alpha A_\nu - \partial_\alpha\partial_\nu A_\beta + \partial_\beta\partial_\nu A_\alpha)\\
&\qquad\qquad - \tfrac{1}{2}(\partial_\nu\partial_\beta A_\nu - \partial_\nu\partial_\alpha A_\nu - \partial_\nu\partial_\alpha A_\nu + \partial_\nu\partial_\beta A_\nu)\big\}\\
&= 0 \quad \text{after cancellations}
\end{aligned}$$

We have only to use $\partial_\mu F^{\mu\alpha}$ to recover (166.1). Proceeding similiary now to (166.2), we have

$$\begin{aligned}
\partial^\mu \tilde{S}_{\mu\nu} = \tfrac{1}{2}\big\{ &- J^\alpha\partial_\nu A_\alpha - A_\alpha\partial_\nu J^\alpha + 2(\partial^\mu J_\mu)A_\nu + 2J^\alpha\partial_\alpha A_\nu\\
&+ \partial^\mu\big[A_\alpha(\partial_\mu\partial_\nu A^\alpha) - (\partial_\mu A_\alpha)(\partial_\nu A^\alpha)\\
&\qquad + (\partial_\nu A_\mu)(\partial_\alpha A^\alpha) - A_\mu(\partial_\nu\partial_\alpha A^\alpha)\big]\big\}
\end{aligned}$$

But $\partial^\mu J_\mu = 0$, and if we exploit

$$\begin{aligned}
A^\alpha\partial_\nu J_\alpha &= A^\alpha\partial_\nu(\Box A_\alpha - \partial_\alpha\partial_\beta A^\beta)\\
2J^\alpha\partial_\alpha A_\nu &= J^\alpha\partial_\alpha A_\nu + (\partial^\alpha A_\nu)(\Box A_\alpha - \partial_\alpha\partial_\beta A^\beta)
\end{aligned}$$

we obtain

$$\begin{aligned}
\partial^\mu \tilde{S}_{\mu\nu} = \tfrac{1}{2}\big\{ &J^\alpha(\partial_\alpha A_\nu - \partial_\nu A_\alpha)\\
&+ \big[- A^\alpha\partial_\nu(\Box A_\alpha - \partial_\alpha\partial_\beta A^\beta) + (\partial^\alpha A_\nu)(\Box A_\alpha - \partial_\alpha\partial_\beta A^\beta)\\
&\quad + (\partial^\alpha A_\beta)(\partial_\alpha\partial_\nu A^\beta) + A_\alpha(\partial_\nu\Box A^\alpha)\\
&\quad - (\partial_\nu A_\alpha)\Box A^\alpha - (\partial^\alpha A_\beta)(\partial_\nu\partial_\alpha A^\beta)\\
&\quad + (\partial_\alpha A^\alpha)(\partial_\nu\partial_\beta A^\beta) + (\partial_\nu A^\alpha)(\partial_\alpha\partial_\beta A^\beta)\\
&\quad - (\partial_\alpha A^\alpha)(\partial_\nu\partial_\beta A^\beta) - A^\alpha(\partial_\alpha\partial_\nu\partial_\beta A^\beta)\big]\big\}
\end{aligned}$$

which after cancellations gives the result we sought to establish:

$$\begin{aligned}
&= \tfrac{1}{2}\big\{J^\alpha(\partial_\alpha A_\nu - \partial_\nu A_\alpha) + (\Box A^\alpha - \partial^\alpha\partial_\beta A^\beta)(\partial_\alpha A_\nu - \partial_\nu A_\alpha)\big\}\\
&= \tfrac{1}{2}\big\{J^\alpha F_{\alpha\nu} + J^\alpha F_{\alpha\nu}\big\}
\end{aligned}$$

Morgenthaler's $\tilde{S}_{\mu\nu}$ is in many respects an odd creature: it is not symmetric and not traceless; its symmetric and antisymmetric components are found (by calculations similiar to that just concluded) not individually to satisfy (166); it is—because it involves naked A_μ factors—not gauge-invariant. Imposition of (for example) the Lorentz gauge condition $\partial^\alpha A_\alpha = 0$ does serve to simplify the description of $\tilde{S}_{\mu\nu}$, and would simplify the proof of (166.2), but the validity of (166.2) hinges on no such gauge specialization. How does such a creature come into being? Valuable insight can be gained when the question is posed within the simplified context provided by "2-dimensional electrodynamics," and the following discussion can be read as a demonstration of the exploratory power of that toy theory. It is a demonstration also of the utility of the "hybrid representation trick" employed in the preceding discussion of (166). Quoting from (151) and using (see the line of text preceeding (127)) $E = -\partial_0 A$ and $B = -\partial_1 A$ to bring into play the "hybrid representation trick," we have

$$\mathbb{S} = \begin{pmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{pmatrix} = -\tfrac{1}{2}\begin{pmatrix} E\partial_0 A + B\partial_1 A & E\partial_1 A + B\partial_0 A \\ E\partial_1 A + B\partial_0 A & E\partial_0 A + B\partial_1 A \end{pmatrix}$$

But

$$\begin{aligned} E\partial_0 A + B\partial_1 A &= E\partial_0 A + \partial_1(BA) - A\partial_1 B \\ &\qquad\qquad \partial_1 B = J + \partial_0 E \qquad \text{by (127)} \\ &= \partial_1(AB) - JA + E\partial_0 A - A\partial_0 E \\ &= -\partial_1(A\partial_1 A) - JA + A\partial_0\partial_0 A - (\partial_0 A)(\partial_0 A) \qquad (169.1) \\ &= -\partial_0(A\partial_0 A) + JA + A\partial_1\partial_1 A - (\partial_1 A)(\partial_1 A) \qquad (169.2) \end{aligned}$$

where (169.2) is achieved by straightforward variation of the argument that gave (169.1). Similarly

$$\begin{aligned} E\partial_1 A + B\partial_0 A &= E\partial_1 A + \partial_0(BA) - A\partial_0 B \\ &\qquad\qquad \partial_0 B = \partial_1 E \qquad \text{by (127)} \\ &= \partial_0(AB) + E\partial_1 A - A\partial_1 E \\ &= -\partial_0(A\partial_1 A) + A\partial_0\partial_1 A - (\partial_0 A)(\partial_1 A) \qquad (169.3) \\ &= -\partial_1(A\partial_0 A) + A\partial_1\partial_0 A - (\partial_1 A)(\partial_0 A) \qquad (169.4) \end{aligned}$$

The implication is that we can, in imitation of (168), write

$$\mathbb{S} = \tilde{\mathbb{S}} - \mathbb{T}$$

where

$$\mathbb{T} = \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix} \equiv -\tfrac{1}{2}\begin{pmatrix} \partial_1(A\partial_1 A) & \partial_1(A\partial_0 A) \\ \partial_0(A\partial_1 A) & \partial_0(A\partial_0 A) \end{pmatrix} \qquad (170)$$

and

$$\tilde{\mathbb{S}} \equiv \tfrac{1}{2}\begin{pmatrix} JA - A\partial_0\partial_0 A + (\partial_0 A)(\partial_0 A) & -A\partial_0\partial_1 A + (\partial_0 A)(\partial_1 A) \\ -A\partial_1\partial_0 A + (\partial_1 A)(\partial_0 A) & -JA - A\partial_1\partial_1 A + (\partial_1 A)(\partial_1 A) \end{pmatrix}$$

More compactly

$$\tilde{S}_{ij} = \tfrac{1}{2}\big\{JAg_{ij} - A(\partial_i\partial_j A) + (\partial_i A)(\partial_j A)\big\} \tag{171}$$

which bears a striking structural resemblance to Morganthaler's construction. One has only to introduce $J = \Box A$ into (171) to confirm that (compare (152)) $\partial^i S_{ij} \equiv \partial_0 S_{0j} - \partial_1 S_{1j} = J\partial_j A = -JF_j$, but there is in fact no need to engage in such calculation, for it follows trivially from the construction of $\mathbb{T}$ that $\partial^i T_{ij} = 0$. We are led thus to the striking conclusion that we could—at least within the context of our toy electrodynamics—create any number of alternatives to Morganthaler's construction by writing

$$\tilde{\mathbb{S}} = \mathbb{S} + \mathbb{T} \qquad \text{with} \qquad \mathbb{T} = \begin{pmatrix} \partial_1 T_1 & \partial_1 T_0 \\ \partial_0 T_1 & \partial_0 T_0 \end{pmatrix} \tag{172}$$

and assigning to T_0 and T_1 any values/meanings we please. It is interesting to observe that the $\mathbb{T}$ encountered in (172) has precisely the structure already anticipated at (165). At work in (169) is the differential analog of an elementary "integration by parts" procedure. Presumably a similar procedure would give rise in a 4-dimensional setting to Morgenthaler's construction; I shall, however, not belabor the tedious details, for they appear to me to be devoid of physical consequence.

To the extent that $\mathbb{T} \equiv \tilde{\mathbb{S}} - \mathbb{S}$ is "devoid of physical consequence" it is released from any well-motivated requirement that it be Lorentz covariant, gauge invariant, or invariant with respect to duality rotations—however much it may offend our habituated sensibilities to write down expressions which do not conform to such formal constraints. Latent "physical consequence" can be assigned to $\mathbb{T}$ only in contexts into which the stress-energy tensor enters "nakedly." General relativity (into which the electromagnetic stress-energy tensor enters as a source term) and the construction of the angular momentum tensor (see CLASSICAL FIELD THEORY (1979), p. 117) come immediately to mind, and in those connections the sacrifice of covariance/invariance properties would appear to be intolerable. It is unknown to me whether the available gravitational data has been examined with an eye to its $\mathbb{S}$-discriminating potential. The angular momentum tensor, on the other hand, emerges from the bowels of Noetherian formalism; to complete a discussion of what angular momentum might have to contribute to the "physicality" of Morgenthaler's $\tilde{\mathbb{S}}$ one would have first to show how—if at all—$\tilde{\mathbb{S}}$ emerges as a natural object from that formalism.

A final word concerning the "conservation of zilch:" In 1964 D. M. Lipkin had occasion to notice that if

$$\begin{aligned} Z^0 &\equiv \mathbf{E}\cdot\text{curl}\,\mathbf{E} + \mathbf{B}\cdot\text{curl}\,\mathbf{B} \\ \mathbf{Z} &\equiv \mathbf{E}\times\partial_0\mathbf{E} + \mathbf{B}\times\partial_0\mathbf{B} \end{aligned}$$

then it is an implication of Maxwell's equations that

$$\partial_\alpha Z^\alpha = 0 \qquad \text{at source-free points in spacetime}$$

Lipkin was led from this curious observation to the discovery of a 10-fold set of unfamiliar conservations laws $\partial_\alpha V^{\mu\nu\alpha} = 0$ with $V^{\mu\nu\alpha} = V^{\nu\mu\alpha}$ which give back his original result in a special case: $Z^\alpha = V^{00\alpha}$. T. A. Morgan noticed almost immediately that $V^{\mu\nu\alpha}$ can be described

$$V^{\mu\nu\alpha} = (\partial^\alpha G^\mu{}_\lambda)F^{\lambda\nu} - (\partial^\alpha F^\mu{}_\lambda)G^{\lambda\nu}$$

and that these are but the leading members of an infinite class of multiply-indexed objects

$$\begin{aligned} &U^{\mu\nu\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q} \\ &\quad\equiv \tfrac{1}{2}\big[(\partial^{\alpha_1}\cdots\partial^{\alpha_p}G^\mu{}_\lambda)(\partial^{\beta_1}\cdots\partial^{\beta_q}F^{\lambda\nu}) - (\partial^{\alpha_1}\cdots\partial^{\alpha_p}F^\mu{}_\lambda)(\partial^{\beta_1}\cdots\partial^{\beta_q}G^{\lambda\nu})\big] \\ &T^{\mu\nu\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q} \\ &\quad\equiv \tfrac{1}{2}\big[(\partial^{\alpha_1}\cdots\partial^{\alpha_p}F^\mu{}_\lambda)(\partial^{\beta_1}\cdots\partial^{\beta_q}F^{\lambda\nu}) + (\partial^{\alpha_1}\cdots\partial^{\alpha_p}G^\mu{}_\lambda)(\partial^{\beta_1}\cdots\partial^{\beta_q}G^{\lambda\nu})\big] \end{aligned}$$

which are conserved in the sense that at source-free points in spacetime

$$\partial_\mu U^{\mu\nu\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q} = \partial_\mu T^{\mu\nu\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q} = 0$$

$U^{\mu\nu\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}$ gives back $V^{\mu\nu\alpha}$ in a special case, while $T^{\mu\nu\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}$ gives back the standard stress-energy tensor when all $\alpha\beta$-indices are absent. In 1965 T. Kibble showed that such constructions are not special to electrodynamics, but occur in the general theory of free fields. All of which has a strongly "exterior" look about it ($G^{\mu\nu}$ is after all the dual of $F^{\mu\nu}$), but on the other hand runs exactly counter to the sprit of the exterior calculus, for $\partial^{\alpha_1}\cdots\partial^{\alpha_p}$ and $\partial^{\beta_1}\cdots\partial^{\beta_q}$ are *symmetric* differential operators. To create a natural home for such constructions—the deep mathematical/physical significance of which no one, after 30 years, yet claims to understand[25] —one has, it would seem, either to "stretch"(much more than I have done) the exterior calculus, or to abandon any claim that "it is not tensor calculus but the exterior calculus that provides the natural language of physics." For further discussion and detailed references, see CLASSICAL ELECTRODYNAMICS (1980), pp. 329–332.

Concluding remarks. These pages are the fruit of a chance encounter with a mathematical colleague who, having introduced his students to the basics of the exterior calculus, wondered where he might look to find a good brief account of the electrodynamical applications of that subject. "How do you even know of those applications?" I asked. "It's common knowledge, I guess" was his response. And—among mathematicians—I suppose it is; it was a Professor Fuchs, of the Cornell University mathematics faculty, who (speaking at a physics seminar in the spring of 1956) whetted my own interest in the field. In any event, I was obliged to inform my mathematical colleague that I could think of no suitable reference, but would try to write something out.

[25] My hunch is that to gain such understanding one should abandon Noether's theorem as one's point of entry, and look to symmetries which inhabit the electrodynamical *phase space*.

Thus did the first few pages of this material come into being, and that fact accounts for the stance and style of the introductory pages; I imagined myself to be writing for the benefit of some second-year mathematicians, whom I had not individually met, and whose command of electrodynamics could be assumed to be minimal. My initial objective in §3 was to spell out my notational conventions and to make plain my conceptual biases; I am entirely self-taught in this field, which I have approached always with the preoccupations of a physicist; I anticipated that my student readers (and their instructor) would find my language quaint, and so lacking in sophistication as to be border on the unintelligible.

But in the early pages of §3 my intended reader experienced a major change of identity, and so does my style. David Griffiths, my immediate colleague and a specialist in the field of electrodynamics, learned of my little project, and announced that "I want a copy when you're done." Suddenly, instead of writing electrodynamics for students of the exterior calculus, I was writing exterior calculus for electrodynamical experts, for my peers.

Concerning my sources: I gave part of a summer in the early 1970s to study of the exterior calculus, working mainly from H. Flanders' *Differential Forms with Applications to the Physical Sciences*, which had appeared in 1963, and which I found difficult. An elaborate re-write of material developed on that occasion was written into my ELECTRODYNAMICS (1972), and much of the material presented here has been adapted from pp. 151–196 of that source. The material presented in §8, §9 and §10—the material having to do with "2p-dimensional electrodynamics," with the systematic exterior construction of constitutive relations and with Morgenthaler's construction—is, however, new, as are miscellaneous observations scattered throughout the text; while working out that material I have at many points drawn stimulation from conversation with David Griffiths. I am especially—and enormously—indebted to David for the elaborately marked-up copy of an earlier draft which he presented to me upon his return from an ostensible "vacation." I have incorporated many of his suggestions into this revision; others will require independent development on another occasion. It is a pleasure to acknowledge also my indebtedness to Donald Knuth for the invention of TEX, and to the good people at Blue Sky Research for the development of Textures. Readers who desire to pursue this subject may want to consult the fairly detailed bibliographic data available in ELECTRODYNAMICS (1972). Related material and references can also be found in Patrick Roberts' *The Principle of Exterior Expressibility* (Reed College thesis, 1983). Also not to be missed (though I personally find it so brazenly idiosyncratic and compulsively aphoristic as to be almost useless) is Chapter 4 of C. W. Misner, K. S. Thorne & J. A. Wheeler, *Gravitation* (W. H. Freeman, 1973).