On some recent electrodynamical work by Thomas Wieting

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Introduction. Yesterday—on Linus Pauling's 100^{th} birthday, as it happens—I was presented with a manuscript which by its title[1] purports to be about physics, but which is written in language so formally ideosyncratic as to be virtually unreadable (or at any rate unskimable) by this physicist, and which in the normal course of events I would cast dismissively (and unread) aside. The author declares an intention to "describe a procedure for solving Maxwell's equations" and (in a sequel) to "describe, quite explicitly, the relation between the various source-free electromagnetic fields and the various quantum states of the photon." But the paper provides almost nothing by way of detailed motivational commentary, and no reference which might help the reader to guess the tradition within which the author imagines himself to be working, what unsolved problem he imagines himself to be solving, what tool or tools to be sharpening or displacing.

But I have high regard for the author of this strange work, whom I know to be a mathematician with a celebratedly crystaline quality of mind, and (if not actually a physicist then) a reflective "physics watcher" of the highest order. So in the knowledge that Tom has labored hard to produce a work that he himself considers to be important ... I sit down to see if I can *figure out* what he is talking about, and why. What follows, therefore, is a record of my effort to *translate into the orthodox language of physics* an obscure statement which (on no explicit evidence) I suspect radiates from Tom's recent interest in what John Wheeler[2] called "Rainich's already unified field theory."

Electromagnetic field as a complex vector field. Electrodynamics came to us—from Maxwell *via* Heaviside and Lorentz—as the theory of a pair of (real-valued) vector fields $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{B}(\boldsymbol{x},t)$ driven by a pair of prescribed source fields $\rho(\boldsymbol{x},t)$ and $\boldsymbol{J}(\boldsymbol{x},t)$. The equations of motion (Maxwell-Lorentz) can be written[3]

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[1] "Fourier Transforms and Maxwell's Equations: Part I"

[2] *Geometrodynamics* (1962), pages 237 *et seq.*

[3] See CLASSICAL ELECTRODYNAMICS (1980/81), page 161.

$$
\begin{aligned}
\nabla\cdot\boldsymbol{E} &= \rho\\
\nabla\times\boldsymbol{B} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{E} &= \tfrac{1}{c}\boldsymbol{J}\\
\nabla\cdot\boldsymbol{B} &= 0\\
\nabla\times\boldsymbol{E} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{B} &= \boldsymbol{0}
\end{aligned}
$$

To render explicit the Lorentz covariance of the theory, and to facilitate certain computations, it proves useful to feed $\boldsymbol{E}$ and $\boldsymbol{B}$ into the design of a 2^{nd}-rank "field tensor" $F^{\mu\nu}$ and to feed ρ and $\boldsymbol{J}$ into the design of a "current vector" J^μ:

$$
\|F^{\mu\nu}\| \equiv \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3\\ E_1 & 0 & -B_1 & B_2\\ E_2 & B_3 & 0 & -B_1\\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad\text{and}\quad \|J^\nu\| = \begin{pmatrix} c\rho\\ J_1\\ J_2\\ J_3\end{pmatrix}
$$

But Tom elects to work *non*-relativistically. No 4-dimensional tensors appear in his work (all vectors are 3-vectors), and he assigns distinct roles to t and $\boldsymbol{x}$. He chooses a different (but almost equally well-trodden) path toward unification and generalization.

First, he posits (with Rainich's "duality rotations" evidently in mind) the existence of magnetic sources: the Maxwell-Lorentz equations become

$$
\left.\begin{aligned}
\nabla\cdot\boldsymbol{E} &= \quad\rho_e\\
+\nabla\times\boldsymbol{B} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{E} &= \tfrac{1}{c}\boldsymbol{J}_e\\
\nabla\cdot\boldsymbol{B} &= \quad\rho_m\\
-\nabla\times\boldsymbol{E} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{B} &= \tfrac{1}{c}\boldsymbol{J}_m
\end{aligned}\right\} \tag{1}
$$

Secondly, he elects to look upon $\boldsymbol{E}$ and $\boldsymbol{B}$ as the real/imaginary parts of a *complex* 3-vector field

$$\boldsymbol{V} \equiv \boldsymbol{E} + i\boldsymbol{B}$$

and in that same spirit to write

$$
\begin{aligned}
\rho &\equiv \rho_e + i\,\rho_m\\
\boldsymbol{J} &\equiv \boldsymbol{J}_e + i\boldsymbol{J}_m
\end{aligned}
$$

Equations (1) can then be rendered

$$
\left.\begin{aligned}
\nabla\cdot\boldsymbol{V} &= \rho\\
-i\,\nabla\times\boldsymbol{V} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{V} &= \tfrac{1}{c}\boldsymbol{J}
\end{aligned}\right\} \tag{2}
$$

If we take the divergence of the latter, draw upon the former and recall that (identically) $\operatorname{div}\operatorname{curl}\boldsymbol{V} = 0$ we obtain the continuity equation

$$\tfrac{\partial}{\partial t}\rho + \nabla\cdot\boldsymbol{J} = 0 \tag{3}$$

as a forced implication of the field equations. The "dangling i" in the second of the field equations (2) serves to couple the real (electrical) and imaginary (magnetic) components of the $\boldsymbol{V}$-field. The absence of such an i in (3) supplies "electro-charge conservation" and "magneto-charge conservation" of separate and distinct implications of (3). Proceeding in the spirit of the preceding remark, we notice that

$$\underbrace{\Big\{+i\,(\boldsymbol{\nabla}\times) - \tfrac{1}{c}\tfrac{\partial}{\partial t}\Big\}\Big\{-i\,(\boldsymbol{\nabla}\times) - \tfrac{1}{c}\tfrac{\partial}{\partial t}\Big\}}_{=\big(\frac{1}{c}\frac{\partial}{\partial t}\big)^2 + \big(\boldsymbol{\nabla}\times\big)^2}\boldsymbol{V} = \tfrac{1}{c}\Big\{+i\,(\boldsymbol{\nabla}\times) - \tfrac{1}{c}\tfrac{\partial}{\partial t}\Big\}\boldsymbol{J}$$

and recall that identically $\big(\boldsymbol{\nabla}\times\big)^2 = \text{grad div} - \text{“vector laplacian”}$. *In the absence of sources* the expression on the right vanishes, and so also (by the first of the field equations) does $\boldsymbol{\nabla}\cdot\boldsymbol{V}$, leaving us with the second order wave equation

$$\Box\boldsymbol{V} = 0 \tag{4}$$

which is really six equations: the components of $\boldsymbol{V}$ have become decoupled, and so have their real from their imaginary parts.

Fourier transformed field equations. For reasons that remain obscure, Tom attaches special importance to the Fourier transform. Proceeding formally (which Tom also does, but only after the apology expected of a mathematician) we write

$$\left.\begin{aligned}
\boldsymbol{V}(\boldsymbol{k},t) &\equiv (2\pi)^{-\frac{3}{2}}\iiint \exp\big\{-i\,\boldsymbol{k}\cdot\boldsymbol{x}\big\}\boldsymbol{V}(\boldsymbol{x},t)\,d^3x\\
\rho(\boldsymbol{k},t) &\equiv (2\pi)^{-\frac{3}{2}}\iiint \exp\big\{-i\,\boldsymbol{k}\cdot\boldsymbol{x}\big\}\,\rho(\boldsymbol{x},t)\,d^3x\\
\boldsymbol{J}(\boldsymbol{k},t) &\equiv (2\pi)^{-\frac{3}{2}}\iiint \exp\big\{-i\,\boldsymbol{k}\cdot\boldsymbol{x}\big\}\,\boldsymbol{J}(\boldsymbol{x},t)\,d^3x
\end{aligned}\right\} \tag{5}$$

with obvious inverse relations.[4] Notice that the t-variable is a shared variable; *i.e.*, that Tom does not contemplate passage from the time domain to the frequency domain: he thinks of his fields as t-parameterized objects—"curves" in fieldspace. Similarly his sources, which have become "t-parameterized curves in source space."

Introduce the inverse of (5) into the field equations (2) and obtain the transformed field equations

$$\left.\begin{aligned}
i\,\boldsymbol{k}\cdot\boldsymbol{V} &= \rho\\
\boldsymbol{k}\times\boldsymbol{V} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{V} &= \tfrac{1}{c}\boldsymbol{J}
\end{aligned}\right\} \tag{6}$$

[4] I refrain from Tom's practices of writing $\boldsymbol{p}$ in place of my $\boldsymbol{k}$, and of calling it "momentum"—impossible already on dimensional grounds in the absence of something like $\hbar$.

Notice that the dangling i has come to rest now in the *first* of the field equations. Tom observes that the second of those equations can be written

$$\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{V} = \mathbb{K}\boldsymbol{V} - \tfrac{1}{c}\boldsymbol{J} \quad \text{with} \quad \mathbb{K} \equiv \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix}$$

or a bit more simply

$$\tfrac{\partial}{\partial t}\boldsymbol{V} = \mathbb{A}\boldsymbol{V} - \boldsymbol{J} \quad \text{with} \quad \mathbb{A} \equiv c\,\mathbb{K} \;\;:\;\; \text{real antisymmetric} \tag{7}$$

The continuity equation (3) has become

$$\tfrac{\partial}{\partial t}\rho + i\,\boldsymbol{k}\cdot\boldsymbol{J} = 0 \tag{8}$$

which is not an equation to be solved, but a condition which the fields $\rho(\boldsymbol{x},t)$ and $\boldsymbol{J}(\boldsymbol{x},t)$ must satisfy if they are to be considered "admissible."

First steps toward solution of the transformed field equations. The beauty of the field equations (7) and the continuity equation (8)—and the point of the Fourier transformational step that produced them—is that they involve only a single kind of derivative: $\frac{\partial}{\partial t}$ survives, but $\boldsymbol{\nabla}$ has been "algebraicized." The equations in question yield immediately to methods standard to the elementary theory of first-order ODE's. The shift rule

$$\tfrac{\partial}{\partial t} - \mathbb{A} = e^{\mathbb{A}\,t}\tfrac{\partial}{\partial t}e^{-\mathbb{A}\,t}$$

supplies

$$\boldsymbol{V}(\boldsymbol{k},t) = e^{\mathbb{A}\,t}\Big\{\boldsymbol{V}(\boldsymbol{k},0) - \int_0^t e^{-\mathbb{A}\,s}\boldsymbol{J}(\boldsymbol{k},s)\,ds\Big\} \tag{9}$$

which demonstrably does give back $\boldsymbol{V}(\boldsymbol{k},0)$ at $t=0$, and does satisfy (7) at all times. The obvious presumption here is that $\boldsymbol{V}(\boldsymbol{k},0)$ and $\boldsymbol{J}(\boldsymbol{k},t)$ are known/prescribed.

From the antisymmetry and $\boldsymbol{k}$-dependent definition of $\mathbb{A}$ it follows that

$$\mathbb{R}(t) \equiv e^{\mathbb{A}\,t} \quad \text{is a rotation matrix}$$

It achieves a rotation through the angle $\varphi = c|\boldsymbol{k}|$ about the unit vector $\hat{\boldsymbol{k}}$, and possesses therefore the property that

$$\mathbb{R}(t)\,\boldsymbol{k} = \boldsymbol{k} \quad : \quad \hat{\boldsymbol{k}} \text{ is the spin axis of } \mathbb{R}(t)$$

With this fact and the yet-unused field equation in mind, Tom dots $i\boldsymbol{k}$ into (9) to obtain

$$\begin{aligned} i\boldsymbol{k}\cdot\boldsymbol{V}(\boldsymbol{k},t) &= i\boldsymbol{k}\cdot e^{\mathbb{A}\,t}\Big\{\boldsymbol{V}(\boldsymbol{k},0) - \int_0^t e^{-\mathbb{A}\,s}\boldsymbol{J}(\boldsymbol{k},s)\,ds\Big\} \\ &= i\boldsymbol{k}\cdot\boldsymbol{V}(\boldsymbol{k},0) - \int_0^t i\boldsymbol{k}\cdot e^{\mathbb{A}\,(t-s)}\boldsymbol{J}(\boldsymbol{k},s)\,ds \\ &= \rho(\boldsymbol{k},0) - \int_0^t i\boldsymbol{k}\cdot\boldsymbol{J}(\boldsymbol{k},s)\,ds \\ &= \rho(\boldsymbol{k},t) \quad \text{by integration of the continuity equation (8)} \end{aligned}$$

Tom seems to attach great importance to this result ("contention" in his odd phrase), but I see the argument—pretty though it is—as simply an alternative demonstration that the field equations jointly imply the continuity equation.

Here then in a nutshell is what I understand Tom's (elegantly efficient) procedure to be:

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \boldsymbol{J}(\boldsymbol{k},t) \text{ assumed to be given} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \boldsymbol{V}(\boldsymbol{k},t) = e^{\mathbb{A}t}\Big\{\boldsymbol{V}(\boldsymbol{k},0) - \int_0^t e^{-\mathbb{A}s}\boldsymbol{J}(\boldsymbol{k},s)\,ds\Big\} \\ \downarrow \\ i\,\boldsymbol{k}\cdot\boldsymbol{V}(\boldsymbol{k},t) = \rho\,(\boldsymbol{k},t) \\ \qquad\qquad\qquad \uparrow \text{automatic compliance with continuity equation} \end{array} \tag{10}$$

Equivalent statements about $\boldsymbol{J}(\boldsymbol{x},t)$, $\boldsymbol{V}(\boldsymbol{x},t)$ and $\rho(\boldsymbol{x},t) = \boldsymbol{\nabla}\cdot\boldsymbol{V}(\boldsymbol{x},t)$ lie just a Fourier transform away.

If, on the other hand, it were $\rho(\boldsymbol{x},t)$ that was considered given/prescribed then $\boldsymbol{J}(\boldsymbol{x},t)$ would (by the continuity equation) be determined only to within the addition of an arbitrary curl:

$$\boldsymbol{J} \longrightarrow \boldsymbol{J} + \text{curl}\,\boldsymbol{\mathfrak{J}} \quad : \quad \boldsymbol{\mathfrak{J}} \text{ arbitrary}$$

Which is to say: if $\rho(\boldsymbol{k},t)$ were given then $\boldsymbol{J}(\boldsymbol{k},t)$ would be determined only to within

$$\boldsymbol{J} \longrightarrow \boldsymbol{J} + i\,\boldsymbol{k}\times\boldsymbol{\mathfrak{J}}$$

where $\boldsymbol{\mathfrak{J}}_\perp \equiv \boldsymbol{k}\times\boldsymbol{\mathfrak{J}}$ stands normal to $\boldsymbol{k}$. The effect, by (9), can be described

$$\begin{aligned} \boldsymbol{V} &\longrightarrow \boldsymbol{V} + \boldsymbol{V}_\perp \\ &\qquad \boldsymbol{V}_\perp(\boldsymbol{k},t) \equiv -i\int_0^t e^{\mathbb{A}(t-s)}\boldsymbol{k}\times\boldsymbol{\mathfrak{J}}(\boldsymbol{k},s)\,ds \end{aligned}$$

We note that the first of the field equations (6)—the Fourier transform of Gauss' law—is insensitive to such transformations; equivalently,

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = \rho \quad \text{is insensitive to} \quad \boldsymbol{V} \longrightarrow \boldsymbol{V} + \text{curl}\,\boldsymbol{\mathcal{V}}$$

It may be (but on the other hand may not be: it is Tom's habit to expunge all motivational commentary) such considerations that motivate Tom to insert at this halfway point in his paper some elementary remarks pertaining to ...

Rotational geometry in 3-dimensional k-space. We have been led to associate time-dependent rotation matrices

$$\mathbb{R}(t) = e^{\mathbb{A}t} \quad : \quad \mathbb{A} \equiv ck\begin{pmatrix} 0 & -\hat{k}_3 & \hat{k}_2 \\ \hat{k}_3 & 0 & -\hat{k}_1 \\ -\hat{k}_2 & \hat{k}_1 & 0 \end{pmatrix}$$

with the points $\boldsymbol{k} = k \cdot \hat{\boldsymbol{k}}$ of $\boldsymbol{k}$-space. Dimensionally we have

$$[\boldsymbol{k}] = (\text{length})^{-1}$$

so $[ck] = (\text{time})^{-1} = \text{frequency}$: we have, in effect, spread an "angular velocity field" $\boldsymbol{\omega}(\boldsymbol{k})$ on "wave vector space," with

$$\begin{aligned}\boldsymbol{\omega}(\boldsymbol{k}) &= \omega(k)\,\hat{\boldsymbol{k}}\\ \omega(k) &\equiv ck\end{aligned}$$

Writing

$$\mathbb{R} = e^{\hat{\mathbb{K}}\omega t}$$

we observe that (by the Cayley-Hamilton theorem) $\hat{\mathbb{K}}(\hat{\mathbb{K}}^2 + \mathbb{I}) = \mathbb{O}$, which leads to the recognition that

$$\mathbb{P}_\parallel \equiv \hat{\mathbb{K}}^2 + \mathbb{I} = \begin{pmatrix} \hat{k}_1\hat{k}_1 & \hat{k}_1\hat{k}_2 & \hat{k}_1\hat{k}_3 \\ \hat{k}_2\hat{k}_1 & \hat{k}_2\hat{k}_2 & \hat{k}_2\hat{k}_3 \\ \hat{k}_3\hat{k}_1 & \hat{k}_3\hat{k}_2 & \hat{k}_3\hat{k}_3 \end{pmatrix} \quad \text{and} \quad \mathbb{P}_\perp \equiv -\hat{\mathbb{K}}^2$$

comprise a complete set of orthogonal projection operators. From $\text{tr}\,\mathbb{P}_\parallel = 1$ we learn that $\mathbb{P}_\parallel$ projects onto a 1-space (the ray defined by $\hat{\boldsymbol{k}}$) while by $\text{tr}\,\mathbb{P}_\perp = 2$ we know that $\mathbb{P}_\perp$ projects onto a 2-space (the plane normal to that ray). Write

$$\mathbb{R} = e^{\vartheta\hat{\mathbb{K}}} = \sum_{n=0}^{\infty} \tfrac{1}{n!}\vartheta^n\hat{\mathbb{K}}^n\big\{\mathbb{P}_\parallel + \mathbb{P}_\perp\big\} \quad \text{with} \quad \vartheta \equiv \omega t$$

observe that (by appeal once again to the Cayley-Hamilton theorem)

$$\begin{aligned}\hat{\mathbb{K}}^n\,\mathbb{P}_\parallel &= \begin{cases}\mathbb{P}_\parallel & : \quad n = 0\\ \mathbb{O} & : \quad n = 1, 2, 3, \ldots\end{cases}\\ \hat{\mathbb{K}}^{2\nu}\,\mathbb{P}_\perp &= (-)^\nu\,\mathbb{P}_\perp \qquad : \quad \nu = 0, 1, 2, 3, \ldots\\ \hat{\mathbb{K}}^{2\nu+1}\,\mathbb{P}_\perp &= (-)^\nu\,\hat{\mathbb{K}}\,\mathbb{P}_\perp \quad : \quad \nu = 0, 1, 2, 3, \ldots\end{aligned}$$

and obtain[5]

$$\mathbb{R} = e^{\vartheta\hat{\mathbb{K}}} = \mathbb{P}_\parallel + \big\{\cos\vartheta\cdot\mathbb{I} + \sin\vartheta\cdot\hat{\mathbb{K}}\big\}\mathbb{P}_\perp \tag{11}$$

Let $\boldsymbol{Z}(\boldsymbol{k})$ be an arbitrary vector field defined on $\boldsymbol{k}$-space. The action of $\mathbb{R}(t)$ upon $\boldsymbol{Z} = \boldsymbol{Z}_\parallel + \boldsymbol{Z}_\perp$ can now be described very simply (see the following figure): $\boldsymbol{Z}_\parallel$ does nothing, while $\boldsymbol{Z}_\perp$ twirls with angular velocity $\omega = ck$.

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[5] I have used a method borrowed from Chapter 1, pages 83–88 of CLASSICAL DYNAMICS (1964/65) and generalized in "What does an N-dimensional rotation look like?" which is reprinted near the end of TRANSFORMATIONAL PHYSICS & PHYSICAL GEOMETRY (1971–1983). Tom uses a vectorial method special to the 3-dimensional case.

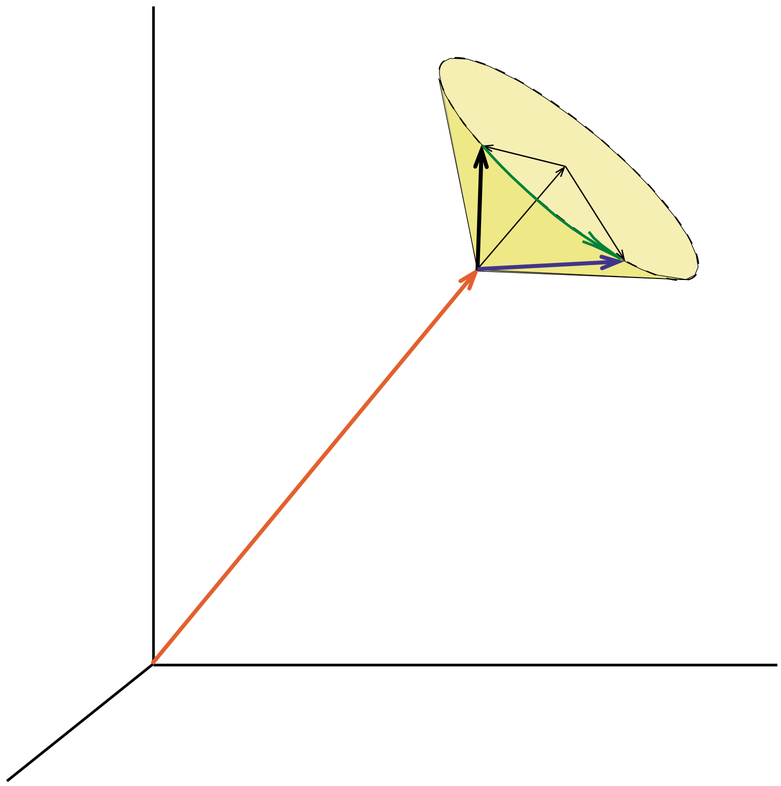

FIGURE 1: *The red vector marks a point in $\boldsymbol{k}$-space, where (as at every such point) dwells a time-dependent rotation matrix. The figure illustrates the action of that rotation matrix on a typical (black) field vector, which in time t assumes the blue position. Farther out along that same $\hat{\boldsymbol{k}}$-ray the action is similar except that the angular velocity is increased.*

It seems to me remarkable that the Maxwell equations serve to deposit on $\boldsymbol{k}$-space such a highly structured continuum of rotation matrices, each of which does its twirly thing quite independently of such specific "physics" (encoded into the design of $\boldsymbol{J}(\boldsymbol{k}, t)$) as may be written onto the the space.

Returning in this light to (10)—to what Tom calls "The Procedure"—his proposal is that we . . .
$\boxed{\text{STEP ZERO}}$ prescribe $\rho(\boldsymbol{k}, t)$ and use the continuity equation (8)—insensitive to $\boldsymbol{J}_\perp$, as has already been noted—to construct

$$\boldsymbol{J}_\parallel(\boldsymbol{k}, t) = i\tfrac{1}{k}\tfrac{\partial}{\partial t}\rho(\boldsymbol{k}, t) \tag{12.0}$$

This step becomes superfluous if $\rho(\boldsymbol{k}, t)$ and $\boldsymbol{J}(\boldsymbol{k}, t)$ were guaranteed to satisfy the continuity condition when shipped from the factory.

$\boxed{\text{STEP ONE}}$ Use Gauss' law (6)—the insensitivity of which to $\boldsymbol{J}_\perp$ has again already been noted—to obtain

$$\boldsymbol{V}_\parallel(\boldsymbol{k},t) = -i\tfrac{1}{k}\rho(\boldsymbol{k},t) \tag{12.1}$$

$\boxed{\text{STEP TWO}}$ Prescribe $\boldsymbol{J}_\perp(\boldsymbol{k},t)$ and $\boldsymbol{V}_\perp(\boldsymbol{k},0)$, and use the integrated form (9) of the other field equation to obtain

$$\boldsymbol{V}_\perp(\boldsymbol{k},t) = e^{\mathbb{A}\,t}\Big\{\boldsymbol{V}_\perp(\boldsymbol{k},0) - \int_0^t e^{-\mathbb{A}\,s}\boldsymbol{J}_\perp(\boldsymbol{k},s)\,ds\Big\} \tag{12.2}$$

Notice that we are *not* free to ascribe value arbitrarily to $\boldsymbol{V}_\parallel(\boldsymbol{k},0)$; that data was, by (12.1), implicit in the specification of $\rho(\boldsymbol{k},t)$. Note also that, while (12.1) relates $\boldsymbol{V}_\parallel(\boldsymbol{k},t)$ to *instantaneous* local source data, the integral in (12.2) signifies that the momentary value of $\boldsymbol{V}_\perp(\boldsymbol{k},t)$ summarizes *historic* local values of $\boldsymbol{J}_\perp(\boldsymbol{k},t)$.

This stuff is pretty, and does seem to cast strange new light on a very old subject. But is it good for anything? Could one use such techniques to recast (say) electrostatics/magnetostatics? Tom admits to no interest in such questions, however natural they may be ...but plunges straight on into the area that clearly does interest him, and for which he apparently considers all else to be mere preparation:

The free field: electrodynamics in the absence of sources. Set $\rho(\boldsymbol{x},t)=0$ and $\boldsymbol{J}(\boldsymbol{x},t)=\boldsymbol{0}$, which in the Fourier transformed formalism entails that we set

$$\rho(\boldsymbol{k},t)=0 \quad\text{and}\quad \boldsymbol{J}(\boldsymbol{k},t)=\boldsymbol{0}$$

The field equations (6) become

$$\left.\begin{aligned} i\,\boldsymbol{k}\cdot\boldsymbol{V} &= 0\\ \boldsymbol{k}\times\boldsymbol{V} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{V} &= \boldsymbol{0}\end{aligned}\right\} \tag{13}$$

and the continuity equation reduces to a triviality. The first field equation supplies

$$\boldsymbol{V}(\boldsymbol{k},t)\perp\boldsymbol{k}\quad\text{universally (each }\boldsymbol{k}\text{, all }t) \tag{14.1}$$

We therefore assume $\boldsymbol{V}_\perp(\boldsymbol{k},0)$ to have been prescribed, and inquire after the evolved free fields $\boldsymbol{V}_\perp(\boldsymbol{k},t)$, concerning which (12.2) and (11) supply

$$\begin{aligned}\boldsymbol{V}_\perp(\boldsymbol{k},t) &= e^{\mathbb{A}\,t}\,\boldsymbol{V}_\perp(\boldsymbol{k},0)\\ &= \cos\omega t\cdot\boldsymbol{V}_\perp(\boldsymbol{k},0)+\sin\omega t\cdot\hat{\mathbb{K}}\,\boldsymbol{V}_\perp(\boldsymbol{k},0)\end{aligned} \tag{14.2}$$

At each point in $\boldsymbol{k}$-space the free field vector $\boldsymbol{V}=\boldsymbol{V}_\perp$ simply <u>twirls about the local $\boldsymbol{k}$ with the locally-determined angular velocity ω</u>. The "integrated local history" effects mentioned above are entirely absent.

The plan now is first to sharpen our understanding of the significance of (14.2), then to retreat from $\boldsymbol{k}$-space to $\boldsymbol{x}$-space.

What I have once called the "plane normal to the $\boldsymbol{k}$-vector" is now more usefully/precisely described

$$\Pi(\boldsymbol{k}) \equiv \begin{cases} \text{2-dimensional complex vector space} \\ \text{tangent at } \boldsymbol{k} \text{ to the sphere of radius } k \end{cases}$$

Every element $\boldsymbol{Z}_\perp \in \Pi(\boldsymbol{k})$ can be developed

$$\boldsymbol{Z}_\perp = \text{linear combination of basis elements } \boldsymbol{Z}^1_\perp(\boldsymbol{k}) \text{ and } \boldsymbol{Z}^2_\perp(\boldsymbol{k})$$

While $\{\boldsymbol{Z}^1_\perp, \boldsymbol{Z}^2_\perp\}$ can be selected arbitrarily (subject only to the requirement that they be linearly independent), it is algebraically most convenient/natural to select the *eigenbasis* of $\hat{\mathbb{K}}$. I reserve for an appendix discussion of how, in general detail, the eigenvectors of $\hat{\mathbb{K}}$ can be constructed/described: for the moment it is sufficient to notice that

$$\text{the real antisymmetry of } \hat{\mathbb{K}} \implies \text{the hermiticity of } \mathbb{H} \equiv -i\,\hat{\mathbb{K}}$$

The eigenvalues of $\mathbb{H}$ are necessarily real (they are in fact real and distinct: $\{0, \pm 1\}$) so those of $\hat{\mathbb{K}}$ are necessarily imaginary: $\{0, \pm i\}$. The eigenvectors of $\mathbb{H}$ are simultaneously eigenvectors of $\hat{\mathbb{K}}$—call them $\boldsymbol{e}_-$, $\boldsymbol{e}_0$ and $\boldsymbol{e}_+$—and are necessarily orthogonal, and can be arranged to be *orthonormal* in the sense standard to complex theory.[6] From $\hat{\mathbb{K}}\,\boldsymbol{e}_0 = 0\,\boldsymbol{e}_0$ (which in Tom's vectorial language reads $\boldsymbol{k} \times \boldsymbol{e}_0 = \boldsymbol{0}$) we know that $\boldsymbol{e}_0 = \hat{\boldsymbol{k}}$, while

$$\left.\begin{aligned} \hat{\mathbb{K}}\,\boldsymbol{e}_+ &= +i\,\boldsymbol{e}_+ \\ \hat{\mathbb{K}}\,\boldsymbol{e}_- &= -i\,\boldsymbol{e}_- \end{aligned}\right\} \tag{15}$$

are the defining characteristics of the eigenbasis in $\Pi(\boldsymbol{k})$.

Let the vector $\boldsymbol{V}_\perp(\boldsymbol{k}, 0)$ be developed

$$\boldsymbol{V}_\perp(\boldsymbol{k}, 0) = V_+(\boldsymbol{k}, 0)\,\boldsymbol{e}_+(\boldsymbol{k}) + V_-(\boldsymbol{k}, 0)\,\boldsymbol{e}_-(\boldsymbol{k})$$

where $V_\pm(\boldsymbol{k}, 0)$ are complex numbers. From (14.2) it now follows by (15) that

$$\boldsymbol{V}_\perp(\boldsymbol{k}, t) = V_+(\boldsymbol{k}, t)\,\boldsymbol{e}_+(\boldsymbol{k}) + V_-(\boldsymbol{k}, t)\,\boldsymbol{e}_-(\boldsymbol{k})$$

with

$$V_\pm(\boldsymbol{k}, t) = e^{\pm i\omega t} \cdot V_\pm(\boldsymbol{k}, 0) \tag{16}$$

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[6] This slight subtlty has heretofore remained invisible because in all prior statements of the form $\boldsymbol{a} \perp \boldsymbol{b}$ either $\boldsymbol{a}$ or $\boldsymbol{b}$ (or both) was real. Note, by the way, that the eigenvectors $\boldsymbol{e}$ are determined only to within arbitrary/independent phase factors $e^{i\alpha}$. The point is elaborated in Appendix A.

The "retreat from $\boldsymbol{k}$-space to $\boldsymbol{x}$-space" is accomplished by inversion of (5):

$$\boldsymbol{V}_\perp(\boldsymbol{x},t) = (2\pi)^{-\frac{3}{2}}\iiint \exp\big\{+i\,\boldsymbol{k}\cdot\boldsymbol{x}\big\}\boldsymbol{V}_\perp(\boldsymbol{k},t)\,d^3k \tag{17}$$

where the $_\perp$ on the lefthand side is vestigial, the original meaning having been lost in the integration process. But let us, with Tom, look to the primitive case in which $\boldsymbol{V}_\perp(\boldsymbol{k},0)$—whence also $\boldsymbol{V}_\perp(\boldsymbol{k},t)$—vanishes except at an isolated point in $\boldsymbol{k}$-space:

$$\boldsymbol{V}_\perp(\boldsymbol{k},0) = (2\pi)^{+\frac{3}{2}}\,\boldsymbol{W}_\perp\delta(\boldsymbol{k}-\boldsymbol{k}_0) \quad : \quad \boldsymbol{W}_\perp \text{ normal to } \boldsymbol{k}_0 \tag{18}$$

The $\iiint$ then trivializes; we have

$$\boldsymbol{V}_\perp(\boldsymbol{x},0) = \boldsymbol{W}_\perp \exp\big\{i\,\boldsymbol{k}_0\cdot\boldsymbol{x}\big\}$$

which is the very opposite of localized: $\boldsymbol{V}_\perp(\boldsymbol{x},0)$ is constant on space-filling planes normal to $\boldsymbol{k}_0$. Now let $\big\{\boldsymbol{e}_+,\boldsymbol{e}_-\big\}$ be orthonormal eigenvectors of $\hat{\mathbb{K}}_0$, resolve $\boldsymbol{W}_\perp$ in the now familar way

$$\boldsymbol{W}_\perp = W_+\,\boldsymbol{e}_+ + W_-\,\boldsymbol{e}_-$$

. . . turn on time and obtain

$$\boldsymbol{V}_\perp(\boldsymbol{x},t) = W_+\exp\big\{i\,(\boldsymbol{k}_0\cdot\boldsymbol{x}+\omega_0 t)\big\}\,\boldsymbol{e}_+ + W_-\exp\big\{i\,(\boldsymbol{k}_0\cdot\boldsymbol{x}-\omega_0 t)\big\}\,\boldsymbol{e}_- \tag{19}$$

The first term on the right fills $\boldsymbol{x}$-space with plane waves rushing antiparallel to $\boldsymbol{k}_0$ with speed $k_0/\omega_0 = c$; the second term with waves rushing parallel to $\boldsymbol{k}_0$. The antipodal construction

$$\boldsymbol{V}_\perp^{\text{antipodal}}(\boldsymbol{k},0) = (2\pi)^{+\frac{3}{2}}\,\boldsymbol{W}_\perp\delta(\boldsymbol{k}+\boldsymbol{k}_0) \quad : \quad \boldsymbol{W}_\perp \text{ normal to } \boldsymbol{k}_0$$

reverses those directional assignments: we obtain[7]

$$\boldsymbol{V}_\perp^{\text{antipodal}}(\boldsymbol{x},t) = W_+\exp\big\{i\,(-\boldsymbol{k}_0\cdot\boldsymbol{x}+\omega_0 t)\big\}\,\boldsymbol{e}_- + W_-\exp\big\{i\,(-\boldsymbol{k}_0\cdot\boldsymbol{x}-\omega_0 t)\big\}\,\boldsymbol{e}_+$$

Let me, for the purposes of this discussion, abandon the retrograde blue terms.

In Appendix A we learn to write

$$\left.\begin{aligned}\boldsymbol{e}_+ &\equiv \tfrac{1}{\sqrt{2}}(\boldsymbol{a}-i\,\boldsymbol{b})\\ \boldsymbol{e}_- &\equiv \tfrac{1}{\sqrt{2}}(\boldsymbol{a}+i\,\boldsymbol{b})\end{aligned}\quad\right\} \tag{20}$$

where $\boldsymbol{a}$ is a real vector $\perp\,\boldsymbol{k}_0$, and so is $\boldsymbol{b}\equiv\boldsymbol{k}_0\times\boldsymbol{a}$. Returning with (20) to (19) we obtain a result which might be notated

$$\begin{aligned}\boldsymbol{V}_\circlearrowleft(\boldsymbol{x},t) &= \mathcal{A}e^{-i\alpha}\exp\big\{i\,(\boldsymbol{k}_0\cdot\boldsymbol{x}-\omega_0 t)\big\}(\boldsymbol{a}+i\,\boldsymbol{b})\\ &= \mathcal{A}\big[\boldsymbol{a}\cos\varphi-\boldsymbol{b}\sin\varphi\big]+i\,\mathcal{A}\big[\boldsymbol{a}\sin\varphi+\boldsymbol{b}\cos\varphi\big]\\ &= \boldsymbol{E}_\circlearrowleft(\boldsymbol{x},t)+i\boldsymbol{B}_\circlearrowleft(\boldsymbol{x},t)\end{aligned} \tag{21.1}$$

[7] Here a subtlty: sign-reversal of $\boldsymbol{k}$ entails that in (20) we reverse also the sign of $\boldsymbol{b}$, so that $\big\{\hat{\boldsymbol{k}},\boldsymbol{a},\boldsymbol{b}\big\}_{\text{new}}$ is again a righthanded frame: this entails interchange of the roles played by $\boldsymbol{e}_+$ and $\boldsymbol{e}_-$, of which I take account in the following equation.

with $\varphi \equiv \boldsymbol{k}_0 \cdot \boldsymbol{x} - \omega_0 t$. In the antipodal case we on the other hand find

$$\begin{aligned}
\boldsymbol{V}_{\circlearrowleft}(\boldsymbol{x},t) &= \mathcal{B}e^{i\beta}\exp\big\{i(-\boldsymbol{k}_0\cdot\boldsymbol{x} + \omega_0 t)\big\}(\boldsymbol{a} + i\,\boldsymbol{b}) \\
&= \mathcal{B}\big[\boldsymbol{a}\cos\varphi + \boldsymbol{b}\sin\varphi\big] + i\,\mathcal{B}\big[-\boldsymbol{a}\sin\varphi + \boldsymbol{b}\cos\varphi\big] \\
&= \boldsymbol{E}_{\circlearrowleft}(\boldsymbol{x},t) + i\boldsymbol{B}_{\circlearrowleft}(\boldsymbol{x},t)
\end{aligned} \tag{21.2}$$

Superposition—construction of

$$\boldsymbol{E}(\boldsymbol{x},t) = \mathcal{A}\big[\boldsymbol{a}\cos\varphi - \boldsymbol{b}\sin\varphi\big] + \mathcal{B}\big[\boldsymbol{a}\cos(\varphi+\delta) + \boldsymbol{b}\sin(\varphi+\delta)\big] \tag{22.1}$$

which physically (but, from a mathematical viewpoint, redundantly) entails construction also of

$$\begin{aligned}
\boldsymbol{B}(\boldsymbol{x},t) = \mathcal{A}\big[\boldsymbol{a}\cos(\varphi+\tfrac{\pi}{2}) - \ \boldsymbol{b}\sin(\varphi+\tfrac{\pi}{2})\big]& \\
+ \mathcal{B}\big[\boldsymbol{a}\cos(\varphi+\delta+\tfrac{\pi}{2}) + \boldsymbol{b}\sin(\varphi+\delta+\tfrac{\pi}{2})\big]&
\end{aligned} \tag{22.2}$$

—yields a description of the most general monochromatic electromagnetic plane wave with propagation vector $\boldsymbol{k}_0$.

It is interesting that Tom's formalism leads most naturally to the synthesis of plane waves by superposition of $\circlearrowright$ / $\circlearrowleft$ <u>*circularly*</u> *polarized* waves, while the more standard line of argument—which has much in common with Tom's argument,[8] but is much swifter—leads with equal naturalness to superposition of $\longleftrightarrow$ / $\updownarrow$ <u>*linearly*</u> *polarized* waves:

$$\boldsymbol{E}(\boldsymbol{x},t) = \mathcal{E}_1\cos\varphi\cdot\boldsymbol{a} + \mathcal{E}_2\cos(\varphi+\delta)\cdot\boldsymbol{b} \tag{23.1}$$
$$\boldsymbol{B}(\boldsymbol{x},t) = \mathcal{E}_1\cos(\varphi+\tfrac{\pi}{2})\cdot\boldsymbol{a} + \mathcal{E}_2\cos(\varphi+\delta+\tfrac{\pi}{2})\cdot\boldsymbol{b} \tag{23.2}$$

Tom's $\circlearrowright$-wave can be recovered from (23) by setting $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{A}$ and $\delta = \frac{\pi}{2}$, while his $\circlearrowleft$-wave results from setting $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{B}$ and $\delta = -\frac{\pi}{2}$.

---

[8] One asks for the conditions imposed upon vectors $\boldsymbol{k}$, $\boldsymbol{E}$ and $\boldsymbol{B}$ by the requirement that the fields

$$\boldsymbol{E}(\boldsymbol{x},t) \equiv \boldsymbol{E}\exp\big\{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\big\} \quad \text{and} \quad \boldsymbol{B}(\boldsymbol{x},t) \equiv \boldsymbol{B}\exp\big\{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\big\}$$

be solutions of the free-field equations

$$\left.\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{V} &= 0 \\
-i\,\boldsymbol{\nabla}\times\boldsymbol{V} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{V} &= \boldsymbol{0}
\end{aligned}\right\} \tag{24}$$

See CLASSICAL ELECTRODYNAMICS (1980/81) page 342 for an account of the elementary details.

The polarizational state of a plane wave is most usefully described by the so-called Stokes parameters

$$\begin{aligned} S_0 &= \mathcal{E}_1^2 + \mathcal{E}_2^2 \\ S_1 &= \mathcal{E}_1^2 - \mathcal{E}_2^2 \\ S_2 &= 2\,\mathcal{E}_1^2\mathcal{E}_2^2 \cos\delta \\ S_3 &= 2\,\mathcal{E}_1^2\mathcal{E}_2^2 \sin\delta \end{aligned}$$

The assignments just described yield

$$\left.\begin{aligned} S_0 &= 2\mathcal{A}^2 \\ S_1 &= 0 \\ S_2 &= 0 \\ S_3 &= +2\mathcal{A}^2 \end{aligned}\right\}\text{in Tom's case } \boldsymbol{E}_{\circlearrowright}\text{, and}$$

and

$$\left.\begin{aligned} S_0 &= 2\mathcal{A}^2 \\ S_1 &= 0 \\ S_2 &= 0 \\ S_3 &= -2\mathcal{A}^2 \end{aligned}\right\}\text{in Tom's case } \boldsymbol{E}_{\circlearrowleft}$$

which can be read (see page 348 in the notes just cited) as a reassertion that Tom has been led to states of left/right *circular* polarization.

So what has Wieting taught us? Tom has reminded us that, in a class of favorable cases, Fourier transform techniques can be used to reduce systems of partial differential equations to systems of *ordinary* differential equations, and thus to purchase some analytical advantages. This is not news to the engineers (beginning, I think, with Heaviside) who study things like transmission lines, or to physicists interested in special solutions of (say) the Dirac equation. The PDE's encountered in those cases are—like the PDE's bequeathed to us by Maxwell/Heaviside/Lorentz—multicomponent linear systems into which $\frac{\partial}{\partial t}$ enters linearly. But if the technique is hardly novel, it has served to cast classical electrodynamics in a light that I find fresh and interesting.

To sharpen the point a bit: Tom has achieved interesting electrodynamical insight by abandoning the (manifest) Lorentz covariance of that theory (and would, at the outset, do the same if he were to undertake discussion of the Dirac equation). That I found to be a bracing reminder that strict adherence to manifest Lorentz covariance, though often quite illuminating, does entail costs . . . and closes doors that might usefully be left ajar.

But abandonment of Lorentz covariance seems to me an unpromising first step if one's objective is (as Tom's appears to be) to establish contact with the Wigner/Bargmann classification of the unitary representations of the Lorentz group.

Tom does not claim to have produced a comprehensive "electrodynamics," so cannot be faulted for having omitted all reference to the theory of potentials (of which he had no computational need), the structure of the stress-energy tensor (which in non-relativistic theory tends to fragment), the means by which one establishes contact with the language of Lagrangian field theory: I will take up some of those topics in my appendices.

Tom does, however, seem to imagine that his work bears directly (if in some unspecified way) on the theory of "photons." This I strenuously deny. His work proceeds without reference to $\hbar$ (a notational adjustment $\boldsymbol{k} \mapsto \boldsymbol{p}/\hbar$ would accomplish nothing of physical substance), is a contribution to *classical* electrodynamics, and bears only in that vaguely latent sense upon the essential phenomenology of QED. The theory, in its present (unquantized) state, has absolutely *nothing* to say about Einstein's "photon"—nothing to say about why the electromagnetic field, when interacting with other systems (matter) exchanges energy in frequency-dependent units of $\hbar\nu$, angular momentum in units of $\hbar$.

Tom's essay could have been written by Maxwell himself, and the thought has occurred to me that it may, in fact, have been intended as homage to Maxwell. Both avoid the notations that since ∼1896 have been standard to the field. Both are content to use different letters of the alphabet to name the components of multicomponent objects.[9] And both are—each for his own reason—content to keep relativity (also quantum mechanics!) "off stage."

[9] Is it possible that the indexed notation that has been commonplace for now more than a century was made possible by a late 19$^{\text{th}}$ Century typographic development?

APPENDIX A: Projective construction of the eigenbasis of $\hat{\mathbb{K}}$. Look first, by way of orientation, to the case

$$\hat{\boldsymbol{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Longrightarrow \quad \hat{\mathbb{K}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The characteristic equation reads $\lambda(\lambda^2+1)=0$; the eigenvalues are $\{0, \pm i\}$ and the associated normalized eigenvectors are

$$\boldsymbol{e}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{e}_+ = \tfrac{1}{\sqrt{2}} \begin{pmatrix} +i \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_- = \tfrac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = (\boldsymbol{e}_+)^*$$

By inspection we have this table of inner products

$$\begin{pmatrix} \boldsymbol{e}_0^{\rm t}\boldsymbol{e}_0 & \boldsymbol{e}_0^{\rm t}\boldsymbol{e}_+ & \boldsymbol{e}_-^{\rm t}\boldsymbol{e}_- \\ \boldsymbol{e}_+^{\rm t}\boldsymbol{e}_0 & \boldsymbol{e}_+^{\rm t}\boldsymbol{e}_+ & \boldsymbol{e}_+^{\rm t}\boldsymbol{e}_- \\ \boldsymbol{e}_-^{\rm t}\boldsymbol{e}_0 & \boldsymbol{e}_-^{\rm t}\boldsymbol{e}_+ & \boldsymbol{e}_-^{\rm t}\boldsymbol{e}_- \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where I have used ${}^{\rm t}$ to signify the conjugated transposition (adjunction). The associated projectors are

$$\begin{aligned} \mathbb{P}_0 &= \boldsymbol{e}_0\,\boldsymbol{e}_0^{\rm t} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathbb{P}_+ &= \boldsymbol{e}_+\boldsymbol{e}_+^{\rm t} = \tfrac{1}{2}\begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \mathbb{P}_- &= \boldsymbol{e}_-\boldsymbol{e}_-^{\rm t} = \tfrac{1}{2}\begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{P}_+^* \end{aligned}$$

which are readily seen to be hermitian/orthogonal/complete. From

$$\hat{\mathbb{K}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\mathbb{K}}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It becomes obvious that we can write

$$\left.\begin{aligned} \mathbb{P}_0 &= \mathbb{I} + \hat{\mathbb{K}}^2 \\ \mathbb{P}_+ &= -\tfrac{1}{2}\big(\hat{\mathbb{K}}^2 + i\,\hat{\mathbb{K}}\big) \\ \mathbb{P}_- &= -\tfrac{1}{2}\big(\hat{\mathbb{K}}^2 - i\,\hat{\mathbb{K}}\big) \end{aligned}\right\} \qquad (A1)$$

Immediately $\mathbb{P}_0 + \mathbb{P}_+ + \mathbb{P}_- = \mathbb{I}$, while each of the statements

$$\left.\begin{array}{lll} \mathbb{P}_0^2 = \mathbb{P}_0 & \text{and} & \mathbb{P}_0\mathbb{P}_+ = \mathbb{P}_0\mathbb{P}_- = \mathbb{O} \\ \mathbb{P}_+^2 = \mathbb{P}_+ & \text{and} & \mathbb{P}_+\mathbb{P}_0 = \mathbb{P}_+\mathbb{P}_- = \mathbb{O} \\ \mathbb{P}_-^2 = \mathbb{P}_- & \text{and} & \mathbb{P}_-\mathbb{P}_0 = \mathbb{P}_-\mathbb{P}_+ = \mathbb{O} \end{array}\right\} \tag{A2}$$

can be extracted directly from the Cayley-Hamilton identity: $\hat{\mathbb{K}}^3 = -\hat{\mathbb{K}}$.

But the Cayley-Hamilton identity holds *generally* (*i.e.*; is not special to the $\hat{\boldsymbol{k}}$ assumed at the outset, but holds for *all* $\hat{\boldsymbol{k}}$). Using

$$\hat{\mathbb{K}} = \begin{pmatrix} 0 & -\hat{k}_3 & \hat{k}_2 \\ \hat{k}_3 & 0 & -\hat{k}_1 \\ -\hat{k}_2 & \hat{k}_1 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\mathbb{K}}^2 = \begin{pmatrix} \hat{k}_1\hat{k}_1 - 1 & \hat{k}_1\hat{k}_2 & \hat{k}_1\hat{k}_3 \\ \hat{k}_2\hat{k}_1 & \hat{k}_2\hat{k}_2 - 1 & \hat{k}_2\hat{k}_3 \\ \hat{k}_3\hat{k}_1 & \hat{k}_3\hat{k}_2 & \hat{k}_3\hat{k}_3 - 1 \end{pmatrix}$$

we could in a moment write down explicit descriptions of the projectors $\mathbb{P}$ *in the general case*, and in terms of them recover the projectors previously denoted

$$\left.\begin{array}{l} \mathbb{P}_\parallel = \mathbb{P}_0 \\ \mathbb{P}_\perp = \mathbb{P}_+ + \mathbb{P}_- \end{array}\right\} \tag{A3}$$

It is evident that $\mathbb{P}_0$ projects onto

$$\boldsymbol{e}_0(\hat{\boldsymbol{k}}) = \hat{\boldsymbol{k}} \tag{A4.1}$$

To discover $\boldsymbol{e}_+$ and $\boldsymbol{e}_-$ we might (with the assistance of *Mathematica*)

construct $\mathbb{P}_0\boldsymbol{s}$ with a "seed vector" given by (say) $\boldsymbol{s} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

and normalize the result; such a procedure leads to

$$\boldsymbol{e}_+(\hat{\boldsymbol{k}}) = \frac{1}{2\sqrt{1-\hat{k}_2\hat{k}_3-\hat{k}_3\hat{k}_1-\hat{k}_1\hat{k}_2}} \begin{pmatrix} 1 - \hat{k}_1(\hat{k}_1+\hat{k}_2+\hat{k}_3) - i(\hat{k}_2-\hat{k}_3) \\ 1 - \hat{k}_2(\hat{k}_1+\hat{k}_2+\hat{k}_3) - i(\hat{k}_3-\hat{k}_1) \\ 1 - \hat{k}_3(\hat{k}_1+\hat{k}_2+\hat{k}_3) - i(\hat{k}_1-\hat{k}_2) \end{pmatrix} \tag{A4.2}$$

$$\boldsymbol{e}_-(\hat{\boldsymbol{k}}) = \frac{1}{2\sqrt{1-\hat{k}_2\hat{k}_3-\hat{k}_3\hat{k}_1-\hat{k}_1\hat{k}_2}} \begin{pmatrix} 1 - \hat{k}_1(\hat{k}_1+\hat{k}_2+\hat{k}_3) + i(\hat{k}_2-\hat{k}_3) \\ 1 - \hat{k}_2(\hat{k}_1+\hat{k}_2+\hat{k}_3) + i(\hat{k}_3-\hat{k}_1) \\ 1 - \hat{k}_3(\hat{k}_1+\hat{k}_2+\hat{k}_3) + i(\hat{k}_1-\hat{k}_2) \end{pmatrix} \tag{A4.3}$$

These vectors pass all orthonormality and other tests, and in the case

$$\hat{\boldsymbol{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are phase-equivalent (multiply by $e^{i\frac{\pi}{2}} = (1+i)/\sqrt{2}\,$) to the eigenvectors which marked our point of departure. But they fail in the case $\hat{\boldsymbol{k}} = \boldsymbol{s}$; one has, in that case, to select a different seed (any seed with a component normal to $\hat{\boldsymbol{k}}$). The seed-adjustment problem leads us to notice this generalization of the result just obtained:

$$\left.\begin{aligned}\boldsymbol{e}_+(\hat{\boldsymbol{k}}) &= \frac{\boldsymbol{s} - (\boldsymbol{s}\cdot\hat{\boldsymbol{k}})\hat{\boldsymbol{k}} - i\,\hat{\boldsymbol{k}}\times\boldsymbol{s}}{\text{length}}\\ \boldsymbol{e}_-(\hat{\boldsymbol{k}}) &= \frac{\boldsymbol{s} - (\boldsymbol{s}\cdot\hat{\boldsymbol{k}})\hat{\boldsymbol{k}} + i\,\hat{\boldsymbol{k}}\times\boldsymbol{s}}{\text{length}}\\ \text{length} &= \sqrt{2(1-(\boldsymbol{s}\cdot\hat{\boldsymbol{k}})^2}\end{aligned}\right\} \tag{A5}$$

The assumption here is that $\boldsymbol{s}$ is a unit vector not parallel to $\hat{\boldsymbol{k}}$, but otherwise arbitrary. Equations (A5) render the entire subject transparent, but contain a surprise: *there are as many ways to assign meaning to* $\big\{\boldsymbol{e}_+(\hat{\boldsymbol{k}}), \boldsymbol{e}_-(\hat{\boldsymbol{k}})\big\}$ *as there are ways to select* $\boldsymbol{s}$. And such a state of affairs is, in fact, quite intelligible:

Equations (A5) are of the form

$$\begin{aligned}\boldsymbol{e}_+ &= \frac{\boldsymbol{a} - i\,\boldsymbol{b}}{\sqrt{a^2+b^2}}\\ \boldsymbol{e}_- &= \frac{\boldsymbol{a} + i\,\boldsymbol{b}}{\sqrt{a^2+b^2}}\end{aligned}$$

where $\boldsymbol{a}$ and $\boldsymbol{b} = \hat{\boldsymbol{k}}\times\boldsymbol{a}$ are real vectors of the same length, normal to each other and also to $\hat{\boldsymbol{k}}$. Without loss of generality we may assume $\boldsymbol{a}$ whence also $\boldsymbol{b}$ to be unit vectors: then we confront

$$\left.\begin{aligned}\boldsymbol{e}_+ &= \tfrac{1}{\sqrt{2}}(\boldsymbol{a} - i\,\hat{\boldsymbol{k}}\times\boldsymbol{a})\\ \boldsymbol{e}_- &= \tfrac{1}{\sqrt{2}}(\boldsymbol{a} + i\,\hat{\boldsymbol{k}}\times\boldsymbol{a})\end{aligned}\right\} \tag{A6}$$

Immediately, the orthonormality properties of $\big\{\boldsymbol{e}_+, \boldsymbol{e}_-\big\}$ mimic those of $\big\{\boldsymbol{a}, \boldsymbol{b}\big\}$. Moreover,

$$\begin{aligned}\hat{\mathbb{K}}\,\boldsymbol{e}_+ = \hat{\boldsymbol{k}}\times\boldsymbol{e}_+ &= \tfrac{1}{\sqrt{2}}\Big\{\hat{\boldsymbol{k}}\times\boldsymbol{a} - i\,\hat{\boldsymbol{k}}\times(\hat{\boldsymbol{k}}\times\boldsymbol{a})\Big\}\\ &= \tfrac{1}{\sqrt{2}}\Big\{\hat{\boldsymbol{k}}\times\boldsymbol{a} - i\big[(\hat{\boldsymbol{k}}\cdot\boldsymbol{a})\hat{\boldsymbol{k}} - (\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}})\boldsymbol{a}\big]\Big\}\\ &= \tfrac{1}{\sqrt{2}}\Big\{\hat{\boldsymbol{k}}\times\boldsymbol{a} + i\,\boldsymbol{a}\Big\}\\ &= +i\,\boldsymbol{e}_+\end{aligned}$$

Similarly (or by simple complex conjugation) $\hat{\mathbb{K}}\,\boldsymbol{e}_- = -i\,\boldsymbol{e}_-$. The simple upshot of the (excessively round about) argument is that

> the $\boldsymbol{e}_\pm$ given by (A6) will be eigenvectors of $\hat{\mathbb{K}}$
> <u>no matter how</u> the real unit vector $\boldsymbol{a}$ is positioned
> on the plane normal to the real unit vector $\hat{\boldsymbol{k}}$

And the $O(2)$ symmetry that lurks within the definitions (A6) is in fact very easily understood. Let $\{\boldsymbol{a},\boldsymbol{b}\}$ be a real orthonormal basis inscribed on the plane $\perp$ to $\hat{\boldsymbol{k}}$, and let $\{\boldsymbol{a},\boldsymbol{b}\}$ be another. Then

$$
\begin{aligned}
\boldsymbol{a} &= \boldsymbol{a}(\boldsymbol{a}\cdot\boldsymbol{a}) + \boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{a}) = \boldsymbol{a}\cos\theta - \boldsymbol{b}\sin\theta\\
\boldsymbol{b} &= \boldsymbol{a}(\boldsymbol{a}\cdot\boldsymbol{b}) + \boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{b}) = \boldsymbol{a}\sin\theta + \boldsymbol{b}\cos\theta
\end{aligned}
$$

which when introduced into

$$
\begin{aligned}
\boldsymbol{e}_+ &= \boldsymbol{e}_+(\boldsymbol{e}_+^{\,\mathrm{t}}\,\boldsymbol{e}_+) + \boldsymbol{e}_-(\boldsymbol{e}_-^{\,\mathrm{t}}\,\boldsymbol{e}_+)\\
\boldsymbol{e}_- &= \boldsymbol{e}_+(\boldsymbol{e}_+^{\,\mathrm{t}}\,\boldsymbol{e}_-) + \boldsymbol{e}_-(\boldsymbol{e}_-^{\,\mathrm{t}}\,\boldsymbol{e}_-)
\end{aligned}
$$

together with the definitions

$$
\begin{aligned}
\boldsymbol{e}_+ &\equiv \tfrac{1}{\sqrt{2}}(\boldsymbol{a} - i\,\boldsymbol{b})\\
\boldsymbol{e}_- &\equiv \tfrac{1}{\sqrt{2}}(\boldsymbol{a} + i\,\boldsymbol{b})
\end{aligned}
\qquad\text{and}\qquad
\begin{aligned}
\boldsymbol{e}_+ &\equiv \tfrac{1}{\sqrt{2}}(\boldsymbol{a} - i\,\boldsymbol{b})\\
\boldsymbol{e}_- &\equiv \tfrac{1}{\sqrt{2}}(\boldsymbol{a} + i\,\boldsymbol{b})
\end{aligned}
$$

are found by quick calculation to give

$$
\begin{pmatrix}\boldsymbol{e}_+\\ \boldsymbol{e}_-\end{pmatrix} = \mathbb{U}\begin{pmatrix}\boldsymbol{e}_+\\ \boldsymbol{e}_-\end{pmatrix}\quad\text{where}\quad \mathbb{U} = \begin{pmatrix} e^{+i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix}\quad\text{is \underline{unitary}}
$$

Unitary transformations are well known to preserve all essential properties of eigenvalues/vectors.

It has come accidentally to my attention that ***Problem 3.17** in David Griffiths' *Introduction to Quantum Mechanics* (1995) captures the essence of what I have had to say.

APPENDIX B: Potentials and duality rotations in the complex formalism. Tom is quite aware that the Maxwell equations (2)

$$\begin{aligned} \boldsymbol{\nabla}\!\cdot\!\boldsymbol{V} &= \rho \\ -i\,\boldsymbol{\nabla}\times\boldsymbol{V} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{V} &= \tfrac{1}{c}\boldsymbol{J} \end{aligned}$$

are—trivially—invariant under the simultaneous adjustments

$$\boldsymbol{V}\mapsto e^{i\theta}\boldsymbol{V} \quad\text{and}\quad \rho\mapsto e^{i\theta}\rho \quad\text{and}\quad \boldsymbol{J}\mapsto e^{i\theta}\boldsymbol{J} \tag{B1}$$

and that this adjustment, when spelled out

$$\begin{aligned} \boldsymbol{E} &\mapsto \boldsymbol{E}\cos\theta - \boldsymbol{B}\sin\theta \\ \boldsymbol{B} &\mapsto \boldsymbol{E}\sin\theta + \boldsymbol{B}\cos\theta \\ \rho_e &\mapsto \rho_e\cos\theta - \rho_m\sin\theta \\ \rho_m &\mapsto \rho_e\sin\theta + \rho_m\cos\theta \\ \boldsymbol{J}_e &\mapsto \boldsymbol{J}_e\cos\theta - \boldsymbol{J}_m\sin\theta \\ \boldsymbol{J}_m &\mapsto \boldsymbol{J}_e\sin\theta + \boldsymbol{J}_m\cos\theta \end{aligned}$$

is precisely what is standardly called a "duality rotation."[10] The relative economy of the complex formalism—of (B1)—in this regard is striking. But why do I couple that remark with an allusion to the theory of electromagnetic potentials?

In ordinary Maxwellian theory (*i.e.*, in the absence of magnetic charges and currents) we write

$$\begin{aligned} &\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \\ &\qquad \boldsymbol{A}\text{ susceptible to gauge: } \boldsymbol{A}\mapsto\boldsymbol{A}+\boldsymbol{\nabla}\!\cdot\chi \end{aligned} \tag{B2.1}$$

as a means of rendering $\boldsymbol{\nabla}\!\cdot\!\boldsymbol{B}=0$ *automatic*. Then

$$\begin{aligned} &\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} \\ &\qquad \phi\text{ susceptible to gauge: } \phi\mapsto\phi-\tfrac{1}{c}\tfrac{\partial}{\partial t}\chi \end{aligned} \tag{B2.2}$$

serves to render automatic also the other sourceless equation: $\boldsymbol{\nabla}\times\boldsymbol{E}+\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}=\boldsymbol{0}$. The sourcey field equations now become

$$\begin{aligned} -\nabla^2\phi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\nabla}\!\cdot\!\boldsymbol{A} &= \rho_e \\ \boldsymbol{\nabla}(\boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}+\tfrac{1}{c}\tfrac{\partial}{\partial t}\phi) - \boldsymbol{\nabla}^2\boldsymbol{A} + \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\boldsymbol{A} &= \tfrac{1}{c}\boldsymbol{J}_e \end{aligned}$$

[10] For a fairly elaborate discussion of this topic, couched in language borrowed from the exterior calculus, see ELECTRODYNAMICS (1972/72); also C. Misner, K. Thorne & J. Wheeler, *Gravitation* (1973), pages 108 & 482.

and if we impose the Lorentz gauge condition

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\phi = 0$$

assume the attractive form

$$\begin{aligned}\Big\{\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \nabla^2\Big\}\phi &= \rho_e\\ \Big\{\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \boldsymbol{\nabla}^2\Big\}\boldsymbol{A} &= \boldsymbol{J}_e\end{aligned}$$

The fields ϕ and $\boldsymbol{A}$ are now *decoupled*, but satisfy PDE's of *second* order. Tom's Fourier transform technique could be adapted straightforwardly to the solution of those equations, and to the reexpression of all the manipulations that led to them.

In the hypothetical presence of magnetic sources one no longer has two "sourceless Maxwell equations," is cut off from what classically was the primary motivation for introducing potentials $\{\phi, \boldsymbol{A}\}$, and cut off also from any obvious means to do so. Yet classically the introduction of potentials had also a second line of motivation (they permit one to construct decoupled field equations, which are easier to solve), and it is in the language of potential theory that one standardly undertakes to quantize the electromagnetic field. One would like, therefore, to be able to retain (some generalization of) the potential concept even in the presence of magnetic sources. Dirac (1931) described one way in which this might be accomplished. An alternative procedure, involving the introduction of a *pair of 4-potentials*, was described by N. Cabibbo & E. Ferrari,[11] but that theory (ditto the standard theory sketched above!) seems to stand skew to natural tendencies of the complex formalism favored by Tom. I follow now, therefore, in the footsteps of M. Y. Han & L. C. Biedenharn ,[12] who themselves proceed form Hertz' observations (1889) that the Lorentz gauge condition becomes automatic if one writes[13]

$$\phi = \boldsymbol{\nabla}\cdot\boldsymbol{H} \quad\text{and}\quad \boldsymbol{A} = -\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{H} + \boldsymbol{\nabla}\times\boldsymbol{G} \quad : \quad \boldsymbol{G}\text{ arbitrary} \tag{B3}$$

and that the charge continuity equation $\frac{\partial}{\partial t}\rho_e + \boldsymbol{\nabla}\cdot\boldsymbol{J}_e = 0$ becomes automatic if one writes

$$c\rho_e = \boldsymbol{\nabla}\cdot\boldsymbol{P}_e \quad\text{and}\quad \boldsymbol{J}_e = -\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{P}_e + \boldsymbol{\nabla}\times\boldsymbol{Q} \quad : \quad \boldsymbol{Q}\text{ arbitrary} \tag{B4}$$

---

[11] "Quantum electrodynamics with magnetic monopoles," Nuovo Cimento **23**, 1147 (1962). In ELECTRODYNAMICS (1972/73), beginning at page 350, I discuss in elaborate detail how the exterior calculus can be used to construct an elegant account of the Cabibbo/Ferrari theory, and at the same time to enlarge upon what might be called the "internal symmetry group" of Maxwellian electrodynamics (*i.e.*, to construct "generalized duality transformations").

[12] "Manifest dyality invariance in electrodynamics and the Cabibbo-Ferrari theory of magnetic monopoles," Nuovo Cimento **2A**, 544 (1971).

[13] I use $\boldsymbol{H}$ (where Hertz himself used $-\boldsymbol{\Pi}$) to suggest "**H**ertz potential." For a good account of Hertz' original idea, see §§**13-4** through **13-7** in Panofsky & Phillips, *Classical Electricity and Magnetism* (1955).

Returning with (B3) to (B2) we obtain

$$\left.\begin{aligned}\boldsymbol{B} &= \boldsymbol{\nabla}\times\Big\{-\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{H}+\boldsymbol{\nabla}\times\boldsymbol{G}\Big\}\\ \boldsymbol{E} &= \boldsymbol{\nabla}\times\Big\{-\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{G}-\boldsymbol{\nabla}\times\boldsymbol{H}\Big\}+\square\boldsymbol{H}\end{aligned}\right\}\tag{$B5$}$$

where in the derivation of the latter equation we have again made use of the identity $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{H}) = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{H}+\boldsymbol{\nabla}^2\boldsymbol{H}$ (here $\boldsymbol{\nabla}^2$ is the "vectorial laplacian") and of $\square \equiv \big\{\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2-\boldsymbol{\nabla}^2\big\}$ which defines the "vectorial wave operator." In this notation

$$\left.\begin{aligned}&\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 &&\text{remains automatic}\\ &\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho_e &&\text{becomes}\quad \boldsymbol{\nabla}\cdot\square\boldsymbol{H} = \boldsymbol{\nabla}\cdot\tfrac{1}{c}\boldsymbol{P}_e\end{aligned}\right\}\tag{$B6.1$}$$

The latter equation would follow from (but does not strictly entail)

$$\square\boldsymbol{H} = \tfrac{1}{c}\boldsymbol{P}_e$$

Similarly (the computational labor becomes now a bit tedious, but involves no new idea, and takes just a minute if done carefully on a large piece of paper)

$$\left.\begin{aligned}&-\boldsymbol{\nabla}\times\boldsymbol{E}-\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{B} = \boldsymbol{0} &&\text{remains automatic}\\ &+\boldsymbol{\nabla}\times\boldsymbol{B}-\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{E} = \tfrac{1}{c}\boldsymbol{J}_e &&\text{becomes}\quad \square\Big\{-\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{H}+\boldsymbol{\nabla}\times\boldsymbol{G}\Big\} = \tfrac{1}{c}\boldsymbol{J}_e\end{aligned}\right\}\tag{$B6.2$}$$

—the latter of which is simply Hertz' notational adjustment of the familiar statement that (in Lorentz gauge) $\square\boldsymbol{A} = \frac{1}{c}\boldsymbol{J}_e$.[14] We now have

$$\begin{aligned}\square\Big\{-\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{H}+\boldsymbol{\nabla}\times\boldsymbol{G}\Big\} &= \tfrac{1}{c}\Big\{-\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{P}_e+\boldsymbol{\nabla}\times\boldsymbol{Q}\Big\}\\ &\downarrow\\ -\tfrac{1}{c}\tfrac{\partial}{\partial t}\cdot\square\boldsymbol{H} &= -\tfrac{1}{c}\tfrac{\partial}{\partial t}\cdot\tfrac{1}{c}\boldsymbol{P}_e \quad\text{if we set } \boldsymbol{G}=\boldsymbol{Q}=\boldsymbol{0}\end{aligned}$$

which once again "would follow from (but does not strictly entail)"

$$\square\boldsymbol{H} = \tfrac{1}{c}\boldsymbol{P}_e \tag{$B7$}$$

Let us step for a moment into Hertz' shoes. We claim no interest in magnetic sources, are content to dismiss $\boldsymbol{G}$ and $\boldsymbol{Q}$ as pointless frivolities, and <u>consider (B7) to be fundamental</u>. We introduce the subsidiary derived fields

$$\phi \equiv \boldsymbol{\nabla}\cdot\boldsymbol{H},\quad \boldsymbol{A} \equiv -\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{H},\quad \rho \equiv \tfrac{1}{c}\boldsymbol{\nabla}\cdot\boldsymbol{P}_e\,,\quad \boldsymbol{J}_e \equiv -\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{P}_e$$

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[14] We note in passing that this, by $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, can be read as the "uncurled precursor" of a statement $\square\boldsymbol{B} = \frac{1}{c}\boldsymbol{\nabla}\times\boldsymbol{J}_e$ that follows directly/easily from Maxwell's equations, but is usually encountered only in source-free applications.

(from which $\boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}+\frac{1}{c}\frac{\partial}{\partial t}\phi=0$ and $\boldsymbol{\nabla}\!\cdot\!\boldsymbol{J}_e+\frac{\partial}{\partial t}\phi_e=0$ follow as corollaries) and, hitting (B7) with $\boldsymbol{\nabla}\cdot$ else $\frac{1}{c}\frac{\partial}{\partial t}$, obtain

$$\Box\phi=\rho_e \quad \text{and} \quad \Box\boldsymbol{A}=\tfrac{1}{c}\boldsymbol{J}_e$$

Next (proceeding backwards through some familiar manipulations) we notice that

$$\begin{aligned}
\Box\phi &= \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\phi-\boldsymbol{\nabla}\!\cdot\!\boldsymbol{\nabla}\phi\\
&= -\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}-\boldsymbol{\nabla}\!\cdot\!\boldsymbol{\nabla}\phi \quad \text{by the Lorentz gauge condition}\\
&= \boldsymbol{\nabla}\!\cdot\!\boldsymbol{E} \quad \text{if we introduce} \quad \boldsymbol{E}\equiv-\boldsymbol{\nabla}\phi-\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A}
\end{aligned}$$

$$\begin{aligned}
\Box\boldsymbol{A} &= \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\boldsymbol{A}+\big\{\boldsymbol{\nabla}(\boldsymbol{\nabla}\!\cdot\!\boldsymbol{A})-\boldsymbol{\nabla}^2\boldsymbol{A}\big\}-\boldsymbol{\nabla}(\boldsymbol{\nabla}\!\cdot\!\boldsymbol{A})\\
&= \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\boldsymbol{A}+\big\{\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\big\}+\boldsymbol{\nabla}\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\phi\big) \quad \text{by the Lorentz gauge condition}\\
&= \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})-\tfrac{1}{c}\tfrac{\partial}{\partial t}\big\{-\boldsymbol{\nabla}\phi-\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A}\big\}\\
&= \boldsymbol{\nabla}\times\boldsymbol{B}-\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{E} \quad \text{if we introduce} \quad \boldsymbol{B}\equiv\boldsymbol{\nabla}\times\boldsymbol{A}
\end{aligned}$$

and so obtain

$$\begin{aligned}
\boldsymbol{\nabla}\!\cdot\!\boldsymbol{E}&=\rho_e\\
\boldsymbol{\nabla}\times\boldsymbol{B}-\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{E}&=\tfrac{1}{c}\boldsymbol{J}_e
\end{aligned}$$

together with the automatic identities

$$\begin{aligned}
\boldsymbol{\nabla}\!\cdot\!\boldsymbol{B}&=0\\
\boldsymbol{\nabla}\times\boldsymbol{E}+\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{B}&=\boldsymbol{0}
\end{aligned}$$

Maxwell's equations—the *physics* of the matter—lie, by this account, "two differentiations down" from the equations that Hertz considers fundamental. Hertz' (B7) masks the Lorentz covariance of the theory,[15] and its relation to formal problems (duality invariance) associated with the conjectured existence of magnetic charge/current remains obscure (but see below); sufficient, in his view, was the recommendation that in (B7) one has only three (instead of the usual four) uncoupled equations to worry about/solve.

In the simultaneous presence of electric and magnetic sources we expect to have

$$\tfrac{\partial}{\partial t}\rho_e+\boldsymbol{\nabla}\!\cdot\!\boldsymbol{J}_e=0 \qquad \text{and} \qquad \tfrac{\partial}{\partial t}\rho_m+\boldsymbol{\nabla}\!\cdot\!\boldsymbol{J}_m=0$$

Write $\rho=\rho_e+i\rho_m$ and $\boldsymbol{J}=\boldsymbol{J}_e+i\boldsymbol{J}_m$, introduce $\boldsymbol{P}=\boldsymbol{P}_e+i\boldsymbol{P}_m$ and observe that the electric/magnetic continuity equations would arise as the real/imaginary parts of

$$\tfrac{\partial}{\partial t}\rho+\boldsymbol{\nabla}\!\cdot\!\boldsymbol{J}=0$$

[15] Should one try to promote $\boldsymbol{H}$ and $\boldsymbol{P}$ to the status of 4-vectors? Find places for them (together with what other six things?) within a pair of antisymmetric second rank tensors?

if we had

$$\rho \equiv \tfrac{1}{c}\boldsymbol{\nabla}\!\cdot\!\boldsymbol{P} \qquad \text{and} \qquad \boldsymbol{J} \equiv -\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{P} \tag{B8.1}$$

Next complexify the Hertz potential

$$\boldsymbol{H} = \boldsymbol{H}_e + i\boldsymbol{H}_m$$

and postulate the persistence of (B7):

$$\Box\boldsymbol{H} = \tfrac{1}{c}\boldsymbol{P} \tag{B8.2}$$

We then have

$$\begin{aligned} \boldsymbol{\nabla}\!\cdot\!\Box\boldsymbol{H} &= \rho \\ \tfrac{1}{c}\tfrac{\partial}{\partial t}\cdot\Box\boldsymbol{H} &= \tfrac{1}{c}\boldsymbol{J} \end{aligned}$$

from which we want, by contrivance, to recover (2):

$$\begin{aligned} \boldsymbol{\nabla}\!\cdot\!\boldsymbol{V} &= \rho \\ -i\,\boldsymbol{\nabla}\times\boldsymbol{V} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{V} &= \tfrac{1}{c}\boldsymbol{J} \end{aligned}$$

Immediately $\boldsymbol{V} = \Box\boldsymbol{H} + \boldsymbol{\nabla}\times\boldsymbol{W}$. Assuming $\boldsymbol{W}$ to be assembled from the only material available

$$\boldsymbol{W} = a\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{H} + b\boldsymbol{\nabla}\times\boldsymbol{H} \quad : \quad a \text{ and } b \text{ are adjustable constants}$$

we have

$$\begin{aligned} -i\,\boldsymbol{\nabla}\times\Big\{\boldsymbol{\nabla}\times\big[a\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{H} + b\boldsymbol{\nabla}\times\boldsymbol{H}\big] + \Box\boldsymbol{H}\Big\} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\Big\{\boldsymbol{\nabla}\times\big[a\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{H} + b\boldsymbol{\nabla}\times\boldsymbol{H}\big]\Big\}& \\ - \tfrac{1}{c}\tfrac{\partial}{\partial t}\Box\boldsymbol{H} = \tfrac{1}{c}\boldsymbol{J}& \end{aligned}$$

and require the red terms to vanish. Elementary manipulations give

$$\begin{aligned} \text{red terms} &= -(b+ia)\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{H} - (a+i)\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\,\boldsymbol{\nabla}\times\boldsymbol{H} - i(b+1)\nabla^2\,\boldsymbol{\nabla}\times\boldsymbol{H} \\ &= 0 \quad \text{if and only if we set } a = -i \text{ and } b = -1 \end{aligned}$$

The implication is that if we set

$$\boldsymbol{V} = \boldsymbol{\nabla}\times\Big\{-\,i\,\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{H} - \boldsymbol{\nabla}\times\boldsymbol{H}\Big\} + \Box\boldsymbol{H} \tag{B9.1}$$

—more explicitly: if we set (compare (B5))

$$\left.\begin{aligned} \boldsymbol{E} &= \boldsymbol{\nabla}\times\Big\{+\,\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{H}_m - \boldsymbol{\nabla}\times\boldsymbol{H}_e\Big\} + \Box\boldsymbol{H}_e \\ \boldsymbol{B} &= \boldsymbol{\nabla}\times\Big\{-\,\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{H}_e - \boldsymbol{\nabla}\times\boldsymbol{H}_m\Big\} + \Box\boldsymbol{H}_m \end{aligned}\right\} \tag{B9.2}$$

—then we can recover the generalized Maxwell equations (2) as corollaries of (B8).[16]

We stand now in possession of a formalism that takes the electromagnetic field to be a complex 3-vector field, that is endowed with a theory of potentials, and that is manifestly duality-invariant.

Duality rotations can be used to kill the magnetic sources *at a point*; the proposition that they can be killed *globally* by such means is physically surprising, but was incorporated implicitly into the theory devised by Maxwell. If we set $\boldsymbol{H}_m = \boldsymbol{J}_m = \mathbf{0}$ and introduce the notations

$$\phi \equiv \boldsymbol{\nabla}\!\cdot\!\boldsymbol{H}_e \quad \text{and} \quad \boldsymbol{A} \equiv -\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{H}_e$$

then (B9.2) give back the statements (B2) that are standard to the textbooks:

$$\begin{aligned} \boldsymbol{E} &= -\boldsymbol{\nabla}\phi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} \\ \boldsymbol{B} &= \boldsymbol{\nabla}\times\boldsymbol{A} \end{aligned}$$

Equation (B8.2)—exquisitely simple as it stands—becomes even simpler in the absence of sources. In either case it yields straightforwardly to solution by Tom's (Lorentz covariance breaking) Fourier transform technique. Also available, however, is a covariant Fourier analytic technique that leads back again to the familiar theory of D-functions.[17]

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[16] The argument just concluded is (in my view) swifter and more natural than that given by Han & Biedenharn, yet very clumbsy compared to the (manifestly covariant) argument presented in some old notes cited earlier[11] and on pages 29–40 of "Electrodynamical applications of the exterior calculus" (1996).

[17] See CLASSICAL ELECTRODYNAMICS (1980/81), pages 382–389.

APPENDIX C (projected): Formulation in the language of Lagrangian field theory.

My intent here will be to explore the possibility of expressing the complex vector field theories described above (the $\boldsymbol{V}$-theory, the $\boldsymbol{H}$-theory) in the canonical language of Lagrangian field theory. Success would place one in position to discuss mechanical properties of the field in a rational way, to study the Noetherian implications of duality symmetry, etc. Han & Biedenharn (near the end of their §2) claim that a "completely satisfactory such theory does not yet exist," and cite a paper by F. Rohrlich (Phys. Rev. **150**, 1104) in which it is argued that the Cabibbo/Ferrari formalism *does not admit* of Lagrangian formulation. They put their paper forward as an improvement upon Cabibbo/Ferrari, but no Lagrangian appears within it. A satisfactory Lagrangian would have to supply not just (say) $\Box\boldsymbol{H} = \frac{1}{c}\boldsymbol{P}$ but also all the subsidiary equations/definitions that lead back to $\boldsymbol{E}$, $\boldsymbol{B}$, ρ and $\boldsymbol{J}$.

APPENDIX D (projected): Broken transversality.

My objective would be to clarify the origin/meaning of transversality by producing models (Lagrangians?) in which the planewave solutions are *not* transverse—models which serve to illustrate what (mass?) must be "turned off" to achieve the transversality characteristic of electromagnetic planewaves.