# 7

# RADIATIVE PROCESSES

**Introduction.** It was established in §4 of the preceding chapter that the leading term on the right side of (459/461)—the acceleration-independent term that falls off as $1/r^2$—admits straightforwardly of interpretation as the Coulomb field of the source, as seen from the field point, where the phrase "as seen from" alludes to

- a "retardation effect:" the field point senses not the "present location" of the source (a notion that relativity declares to be meaningless) but the location of the puncture point—the point at which the worldline of the source punctured the lightcone that extends backward from the field point (a notion that *does* make relativistic good sense);
- the fact that if the field point sees the source to be *moving* at the moment of puncture then it sees not the familiar "Coulomb field of a charge at rest" but a *Lorentz transform* of that field.

We turn now to discussion of the structure and physical ramifications of the remaining term on the right side of (459/461)—the acceleration-dependent term that falls off as $1/r^1$. This is physics for which elementary experience provides no sharp intuitive preparation, but which lies at the base of much that is most characteristic of classical electrodynamics. The details are occasionally a bit intricate, and their theoretical/phenomenological/technological consequences remarkably diverse . . . which is why I give the subject a chapter of its own.

**1. Radiation fields.** Dropping the Coulombic component from the field (459) of a moving charge we obtain the radiation field

$$F^{\mu\nu} = \frac{e}{4\pi c^2}\Big[\frac{1}{r}\big\{(b^\mu a^\nu - b^\nu a^\mu) + (w^\mu a^\nu - w^\nu a^\mu) - (aw)(w^\mu b^\nu - w^\nu b^\mu)\big\}\Big]_0$$

But (see again page 359)

$$w_0^\mu = \left[\frac{R^\mu}{r} - b^\mu\right]_0$$
$$b^\mu \equiv \tfrac{1}{c}u^\mu$$
$$r \equiv \tfrac{1}{c}R_\alpha u^\alpha = (Rb) = \gamma(1-\beta_\parallel)R$$

so after a short calculation we find

$$\begin{aligned} F^{\mu\nu} &= \frac{e}{4\pi}\left[\frac{1}{(Ru)^2}\Big\{(R^\mu a^\nu - R^\nu a^\mu) - \frac{(Ra)}{(Ru)}(R^\mu u^\nu - R^\nu u^\mu)\Big\}\right]_0 \\ &= \frac{e}{4\pi}\left[\frac{1}{(Ru)^2}(R^\mu a_\perp^\nu - R^\nu a_\perp^\mu)\right]_0 \end{aligned} \tag{464.1}$$

where
$$a_\perp^\mu \equiv a^\mu - \frac{(Ra)}{(Ru)}u^\mu \tag{464.2}$$

is (in the Lorentzian sense) $\perp$ to $R^\mu$: $(Ra_\perp) = 0$. Note the manifest covariance of this rather neat result.

3-vector notation—though contrary to the spirit of the principle of manifest covariance, and though always uglier—is sometimes more useful. Looking back again, therefore, to (461), we observe that[279]

$$\hat{\boldsymbol{R}}\times\big((\hat{\boldsymbol{R}}-\boldsymbol{\beta})\times\boldsymbol{a}\big) = -(1-\hat{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{\beta})\underbrace{\Big\{\boldsymbol{a} - \frac{\hat{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{a}}{1-\hat{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{\beta}}(\hat{\boldsymbol{R}}-\boldsymbol{\beta})\Big)\Big\}}_{\perp\,\boldsymbol{R}}$$

and that on this basis the radiative part of (461) can be written[280]

$$\boldsymbol{E} = -\frac{e}{4\pi c^2}\left[\frac{1}{R(1-\hat{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{\beta})^2}\Big\{\boldsymbol{a} - \frac{\hat{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{a}}{1-\hat{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{\beta}}(\hat{\boldsymbol{R}}-\boldsymbol{\beta})\Big)\Big\}\right]_0 \tag{465.1}$$

$$\boldsymbol{B} = \Big[\hat{\boldsymbol{R}}\times\boldsymbol{E}\Big]_0 \tag{465.2}$$

Equations (464) & (465) provide notationally distinct but physically equivalent descriptions of the radiation field generated by an accelerated point charge.

It is instantaneously possible to have $\boldsymbol{v} = \boldsymbol{0}$ but $\boldsymbol{a} \neq \boldsymbol{0}$; *i.e.*, for a point momentarily at rest to be accelerating. In such a circumstance (465.1) becomes

$$\begin{aligned} \boldsymbol{E} &= -\frac{e}{4\pi c^2}\left[\frac{1}{R}\,\boldsymbol{a}_\perp\right]_0 \\ &\qquad \boldsymbol{a}_\perp = \boldsymbol{a} - (\hat{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{a})\hat{\boldsymbol{R}} = -\hat{\boldsymbol{R}}\times(\hat{\boldsymbol{R}}\times\boldsymbol{a}) \\ &= \frac{e}{4\pi c^2}\Big[\hat{\boldsymbol{R}}\times(\hat{\boldsymbol{R}}\times\boldsymbol{a})\Big]_0 \end{aligned} \tag{466}$$

with consequences which are illustrated in Figures 123 & 124.

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[279] PROBLEM 75.

[280] We make use here of $r \equiv \gamma(1-\hat{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{\beta})R$: see again page 359.

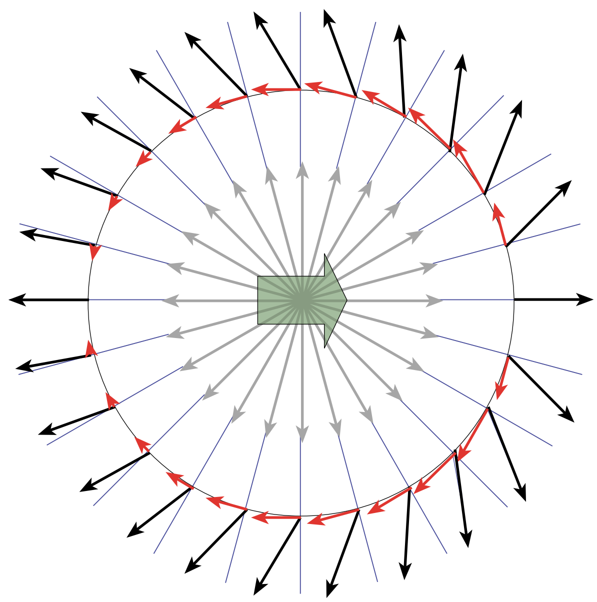

FIGURE 123: *Electric field at points that look back to the same puncture point, where they see the charge to be momentarily at rest but accelerating (in the direction indicated by the green arrow). The red* $\boldsymbol{E}$*-vectors arise from the radiative term (466). Addition of the Coulombic component produces the black* $\boldsymbol{E}$*-vectors. The grey arrows are unit vectors* $\hat{\boldsymbol{R}}$. *The figure is deceptive in one respect: every* $\boldsymbol{E}$*-vector on the left should, according to (466), have the same length as its counterpart on the right.*

The intricate details of (461) are well-adapted to computer-graphic analysis. In this connection every student of electrodynamics should study the classic little paper by R. Y. Tsien,[281] from which I have taken Figures 125–128. Tsien assumes the source orbit to lie in every case in a plane, and it is in that plane that he displays the "electric lines of force." From his figures one can read off the *direction* of the retarded $\boldsymbol{E}$-field, but information pertaining directly to the *magnitude* of the $\boldsymbol{E}$-field (and all information pertaining to the $\boldsymbol{B}$-field) has been discarded. Nor does Tsien attempt to distinguish the radiative from the Coulombic component of $\boldsymbol{E}$.

[281] "Pictures of Dynamic Electric Fields," AJP **40**, 46 (1972). Computers and software have come a very long way in thirty years: the time is ripe for someone to write (say) a *Mathematica* program that would permit students to do interactively/experimentally what Tsien labored so hard to do with relatively primitive resources. Tsien, by the way, is today a well-known biophysicist, who in 1972 was still an undergraduate at Harvard, a student of E. M. Purcell, whose influential *Electricity & Magnetism* (Berkeley Physics Course, Volume II) was then recent.

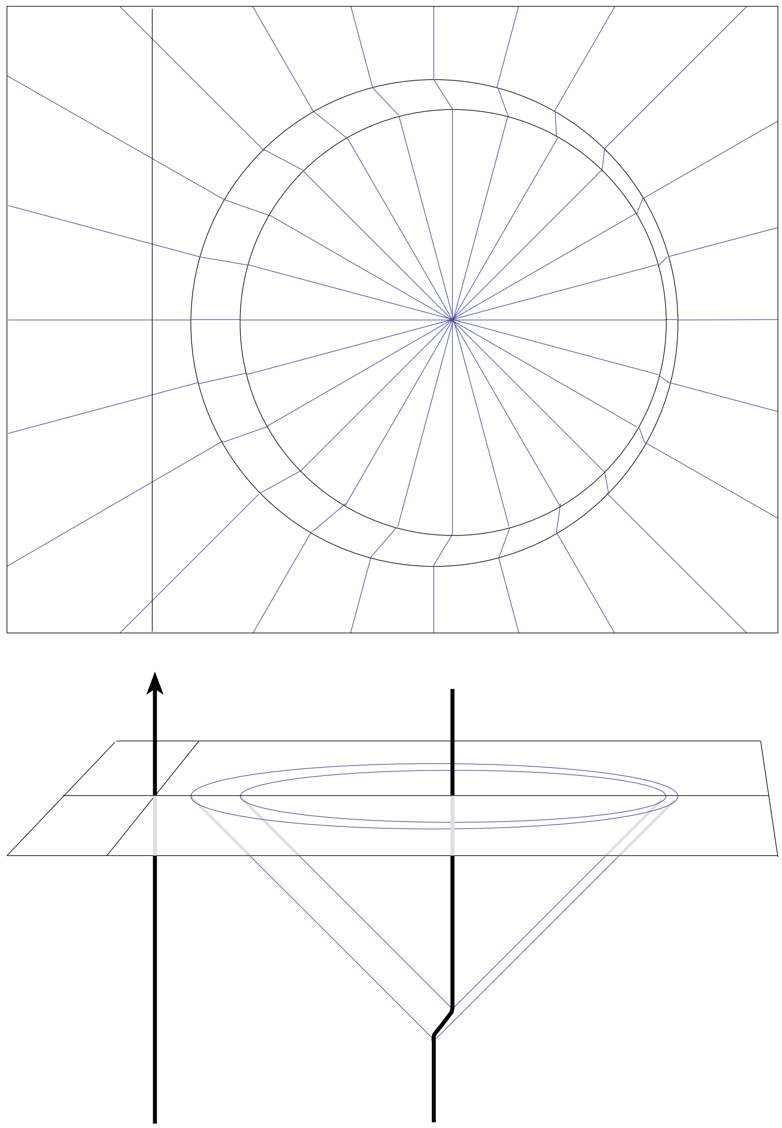

FIGURE 124: *Shown below: the worldline of a charged particle—initially at rest—that begins abruptly to accelerate to the right, then promptly decelerates, returning again to rest. Shown above is the resulting* $\boldsymbol{E}$*-field. The remote radial section is concentric about the original position, the inner radial section is concentric about the altered position. The acceleration-dependent interpolating field has the form shown in Figure 123. Indeed: it was from this figure—not (466)—that I took the details of Figure 123. The next figure speaks more precisely to the same physics.*

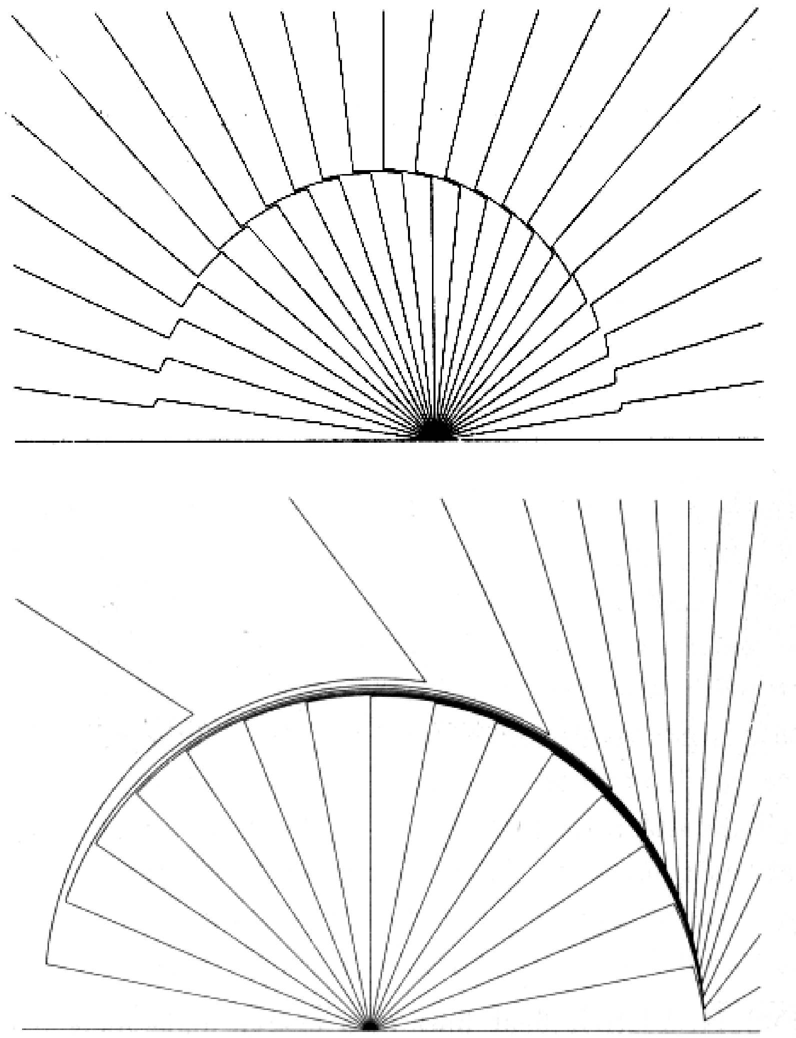

Figure 125: *Snapshots of electric field lines derived from the* $\boldsymbol{E}$*-field generated by a charge which abruptly decelerates while moving in the* → *direction. The initial velocity was* $\beta = 0.20$ *in the upper figure,* $\beta = 0.95$ *in the lower figure. I am indebted to Fred Lifton for the digitization of Tsien's figures, and regret that the available technology so seriously degraded the quality of Tsien's wonderfully sharp images. See the originals in Tsien's paper*[264]*... or better: run Tsien's algorithm on* Mathematica *to produce* <u>*animated*</u> *versions of the figures.*

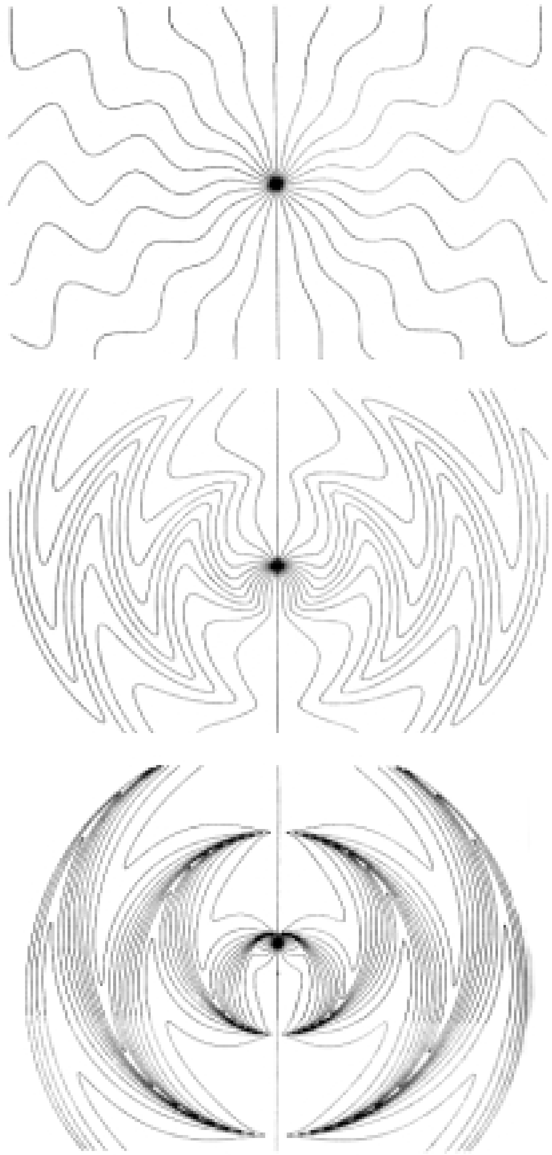

FIGURE 126: *Snapshots of the electric field lines generated by a charge undergoing simple harmonic motion in the* $\updownarrow$ *direction. In the upper figure* $\beta_{\rm max} = 0.10$*, in the middle figure* $\beta_{\rm max} = 0.50$*, in the lower figure* $\beta_{\rm max} = 0.90$.

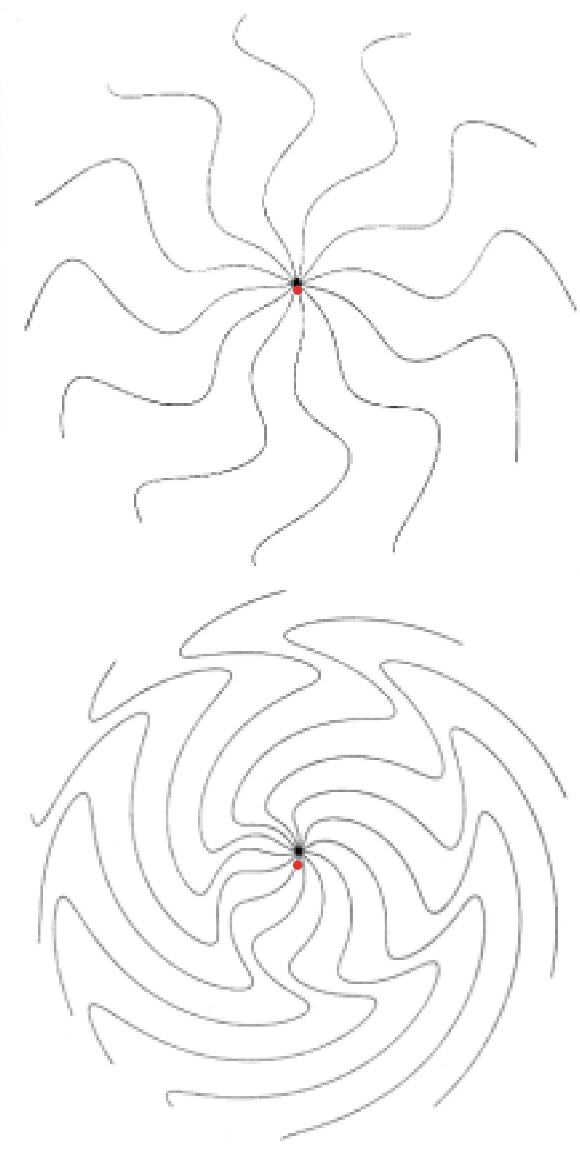

FIGURE 127: *Snapshots (not to the same scale) of the electric field lines generated by a charge undergoing uniform circular motion* ↻ *about the point marked* •. *In the upper figure* $\beta = 0.20$, *in the lower figure* $\beta = 0.50$. *In the upper figure the field is—pretty evidently—dominated by the Coulombic component of* (459/461).

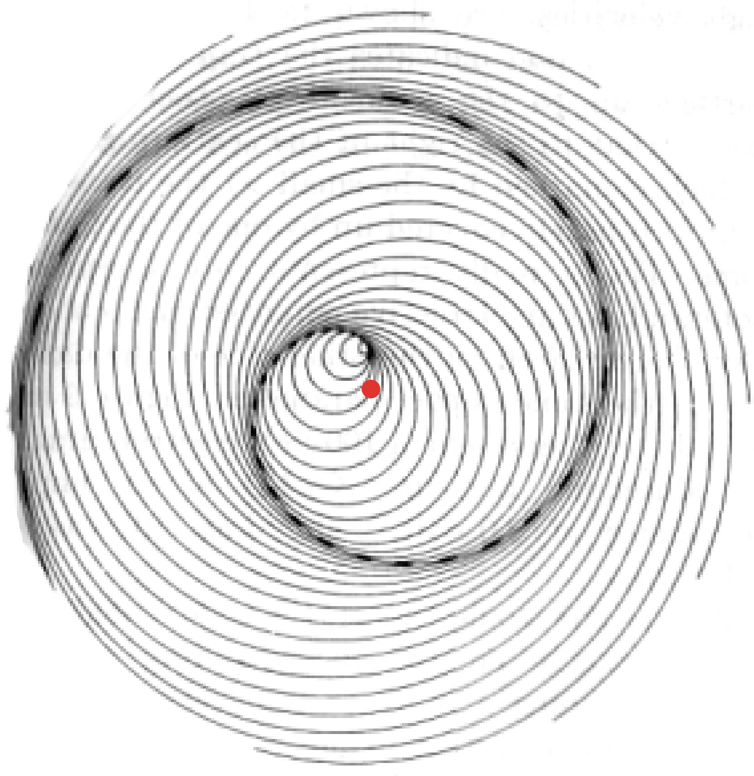

FIGURE 128: *Enlargement of the same physics as that illustrated in Figure 127, except that now* $\beta = 0.95$. *The figure can be animated by placing it on a phonograph turntable: since phonographs turn* ↻ *the spiral will appear to expand. Beyond a certain radius the field lines will appear to move faster than the speed of light. That violates no physical principle, since the field lines themselves are diagramatic fictions: marked features of the field (for example: the kinks) are seen* not *to move faster than light. At such high speeds the field is dominated by the radiative part of* (459/461). *This is "synchrotron radiation," and (as Tsien remarks) the kinks account for the rich harmonic content of relativistic synchrotron radiation.*

**2. Energetics of fields produced by a single source.** To discuss this topic all we have in principle to do is to introduce (459/461)—which describe the *field generated by a point charge in arbitrary motion*—into (309, page 215)—which describes the *stress/energy/momentum associated with an arbitrarily prescribed electromagnetic field*. The program is clear-cut, but the details can easily become overwhelming . . . and we are forced to look only at the physically most characteristic/revealing features of the physically most important special cases.

The experience thus gained will, however, make it relatively easy to think *qualitatively* about more realistic/complex problems.

We will need to know (see again page 216) that

$$\begin{aligned}
\mathcal{E} &= \tfrac{1}{2}(E^2 + B^2) && \text{describes } \textbf{energy density}\\
\boldsymbol{S} &= c(\boldsymbol{E}\times\boldsymbol{B}) && \text{describes } \textbf{energy flux}\\
\boldsymbol{\mathcal{P}} &= \tfrac{1}{c}(\boldsymbol{E}\times\boldsymbol{B}) && \text{describes } \textbf{momentum density}
\end{aligned}$$

but will have no direct need of the other nine components $\mathbb{T}$ of the stress-energy tensor $S^{\mu\nu}$. Mechanical properties of the fields generated by *accelerated sources* lie at the focal point of our interest, but to place that physics in context we look first to a couple of simpler special cases:

FIELD ENERGY/MOMENTUM OF A CHARGE AT REST In the rest frame of an unaccelerated charge $e$ we have

$$\boldsymbol{E} = \frac{e}{4\pi}\frac{1}{R^2}\hat{\boldsymbol{R}} \qquad\text{and}\qquad \boldsymbol{B} = \boldsymbol{0}$$

giving

$$\mathcal{E} = \tfrac{1}{2}\Big(\frac{e}{4\pi}\Big)^2\frac{1}{R^4} \qquad\text{and}\qquad \boldsymbol{S} = \boldsymbol{\mathcal{P}} = \boldsymbol{0}$$

If (as in PROBLEM 10) we center a (mental) sphere of radius $a$ on the charge we find the field energy exterior to the sphere to be given by

$$W(a) = \int_a^\infty \mathcal{E}(R)4\pi R^2\,dR = \frac{e^2}{8\pi a} \tag{467}$$

. . . which—“self-energy problem”—becomes infinite as $a \downarrow 0$, and which when we set

$$= mc^2$$

gives rise to the “classical radius” $a = e^2/8\pi mc^2$ of the massive point charge $e$.

FIELD ENERGY/MOMENTUM OF A CHARGE IN UNIFORM MOTION Drawing now upon (463) we have

$$\boldsymbol{E} = \frac{e}{4\pi\gamma^2}\frac{1}{\big(1-\beta^2\sin^2\alpha\big)^{\frac{3}{2}}}\frac{1}{R^2}\hat{\boldsymbol{R}} \qquad\text{and}\qquad \boldsymbol{B} = \boldsymbol{\beta}\times\boldsymbol{E}$$

But $B^2 = (\boldsymbol{\beta}\times\boldsymbol{E})\cdot(\boldsymbol{\beta}\times\boldsymbol{E}) = (\boldsymbol{\beta}\cdot\boldsymbol{\beta})(\boldsymbol{E}\cdot\boldsymbol{E}) - (\boldsymbol{\beta}\cdot\boldsymbol{E})^2 = \beta^2E^2\sin^2\alpha$, so

$$\mathcal{E} = \tfrac{1}{2}\Big(\frac{e}{4\pi\gamma^2}\Big)^2\frac{1+\beta^2\sin^2\alpha}{\big(1-\beta^2\sin^2\alpha\big)^3}\frac{1}{R^4}$$

The momentum density $\boldsymbol{\mathcal{P}} = \frac{1}{c}(\boldsymbol{E}\times\boldsymbol{B})$ is oriented as shown in the first of the following figures. From

$$\mathcal{P}^2 = \tfrac{1}{c^2}\big\{(\boldsymbol{E}\cdot\boldsymbol{E})(\boldsymbol{B}\cdot\boldsymbol{B}) - (\boldsymbol{E}\cdot\boldsymbol{B})^2\big\} = \tfrac{1}{c^2}E^2B^2 = \tfrac{1}{c^2}\beta^2E^4\sin^2\alpha$$

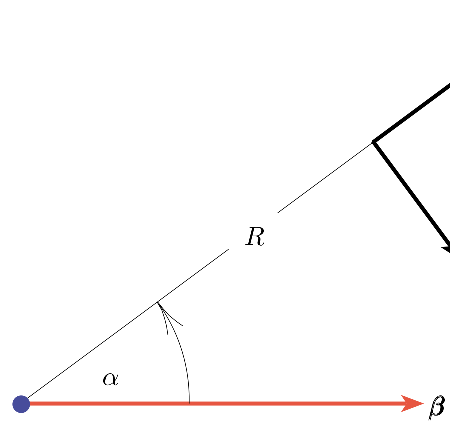


FIGURE 127: *The solinoidal* $\boldsymbol{B}$ *field is up out of page at the point shown, so* $\boldsymbol{\mathcal{P}} = \frac{1}{c}(\boldsymbol{E}\times\boldsymbol{B})$ *lies again on the page. Only* $\mathcal{P}_{\parallel}$*—the component parallel to* $\boldsymbol{\beta}$*—survives integration over all of space..*

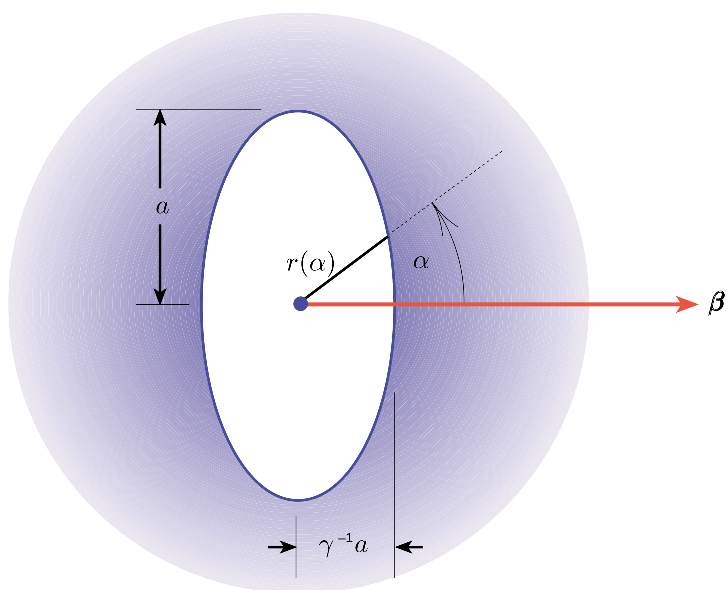


FIGURE 130: *Lorentz contracted geometry of what in the rest frame of the charge was the familiar "sphere of radius a," exterior to which we compute the total energy and total momentum. The figure is rotationally symmetric about the* $\boldsymbol{\beta}$*-axis.*

we find that the magnitude of $\mathcal{P}$ is given by

$$\mathcal{P} = \tfrac{1}{c}\beta E^2 \sin\alpha = \tfrac{1}{c}\beta\Big(\frac{e}{4\pi\gamma^2}\Big)^2 \frac{1}{\big(1-\beta^2\sin^2\alpha\big)^3}\frac{1}{R^4}$$

Turning now to the evaluation of the integrated *field energy and field momentum exterior to the spherical region considered previously*—a region which appears now to be Lorentz contracted (see the second of the figures on the preceding page)—we have

$$W = \int_0^\pi \int_{r(\alpha)}^\infty \mathcal{E}\cdot 2\pi R^2 \sin\alpha\, dRd\alpha \tag{468.1}$$

and $\boldsymbol{P} = P\hat{\boldsymbol{\beta}}$ with

$$P = \int_0^\pi \int_{r(\alpha)}^\infty \mathcal{P}\sin\alpha\cdot 2\pi R^2 \sin\alpha\, dRd\alpha \tag{468.2}$$

where $r(\alpha)$, as defined by the figure, is given[282] by

$$r(\alpha) = \frac{a}{\gamma}\frac{1}{\sqrt{1-\beta^2\sin^2\alpha}}$$

The $R$-integrals are trivial: we are left with

$$\begin{aligned} W &= \quad \pi\Big(\frac{e}{4\pi\gamma^2}\Big)^2\frac{\gamma}{a}\int_0^\pi \frac{1}{(1-\beta^2\sin^2\alpha)^{\frac{5}{2}}}\big\{\sin\alpha+\beta^2\sin^3\alpha\big\}\,d\alpha \\ P &= \frac{\beta}{c}2\pi\Big(\frac{e}{4\pi\gamma^2}\Big)^2\frac{\gamma}{a}\int_0^\pi \frac{1}{(1-\beta^2\sin^2\alpha)^{\frac{5}{2}}}\sin^3\alpha\,d\alpha \end{aligned}$$

Entrusting the surviving integrals to *Mathematica*, we are led to results that can be written[283]

$$W = \big(1-\tfrac{1}{4\gamma^2}\big)\cdot\gamma M c^2 \tag{469.1}$$

$$\boldsymbol{P} = \gamma M\boldsymbol{v} \tag{469.2}$$

with

$$M \equiv \frac{4}{3}\frac{e^2}{8\pi a c^2} = \tfrac{4}{3}m \tag{470}$$

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[282] The argument runs as follows: we have

$$\frac{x^2}{(a/\gamma)^2}+\frac{y^2}{a^2} = 1 \quad\text{whence}\quad \gamma^2(r\cos\alpha)^2+(r\sin\alpha)^2 = a^2$$

Divide by $\gamma^2$ and obtain

$$r^2(1-\sin^2\alpha)+(1-\beta^2)r^2\sin^2\alpha = (a/\gamma)^2$$

Simplify, solve for $r$.

[283] PROBLEM 76.

The curious velocity-dependent factor

$$\left(1-\tfrac{1}{4\gamma^2}\right)=\begin{cases}\tfrac{3}{4} & : \quad \beta=0\\[1em] 1 & : \quad \beta=1\end{cases}$$

Were that factor absent (which is to say: in the approximation that $\left(1-\frac{1}{4\gamma^2}\right)\sim 1$) we would have

$$P^0 \equiv \tfrac{1}{c}W=\tfrac{4}{3}m\cdot\gamma c \qquad \text{and} \qquad \boldsymbol{P}=\tfrac{4}{3}m\cdot\gamma\boldsymbol{v}$$

which (see again (276) page 193) we recognize to be the relativistic relationship between the energy and momentum of a free particle with mass $\frac{4}{3}m$. This fact inspired an ill-fated attempt by M. Abraham, H. Poincaré, H. A. Lorentz and others ($\sim$ 1900, immediately *prior* to the invention of relativity) to develop an "**electromagnetic theory of mass**,"[284] distant echos of which can be detected in modern theories of elementary particles. We note in passing that

- (469.1) gives back (467) in the limit $v\downarrow 0$: the $\frac{3}{4}$ neatly cancels the curious $\frac{4}{3}$, which would not happen if (on some pretext) we yielded to the temptation to drop the otherwise unattractive $\left(1-\frac{1}{4\gamma^2}\right)$-factor.
- Equations (469) and (467) are not boost-equivalent:

$$\begin{pmatrix}W/c\\ \boldsymbol{P}\end{pmatrix}\neq\mathbb{A}(\boldsymbol{v})\begin{pmatrix}mc\equiv e^2/8\pi ac\\ \boldsymbol{0}\end{pmatrix}$$

  The reason is that $P^0\equiv W/c$ and $\boldsymbol{P}$ arise by integration from a *subset* $S^{\mu 0}$ of the sixteen components of the $S^{\mu\nu}$ tensor, and the four elements of the subset are not transformationally disjoint from the other twelve components.
- It becomes rather natural to ask: Could a more satisfactory result be achieved if we assumed that Maxwell's equations must be modified in the close proximity of charges? That relativity breaks down at small distances?

**3. Energy radiated by an accelerated charge momentarily at rest.** It is in the interest mainly of analytical simplicity that we now assume $\boldsymbol{v}=\boldsymbol{0}$, a condition that (when $\boldsymbol{a}\neq\boldsymbol{0}$) can hold only instantaneously. But the calculation is less artificial than might at first appear: it leads to results that are *nearly exact in the non-relativistic regime* $v\ll c$.

[284] For a good general review—with bibliography—see R. L. Dendy, "A history of the Abraham–Lorentz electromagnetic theory of mass" (Reed College, 1964). See also Chapter 2 in F. Rohrlich, *Classical Charged Particles* (1965) and R. P. Feynman's *Lectures on Physics* (1964), Volume II, Chapter 28.

Borrowing now from (461) we have (set $\boldsymbol{\beta} = \mathbf{0}$)

$$\begin{aligned}\boldsymbol{E} &= \frac{e}{4\pi}\Big[\frac{1}{R^2}\hat{\boldsymbol{R}}\Big]_0 + \frac{e}{4\pi c^2}\Big[\frac{1}{R}\hat{\boldsymbol{R}}\times(\hat{\boldsymbol{R}}\times\boldsymbol{a})\Big]_0 \equiv \boldsymbol{E}^{\mathsf{C}} + \boldsymbol{E}^{\mathsf{R}}\\ \boldsymbol{B} &= \Big[\hat{\boldsymbol{R}}\times\boldsymbol{E}\Big]_0 \equiv \boldsymbol{B}^{\mathsf{C}} + \boldsymbol{B}^{\mathsf{R}}\end{aligned}$$

where the superscript $^{\mathsf{C}}$ identifies the "Coulombic component," and $^{\mathsf{R}}$ the "radiative component." We want to study energy loss (radiation from the vicinity of the charge) so we look not to $\mathcal{E}$ or $\boldsymbol{\mathcal{P}}$ but the energy *flux* vector

$$\begin{aligned}\boldsymbol{S} &= c(\boldsymbol{E}\times\boldsymbol{B})\\ &= \boldsymbol{S}^{\mathsf{CC}} + \boldsymbol{S}^{\mathsf{CR}} + \boldsymbol{S}^{\mathsf{RC}} + \boldsymbol{S}^{\mathsf{RR}} \quad\text{where}\quad \begin{cases}\boldsymbol{S}^{\mathsf{CC}} \equiv c(\boldsymbol{E}^{\mathsf{C}}\times\boldsymbol{B}^{\mathsf{C}}) \sim 1/R^4\\ \boldsymbol{S}^{\mathsf{CR}} \equiv c(\boldsymbol{E}^{\mathsf{C}}\times\boldsymbol{B}^{\mathsf{R}}) \sim 1/R^3\\ \boldsymbol{S}^{\mathsf{RC}} \equiv c(\boldsymbol{E}^{\mathsf{R}}\times\boldsymbol{B}^{\mathsf{C}}) \sim 1/R^3\\ \boldsymbol{S}^{\mathsf{RR}} \equiv c(\boldsymbol{E}^{\mathsf{R}}\times\boldsymbol{B}^{\mathsf{R}}) \sim 1/R^2\end{cases}\end{aligned}$$

$\boldsymbol{S}^{\mathsf{CC}}$, $\boldsymbol{S}^{\mathsf{CR}}$ and $\boldsymbol{S}^{\mathsf{RC}}$ may be of importance—even dominant importance—in the "near zone," but they *fall off faster than geometrically*: only $\boldsymbol{S}^{\mathsf{RR}}$ can pertain to the "transport of energy to infinity"—the process of present concern. We look therefore to

$$\boldsymbol{S}^{\mathsf{RR}} = c(\boldsymbol{E}^{\mathsf{R}}\times\boldsymbol{B}^{\mathsf{R}}) \tag{471}$$

with

$$\begin{aligned}\boldsymbol{B}^{\mathsf{R}} &= \Big[\hat{\boldsymbol{R}}\times\boldsymbol{E}^{\mathsf{R}}\Big]_0\\ \boldsymbol{E}^{\mathsf{R}} &= \frac{e}{4\pi c^2}\Big[\frac{1}{R}\hat{\boldsymbol{R}}\times(\hat{\boldsymbol{R}}\times\boldsymbol{a})\Big]_0\end{aligned}$$

Clearly $\hat{\boldsymbol{R}}\cdot\boldsymbol{E} = 0$ so $\boldsymbol{E}\times(\hat{\boldsymbol{R}}\times\boldsymbol{E}) = (\boldsymbol{E}\cdot\boldsymbol{E})\hat{\boldsymbol{R}} - (\hat{\boldsymbol{R}}\cdot\boldsymbol{E})\boldsymbol{E}$ gives[285]

$$\begin{aligned}\boldsymbol{S} &= S\hat{\boldsymbol{R}}\\ S &= c(\boldsymbol{E}\cdot\boldsymbol{E}) = \frac{1}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)\Big(\frac{a}{R}\Big)^2\sin^2\vartheta\end{aligned} \tag{472}$$

where $\vartheta \equiv$ (angle between $\hat{\boldsymbol{R}}$ and $\boldsymbol{a}$). The temporal rate at which field energy is seen ultimately to stream through the remote surface differential $\boldsymbol{d\sigma}$ is given by $dP = \boldsymbol{S}\cdot\boldsymbol{d\sigma}$. But $d\Omega \equiv R^{-2}\hat{\boldsymbol{R}}\cdot\boldsymbol{d\sigma}$ is just the *solid angle* subtended (at $e$) by $\boldsymbol{d\sigma}$. We conclude that the power radiated into the solid angle $d\Omega$ is given by

$$dP = \underbrace{\Big\{\frac{1}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)a^2\sin^2\vartheta\Big\}}_{\text{so-called ``sine squared distribution''}}d\Omega \tag{473}$$

The "sine squared distribution" will be shown to be characteristic of *dipole radiation*, and has the form illustrated in the first of the following figures.

[285] PROBLEM 77. Here and henceforth I drop the superscripts $^{\mathsf{R}}$.

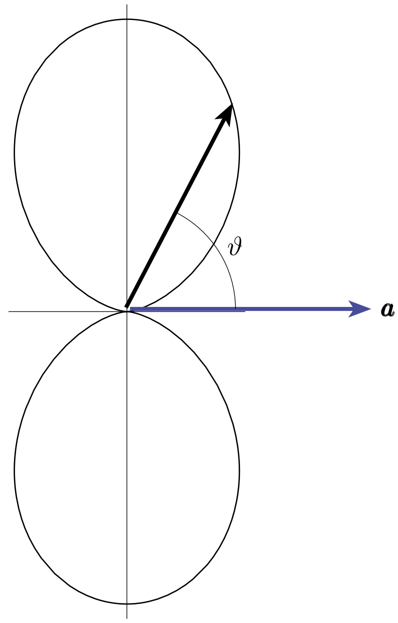



FIGURE 131: *The "sine squared distribution" arises when* $\boldsymbol{v} \sim \mathbf{0}$ *but* $\boldsymbol{a} \neq \mathbf{0}$. *The distribution is axially symmetric about the* $\boldsymbol{a}$*-vector, and describes the relative amounts of energy dispatched in various* $\vartheta$*-directions. The radiation is predominantly* $\perp$ *to* $\boldsymbol{a}$.

Integrating over the "sphere at infinity" we find the instantaneous *total radiated power* to be given by[286]

$$P = \frac{1}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)a^2 \cdot 2\pi \int_0^\pi \sin^2\vartheta\, d\vartheta = \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{a^2}{c^3} \tag{474}$$

This is the famous **Larmor formula**, first derived by Joseph Larmor in 1897. The following figure schematizes the physical assumptions which underlie (474). We note that while energy may also be dispatched into the solid angle $d\Omega$ by the $\boldsymbol{S}^{\mathsf{CC}}$, $\boldsymbol{S}^{\mathsf{CR}}$ and $\boldsymbol{S}^{\mathsf{RC}}$ it is attenuated too rapidly to contribute to the net "energy flux across the sphere at infinity."

From the $c^{-3}$-dependence of $P_{\text{Larmor}}$ we conclude that *it is not easy to radiate*. Finally, I would emphasize once again that we can expect Larmor's formula to pertain in good approximation *whatever* the non-relativistic (!) motion of the source.

**4. Energy radiated by a charge in arbitrary motion.** When one turns to the general case the basic strategy (study $\boldsymbol{S}^{\mathsf{RR}}$ in the far zone) is unchanged, but the details[287] become a good deal more complicated. In the interests of brevity

[286] PROBLEM 78.
[287] See CLASSICAL RADIATION (1974), pages 558–571.

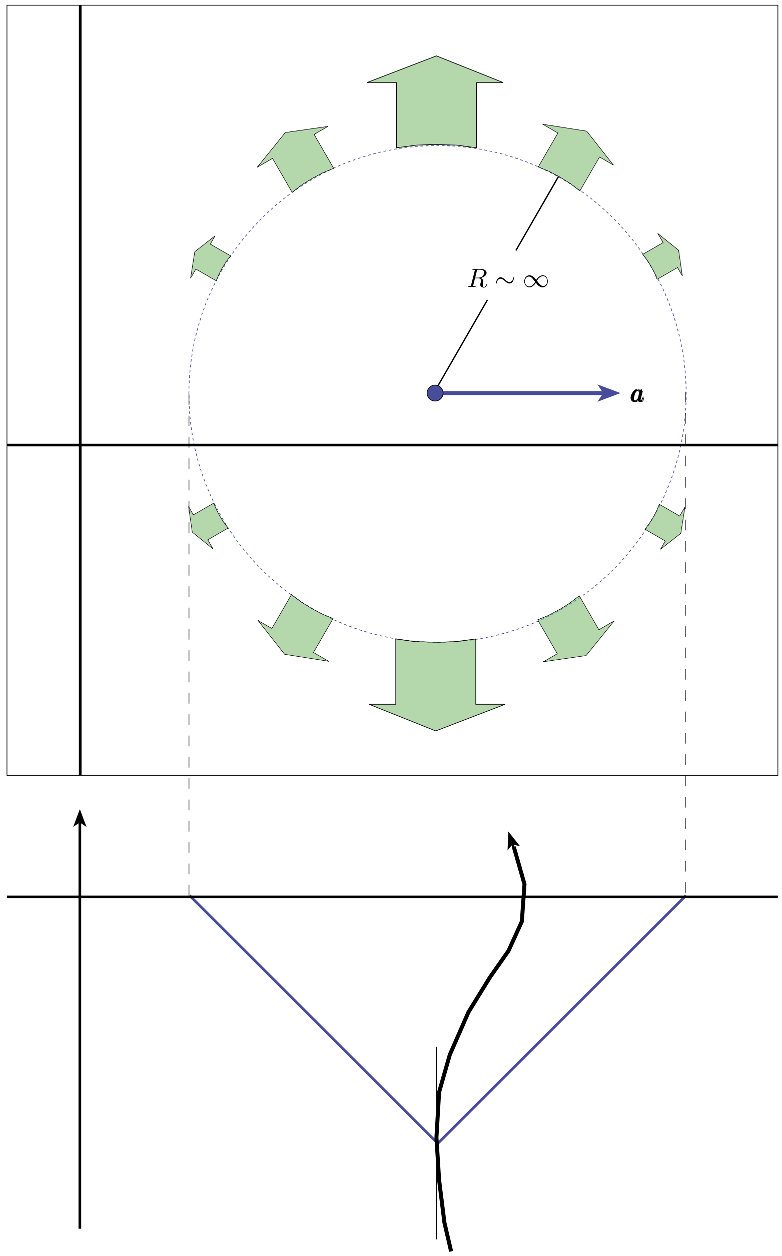



FIGURE 132: *Above: representation of the sine-squared radiation pattern produced by a charge seen (below) at the moment of puncture to have* $\boldsymbol{v} \sim \boldsymbol{0}$ *but* $\boldsymbol{a} \neq \boldsymbol{0}$.

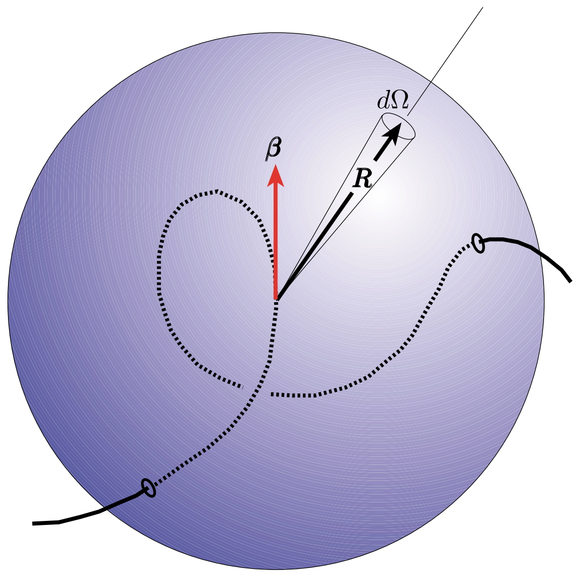



FIGURE 133: *A charged particle $e$ pursues an arbitrary path in physical 3-space. We are concerned with the energy radiated into the solid angle $d\Omega$ identified by the direction vector $\boldsymbol{R}$. The vector $\boldsymbol{\beta}$ refers to the particle's velocity at the radiative moment, and—adhering to the convention introduced in Figures 127 & 128—we write*

$$\alpha \equiv \textit{angle between } \boldsymbol{R} \textit{ and } \boldsymbol{\beta}$$

*No attempt has been made here to represent the instantaneous acceleration vector $\boldsymbol{a}$.*

and clarity I must therefore be content to report and discuss here only the *results* of the detailed argument. It turns out that (see the preceding figure) an accelerated charge $e$ radiates energy into the solid angle $d\Omega$ (direction $\hat{\boldsymbol{R}}$) at—in $\tau$-time—a temporal rate given by

$$dP = \frac{1}{(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})^5}\cdot\frac{1}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)\big|\hat{\boldsymbol{R}}\times\big((\hat{\boldsymbol{R}}-\boldsymbol{\beta})\times\boldsymbol{a}\big)\big|^2 d\Omega \tag{475}$$

. . . which gives back (473) when $\boldsymbol{\beta}=\boldsymbol{0}$.

The "Dopplerean prefactor"

$$D(\alpha) \equiv \frac{1}{(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})^5} = \frac{1}{(1-\beta\cos\alpha)^5}$$

is plotted in Figure 134. Evidently

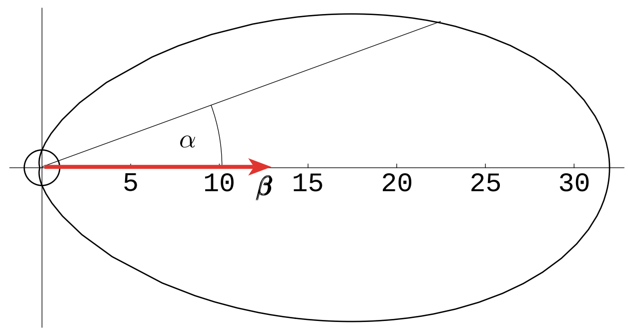



FIGURE 134: *Graph of the Dopplerean factor* $D(\alpha)$*, the cross-section of a figure of revolution about the* $\boldsymbol{\beta}$*-axis. Also shown, for purposes of comparison, is the unit circle. The figure refers to the specific case* $\beta = 0.5$*.*

$$D(\alpha)_{\max} = D(0) = \frac{1}{(1-\beta)^5} \longrightarrow \infty \quad \text{as} \quad \beta \uparrow 1$$

$$D(\alpha)_{\min} = D(\pi) = \frac{1}{(1+\beta)^5} \longrightarrow \tfrac{1}{32} \quad \text{as} \quad \beta \uparrow 1$$

and

$$D(\tfrac{\pi}{2}) = 1 \quad : \quad \text{all } \beta$$

We conclude that the ($\boldsymbol{a}$-independent) Doppler factor serves to favor the forward hemisphere:

*Fast charges tend to throw their radiation forward.*

Looking back again to (475), we see that the $D(\alpha)$-factor competes with (or modulates) a factor of the form $\big|\hat{\boldsymbol{R}}\times\big((\hat{\boldsymbol{R}}-\boldsymbol{\beta})\times\boldsymbol{a}\big)\big|^2$. A simple argument shows that the latter factor vanishes if and only if $(\hat{\boldsymbol{R}}-\boldsymbol{\beta}) \parallel \boldsymbol{a}$. This entails that $\hat{\boldsymbol{R}}$ lie in the $(\boldsymbol{\beta},\boldsymbol{a})$-plane, and that within that plane it have one or the other of the values $\hat{\boldsymbol{R}}_1$ and $\hat{\boldsymbol{R}}_2$ described in Figure 135. $\hat{\boldsymbol{R}}_1$ and $\hat{\boldsymbol{R}}_2$ describe the so-called "nodal directions" which are instantaneously radiation-free. Reading from the figure, we see that

- in the **non-relativistic limit** $\hat{\boldsymbol{R}}_1$ and $\hat{\boldsymbol{R}}_2$ lie fore and aft of the $\boldsymbol{a}$-vector, independently (in lowest order) of the magnitude/direction of $\boldsymbol{\beta}$: this is a property of the "sine squared distribution" evident already in Figure 131.
- in the **ultra-relativistic limit** $\hat{\boldsymbol{R}}_1 \to \boldsymbol{\beta}$ while $\hat{\boldsymbol{R}}_2$ gives rise to a "dangling note," the location of which depends conjointly upon $\boldsymbol{\beta}$ and $\boldsymbol{a}$.

From preceding remarks we conclude that the distribution function that describes the rate at which a charge "sprays energy on the sphere at $\infty$" is (in the general case) quite complicated. Integration over the sphere can, however,

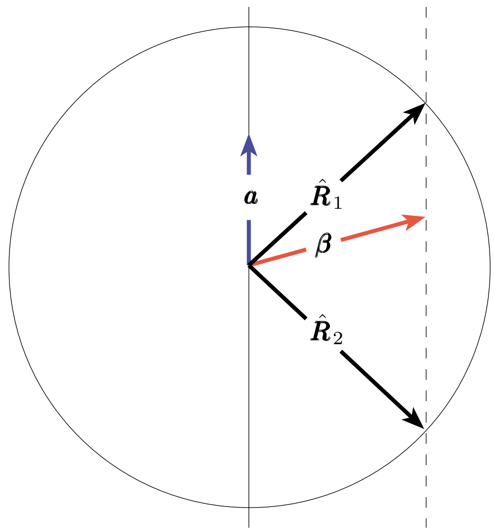



FIGURE 135: *Geometrical construction of the vectors* $\hat{\boldsymbol{R}}_1$ *and* $\hat{\boldsymbol{R}}_2$ *that locate the nodes of the radiative distribution in the general case.*

be carried out in closed form ... and gives rise (compare (474)) to the following description of the **total power instantaneously radiated by an arbitrarily moving source**:

$$\begin{aligned} P &= -\frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{(aa)}{c^3} \qquad (476)\\ &= \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\cdot\Big\{\gamma^4(\boldsymbol{a}\cdot\boldsymbol{a})+\gamma^6(\boldsymbol{a}\cdot\boldsymbol{\beta})^2\Big\}\\ &= \text{———————}\cdot\ \gamma^6\Big\{(\boldsymbol{a}\cdot\boldsymbol{a})-(\boldsymbol{a}\times\boldsymbol{\beta})\cdot(\boldsymbol{a}\times\boldsymbol{\beta})\Big\} \end{aligned}$$

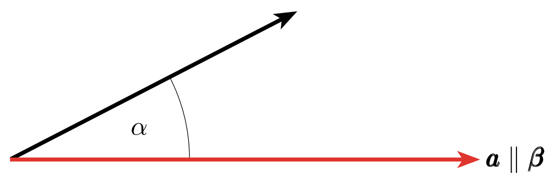
Equation (476) is *manifestly Lorentz covariant*, shows explicitly the sense in which Larmor's formula (474) is a "non-relativistic approximation," and has been extracted here from the relativistic bowels of electrodynamics ... but was first obtained by A. Liénard in 1898, only one year after the publication of Larmor's result, and *seven years prior to the invention of special relativity*!

More detailed commentary concerning the physical implications of (473–476) is most usefully presented in terms of **special cases & applications** ... as below:

CASE $\boldsymbol{a}\parallel\boldsymbol{\beta}$

This is the "most favorable case" in the sense that it is parallelism ($\boldsymbol{a}\times\boldsymbol{\beta}=\mathbf{0}$) that (see the last of the equations just above) maximizes $P$. The distribution itself can in this case be described

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{\sin^2\alpha}{(1-\beta\cos\alpha)^5}\frac{1}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)\,\boldsymbol{a}\cdot\boldsymbol{a} \\ &= D(\alpha)\cdot\Big[\text{sine squared distribution}\Big]\end{aligned} \tag{477}$$




The distribution is symmetric about the $(\boldsymbol{a}\|\boldsymbol{\beta})$-axis (the nodes lie fore and aft), and has the cross section illustrated below:

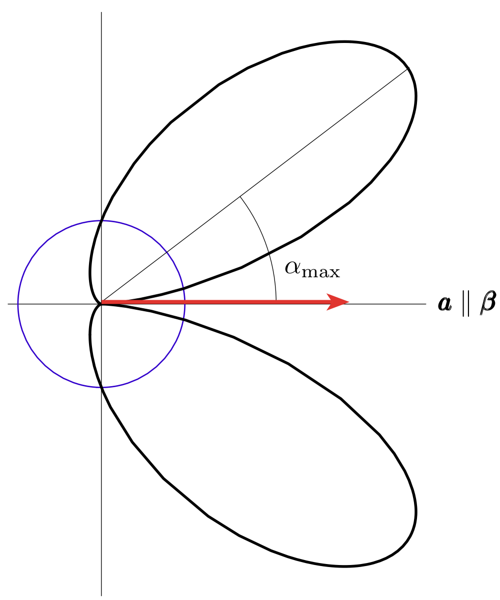



FIGURE 136: *Radiation pattern in the case $\boldsymbol{a}\|\boldsymbol{\beta}$, to be read as the cross section of a figure of revolution. The figure as drawn refers to the specific case $\beta = 0.5$. The circle has radius $\frac{1}{4\pi c^3}(\frac{e^2}{4\pi})a^2$, and sets the scale. The ears of the sine squared distribution (Figure 131) have been thrown forward (independently of whether $\boldsymbol{a}$ is parallel or antiparallel to $\boldsymbol{\beta}$).*

The ears of the sine squared distribution (Figure 131) have been thrown forward (independently of whether $\boldsymbol{a}$ is parallel or antiparallel to $\boldsymbol{\beta}$) by action of the

Doppler factor $D(\alpha)$. *How much* they are thrown forward is measured by

$$\begin{aligned}\alpha_{\text{max}} &= \cos^{-1}\left\{\frac{\sqrt{1+15\beta^2}-1}{3\beta}\right\} = \tfrac{\pi}{2} - \tfrac{5}{2}\beta + \tfrac{325}{48}\beta^3 - \cdots \\ &= \cos^{-1}\left\{\frac{4\sqrt{1-\frac{15}{16}\gamma^{-2}}-1}{3\sqrt{1-\gamma^{-2}}}\right\} = \tfrac{1}{2}\gamma^{-1} + \tfrac{133}{768}\gamma^{-3} + \cdots\end{aligned}$$

where the former equation speaks to the non-relativistic limit $\beta \downarrow 0$, and the latter to the ultra-relativistic limit $\gamma^{-1} \downarrow 0$. In the latter limit, the smallness of $\gamma^{-1}$ implies that of $\alpha$: double expansion of (477)—use $\beta = \sqrt{1-\gamma^{-2}}$—gives[288]

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{a^2}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)32\gamma^8\Big\{(\gamma\alpha)^2 - 5(\gamma\alpha)^4 + \cdots\Big\} \\ &\sim \frac{a^2}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)32\gamma^8\frac{(\gamma\alpha)^2}{[1+(\gamma\alpha)^2]^5}\end{aligned}$$

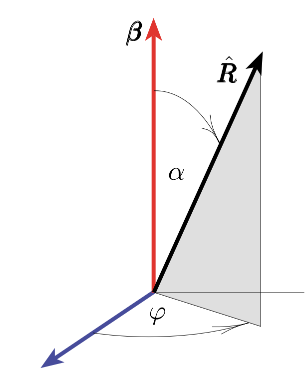
CASE $\boldsymbol{a} \perp \boldsymbol{\beta}$

This is the "least favorable case" in the sense that it is perpendicularity that minimizes $P$: reading from (476) we have (use $1 + \gamma^2\beta^2 = \gamma^2$ )

$$P = \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{a^2}{c^3}\cdot\begin{cases}\gamma^6 & \text{when} \quad \boldsymbol{a}\parallel\boldsymbol{\beta} \\ \gamma^4 & \text{when} \quad \boldsymbol{a}\perp\boldsymbol{\beta}\end{cases}$$

Working from (475) we find that the angular distribution in the special case at hand can be described

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})^3}\Big\{\boldsymbol{a}\cdot\boldsymbol{a} - \frac{1}{\gamma^2}\Big(\frac{\hat{\boldsymbol{R}}\cdot\boldsymbol{a}}{1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta}}\Big)^2\Big\} \\ &= \frac{1}{4\pi}\frac{e^2}{4\pi}\frac{a^2}{c^3}\frac{1}{(1-\beta\cos\alpha)^3}\Big\{1 - \frac{1}{\gamma^2}\frac{\sin^2\alpha\cos^2\varphi}{(1-\beta\cos\alpha)^2}\Big\} \qquad (478)\end{aligned}$$




288 PROBLEMS 79 & 80.

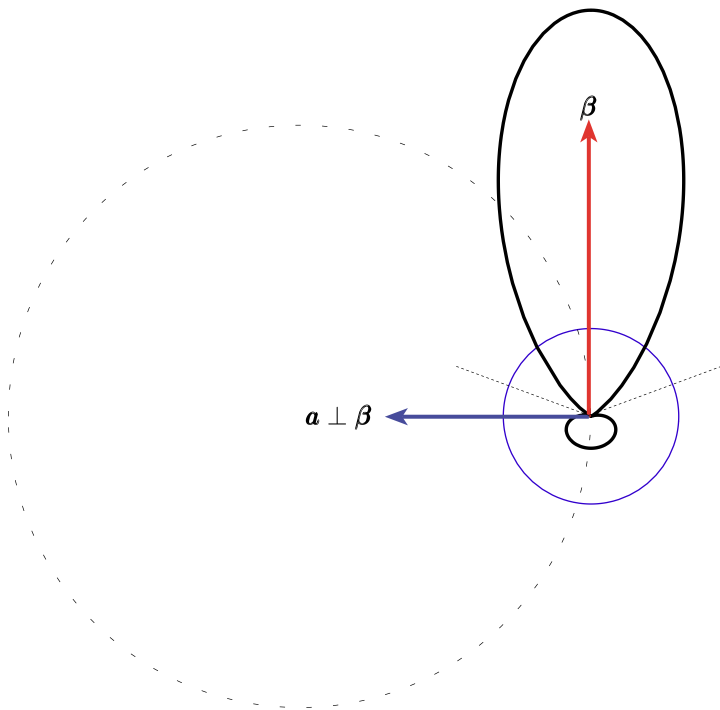
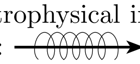



FIGURE 137: *A charge traces a circular orbit (large dashed circle) with constant speed. The figure shows a cross section of the resulting radiation pattern, which is now* not *a figure of revolution. The short dotted lines on left and right indicate the radiation-free nodal directions, which in a 3-dimensional figure would look like dimples on the cheeks of an ellipsoid. The small blue circle sets the scale, here as in Figure 136. The figure was extracted from* (478) *with* $\varphi = 0$ *and, as drawn, refers to the specific case* $\beta = 0.4$.

where the diagram at the bottom of the preceding page indicates the meanings of the angles $\alpha$ and $\varphi$. Shown above is a cross section of the associated radiation pattern. Notice that the nodal directions do *not* lie fore and aft: both are tipped forward, and stand in an angular relationship to $\boldsymbol{\beta}$ that can be extracted from Figure 135:

$$\tan(\text{angle between } \boldsymbol{\beta} \text{ and node}) = a/\beta$$

The $D(\alpha)$-factor has now enhanced the leading lobe of the radiation pattern, and attenuated the trailing lobe . . . giving rise to the "**synchrotron searchlight**," in which connection one might also look back again to Figure 128.

The radiative process just described is of major astrophysical importance (arising when electrons spiral about magnetic field lines:  ) and sets a limit on the energy which can be achieved by particle accelerators of toroidal geometry (whence the linear design of SLAC: today many of the toroidal accelerators scattered about the world are *dedicated* to the production of synchrotron radiation—serve, in effect, as fancy "lightbulbs"). It is therefore not surprising that the properties of synchrotron radiation have been studied very closely—initially by Julian Schwinger, who asks (for example) "What are the distinguishing spectral and polarization characteristics of the radiation seen by an observer who looks into the synchrotron beam as it sweeps past?" For a detailed account of the theory see Chapters 39–40 in J. Schwinger *et al*, *Classical Electrodynamics* (1998).

Synchrotron radiation would lead also to the

RADIATIVE COLLAPSE OF THE BOHR ATOM

if quantum mechanical constraints did not intervene. To study the details of this topic (which is of mainly historical interest) we look specifically to the Bohr model of hydrogen. In the ground state the electron is imagined to pursue a circular orbit of radius[289]

$$R = \frac{\hbar^2}{me^2} = 5.292 \times 10^{-9}\text{cm}$$

with velocity

$$v = \frac{e^2}{\hbar} = \frac{1}{137}c = 2.188 \times 10^{8}\text{cm/sec}$$

The natural time characteristic of the system is

$$\tau = \frac{R}{v} = \frac{\hbar^3}{me^4} = 2.419 \times 10^{-17}\text{sec}$$

Reproduced below is the 3$^{\text{rd}}$ paragraph (§1) of Bohr's original paper ("On the constitution of atoms and molecules," Phil. Mag. **26**,1 (1913)):

> *"Let us now, however, take the effect of energy radiation into account, calculated in the ordinary way from the acceleration of the electron. In this case the electron will no longer describe stationary orbits. W will continuously increase, and the electron will approach the nucleus describing orbits of smaller and smaller dimensions, and with greater and greater frequency; the electron on the average gaining in kinetic energy at the same time as the whole system loses energy. This process will go on until the dimensions of the orbit are of the same order of magnitude as the dimensions of the electron or those of the nucleus. A simple calculation shows that the energy radiated out during the process considered will be enormously great compared with that radiated out by ordinary molecular processes.*

To make his model work Bohr simply/audaciously assumed the (classical) physical ideas thus described to be "microscopically inoperative." But I want

[289] See, for example, QUANTUM MECHANICS (1967), Chapter 2, pages 138–139. For the duration of the present discussion I adopt rationalized units: $e^2/4\pi \rightarrow e^2$.

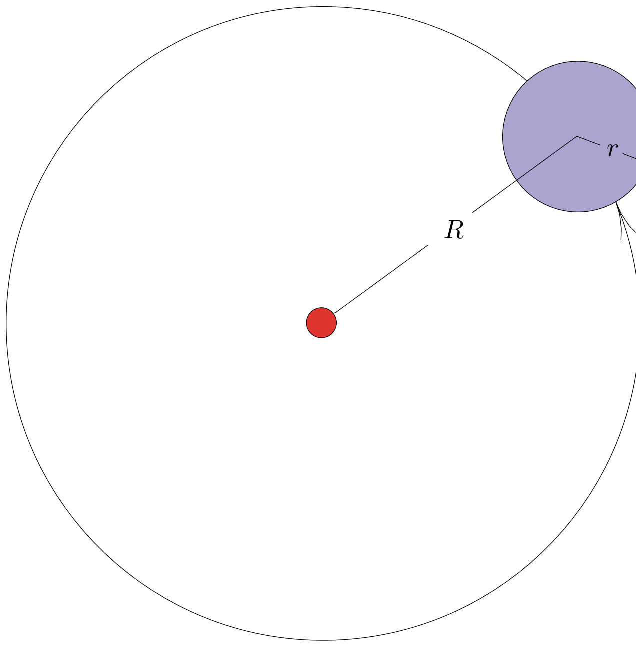



FIGURE 138: *Bohr atom, in which the nuclear proton and orbital electron have been assigned their classical radii. We study the "collapse" of the system which would follow from classical radiation theory if quantum mechanics did not intervene.*

here to pursue the issue—to inquire into the details of the "simple calculation" to which Bohr is content merely to allude. We ask: *How much energy would be released by the radiative collapse of a Bohr atom, and how long would the process take*?

If the electron and proton were literally point particles then, clearly, the energy released would be infinite ... which is unphysical. So (following Bohr's own lead) let us assume the electron and proton to have "classical radii" given by

$$r = e^2/2mc^2 \quad \text{and} \quad r_p = r/1836.12 \ll r$$

respectively, and the collapse "proceeds to contact." The elementary physics of Keplerean systems[290] leads then to the conclusion that the energy released can be described

$$\begin{aligned} E = \frac{1}{2}e^2\Big\{\frac{1}{r+r_p} - \frac{1}{R}\Big\} \sim \frac{1}{2}e^2\Big\{\frac{1}{r} - \frac{1}{R}\Big\} &= \frac{1}{2}e^2\Big\{\frac{2mc^2}{e^2} - \frac{me^2}{\hbar^2}\Big\} \\ &= mc^2\Big\{1 - \frac{1}{2}\Big(\frac{e^2}{\hbar c}\Big)^2\Big\} \sim mc^2 \end{aligned}$$

---

[290] See, for example, H. Goldstein, *Classical Mechanics* ($2^{\text{nd}}$ edition 1980), page 97.

The atom radiates at a rate given initially (Larmor's formula) by

$$P = \frac{2}{3}\frac{e^2}{c^3}a^2$$

with
$$a = \frac{v^2}{R} = \Big(\frac{1}{137}\frac{e}{\hbar}\Big)^2 mc^2$$

and has therefore a lifetime given in first approximation by

$$\begin{aligned}\mathcal{T} = \frac{E}{P} &= mc^2 \Big/ \frac{2}{3}\frac{e^2}{c^3}\Big(\frac{1}{137}\frac{e}{\hbar}\Big)^4 (mc^2)^2 \\ &= \tfrac{3}{2}(137)^5\,\tau \\ &= (7.239\times 10^{10})\,\tau \\ &= 1.751\times 10^{-6}\text{sec}\end{aligned}$$

Despite the enormous accelerations experienced by the electron, the radiation rate is seen thus to be "small": the orbit shrinks in a gentle spiral and the atom lives for a remarkably long time ($10^{10}$ revlolutions corresponds, in terms of the earth-sun system, to roughly the age of the universe!)... but not long enough. The preceding discussion is, of course, declared to be "naively irrelevant" by the quantum theory (which, in the first instance, means: by Bohr) ...which is seen now to be "super-stabilizing" in some of its corollary effects. It can, in fact, be stated quite generally that the stability of matter is an intrinsically quantum mechanical phenomenon, though the "proof" of this "meta-theorem" is both intricate and surprisingly recent.[291]

**5. Collision-induced radiation.** In many physical contexts charges move freely except when experiencing *abrupt scattering processes*, as illustrated in the figure on the facing page. We expect the *energy radiated per scatter* to be given in leading approximation by

$$E_{\text{per scatter}} = \frac{2}{3}\frac{e^2}{4\pi c^3}\Big(\frac{\Delta v}{\tau}\Big)^2\tau$$

where $\Delta v \equiv v_{\text{out}} - v_{\text{in}}$ and where $\tau$ denotes the characteristic duration of each scattering event. Suppose we had a *confined population of $N$ such charges*, and that each charge experiences (on average) $n$ collisions per unit time. We expect to have $\tau \sim 1/v$ and $n \sim v$. The rough implication is that the population should radiate at the rate

$$P \sim NnE_{\text{per scatter}} \sim (\Delta v)^2 v^2$$

If we could show that $\Delta v$ ($\sim$ momentum transfer per collision) is $v$-independent we would (by $v^2 \sim$ temperature) have established the upshot of *Newton's law of cooling*. The point I want to make is that radiative cooling is a (complicated) *radiative process*. The correct theory is certainly quantum mechanical (and probably system-dependent), but the gross features of the process appear to be within reach of classical analysis. A much more careful account of the radiation produced by impulsive scattering processes can be found in Chapter 37 of the Schwinger text cited on page 392.

[291] See F. J. Dyson & A. Lenard, "Stability of matter. I," J. Math. Phys. **8**, 423 (1967) and subsequent papers.

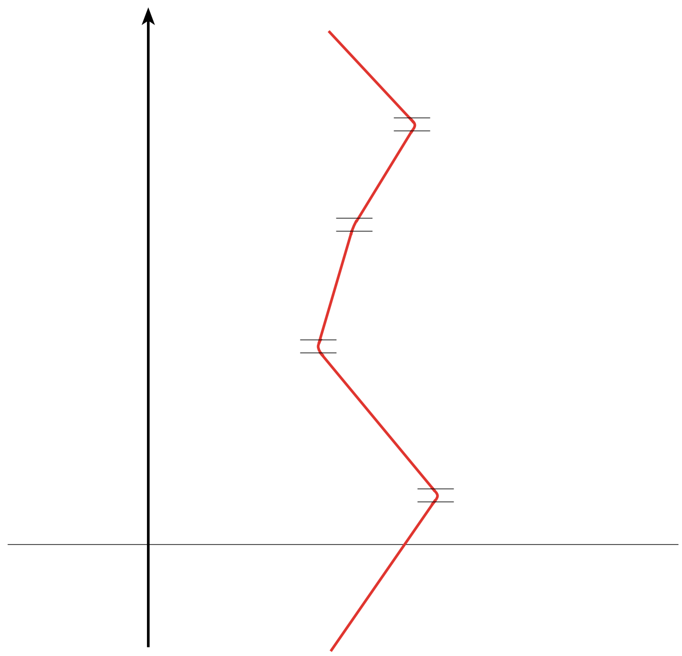


FIGURE 139: *Worldline of a charged particle subject to recurrent scattering events. Brackets mark the intervals during which the particle is experiencing non-zero acceleration.*

We have concentrated thus far mainly on single-source radiative processes, though the theory of cooling invited us to contemplate the radiation produced by random *populations* of accelerated charges. And we will want later to study the radiation produced when multiple sources act in concert (as in an antenna). But there are some important aspects and manifestations of single-source radiation theory which remain to be discussed, and it is to these that I now turn.

**6. The self-interaction problem.** We know that charges feel—and accelerate in response to—*impressed* electromagnetic fields. But **do charges feel their own fields?**. . . as (say) a motorboat may interact with the waves generated by its own former motion? Thought about the dynamics of a free charge at rest makes it appear semi-plausible that *charges do* <u>*not*</u> *feel their own* <u>*Coulomb*</u> *fields*. But the situation as it pertains to *radiation* fields is much less clear . . . for when a charge "radiates" it (by definition) "mails energy/momentum to infinity" and thus acquires a debt which (by fundamental conservation theorems) must somehow be paid. One might suppose that the responsibility for payment would fall to the agency which stimulated the charge to accelerate. But theoretical/observational arguments will be advanced which suggest that there is a sense in which **accelerated charges do feel—and recoil from—their own radiative acts**.

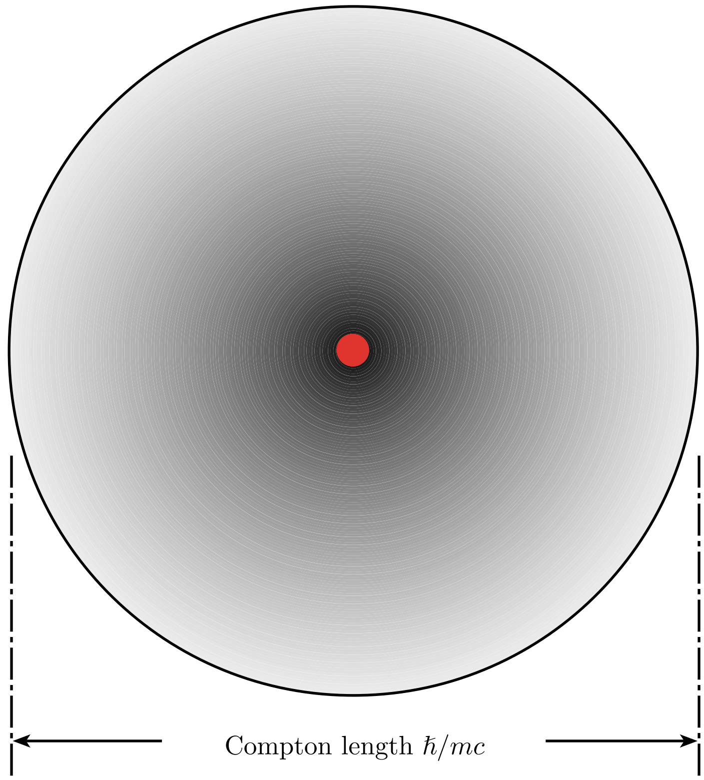



FIGURE 140: *The "classical electron"* • *is not, as one might expect, larger than but much* <u>*smaller*</u> *than the "quantum electron." A photon with wavelength* $\lambda = e^2/2mc^2$ *short enough to permit one to see the* • *would carry energy* $E = h\nu = hc/\lambda = (hc/e^2)2mc^2 = 137 \cdot 2mc^2$ *enough to create 137 electron-positron pairs ... and in the clutter the intended object of the measurement process would be lost!*

The point at issue is made complicated by at least three interrelated circumstances. The first stems from the fact that the structural properties which distinguish "radiation fields" become manifest only in the "far zone," but *it is in the "near zone" that (in a local theory like electrodynamics) any particle/self-field interaction must occur*. The second derives from the truism that "to describe the motorboat-wake interaction one must know something about the geometry of motorboats": similarly, *to study the electrodynamical self-interaction problem one must be prepared to make assumptions concerning the "structure oif charged particles."* Classical theory speaks of "point particles" and—in the next breath—of "charged balls" of classical radius $e^2/2mc^2$, but (as Abraham/Lorentz/Poincaré discovered: see again page 382) seems incapable of generating a seriously-intended electron model. Which is hardly surprising, for electrons (and charged particles generally) are <u>quantum mechanical objects</u>. In this connection it is illuminating to note that the "quantum radius" of a mass point is (irrespective of its charge) given by $\hbar/mc$. But

$$\text{“classical radius”} \equiv \frac{e^2}{mc^2} = \frac{e^2}{\hbar c}\cdot\frac{\hbar}{mc} = \frac{\text{“quantum radius”}}{137}$$

...so the "classical electron" is much *smaller* than the "quantum electron."[292] Which brings us to the third complicating circumstance (Figure 140): *we seek a classical theory of processes which are buried so deeply within the quantum regime as to make the prospects of a formally complete and self-consistent theory seem extremely remote*. From this point of view the theory described below—imperfect though it is—acquires a semi-miraculous quality.

Limited success in this area was first achieved (1904) by M. Abraham, who argued *non-relativistically—from energy conservation*. We have

$$\boldsymbol{F} + \boldsymbol{F}_\mathsf{R} = m\boldsymbol{a} \quad \text{where} \quad \begin{cases} \boldsymbol{F} \ \equiv \text{impressed force} \\ \boldsymbol{F}_\mathsf{R} \equiv \text{self-force, the nature of which we seek to determine} \end{cases}$$

$\boldsymbol{F}$ may act to change the energy of the (charged) particle, but we semi-expect $\boldsymbol{F}_\mathsf{R}$ to conform to the **energy balance condition**

$$(\text{work on particle by } \boldsymbol{F}_\mathsf{R}) + (\text{energy radiated}) = 0$$

Drawing upon Larmor's formula (474) we are led thus to write (on a typical time interval $t_1 \leqslant t \leqslant t_2$)

$$\int_{t_1}^{t_2} \boldsymbol{F}_\mathsf{R}\cdot\boldsymbol{v}\,dt + \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\underbrace{\int_{t_1}^{t_2}\boldsymbol{a}\cdot\boldsymbol{a}\,dt} = 0$$

Integration by parts gives

$$= \boldsymbol{a}\cdot\boldsymbol{v}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\dot{\boldsymbol{a}}\cdot\boldsymbol{v}\,dt$$

If it may be assumed (in consequence of periodicity or some equivalent condition) that

$$\boldsymbol{a}\cdot\boldsymbol{v}\Big|_{t_1}^{t_2} = 0$$

then

$$\int_{t_1}^{t_2}\Big\{\boldsymbol{F}_\mathsf{R} - \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\dot{\boldsymbol{a}}\Big\}\cdot\boldsymbol{v}\,dt = 0$$

[292] Nor is this fact special to electrons. Since $m$ enters identically on left and right, it pertains also to protons, to *every* particle species.

This suggests—but does not strictly entail—that $\boldsymbol{F}_{\mathsf{R}}$ may have the form

$$\boldsymbol{F}_{\mathsf{R}} = \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\dddot{\boldsymbol{x}} \tag{479.1}$$

More compactly,

$$= m\tau\dddot{\boldsymbol{x}} \tag{479.2}$$

where the parameter $\tau$ can be described

$$\begin{aligned}\tau \equiv \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{mc^3} &= \frac{4}{3}\Big(\frac{e^2}{8\pi mc^2}\Big)\frac{1}{c}\\ &= \frac{4}{3}\frac{\text{classical particle radius}}{c}\\ &\sim \begin{cases}\text{time required for light to transit from}\\ \text{one side of the particle to the other}\end{cases}\end{aligned}$$

The non-relativistic motion of a charged particle can—on the basis of the assumptions that led to (479)—be described

$$\boldsymbol{F} + m\tau\dddot{\boldsymbol{x}} = m\ddot{\boldsymbol{x}} \tag{480.1}$$

or again

$$\boldsymbol{F} = m(\ddot{\boldsymbol{x}} - \tau\dddot{\boldsymbol{x}}) \tag{480.2}$$

. . . which is the so-called "**Abraham-Lorentz equation**." This result has several remarkable features:

- It contains—which is uncommon in dynamical contexts—an allusion to the 3rd derivative. This, by the way, seems on its face to entail that *more than the usual amount of initial data is required to specify a unique solution*.
- The Abraham-Lorentz equation contains *no overt allusion to particle structure* beyond that latent in the definition of the parameter $\tau$.
- The "derivation" is susceptible to criticism at so many points[293] as to have the status of hardly more than a heuristic plausibility argument. It is, in this light, interesting to note that the work of 75 years (by Sommerfeld, Dirac, Rohrlich and many others) has done much to "clean up the derivation," to expose the "physical roots" of (480) . . . but has at the same time *shown the Abraham-Lorentz equation to be essentially correct as it stands* . . . except that
- The Abraham-Lorentz equation (480) is *non-relativistic*, but this is a formal blemish which (see below) admits easily of rectification.

[293] Most critically, the argument *draws upoon the Larmor formula—a "far field result"—to obtain information about "near field physics."* The first of the "complicating circumstances" mentioned on page 396 is not only not illuminated/resolved, it is not even addressed.

We recall from page 192 that the 4-acceleration of a moving point can be described

$$a(\tau) \equiv \tfrac{d^2}{d\tau^2}x(\tau) = \begin{pmatrix} \frac{1}{c}\gamma^4(\boldsymbol{a}\cdot\boldsymbol{v}) \\ \gamma^2\boldsymbol{a} + \frac{1}{c^2}\gamma^4(\boldsymbol{a}\cdot\boldsymbol{v})\,\boldsymbol{v} \end{pmatrix}$$

where $\boldsymbol{v}$ and $\boldsymbol{a}$ are "garden variety" kinematic 3-variables: $\boldsymbol{v} \equiv d\boldsymbol{x}/dt$ and $\boldsymbol{a} \equiv d\boldsymbol{v}/dt$. We know also (page 192/193) that

$$(u,a) = c^2 \tag{481.1}$$

$$(u,a) = 0 \tag{481.2}$$

and can sho by direct computation that

$$(a,a) = -\gamma^4\Big\{(\boldsymbol{a}\cdot\boldsymbol{a}) + \tfrac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2\Big\} \tag{481.3}$$

while a somewhat more tedious computation gives

$$\begin{aligned} b(\tau) &\equiv \tfrac{d}{d\tau}a(\tau) \qquad\qquad (482)\\ &= \gamma^3\begin{pmatrix} \frac{1}{c}\,\gamma^2\big[(\dot{\boldsymbol{a}}\cdot\boldsymbol{v}) + (\boldsymbol{a}\cdot\boldsymbol{a}) + 4\frac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2\big] \\ \dot{\boldsymbol{a}} + 3\frac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{a} + \frac{1}{c^2}\gamma^2\big[(\dot{\boldsymbol{a}}\cdot\boldsymbol{v}) + (\boldsymbol{a}\cdot\boldsymbol{a}) + 4\frac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2\big]\boldsymbol{v} \end{pmatrix} \end{aligned}$$

where $\qquad \dot{\boldsymbol{a}} \equiv \frac{d}{dt}\boldsymbol{a} = \dddot{\boldsymbol{x}}$

A final preparatory computation gives

$$(u,b) = \gamma^4\Big\{(\boldsymbol{a}\cdot\boldsymbol{a}) + \tfrac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2\Big\} = -(a,a) \tag{481.4}$$

We are in position also to evaluate $(a,b)$ and $(b,b)$, but have no immediate need of such information . . . so won't.[294] Our immediate objective is to proceed from $\boldsymbol{F}_{\mathsf{R}} = \frac{2}{3}(e^2/4\pi)\frac{1}{c^3}\dddot{\boldsymbol{x}}$ to its "most natural" relativistic counterpart—call it $K^\mu_{\mathsf{R}}$. It is tempting to set $K_{\mathsf{R}} = \frac{2}{3}(e^2/4\pi)\frac{1}{c^3}b$, but such a result would—by (481.4)—be inconsistent with the general requirement (see again page ???) that $(K,u) = 0$. We are led thus—tentatively—to set

$$K_{\mathsf{R}} = \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\,b_\perp \tag{483}$$

$$\begin{aligned} b_\perp &\equiv b - \frac{(b,u)}{(u,u)}u \\ &= b + \frac{(a,a)}{c^2}u \\ &= \gamma^3\begin{pmatrix} \frac{1}{c}\,\gamma^2\big[(\dot{\boldsymbol{a}}\cdot\boldsymbol{v}) + 3\frac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2\big] \\ \dot{\boldsymbol{a}} + 3\frac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{a} + \frac{1}{c^2}\gamma^2\big[(\dot{\boldsymbol{a}}\cdot\boldsymbol{v}) + 3\frac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2\big]\boldsymbol{v} \end{pmatrix} \end{aligned}$$

in which connection we note that

$$\downarrow \atop = \begin{pmatrix} 0 \\ \boldsymbol{F}_{\mathsf{R}} \end{pmatrix} \quad \text{in the non-relativistic limit (as required)}$$

---

[294] PROBLEM 81.

Now, the spatial part of Minkowski's equation $K^\mu = md^2x/d\tau^2$ can (see again (288) page 197) be written $(1/\gamma)\boldsymbol{K} = \frac{d}{dt}(\gamma m\boldsymbol{v})$, and in this sense it is (not $\boldsymbol{K}$ but) $(1/\gamma)\boldsymbol{K}$ which one wants to call the "relativistic force." We are led thus from (483) to the conclusion that the **relativistic self-force**

$$\boldsymbol{\mathfrak{F}}_{\mathsf{R}} = \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\gamma^2\Big\{\dot{\boldsymbol{a}} + 3\tfrac{1}{c^2}\gamma^2(\boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{v})\boldsymbol{a} + \tfrac{1}{c^2}\gamma^2\big[(\dot{\boldsymbol{a}}\boldsymbol{\cdot}\boldsymbol{v}) + 3\tfrac{1}{c^2}\gamma^2(\boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{v})^2\big]\boldsymbol{v}\Big\} \tag{484.1}$$

This result was first obtained (1905) by Abraham, who however argued not from relativity but from a marginally more physical refinement of the "derivation" of (479). The "argument from relativity" was first accomplished by M. von Laue (1909). The pretty notation

$$\begin{aligned}\boldsymbol{\mathfrak{F}}_{\mathsf{R}} &= \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\gamma^4\Big\{\boldsymbol{g} + \tfrac{1}{c^2}\boldsymbol{v}\times(\boldsymbol{v}\times\boldsymbol{g})\Big\} \\ \boldsymbol{g} &\equiv \dot{\boldsymbol{a}} + 3\tfrac{1}{c^2}\gamma^2(\boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{v})\boldsymbol{a}\end{aligned} \tag{484.2}$$

was introduced into the modern literature by David Griffiths,[295] but was reportedly original to Abraham.[296]

All modern self-interaction theories[297] hold (483)—which can be notated

$$\begin{aligned}K^\mu_{\mathsf{R}} &= \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\Big\{\frac{d^3x^\mu}{d\tau^3} + \frac{1}{c^2}(a^\alpha a_\alpha)\frac{dx^\mu}{d\tau}\Big\} \\ a^\alpha &\equiv \frac{d^2x^\alpha}{d\tau^2}\end{aligned}$$

—to be *exact* (so far as classical theory allows). Which is surprising, for *we have done no new physics*, addressed none of the conceptual difficulties characteristic of this topic. We note with surprise also that *we can, in the relativistic regime, have* $\boldsymbol{\mathfrak{F}}_{\mathsf{R}} \neq \boldsymbol{0}$ *even when* $\dot{\boldsymbol{a}} = \boldsymbol{0}$.

To study the physical implications of the results now in hand we retreat (in the interest of simplicity) to the non-relativistic case: (480). If (also for simplicity) we assume $\boldsymbol{F}$ to be $\boldsymbol{x}$-independent (*i.e.*, to be some arbitrarily prescribed function of $t$ alone) then the Abraham-Lorentz equation (480) reads

$$\dddot{\boldsymbol{x}} - \tfrac{1}{\tau}\ddot{\boldsymbol{x}} = -\tfrac{1}{m\tau}\boldsymbol{F}(t) \tag{485}$$

and entails

$$\ddot{\boldsymbol{x}}(t) = e^{t/\tau}\Big\{\underbrace{\boldsymbol{a}}_{\text{constant of integration}} - \tfrac{1}{m\tau}\int_0^t e^{-s/\tau}\boldsymbol{F}(s)\,ds\Big\} \tag{486.1}$$

[295] "Dumbbell model for the classical radiation reaction," AJP **46** 244 (1978).
[296] PROBLEM 82.
[297] For references see the Griffiths paper just cited.

Successive integrations give

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{v} + \int_0^t \ddot{\boldsymbol{x}}(s)\,ds \tag{486.2}$$

and

$$\boldsymbol{x}(t) = \boldsymbol{x} + \int_0^t \dot{\boldsymbol{x}}(s)\,ds \tag{486.3}$$

where $\boldsymbol{v}$ and $\boldsymbol{x}$ are additional constants of integration.[298]

In the $\boxed{\text{FORCE-FREE CASE } \boldsymbol{F}(t) \equiv \boldsymbol{0}}$ equations (486) promptly give

$$\boldsymbol{x}(t) = \boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}\tau^2 e^{t/\tau}$$

This entails $\dot{\boldsymbol{x}}(t) = \boldsymbol{v} + \boldsymbol{a}\tau e^{t/\tau}$, which is <u>asymptotically infinite unless $\boldsymbol{a} = \boldsymbol{0}$</u>. So we encounter right off the bat an instance of the famous **run-away solution problem**, which bedevils all theories of self-interaction. It is dealt with by conjoining to (485) the stipulation that

**Run-away solutions are to be considered "unphysical" . . . and discarded.** (487)

One (not immediately obvious) effect of the *asymptotic side-condition* (487) is to reduce to its familiar magnitude the amount of initial data needed to specify a particular particle trajectory.

To gain some sense of the practical effect of (487) we look next to the $\boxed{\text{CASE OF AN IMPULSIVE FORCE } \boldsymbol{F}(t) \equiv m\tau\boldsymbol{A}\delta(t - t_0)}$. Immediately

$$\ddot{\boldsymbol{x}}(t) = \begin{cases} e^{t/\tau}\boldsymbol{a} & : \quad t < t_0 \\ e^{t/\tau}\big[\boldsymbol{a} - \boldsymbol{A}e^{-t_0/\tau}\big] & : \quad t > t_0 \end{cases}$$

Thbe requirement—(487)—that $\ddot{\boldsymbol{x}}(t)$ remain asymptotically finite entails that the adjustable constant $\boldsymbol{a}$ be set equal to $\boldsymbol{A}e^{-t_0/\tau}$. Then

$$\ddot{\boldsymbol{x}}(t) = \begin{cases} \boldsymbol{A}e^{(t-t_0)/\tau} & : \quad t < t_0 \\ \boldsymbol{0} & : \quad t > t_0 \end{cases} \tag{488}$$

The situation is illustated in Figure 141. The most striking fact to emerge is that <u>the particle starts to accelerate before it has been kicked</u>! This is an instance of the famous **preacceleration** phenomenon. It is not an artifact of the $\delta$-function, not a consequence of the fact that we are working at the moment in the non-relativistic approximation . . . but a systemic feature of the classical self-interaction problem. Roughly, preacceleration may be considered to arise

[298] PROBLEM 83.

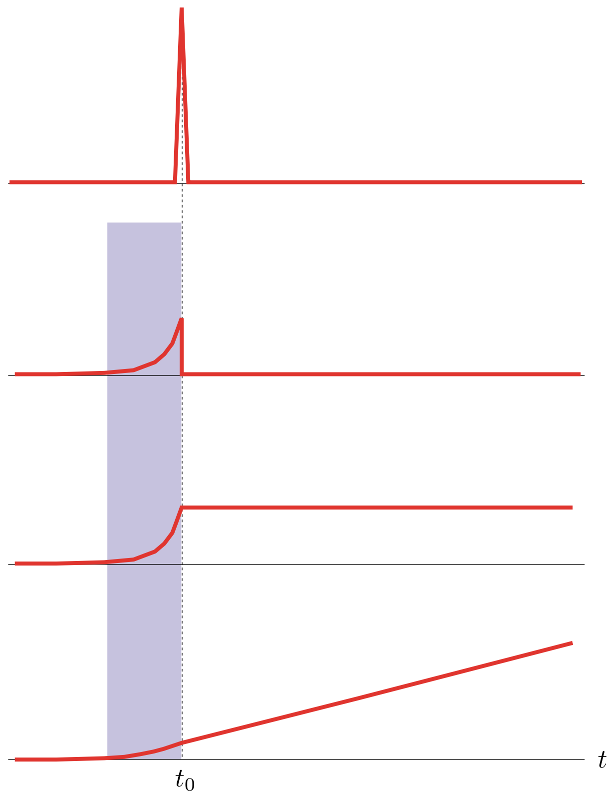



FIGURE 141: *Graphs of (reading from top to bottom) the impulsive force* $\boldsymbol{F}(t) \equiv m\tau\boldsymbol{A}\delta(t - t_0)$ *and of the resulting acceleration* $\ddot{\boldsymbol{x}}(t)$*, velocity* $\dot{\boldsymbol{x}}(t)$ *and position* $\boldsymbol{x}(t)$*. The shaded rectangle identifies the "preacceleration interval."*

because "the leading edge of the extended classical source makes advance contact with the force field." The characteristic preacceleration time is—consistently with this picture—small, being given by $\tau$ ($\sim 10^{-24}$ seconds for an electron). On its face, preacceleration represents a *microscoptic violation of causality* . . . and so it is, but the phenomenon lies so deep within the quantum regime as to be (or so I believe) *classical unobservable in every instance.* Preacceleration is generally considered to be (not a physical but) a merely "mathematical phenomenon," a symptom of an attempt to extend classical physics beyond its natural domain of applicability.

We may "agree not to be bothered" by the preacceleration "phenomenon." But preacceleration comes about as a forced consequence of implementation of the asymptotic condition (487) . . . and the fact that the equation of motion (485) cannot stand on its own feet, but must be propped up by such a side condition, *is* bothersome. Can one modify the equation of motion so as to make the make the asymptotic condition *automatic*?. . . so that "run-away solutions" simply do not arise? The question provokes the following formal manipulation. Let (485) be written

$$(1 - \tau D)m\ddot{\boldsymbol{x}}(t) = \boldsymbol{F}(t)$$

or again

$$m\ddot{\boldsymbol{x}}(t) = \frac{1}{1 - \tau D}\boldsymbol{F}(t) \tag{489}$$

where $D \equiv \frac{d}{dt}$. Recalling $\frac{1}{\lambda} = \displaystyle\int_0^\infty e^{-\lambda\theta}\,d\theta$, we presume to write

$$\frac{1}{1 - \tau D} = \int_0^\infty e^{-(1-\tau D)\theta}\,d\theta$$

even though $D$ is here not a number but a differential operator (this is heuristic mathematics in the noble tradition of Heaviside). Then

$$m\ddot{\boldsymbol{x}}(t) = \int_0^\infty e^{-\theta}e^{\theta\tau D}\boldsymbol{F}(t)\,d\theta$$

But $e^{\theta\tau D}\boldsymbol{F}(t) = \boldsymbol{F}(t + \theta\tau)$ by Taylor's theorem, so

$$= \int_0^\infty \boldsymbol{F}(t + \theta\tau)\,d\theta \tag{490}$$

Notice that, since $c \uparrow \infty$ entails $\tau \downarrow 0$, we can use $\displaystyle\int_0^\infty e^{-\theta\,d\theta} = 1$ to recover Newton's $m\ddot{\boldsymbol{x}}(t) = \boldsymbol{F}(t)$ in the non-relativistic limit. Equation (490) states that $\ddot{\boldsymbol{x}}(t)$ is determined by a <u>weighted average of future force values</u>, and therefore provides a relatively sharp and general characterization of the preacceleration phenomenon—encountered thus far only in connection with a single example. Returning to that example ... insert $\boldsymbol{F}(t) \equiv m\tau\boldsymbol{A}\delta(t-t_0)$ into (490) and obtain

$$\ddot{\boldsymbol{x}}(t) = \int_0^\infty \boldsymbol{A}\delta(t - t_0 + \theta\tau)\tau d\theta = \begin{cases} \boldsymbol{A}e^{(t-t_0)/\tau} & : \quad t < t_0 \\ \boldsymbol{0} & : \quad t > t_0 \end{cases}$$

We have recovered (488), but by an argument that is *free from any explicit reference to the asymptotic condition*. In (490) we have a formulation of the Abraham-Lorentz equation (480) in which the "exotic" features have been translocated into the force term ... but we have actually come out ahead: we have managed to describe the dynamics of a self-interacting charge by means of an <u>integrodifferential equation of motion</u> that *stands alone, without need of a side condition* such as (487). The general solution of (490) has, by the way, the familiar number of adjustable constants of integration, so <u>standard initial data serves to identify particular solutions</u>.

If in place of the "integral representation of $1/(1 - \tau D)$" we use

$$\frac{1}{1 - \tau D} = 1 + \tau D + (\tau D)^2 + \cdots$$

then in place of (490) we obtain

$$
\begin{aligned}
m\ddot{\boldsymbol{x}}(t) &= \boldsymbol{F}(t) + \tau\boldsymbol{F}'(t) + \tau^2\boldsymbol{F}''(t) + \cdots \\
&= \text{Newtonian force} + \text{Radiative corrections}
\end{aligned}
\tag{491}
$$

Equations (490) and (491) are equivalent. The latter masks preacceleration (acausality), but makes explicit the Newtonian limit.[299]

Having thus exposed the central issues, I must refer my readers to the literature for discussion of the technical details of modern self-interaction theory: this is good, deep-reaching physics, which has engaged the attention of some first-rate physicists and very much merits close study.[300] I turn now to discussion of some of the observable physical consequences of self-interaction:

**7. Thomson scattering.** An electron in a microwave cavity or laser beam experiences a Lorentz force of the form

$$
\begin{aligned}
\boldsymbol{F}(t) &= e(\boldsymbol{E} + \tfrac{1}{c}\boldsymbol{v}\times\boldsymbol{B})\cos\omega t \\
&\downarrow \\
&= e\boldsymbol{E}\cos\omega t \quad \text{in the non-relativistic limit}
\end{aligned}
$$

For such a *harmonic driving force* (486.1) becomes

$$
\ddot{\boldsymbol{x}}(t) = e^{\Omega t}\Big\{\boldsymbol{a} - \tfrac{e}{m}\boldsymbol{E}\Omega\underbrace{\int_0^t e^{-\Omega s}\cos\omega s\,ds}\Big\}
$$

where $\Omega \equiv \frac{1}{\tau} = \frac{3}{2}\frac{4\pi}{e^2}mc^3$. But

$$
= \frac{e^{-\Omega s}}{\Omega^2+\omega^2}\Big[-\Omega\cos\omega s + \omega\sin\omega s\Big]_0^t
$$

so

$$
= \frac{e}{m}\boldsymbol{E}\frac{\Omega^2\cos\omega t - \Omega\omega\sin\omega t}{\Omega^2+\omega^2} + e^{\Omega t}\Big\{\boldsymbol{a} - \frac{e}{m}\boldsymbol{E}\frac{\Omega^2}{\Omega^2+\omega^2}\Big\}
$$

The asymptotic condition (487) requires that we set $\{\text{etc.}\} = \mathbf{0}$, so after some

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[299] For a much more elaborate discussion of the ideas sketched above see CLASSICAL RADIATION (1974), pages 600–605.

[300] F. Rohrlich's *Classical Charged Particles* (1965), Chapters 2 & 6 and J. D. Jackson's *Classical Electrodynamics* (3rd edition 1998), Chapter 16 are good places to start. See also T. Erber, "The classical theories of radiation reaction," Fortschritte der Physik **9**, 343 (1961) and G. N. Plass, "Classical electrodynamic equations of motion with radiative reaction," Rev. Mod. Phys. **33**, 37 (1961) …which are excellent general reviews and provide good bibliographies. Students should also not neglect to examine the classics: Dirac (1938), Wheeler-Feynman (1945).

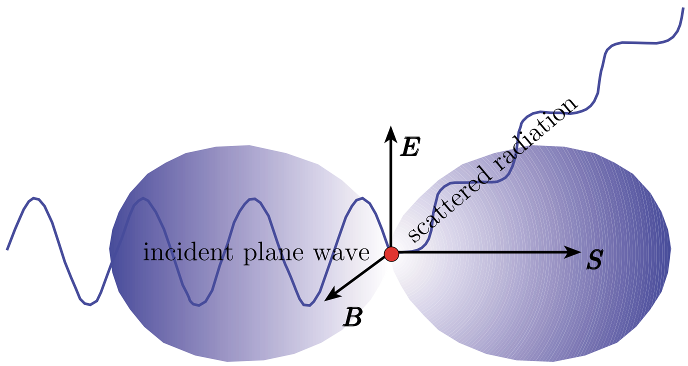


FIGURE 142: *A monochromatic plane wave is incident upon a free electron* •, *which is stimulated to oscillate* ↕ *and therefore to radiate in the characteristic sine-squared pattern. The electron drinks energy from the incident beam and dispatches energy in a variety of other directions: in short, it* scatters *radiant energy. Scattering by this classical mechanism—by* free *charges—is called* **Thomson scattering**.

elementary algebra we obtain

$$\ddot{\boldsymbol{x}}(t) = \frac{1}{\sqrt{1+(\omega/\Omega)^2}}\frac{e}{m}\boldsymbol{E}\cos(\omega t+\delta) \tag{492}$$

where the phase shift

$$\delta = \arctan(\omega/\Omega)$$

is the disguise now worn by the preacceleration phenomenon. We note in passing that

$$\downarrow$$
$$= \frac{e}{m}\boldsymbol{E}\cos\omega t \quad \text{in the non-relativistic limit: } \Omega \gg \omega$$

It is upon (492) that the classical theory of the scattering of electromagnetic radiation by free electrons—"Thomson scattering"—rests. We inquire now into the most important details of this important process.

Using (492) in conjunction with the Larmor formula (474) we conclude that the energy radiated per period by the harmonically stimulated electron (see the preceding figure) can be described

$$\begin{aligned}\int_0^T P\,dt &= \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\Big(\frac{e}{m}E\Big)^2\frac{1}{1+(\omega/\Omega)^2}\int_0^T\cos^2\omega t\,dt \quad \text{with} \quad T\equiv 2\pi/\omega\\ &= \Big(\frac{cE^2\pi}{\omega}\Big)\cdot\frac{8\pi}{3}\Big(\frac{e^2}{4\pi mc^2}\Big)^2\frac{1}{1+(\omega/\Omega)^2}\end{aligned}$$

On the other hand, we know from work on page 305 that the (time-averaged energy flux or) *intensity* of the incident plane wave can be described $I = \frac{1}{2}cE^2$ so the energy incident (per period) upon an area $A$ becomes

$$ITA = \tfrac{1}{2}cE^2(2\pi/\omega)A = \left(\frac{cE^2\pi}{\omega}\right)\cdot A$$

We conclude that

> A free electron absorbs (only to re-radiate) energy from an incident monochromatic wave as though it had a cross-sectional area given by

$$\sigma_{\text{Thomson}} = \tfrac{8\pi}{3}(\text{classical electron radius})^2 \cdot \frac{1}{1+(\omega/\Omega)^2}$$

The final factor can and should be dropped: it differs from unity only if

$$\hbar\omega \gg \hbar\Omega = \tfrac{3}{2}\big(\tfrac{4\pi\hbar c}{e^2}\big)mc^2 = 205mc^2$$

and this carries us so far into the relativistic regime that we must expect our classical results long since to have become meaningless. *Neglect of the factor amounts to neglect of the self-interaction*: it entails $\delta = \arctan(\omega/\Omega) \to \frac{\pi}{2}$ and causes the **Thomson scattering cross-section**

$$\sigma_{\text{Thomson}} = \tfrac{8\pi}{3}\left[e^2/4\pi mc^2\right]^2 \tag{493}$$

to become $\omega$-independent. Thomson scattering—which in the respect just noted is quite atypical—may be considered to comprise the classical limit of Compton scattering, the relativistic *quantum* process diagramed below. The radiation

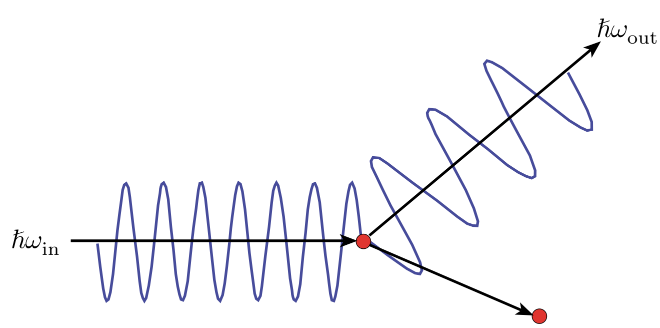


FIGURE 143: *In view of the fact that* **Compton scattering** *yields scattered photons that have been* <u>*frequency-shifted*</u> *it is remarkable that no frequency shift is associated with the Thomson scattering process.*

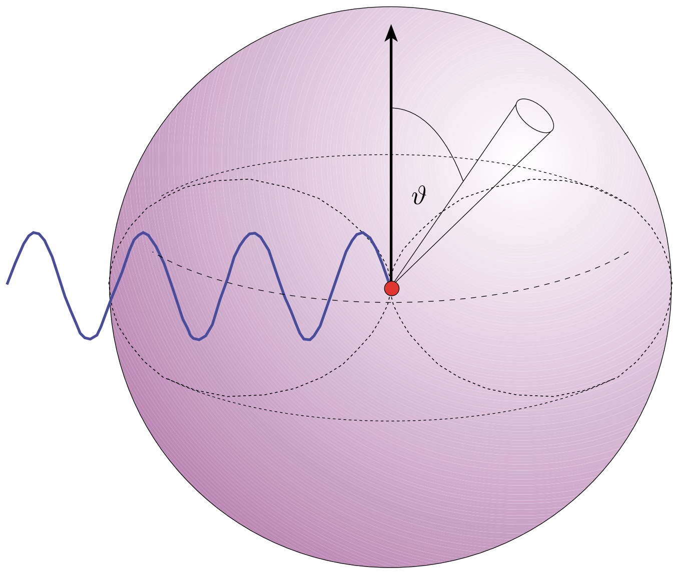


FIGURE 144: *Representation of the axially-symmetric sine-squared character of the Thomson scattering pattern. I invite the reader to consider what would be the pattern if the incidentg radiation were elliptically polarized.*

field generated by a harmonically stimulated free electron has the structure illustrated in Figure 126. The **differential Thomson cross-section** (Figure 144) is readily seen to have the sine-squared structure

$$\frac{d\sigma}{d\Omega}\bigg|_{\text{Thomson}} = \big[e^2/4\pi mc^2\big]^2 \sin^2\vartheta$$

**8. Rayleigh scattering.** Let our electron—formerly free—be considered now to be attached to a spring, part of a "classical molecule." If the spring force is written $\boldsymbol{f} = -m\omega_0^2\boldsymbol{x}$ then the Abraham-Lorentz equation (480) becomes

$$\ddot{\boldsymbol{x}} - \tau\dddot{\boldsymbol{x}} + \omega_0^2\boldsymbol{x} = \frac{e}{m}\boldsymbol{E}\cos\omega t \tag{494}$$

We expect the solution of (494) to have (after transcients have died out) the form

$$\boldsymbol{x}(t) = \boldsymbol{\mathfrak{X}}\cos(\omega t + \delta)$$

with $\boldsymbol{\mathfrak{X}} \parallel \boldsymbol{E}$, and will proceed on the basis of that assumption—an assumption which, by the way,

- renders the asymptotic condition (487) superfluous
- entails $\dddot{\boldsymbol{x}} = -\omega^2\dot{\boldsymbol{x}}$.

Our initial task, therefore, is to describe the solution

$$x(t) = \mathcal{X} e^{i(\omega t - \delta)}$$

of

$$\ddot{x} + 2b\dot{x} + \omega_0^2 x = \tfrac{e}{m} E e^{i\omega t}$$
$$b \equiv \tfrac{1}{2}\tau\omega^2$$

But this is precisely the *harmonically driven damped oscillator* problem—painfully familiar to every sophomore—the only novel feature being that the "radiative damping coefficient" $b$ is now $\omega$-dependent. Immediately

$$\underbrace{(-\omega^2 + 2ib\omega + \omega_0^2)}\mathcal{X}e^{-i\delta} = \tfrac{e}{m}E$$
$$= \sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}\,\exp\Big\{i\tan^{-1}\frac{2b\omega}{\omega_0^2 - \omega^2}\Big\}$$

which gives

$$\begin{aligned}\mathcal{X}(\omega) &= \frac{(e/m)E}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}}\\ &= \frac{eE}{m\omega_0^2}\frac{1}{\sqrt{(1-\xi^2)^2 + k^2\xi^6}} \equiv \frac{eE}{m\omega_0^2}\mathcal{X}(\xi, k)\\ \delta(\omega) &= \tan^{-1}\frac{2b\omega}{\omega_0^2 - \omega^2}\\ &= \tan^{-1}\frac{k\xi^3}{1-\xi^2} \equiv \delta(\xi, k)\end{aligned}$$

where

$$\xi \equiv \omega/\omega_0 \qquad \text{and} \qquad k \equiv \tau\omega_0$$

are dimensionless parameters. It is useful to note that $k$ is, in point of physical fact, typically quite small:

$$\begin{aligned}k &= \frac{\text{period of optical reverberations within the classical electron}}{\text{period of molecular vibrations}}\\ &\sim \frac{e^2/mc^3}{\hbar^3/me^4} = \Big(\frac{e^2}{\hbar c}\Big)^3\\ &= \Big(\frac{1}{137}\Big)^3 = 3.89 \times 10^{-7}\end{aligned}$$

Precisely the argument that led to (493) now leads to the conclusion that the **Rayleigh scattering cross-section** can be described[301]

$$\begin{aligned}\sigma_{\text{Rayleigh}}(\omega) &= \sigma_0 \cdot \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}\\ &= \sigma_0\frac{\xi^4}{(1-\xi^2)^2 + k^2\xi^6}\end{aligned} \tag{495}$$

$$\sigma_0 \equiv \sigma_{\text{Thomson}} = \tfrac{8\pi}{3}\big[e^2/4\pi mc^2\big]^2$$

[301] PROBLEM 84.

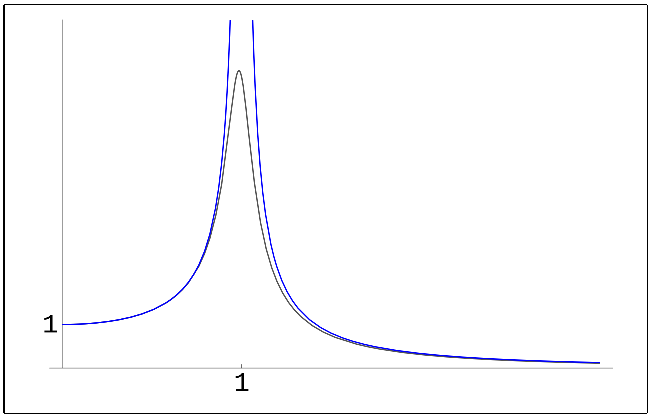


FIGURE 145: *Graphs of* $\mathfrak{X}(\xi,k)$ *in which, for clarity,* $k$ *has been assigned the artificially large values* $k = 0.15$ *and* $k = 0.05$*. An easy calculation shows that the resonant peak stands just to the left of unity:*

$$\frac{\partial}{\partial\xi}\mathfrak{X}(\xi,k) = 0 \quad \textit{at} \quad \xi = \left[\frac{\sqrt{1+6k^2}-1}{3k^2}\right]^{\frac{1}{2}} = 1 - \tfrac{3}{4}k^2 + \tfrac{63}{32}k^4 - \cdots$$

*and that*

$$\mathfrak{X}_{\text{max}} = k^{-1} + \tfrac{9}{8}k - \tfrac{189}{128}k^3 + \cdots$$

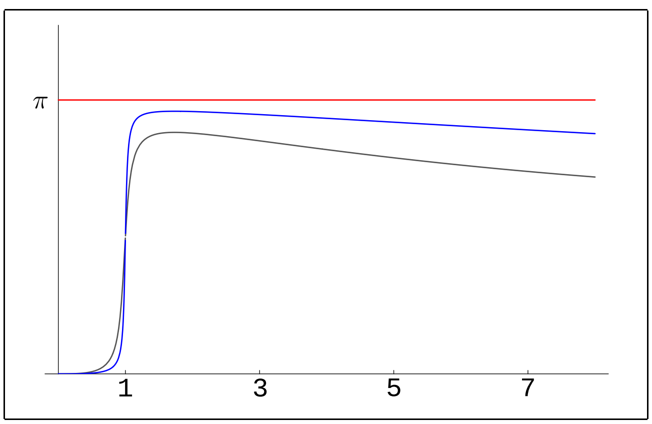


FIGURE 146: *Graphs of* $\delta(\xi,k)$ *in which* $k$ *has been assigned the same artificially large values as described above. As* $k$ *becomes smaller the phase jump becomes steeper,* $\delta$ *approaches* $\pi$ *more closely, and hangs there longer before—at absurdly/unphysically high frequencies* $\omega \gg \Omega$*—dropping to* $\frac{\pi}{2}$*:*

$$\lim_{\xi\uparrow\infty} \tan^{-1}\frac{k\xi^3}{1-\xi^2} = \lim_{\xi\uparrow\infty} \tan^{-1}(-k\xi) = \tfrac{\pi}{2}$$

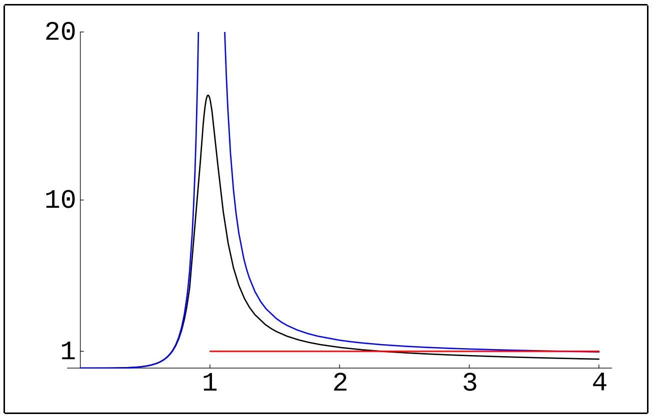


FIGURE 147: *Graphs of the Rayleigh distribution function. In* (495) *I have set* $\sigma_0 = 1$ *and have assigned to* $k$ *the artificially large values* $k = 0.25$ *and* $k = 0.10$. *The red line at unity has been inserted to emphasize the high-frequency asymptote. The resonant peak lies in the very near neighborhood of* $\xi \equiv \omega/\omega_0 = 1$ *and its height becomes infinite when self-interactive effects are turned off*: $k \downarrow 0$. *The physical short of it*: *The apparent size of a "classical molecule" depends upon the color of the light in which it is viewed*.

What we have learned is that **Rayleigh scattering**—energy absorption and reemission by a monochromatically stimulated and self-interactively damped "classical molecule" (charged particle on a spring)—is frequency-dependent. Looking to the qualitative details of that $\omega$-dependence (Figure 147), we find it natural to distinguish three regimes:

LOW-FREQUENCY REGIME $\xi \equiv \omega/\omega_0 \ll 1$ so with *Mathematica*'s aid we expand about $\xi = 0$, obtaining

$$\frac{\xi^4}{(1-\xi^2)^2 + k^2\xi^6} = \xi^4 + 2\xi^6 + 3\xi^8 + (4-k^2)\xi^{10} + (5-4k^2)\xi^{12} + \cdots$$

Thus are we led to the so-called "$4^{\text{th}}$ power law"

$$\sigma_{\text{Rayleigh}}(\omega) \sim \sigma_0(\omega/\omega_0)^4 \quad : \quad \omega \ll \omega_0 \tag{496}$$

The accuracy of the approximation is evident in Figure 148.

It is a familiar fact that (if we may allow ourselves to speak classically in such a connection) slight conformational/dynamical adjustments of atomic/molecular state can result in the emission (or from the absorption) of visible light: $[\Delta E \approx \hbar\Delta\omega_0] = \hbar\omega$. From this we infer that the characteristic atomic/molecular vibrational frequencies $\omega_0$ are themselves $\gg$ than the frequencies

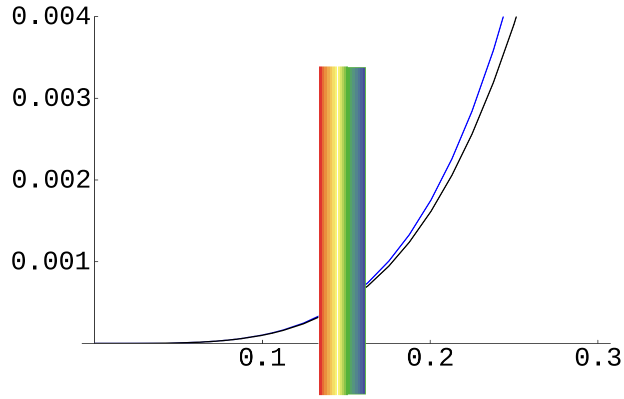


FIGURE 148: *Graph—based upon* (495)*—of* $\sigma_{\text{Rayleigh}}$ *with* $\xi \ll 1$*, compared with the scattering cross-section asserted by the* $4^{\text{th}}$ *power law* (496)*. In both cases I have set* $\sigma_0 = 1$*, and in the former case I have taken* $k = 0.00001$*. Naive arguments developed in the text suggest that atomic/molecular rotational/vibrational frequencies* $\omega_0$ *are typically* $\gg$ *than the frequencies present in the visible spectrum.*

characteristic of visible light,[302] and that the scattering of sunlight by air is therefore a "low-frequency phenomenon."[303]

RESONANCE REGIME Here $\xi \sim 1$ (*i.e.*, $\omega \sim \omega_0$) $\Rightarrow \sigma \sim \sigma_{\text{max}}$ and provides a classical intepretation of the phenomenon of *resonance florencence*. Let (495) be written

$$\begin{aligned}\sigma_{\text{Rayleigh}} &= \sigma_0 \frac{\xi^4}{(1+\xi)^2(1-\xi)^2 + k^2\xi^6} \\ &\approx \tfrac{1}{4}\sigma_0 \frac{1}{(\xi-1)^2 + (\tfrac{1}{2}k)^2} \end{aligned} \tag{497}$$

For a comparison of the exact Rayleigh distribution function with its resonant approximation (497), see Figure 149. The nearly Gaussian appearance of the appoximating function leads us to observe that

$$\int_{-\infty}^{+\infty} \tfrac{1}{4} \frac{1}{(\xi-\xi_0)^2 + (\tfrac{1}{2}k)^2}\, d\xi = \tfrac{\pi}{2}k^{-1} \quad : \quad \text{all } \xi_0$$

and on the basis of that information to introduce the definition

$$L(\xi-\xi_0, k) \equiv \tfrac{1}{2\pi} \frac{k}{(\xi-\xi_0)^2 + (\tfrac{1}{2}k)^2} \quad : \quad k > 0 \tag{498}$$

---

[302] For the former we might borrow $\omega_0 = 2\pi(me^4/\hbar^3) = 2.60 \times 10^{17}$Hz from the Bohr theory of hydrogen (see again page 392). For visible light one has $4.0 \times 10^{14}\text{Hz} < \omega < 7.5 \times 10^{14}$Hz.

[303] PROBLEM 85.

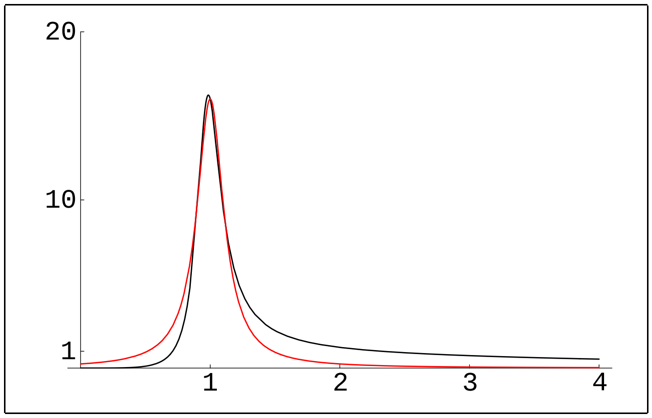


FIGURE 149: *Comparison of the exact Rayleigh cross-section with its resonant approximation (497). In constructing the figure I have assigned $k$ the unphysically large value $k = 0.25$. The fit—already quite good—becomes ever better as $k$ gets smaller.*

We will soon (in §9) have unexpected occasion to inquire more closely into properties of the "Lorenz distribution function" $L(\xi, k)$,[304] but for the moment are content to observe that in this notation

$$\sigma \approx (\pi/2k)\sigma_0 \cdot L(\xi - 1, k) \quad \text{at resonance: } \omega \sim \omega_0$$

and that $L(\xi, k)$ assumes its maximal value at $\xi = 0$: $L(0, k) = \frac{2}{\pi}k^{-1}$ so

$$\sigma_{\text{max}} = \sigma_0/k^2 = (\sigma_0/\tau^2)/\omega_0^2 = (\sigma_0/\tau^2)/(2\pi\nu_0)^2 \tag{499.1}$$

where $\nu_0$ is the literal *frequency* of the resonant radiation and (below) $\lambda_0 = c/\nu_0$ its *wavelength*. But (look back again to pages 398 and 406 for the definitions of $\tau$ and $\sigma_0$)

$$\sigma_0/\tau^2 = \tfrac{8\pi}{3}\big[e^2/4\pi mc^2\big]^2 \Big/ \big[\tfrac{2}{3}e^2/4\pi mc^3\big]^2 = 6\pi c^2$$

so

$$\sigma_{\text{max}} = 6\pi(c/2\pi\nu_0)^2 = \tfrac{3}{2\pi}\lambda_0^2 \tag{499.2}$$

$$\sim \begin{cases} \text{cross-sectional area of the smallest object} \\ \text{visible in radiation of resonant frequency} \end{cases}$$

Radiation of resonant frequency, when incident upon a "gas" made of such "classical molecules," is scattered profusely (the gas becomes"florescent," and

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[304] Also—and with better reason—called the "Cauchy distribution function." See Abramowitz & Stegun, *Handbook of Mathematical Functions* (1964), page 930.

loses its transparency). Classically, we expect a molecule to possess a *variety* of normal modes . . . a variety of "characteristic frequencies," and resonance florescence to occur at each. Notice that if we were to neglect the self-interaction (formally: let $\tau \downarrow 0$ in (499.1)) then the resonant scattering cross-section would become infinite: $\sigma_{\text{max}} \uparrow \infty$. Here as in (for example) the elementary theory of forced damped oscillators, it is damping that accounts for finiteness at resonance.

HIGH-FREQUENCY REGIME If $\xi \gg 1$ then (495) becomes

$$\sigma_{\text{Rayleigh}} = \sigma_0 \cdot \frac{1}{1 + k^2\xi^2}$$

But $k\xi = (\tau\omega_0)(\omega/\omega_0) = \omega/\Omega \ll 1$ except when—as previously remarked—$\omega$ is so large as to render the classical theory meaningless. So the factor $(1 + k^2\xi^2)^{-1}$ can/should be abandoned. The upshot: *Rayleigh scattering reverts to Thomson scattering at frequencies* $\omega \gg$ *the molecular resonance frequency* $\omega_0$. Physically, the charge is stimulated so briskly that it does not feel its attachment to the slow spring, and responds like a *free* particle. It was to represent this fact that the red asymptote was introduced into Figure 147.

**9. Radiative decay.** Suppose now that the incident light beam is abruptly switched off. We expect the oscillating electrona to radiate its energy away, coming finally to rest. This is the process which, as explained below, gives rise to the classical theory of spectral line shape. The radiative relaxation of a harmonically bound classical electron is governed by

$$\ddot{\boldsymbol{x}} - \tau\dddot{\boldsymbol{x}} + \omega_0^2\boldsymbol{x} = \boldsymbol{0} \tag{500}$$

which is just the homogeneous counterpart of (494). Borrowing $\tau = k/\omega_0$ from page 408 and multiplying by $\omega_0$ we obtain

$$\omega_0\ddot{\boldsymbol{x}} - k\dddot{\boldsymbol{x}} + \omega_0^3\boldsymbol{x} = \boldsymbol{0}$$

which proves more convenient for the purposes at hand. Looking for solutions of the form $e^{i\omega t}$ we find that $\omega$ must be a root of the cubic polynomial

$$ik\omega^3 - \omega_0\omega + \omega_0^3 = 0$$

*Mathematica* provides complicated closed-form descriptions of those roots, which when expanded in powers of the dimensionless parameter $k$ become

$$\begin{aligned}
\omega_1 &= +\omega_0 + i\tfrac{1}{2}\omega_0 k - \tfrac{5}{8}\omega_0 k^2 - i\omega_0 k^3 + \cdots \\
\omega_2 &= -\omega_0 + i\tfrac{1}{2}\omega_0 k + \tfrac{5}{8}\omega_0 k^2 - i\omega_0 k^3 - \cdots \\
\omega_3 &= -i\omega_0\big\{k^{-1} + k - 2k^3 + 7k^5 - \cdots\big\}
\end{aligned}$$

The root $\omega_3$ we abandon as an unphysical artifact because

$$e^{i\omega_3 t} = \exp\Big[\omega_0\big\{k^{-1} + k - \cdots\big\}t\Big] \quad \text{very rapidly blows up}$$

That leaves us with two linearly independent solutions

$$e^{-\omega_0(\frac{1}{2}k-k^3+\cdots)\,t}\cdot e^{\pm\, i\,\omega_0(1-\frac{5}{8}\omega_0 k^2+\cdots)\,t}$$

and with the implication that

$$\boldsymbol{x}(t) = \mathfrak{X}\, e^{-\frac{1}{2}\omega_0 k\, t} \cos\big[(\omega_0 - \tfrac{5}{8}\omega_0 k^2)\, t\big]$$

is *in excellent approximation*[305] a particular solution of (500), and that so also is the function got by cos $\mapsto$ sin. In a standard notation

$$= \mathfrak{X}\, e^{-\frac{1}{2}\Gamma\, t} \cos\big[(\omega_0 - \Delta\omega)\, t\big] \tag{501}$$

where

$$\begin{aligned} \Gamma &\equiv \omega_0 k && \text{describes the } \textit{damping coefficient}\\ \Delta\omega &\equiv \tfrac{5}{8}\omega_0 k^2 && \text{describes a small } \textit{downward frequency shift} \end{aligned}$$

A function of the familiar design (501) is plotted in Figure 150.

Notice that it is *self-interaction*, as described by the small dimensionless parameter $k$, that is responsible both for the slow attenuation $e^{-\frac{1}{2}\Gamma\, t}$ and for the slight frequency shift $\Delta\omega$, and that attenuation causes the electronic oscillation (whence also the resulting radiation) to be not quite monochromatic. Turning to the Fourier transform tables (which in this instance serve better than *Mathematica*) we find[306]

$$e^{-\beta y}\cos\alpha y = (\beta/\pi)\int_0^\infty \Big\{\frac{1}{(x-\alpha)^2+\beta^2} + \frac{1}{(x+\alpha)^2+\beta^2}\Big\}\cos yx\, dx$$

The implication is that (501) can be expressed

$$\boldsymbol{x}(t) = \mathfrak{X}\int_0^\infty S(\omega)\cos\omega t\, d\omega \tag{502.1}$$

$$S(\omega) \equiv \frac{\Gamma}{2\pi}\Big\{\frac{1}{[\omega-(\omega_0-\Delta\omega)]^2+(\frac{1}{2}\Gamma)^2} + \frac{1}{[\omega+(\omega_0-\Delta\omega)]^2+(\frac{1}{2}\Gamma)^2}\Big\}$$

The second term is small even for $\omega = 0$ and dies rapidly as $\omega$ increases. We therefore *abandon* that term, and work in the good approximation that

$$S(\omega) \approx \frac{\Gamma}{2\pi}\frac{1}{[\omega-(\omega_0-\Delta\omega)]^2+(\frac{1}{2}\Gamma)^2} \tag{502.2}$$

---

[305] How excellent? *Mathematica* supplies

$$\Big\{\omega_0\tfrac{d^2}{dt^2} - k\tfrac{d^3}{dt^3} + \omega_0^3\Big\}e^{-\frac{1}{2}\omega_0 k\, t}e^{\pm\, i(\omega_0-\frac{5}{8}\omega_0 k^2)\, t}$$
$$= 0 + 0k + 0k^2 - i2\omega_0^3 k^3 + \tfrac{103}{64}\omega_0^3 k^4 + i\tfrac{105}{64}\omega_0^3 k^5 - \cdots$$

[306] A. Erdélyi *et al* (editors), *Tables of Integral Transforms* (1954), Volume I, Table 1.2#13 (page 8) and Table 1.6#19 (page 21).

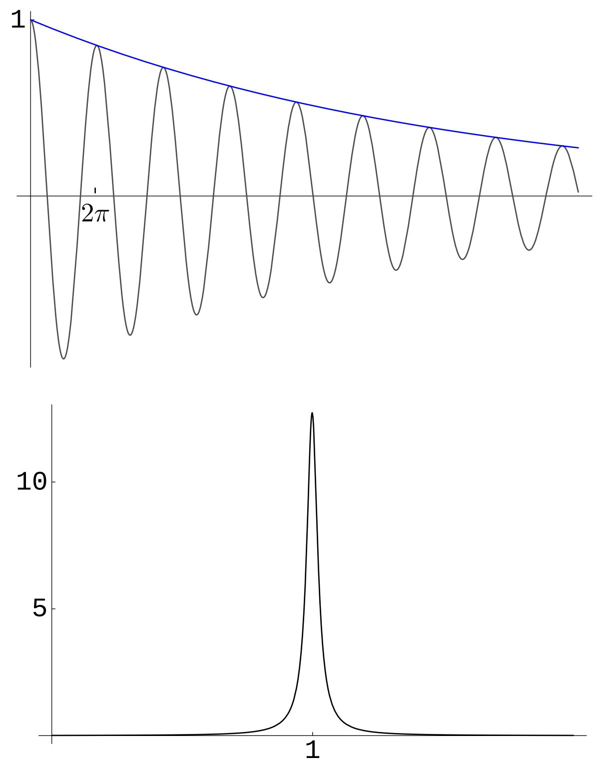


FIGURE 150: *Above: diagram of the motion of a charge-on-a-spring (Rayleigh's "classical molecule") that, because it experiences periodic acceleration, slowly radiates away its initial store of energy. The figure derives from* (501) *with* $\omega_0 = 1$ *and* $k = 0.05$. *The modulating exponential factor* $e^{-\frac{1}{2}\Gamma t}$ *is shown in blue. The Fourier transform of that curve (below) can be interpreted as a description what would be seen by a physicist who examines the emitted radiation with the aid of a spectroscope. The "spectral line" has a "Lorentzian" profile.*

At (502.2) we encounter once again—but this time in the frequency domain—precisely the Lorentz distribution

$$S(\omega) \approx L(\omega - [\omega_0 - \Delta\omega], \Gamma)$$

first encountered at (498), and the basis for the statement that

$$\text{Classical line shape is Lorentzian} \tag{503}$$

We digress to acquire familiarity with some of the basic properties of the Lorentz distribution function $L(x, \Gamma) \equiv \frac{\Gamma}{2\pi}[x^2 + (\frac{1}{2}\Gamma)^2]^{-1}$. Figure 151 shows the

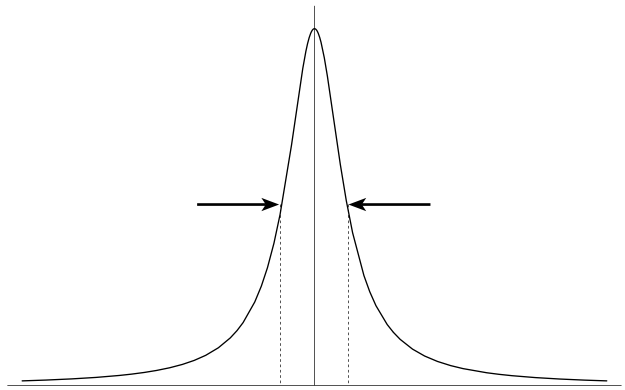

FIGURE 151: *Characteristic shaped of what physicists usually call the "Lorentz distribution" but mathematicians know as the "Cauchy distribution." Arrows mark the half-max points, and $\Gamma$ is shown in the text to be the distance between those points.*

characteristic shape of the Lorentz distribution. It is elementary that

$$L(x,\Gamma) \leqslant L_{\text{max}} = L(0,\Gamma) = \tfrac{2}{\pi\Gamma}$$

and that

$$L(x,\Gamma) = \tfrac{1}{2}L_{\text{max}} \quad\Longrightarrow\quad x = \pm\tfrac{1}{2}\Gamma$$

so the parameter $\Gamma$ can be interpreted

$$\Gamma = \text{width at half-max} \tag{504}$$

On casual inspection (Figure 152) the graphs of the Lorentz and Gaussian (or "normal") distributions appear quite similar, though the former has a noticeably sharper central peak and relatively wide hips. Richard Crandall's "The Lorentz distribution is a pig—too fat!" might seem uncharitable . . . until one looks to the *moments* of the two distributions. For the Gaussian the sequence

$$\langle x^0\rangle, \langle x^1\rangle, \langle x^2\rangle, \langle x^3\rangle, \langle x^4\rangle, \langle x^5\rangle, \langle x^6\rangle, \langle x^7\rangle, \langle x^8\rangle, \ldots$$

proceeds unremarkably

$$1,\ \ 0,\ \ \tfrac{1}{2}a^2,\ \ 0,\ \ \tfrac{3}{4}a^4,\ \ 0,\ \ \tfrac{15}{8}a^6,\ \ 0,\ \ \tfrac{105}{16}a^2, \ldots$$

but in the case of the Lorentz distribution even the *definition* of the moments is a bit problematic (as *Mathematica* is quick to remind us): if we proceed from the definition $\langle x^n\rangle \equiv \lim_{z\uparrow\infty}\int_{-z}^{+z} x^n L(x,\Gamma)\,dx$ we obtain

$$1,\quad 0,\quad \infty,\quad 0,\quad \infty,\quad 0,\quad \infty,\quad 0,\quad \infty, \ldots$$

So wide are the hips of the Lorentz distribution that (in particular)

$$\Delta x \equiv \sqrt{\langle (x - \langle x\rangle)^2\rangle} = \infty$$

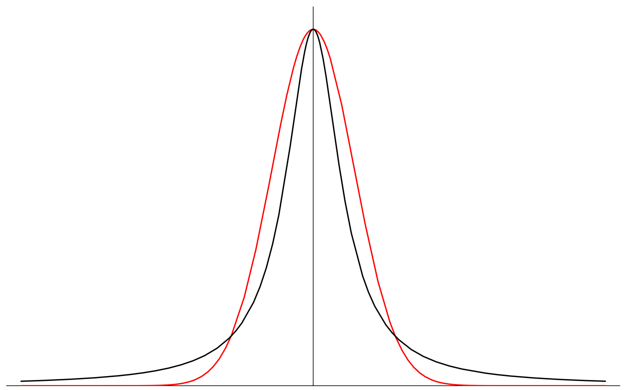

FIGURE 152: *The Lorentz distribution* $L(x, \mathit{\Gamma}) \equiv \frac{\mathit{\Gamma}}{2\pi}[x^2 + (\frac{1}{2}\mathit{\Gamma})^2]^{-1}$ *has here been superimposed upon the Gaussian distribution*

$$G(x, a) \equiv \tfrac{1}{a\sqrt{\pi}}e^{-(x/a)^2}$$

*of the same height (set* $a = \frac{\sqrt{\pi}}{2}\mathit{\Gamma}$*). The Lorentz distribution is seen to have a relatively sharp peak, but relatively broader flanks.*

The standard descriptor of the "width" of the distribution is therefore not available: to provide such information one is *forced* to adopt (504). It is remarkable that, of two distributions that—when plotted—so nearly resemble one another,

- one is arguably "the best behaved in the world," and
- the other one of the worst behaved.[307]

And it is in that light remarkable that in some other respects the Lorentz distribution is quite unexceptional: for example, it leads straightforwardly to a representation of the $\delta$-function

$$\delta(x - x_0) = \lim_{\mathit{\Gamma}\downarrow 0} L(x - x_0, \mathit{\Gamma}) = \lim_{\epsilon\downarrow 0} \frac{\epsilon/\pi}{(x - x_0)^2 + \epsilon^2}$$

that often proves useful in applications. Returning now to the physics . . .

The classical theory of spectral line shape marks an interesting point in the *history* of physics, but leads to results which are of enduring interest only as zeroth approximations to their quantum counterparts. As such, they are

[307] It was known to Poisson already in 1824 that what came to be called the "Cauchy distribution" is a distribution to which the fundamental "central limit theorem" does not pertain. Cauchy himself entered the picture only in 1853—the year of Lorentz' birth. My source here has been the footnote that appears on page 183 of S. M. Stigler's *The History of Statistics* (1986).

remarkably good. To illustrate the point: Reading from (501) we see that our "classical molecule" has a

$$\text{characteristic lifetime} = 2/\Gamma$$

while its

$$\text{spectral linewidth} = \Gamma/2$$

Evidently

$$(\text{linewidth})\cdot(\text{lifetime}) = 1 \tag{505}$$

*Quantum* mechanically, spectral line shape arises in first approximation (*via* $E = h\nu = \hbar\omega$) from an instance of the *Heisenberg uncertainty principle*, according to which

$$\Delta E \cdot (\text{lifetime}) \gtrsim \hbar$$

But $\Delta E = \hbar \cdot (\text{linewidth})$ so we are, in effect, led back again to the classical relation (505). Similar parallels could be drawn from the quantum theory of electromagnetic scattering processes.[308]

**10. Concluding remarks.** Classical radiation theory, though latent in Maxwell's equations, is a subject of which Maxwell himself knew nothing. Its development was stimulated by Hertz' experimental production/detection of electromagnetic waves—a development which Maxwell anticipated, but did not live long enough to see—and especially by the technological effort which attended the invention of radio. It is a subject of which we have only scratched the surface: we have concentrated on the radiation produced by individual accelerated charges, and remain as innocent as babies concerning the fields produced by the currents that flow in the antenna arrays that several generations of radio engineers have worked so ingeniously to devise.

The subject leads, as we have witnessed, to mathematical relationships notable for their complexity. But those intricate relationships among $\boldsymbol{E}$'s, $\boldsymbol{B}$'s, the elements of $S^{\mu\nu}$ ... sprang from relatively simple properties of the potentials $A^\mu$. Indeed, the work of this entire chapter (chapter in the text, chapter in the history of pure/applied physics) can be viewed as an exercise in applied potential theory. It is curious that—in electrodynamics most conspicuously, but also elsewhere in physics—*it appears to be the spooks who speak the language of God, and is in any event certainly the spooks who coordinate our effort to account for and describe the complexity evident in the observable/tangible world of direct experience.*

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Our progress thus far has (in 418 pages and $\sim 60$ hours) taken us in a fairly direct path from the "beginning" or our subject to within sight of its "end" ... from a discussion of first principles and historical roots into the realm where

[308] See, for example, W. Heitler, *Quantum Theory of Radiation* (1954).

electrodynamics shows an ever-stronger tendency to break down. Along the way, electrodynamics gave birth to special relativity (who has long since left home to lead an independent existence elsewhere) . . . and as we take leave of the lady she is clearly once again pregnant (with quantum mechanics, elementary particle physics, general relativity, . . . ). Her best years—if no longer as a dancer, then as a teacher of dance—lie still ahead. But that is another story for another day. In the pages that follow we will be *backtracking*—discussing miscellaneeous issues that, for all their theoretical/technological importance, were judged to be peripheral to our initial effort.