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MECHANICAL PROPERTIES OF THE ELECTROMAGNETIC FIELD

Densities, fluxes & conservation laws

Introduction. Energy, momentum, angular momentum, center of mass, moments of inertia . . . these are concepts which derive historically from the mechanics of particles. And it is from particle mechanics that—for reasons that are interesting to contemplate—they derive their intuitive force. But these are concepts which are now recognized to pertain, if in varying degrees, to the *totality* of physics. My objective here will be to review how the mechanical concepts listed above pertain, in particular, to the electromagnetic field. The topic is of great practical importance. But it is also of some philosophical importance . . . for it supplies the evidence on which we would assess the ontological question: *Is the electromagnetic field "real"*?

How to proceed? Observe that in particle mechanics the concepts in question arise not as "new physics" but as *natural artifacts implicit in the design of the equations of motion*. We may infer that the definitions we seek

i) will arise as "natural artifacts" from Maxwell's equations

ii) must mesh smoothly with their particulate counterparts.

But again: how—within those guidelines—to proceed? The literature provides many alternative lines of argument, the most powerful of which lie presently beyond our reach.[166] In these pages I will outline two complementary

[166] I am thinking here of the Lagrangian formulation of the classical theory of fields, which is usually/best studied as an antonomous subject, then *applied* to electrodynamics as a (rather delicate) special case.

approaches to the electrodynamical concepts of *energy* and *momentum*. The first approach is inductive, informal. The second is deductive, and involves formalism of a relatively high order. Both approaches (unlike some others) draw explicitly on the spirit and detailed substance of relativity. The discussion will then be extended to embrace *angular momentum* and certain more esoteric notions.

1. Electromagnetic energy/momentum: first approach. We know from prior work of an elementary nature[167] that it makes a certain kind of sense to write

$$\left.\begin{aligned} \tfrac{1}{2}\boldsymbol{E}\cdot\boldsymbol{E} &= \text{energy density of an electrostatic field} \\ \tfrac{1}{2}\boldsymbol{B}\cdot\boldsymbol{B} &= \text{energy density of a magnetostatic field} \end{aligned}\right\} \tag{302}$$

But what should we write to describe the energy density $\mathcal{E}$ of an unspecialized electro*dynamical* field? Relativity suggests that we should consider this question in intimate association with a second question: What should we write to describe the *momentum* density $\boldsymbol{\mathcal{P}}$ of an arbitrary electromagnetic field? We are led thus to anticipate[168] the theoretical importance of a *quartet of densities*

$$\mathcal{P} = \begin{pmatrix} \mathcal{P}^0 \\ \mathcal{P}^1 \\ \mathcal{P}^2 \\ \mathcal{P}^3 \end{pmatrix} \quad \text{with } \mathcal{P}^0 \equiv \tfrac{1}{c}\mathcal{E} \tag{303}$$

where $[\mathcal{P}^\mu] = \text{momentum/3-volume}$.

Intuitively we expect *changes* in the energy/momentum at a spacetime point to arise from a combination of

1) the corresponding fluxes (or energy/momentum "currents")
2) the local action of charges (or "sources")

so *at source-free points* we expect[169] to have

$$\begin{aligned} \tfrac{\partial}{\partial t}\mathcal{E} + \boldsymbol{\nabla}\cdot(\text{energy flux vector}) &= 0 \\ \tfrac{\partial}{\partial t}\mathcal{P}^1 + \boldsymbol{\nabla}\cdot(\text{flux vector associated with } 1^{\text{st}} \text{ component of momentum}) &= 0 \\ \tfrac{\partial}{\partial t}\mathcal{P}^2 + \boldsymbol{\nabla}\cdot(\text{flux vector associated with } 2^{\text{nd}} \text{ component of momentum}) &= 0 \\ \tfrac{\partial}{\partial t}\mathcal{P}^3 + \boldsymbol{\nabla}\cdot(\text{flux vector associated with } 3^{\text{rd}} \text{ component of momentum}) &= 0 \end{aligned}$$

This quartet of conservation laws would be expressed quite simply

$$\partial_\mu S^{\mu\nu} = 0 \quad : \quad (\nu = 0, 1, 2, 3) \tag{304}$$

[167] The argument proceeded from elementary mechanics in the electrostatic case (pages 19–24), but was more formal/tentative (page 60) and ultimately more intricate (pages 97–98) in the magnetostatic case.
[168] See again pages 193 and 194.
[169] See again pages 36–37.

if we were to set (here the Roman indices i and j range on $\{1, 2, 3\}$)

$$\left.\begin{aligned} \mathcal{E} \equiv c\mathcal{P}^0 \equiv S^{00} &\equiv \text{energy density} \\ S^{i\,0} &\equiv \tfrac{1}{c}(i^{\text{th}} \text{ component of the energy flux vector}) \\ c\mathcal{P}^j \equiv S^{0j} &\equiv c\,(j^{\text{th}}\text{-component-of-momentum density}) \\ S^{i\,j} &\equiv \quad (i^{\text{th}} \text{ component of the } \mathcal{P}^j \text{ flux vector}) \end{aligned}\right\} \tag{305}$$

where c-factors have been introduced to insure that the $S^{\mu\nu}$ all have the same dimensionality—namely that of $\mathcal{E}$.

Not only are equations (304) wonderfully compact, they seem on their face to be "relativistically congenial." They become in fact *manifestly Lorentz covariant* if it is assumed that

$$S^{\mu\nu} \text{ transforms as a } 2^{\text{nd}} \text{ rank tensor} \tag{306}$$

of presently unspecified weight. This natural assumption carries with it the notable consequence that

$$\text{The } \mathcal{P}^\mu \equiv \tfrac{1}{c}S^{0\mu} \text{ do } \textit{not} \text{ transform as components of a 4-vector}$$

or even (as might have seemed more likely) as components of a 4-vector *density*.

The question from which we proceeded—How to describe $\mathcal{E}$ as a function of the dynamical field variables?—has now become sixteen questions: *How to describe* $S^{\mu\nu}$? But our problem is not on this account sixteen times harder, for (304) and (306) provide powerful guidance. Had we proceeded naively (*i.e.*, without reference to relativity) then we might have been led from the structure of (302) to the conjecture that $\mathcal{E}$ depends in the general case upon $\boldsymbol{E}\cdot\boldsymbol{E}$, $\boldsymbol{B}\cdot\boldsymbol{B}$, maybe $\boldsymbol{E}\cdot\boldsymbol{B}$ and upon scalars formed from $\dot{\boldsymbol{E}}$ and $\dot{\boldsymbol{B}}$ (terms that we would not see in static cases). Relativity suggests that $\mathcal{E}$ should then depend also upon $\boldsymbol{\nabla}\cdot\boldsymbol{E}$, $\boldsymbol{\nabla}\cdot\boldsymbol{B}$, $\boldsymbol{\nabla}\times\boldsymbol{E}$, $\boldsymbol{\nabla}\times\boldsymbol{B}$, ... but such terms are—surprisingly—absent from (302). Equations (304) and (306) enable us to recast this line of speculation ... as follows:

1) We expect $S^{\mu\nu}$ to be a tensor-valued function of $g_{\mu\nu}$, $F_{\mu\nu}$, $F^{\bigstar}_{\mu\nu}$ and possibly of $\partial_\alpha F_{\mu\nu}$, $\partial_\alpha\partial_\beta F_{\mu\nu}$, ... with the property that
2) S^{00} gives back (302) in the electrostatic and magnetostatic cases. We require, moreover, that
3) In source-free regions it shall be the case that Maxwell's equations

$$\partial_\mu F^{\mu\nu} = 0 \text{ and } \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \quad\Longrightarrow\quad \partial_\mu S^{\mu\nu} = 0$$

Two further points merit attention:

4) Dimensionally $[S^{\mu\nu}] = [F^{\mu\nu}]^2$: $S^{\mu\nu}$ is in this sense a *quadratic* function of $F^{\mu\nu}$.
5) Source-free electrodyanmics contains but a single physical constant, namely c: it contains in particular *no natural length*[170]... so one must make do with *ratios* of ∂F-terms, which are transformationally unnatural.

[170] That's a symptom of the *conformal covariance* of the theory.

Motivated now by the 2nd and 4th of those points, we look to the explicit descriptions (159) and (161) of $\|F^{\mu\nu}\|$ and $\|G^{\mu\nu}\|$ and observe that by direct computation[171]

$$\left.\begin{aligned}
\|F^\mu{}_\alpha F^{\alpha\nu}\| &= \begin{pmatrix} \boldsymbol{E}\cdot\boldsymbol{E} & (\boldsymbol{E}\times\boldsymbol{B})_1 & (\boldsymbol{E}\times\boldsymbol{B})_2 & (\boldsymbol{E}\times\boldsymbol{B})_3 \\ (\boldsymbol{E}\times\boldsymbol{B})_1 & C_{11}+\boldsymbol{B}\cdot\boldsymbol{B} & C_{12} & C_{13} \\ (\boldsymbol{E}\times\boldsymbol{B})_2 & C_{21} & C_{22}+\boldsymbol{B}\cdot\boldsymbol{B} & C_{23} \\ (\boldsymbol{E}\times\boldsymbol{B})_3 & C_{31} & C_{32} & C_{33}+\boldsymbol{B}\cdot\boldsymbol{B} \end{pmatrix} \\
\|F^\mu{}_\alpha G^{\alpha\nu}\| &= -\boldsymbol{E}\cdot\boldsymbol{B}\|g^{\mu\nu}\| \\
\|G^\mu{}_\alpha G^{\alpha\nu}\| &= \begin{pmatrix} \boldsymbol{B}\cdot\boldsymbol{B} & (\boldsymbol{E}\times\boldsymbol{B})_1 & (\boldsymbol{E}\times\boldsymbol{B})_2 & (\boldsymbol{E}\times\boldsymbol{B})_3 \\ (\boldsymbol{E}\times\boldsymbol{B})_1 & C_{11}+\boldsymbol{E}\cdot\boldsymbol{E} & C_{12} & C_{13} \\ (\boldsymbol{E}\times\boldsymbol{B})_2 & C_{21} & C_{22}+\boldsymbol{E}\cdot\boldsymbol{E} & C_{23} \\ (\boldsymbol{E}\times\boldsymbol{B})_3 & C_{31} & C_{32} & C_{33}+\boldsymbol{E}\cdot\boldsymbol{E} \end{pmatrix} \\
&= \|F^\mu{}_\alpha F^{\alpha\nu}\| - (\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B})\cdot\|g^{\mu\nu}\|
\end{aligned}\right\} \tag{307}$$

where $C_{ij} \equiv -E_iE_j - B_iB_j$.[172] The arguments that gave (302) assumed in the first instance that $\boldsymbol{B} = \boldsymbol{0}$ and in the second instance that $\boldsymbol{E} = \boldsymbol{0}$, so provide no evidence whether we should in the general case expect the presence of an $\boldsymbol{E}\cdot\boldsymbol{B}$ term. If we assume tentatively that in the general case

$$S^{00} \equiv \mathcal{E} = \tfrac{1}{2}\boldsymbol{E}\cdot\boldsymbol{E} + \tfrac{1}{2}\boldsymbol{B}\cdot\boldsymbol{B} + \lambda\boldsymbol{E}\cdot\boldsymbol{B} \quad : \quad \lambda \text{ an adjustable constant}$$

then we are led by (307) to write

$$\begin{aligned}
S^{\mu\nu} &= \tfrac{1}{2}F^\mu{}_\alpha F^{\alpha\nu} + \tfrac{1}{2}G^\mu{}_\alpha G^{\alpha\nu} - \lambda F^\mu{}_\alpha G^{\alpha\nu} \\
&= \tfrac{1}{2}F^\mu{}_\alpha F^{\alpha\nu} + \tfrac{1}{2}\big[F^\mu{}_\alpha F^{\alpha\nu} - \tfrac{1}{2}(F^{\alpha\beta}F_{\beta\alpha})g^{\mu\nu}\big] - \lambda\tfrac{1}{4}(F^{\alpha\beta}G_{\beta\alpha})g^{\mu\nu} \\
&= F^\mu{}_\alpha F^{\alpha\nu} - \tfrac{1}{4}F^{\alpha\beta}(F_{\beta\alpha} + \lambda G_{\beta\alpha})g^{\mu\nu}
\end{aligned} \tag{308}$$

We come now to the critical question: Does the $S^{\mu\nu}$ of (308) satisfy (304)? The answer can be discovered only by computation: we have

$$\partial_\mu S^\mu{}_\nu = \underbrace{(\partial_\mu F^{\mu\alpha})F_{\alpha\nu}}_{\textbf{a}} + \underbrace{F^{\mu\alpha}\partial_\mu F_{\alpha\nu}}_{\textbf{b}} - \underbrace{\tfrac{1}{4}\partial_\nu(F^{\alpha\beta}F_{\beta\alpha})}_{\textbf{c}} - \lambda\tfrac{1}{4}\partial_\nu(F^{\alpha\beta}G_{\beta\alpha})$$

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171 PROBLEM 54.

172 Recall in this connection that the Lorentz invariance of

$$\begin{aligned}
\tfrac{1}{2}F^{\alpha\beta}F_{\beta\alpha} &= \boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B} = -\tfrac{1}{2}G^{\alpha\beta}G_{\beta\alpha} \\
\tfrac{1}{4}F^{\alpha\beta}G_{\beta\alpha} &= -\boldsymbol{E}\cdot\boldsymbol{B}
\end{aligned}$$

was established already in PROBLEM 48b.

But

$$
\begin{aligned}
\textbf{a} &= 0 \quad \text{by Maxwell: } \partial_\mu F^{\mu\alpha} = \tfrac{1}{c}J^\alpha \text{ and we have assumed } J^\alpha = 0\\
\textbf{b} &= \tfrac{1}{2}F^{\mu\alpha}(\partial_\mu F_{\alpha\nu} - \partial_\alpha F_{\mu\nu}) \quad \text{by antisymmetry of } F^{\mu\alpha}\\
&= \tfrac{1}{2}F^{\mu\alpha}(\partial_\mu F_{\alpha\nu} + \partial_\alpha F_{\nu\mu}) \quad \text{by antisymmetry of } F_{\mu\nu}\\
&= -\tfrac{1}{2}F^{\mu\alpha}\partial_\nu F_{\mu\alpha} \quad \text{by Maxwell: } \partial_\mu F_{\alpha\nu} + \partial_\alpha F_{\nu\mu} + \partial_\nu F_{\mu\alpha} = 0\\
&= \tfrac{1}{4}\partial_\nu(F^{\alpha\beta}F_{\beta\alpha})\\
&= \textbf{c}
\end{aligned}
$$

so we have

$$
\begin{aligned}
\partial_\mu S^\mu{}_\nu &= -\lambda\tfrac{1}{4}\partial_\nu(F^{\alpha\beta}G_{\beta\alpha})\\
&= \lambda\partial_\nu(\boldsymbol{E}\cdot\boldsymbol{B})
\end{aligned}
$$

It is certainly not in general the case that $\boldsymbol{E}\cdot\boldsymbol{B}$ is x-independent (as $\partial_\nu(\boldsymbol{E}\cdot\boldsymbol{B})=0$ would require) so to achieve

$$= 0$$

we are <u>forced to set $\lambda = 0$</u>. Returning with this information to (308) we obtain

$$
\begin{aligned}
S^{\mu\nu} &= \tfrac{1}{2}(F^\mu{}_\alpha F^{\alpha\nu} + G^\mu{}_\alpha G^{\alpha\nu})\\
&= F^\mu{}_\alpha F^{\alpha\nu} - \tfrac{1}{4}(F^{\alpha\beta}F_{\beta\alpha})g^{\mu\nu}
\end{aligned}
\tag{309}
$$

. . . which possesses all of the anticipated/required properties (see again the list on page 213), and in addition posses two others: $S^{\mu\nu}$ is *symmetric*

$$S^{\mu\nu} = S^{\nu\mu} \tag{310}$$

and *traceless*

$$S^\alpha{}_\alpha = 0 \tag{311}$$

Equation (309) describes the elements of what is called the *electromagnetic <u>stress-energy tensor</u>*. Reading from (307) we obtain

$$
\|S^{\mu\nu}\| = \begin{pmatrix} \tfrac{1}{2}(E^2+B^2) & (\boldsymbol{E}\times\boldsymbol{B})^{\mathsf{T}}\\ (\boldsymbol{E}\times\boldsymbol{B}) & \mathbb{T}\end{pmatrix}
$$

where $E^2 \equiv \boldsymbol{E}\cdot\boldsymbol{E}$, $B^2 \equiv \boldsymbol{B}\cdot\boldsymbol{B}$ and where

$$\mathbb{T} \equiv \|(\tfrac{1}{2}E^2\delta_{ij} - E_iE_j) + (\tfrac{1}{2}B^2\delta_{ij} - B_iB_j)\|$$

is the negative of what is—for historical reasons—called the "Maxwell stress tensor" (though it is, with respect to non-rotational elements of the Lorentz group, *not* a tensor!). Writing

$$
= \begin{pmatrix} \mathcal{E} & c\boldsymbol{\mathcal{P}}^{\mathsf{T}}\\ \tfrac{1}{c}\boldsymbol{S} & \mathbb{T}\end{pmatrix}
$$

we conclude (see again page 212) that

$\mathcal{E} = \frac{1}{2}(E^2 + B^2)$ describes **energy density**. This construction was first studied by W. Thompson (Lord Kelvin) in 1853.

$\boldsymbol{S} = c(\boldsymbol{E}\times\boldsymbol{B})$ describes **energy flux**. This construction was discovered by J. H. Poynting and (independently) by O. Heaviside in 1884. It is called the "Poynting vector" (though it is vectorial only with respect to the rotation group).

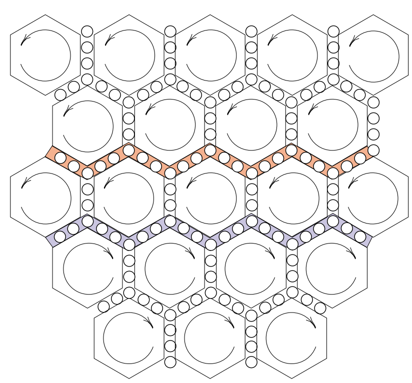
$\boldsymbol{\mathcal{P}} = \frac{1}{c}(\boldsymbol{E}\times\boldsymbol{B})$ describes **momentum density**, and was discovered by J. J. Thompson in 1893.

The successive columns in $\mathbb{T}$ are **momentum fluxes** associated with the successive elements of $\boldsymbol{\mathcal{P}}$. The "stress tensor" was introduced by Maxwell, but to fill quite a different formal need.[173]

It is remarkable that the individual elements of the stress-energy tensor issued historically from so many famous hands . . . and over such a protracted period of time.

The following comments draw attention to aspects of the specific design (309) of the electromagnetic stress-energy tensor $S^{\mu\nu}$:

[173] Maxwell considered it to be his job to describe the "mechanical properties of the æther," and so found it natural to borrow concepts from fluid dynamics and the theory of elastic media. The following design—taken from his "On



physical lines of force" (1861)—illustrates how fantastic he allowed his mechanical imagination to become [see R. Tricker, *Contributions of Faraday & Maxwell to Electrical Science* (1966) page 118 or C. Everitt, *James Clerk Maxwell: Physicist & Natural Philosopher* (1975) page 96 for accounts of the idea the figure was intended to convey]. In his *Treatise* Maxwell writes that he was "only following out the conception of Faraday, that lines of force tend to shorten themselves, and that they repel each other when placed side by side: all that we have done is express the value of the tension along the lines, and the pressure at right angles to them, in mathematical language . . . "

1. Though we have already noted (page 213) that—in view of the facts that $[F^{\mu\nu}] = \sqrt{\text{energy density}}$ and electrodynamics supplies no "natural length"—it would be difficult to build ∂F-dependence into the design of $S^{\mu\nu}$, it still seems remarkable that we have achieved success with a design that *depends not at all on the derivatives of the field* . . . for elsewhere in physics *energy and momentum typically depend critically upon time-derivatives of the dynamical variables*. It was on account of this electrodynamical quirk that the static arguments that gave (302) led us to an $\mathcal{E}$ found to pertain also to *dynamical* fields.

2. It is gratifying that energy density (and therefore also the integrated total energy) is <u>bounded below</u>:

$$S^{00} \equiv \mathcal{E} \geqslant 0 \quad : \quad \text{vanishes if and only if } F^{\mu\nu} \text{ vanishes}$$

For otherwise the electromagnetic field would be an insatiable energy sink (in short: a "rat hole") and would de-stabilize the universe.

3. From the fact that $S^{\mu\nu}$ is a *quadratic* function of $F^{\mu\nu}$ it follows (see again (45) on page 24) if follows that <u>stress-energy does not superimpose</u>:

$$\begin{array}{l} F^{\mu\nu} = F_1^{\mu\nu} + F_2^{\mu\nu} \quad : \quad \text{superimposed fields} \\ \qquad\downarrow \\ S^{\mu\nu} = S_1^{\mu\nu} + S_2^{\mu\nu} + (\text{cross term}) \end{array}$$

4. From the symmetry of $S^{\mu\nu}$ it follows rather remarkably that

$$\text{energy flux} \sim \text{momentum density} \quad : \quad \boldsymbol{S} = c^2\boldsymbol{\mathcal{P}}$$

The discussion that led from (302) to (309) can be read as a further example of the "bootstrap method in theoretical physics," but has been intended to illustrate the *theory-shaping power of applied relativity*. With a little physics and a modest amount of relativity one can often go a remarkably long way. In the present instance—taking a conjectured description of S^{00} as our point of departure—we have managed to deduce the design of all fifteen of the other elements of $S^{\mu\nu}$, and to achieve at (309) a highly non-obvious result of fundamental physical importance.

Suppose now we were to **abandon our former assumption** that $F^{\mu\nu}$ moves "freely;" *i.e.*, that $J^\nu = 0$. The argument that led from the bottom of page 214 to (309) then supplies

$$\begin{array}{l} \partial_\mu S^\mu{}_\nu = \frac{1}{c}J^\alpha F_{\alpha\nu} + \underbrace{\mathbf{b} - \mathbf{c}}_{0 \text{ by previous argument}} \\ \qquad\downarrow \\ \partial_\mu S^{\mu\nu} = -\frac{1}{c}F^\nu{}_\alpha J^\alpha \end{array} \tag{312}$$

The flux components $\bullet$ of the stress-energy tensor

$$\mathbb{S} = \begin{pmatrix} \circ & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

describe how energy/momentum are sloshing about in spacetime, causing local adjustments of the energy/momentum densities ◦. It becomes in this light natural to suppose that the expression on the right side of (312)

$$-\tfrac{1}{c}F^{\nu}{}_{\alpha}J^{\alpha}\begin{cases}\text{describes locally the rate at which energy/momentum}\\ \text{are being } \textit{exchanged} \text{ between the electromagnetic field}\\ F^{\mu\nu} \text{ and the source field } J^{\mu}\end{cases}$$

We turn now to a discussion intended to lend substance to that interpretation.

2. Electromagnetic energy/momentum: second approach. We know that in the presence of an *impressed* electromagnetic field $F^{\mu\nu}$ a charged particle feels a Minkowski force given (see again page 198) by

$$K^{\mu} = (q/c)F^{\mu}{}_{\nu}u^{\nu} \tag{295}$$

. . . to which the particle responds by changing its energy/momentum; *i.e.*, by exchanging energy/momentum with—ultimately—the agent who impressed the field (the field itself acting here as intermediary). I propose to adjust the image—to remove the puppeteer ("agent") and let the puppets themselves (electromagnetic field on the one hand, charged matter on the other) battle it out. For formal reasons—specifically: to avoid the conceptual jangle that tends to arise when *fields* rub elbows with *particles*—it proves advantageous in this context to *consider the source to be spatially distributed*, having the nature of a charged fluid/gas/dust cloud, from which we recover particulate sources as a kind of degenerate limit: "lumpy gas." But to carry out such a program we must have some knowledge of the basic rudiments of fluid mechanics—a subject which was, by the way, well-known to Maxwell,[174] and from which (see again the words quoted in footnote #173) he drew some of his most characteristic images and inspiration.

DIGRESSION: ELEMENTARY ESSENTIALS OF FLUID DYNAMICS

Fluid dynamics is a phenomenological theory, formulated without explicit reference to the underlying microscopic physics.[175] It seeks to develop the $(\boldsymbol{x}, t)$-dependence of

- $\rho(\boldsymbol{x}, t)$, a scalar field which describes *mass density*, and
- $\boldsymbol{v}(\boldsymbol{x}, t)$, a vector field which describes *fluid velocity*.

The product of these admits of two modes of interpretation:

$$\rho\,\boldsymbol{v} \equiv \text{mass current} = \text{momentum density}$$

[174] G. G. Stokes (1819–1903) was twelve years older than Maxwell, and had completed most of his fluid dynamical work by 1850.

[175] . . . Imagined by Navier to be "atomic." Stokes, on the other hand, was not yet convinced of the reality of atoms, and contrived to do without the assistance that might be gained from an appeal to the "atomic hypothesis."

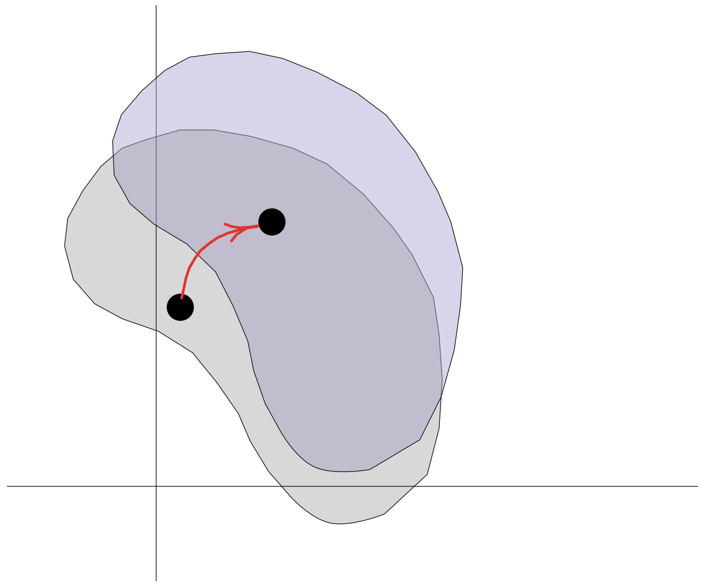


FIGURE 77: *A designated drop of liquid (think of a drop of ink dripped into a glass of water) shown at times t and $t + dt$. Every point in the evolved drop originated as a point in the initial drop. Not shown is the surrounding fluid.*

(in which connection it is instructive to recall that two pages ago we encountered

$$\tfrac{1}{c^2}\boldsymbol{S} = \boldsymbol{\mathcal{P}} \quad : \quad \text{mass flux} \equiv \frac{\text{energy flux}}{c^2} = \text{momentum density}$$

as an expression of the symmetry of a stress-energy tensor). The first of those interpretations supplies

$$\tfrac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\,\boldsymbol{v}) = 0 \tag{313}$$

as an expression of <u>mass conservation</u> . . . while from the second interpretation we infer that the i^{th} component of momentum of a designated drop V of fluid can at times t and $t + dt$ be described

$$\underbrace{\int_V \rho(\boldsymbol{x}, t + dt)v_i(\boldsymbol{x}, t + dt)d^3x}_{\mathbf{a}_i} \quad \text{and} \quad \int_V \rho(\boldsymbol{x}, t)v_i(\boldsymbol{x}, t)d^2x$$

The integrals (see Figure 77) range over distinct domains, but can be made to range over the same domain by a change of variables:

$$\mathbf{a}_i = \int_V \rho(\boldsymbol{x} + \boldsymbol{v}\,dt, t + dt)v_i(\boldsymbol{x} + \boldsymbol{v}\,dt, t + dt)\left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}}\right| d^3x$$

Expanding $\rho(\boldsymbol{x}+\boldsymbol{v}\,dt, t+dt)$, $v_i(\boldsymbol{x}+\boldsymbol{v}\,dt, t+dt)$ and the Jacobian[176]

$$\left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}}\right| = \begin{vmatrix} 1+v_{11}dt & v_{12}dt & v_{13}dt \\ v_{21}dt & 1+v_{22}dt & v_{23}dt \\ v_{31}dt & v_{32}dt & 1+v_{33}dt \end{vmatrix} = 1 + \underbrace{(v_{11}+v_{22}+v_{33})}_{\boldsymbol{\nabla}\cdot\boldsymbol{v}}\,dt + \cdots$$

we obtain

$$\mathsf{a}_i = \int_V \Big\{\rho v_i + \big[\big(\tfrac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\big)\rho v_i + \rho v_i \boldsymbol{\nabla}\cdot\boldsymbol{v}\big]\,dt + \cdots\Big\}\,d^3x$$

From this it follows that the *temporal rate of change of the* i^{th} *component of the momentum of our representative drop* can be described

$$\dot{P}_i(\text{drop}) = \int_V \Big[\big(\tfrac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\big)\rho v_i + \rho v_i \boldsymbol{\nabla}\cdot\boldsymbol{v}\Big]\,d^3x \tag{314.1}$$

This quantity arises physically from *forces experienced by* our drop, which can be considered to be of two types:

$$\textbf{impressed volume forces} \quad : \quad \int_V f_i(\boldsymbol{x},t)\,d^3x$$
$$\textbf{surface forces} \quad : \quad \int_{\partial V} \boldsymbol{\sigma}_i\cdot \boldsymbol{dS} = \int_V \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_i\,d^3x$$

The latter describe *interaction of the drop with adjacent fluid elements*. So we have

$$= \int_V \Big[f_i + \sum_j \frac{\partial \sigma_{ij}}{\partial x^j}\Big]\,d^3x \tag{314.2}$$

where σ_{ij} refers to the j^{th} component of $\boldsymbol{\sigma}_i$. The right sides of equations (314) are equal *for all* V so evidently

$$\underbrace{\big(\tfrac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\big)\rho v_i + \rho v_i \boldsymbol{\nabla}\cdot\boldsymbol{v}} = f_i + \sum_j \frac{\partial \sigma_{ij}}{\partial x^j}$$

These are **Euler's equations of fluid motion**, and can be notated in a great variety of ways: from

$$\begin{aligned} &= \tfrac{\partial}{\partial t}(\rho v_i) + \partial_j(\rho v_i v_j) \\ &= v_i \underbrace{\big[\tfrac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{v})\big]}_{0 \text{ by mass conservation (313)}} + \rho\big(\tfrac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\big)v_i \end{aligned}$$

we see that we can, in particular, write

[176] Here $v_{i\,j} \equiv \partial v_i/\partial x^j$.

$$\tfrac{\partial}{\partial t}(\rho v_i) + \partial_j(\rho v_i v_j - \sigma_{ij}) = f_i \tag{315}$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \llcorner$ impressed *force density*

...but any attempt to *solve* equations (313) and (315) must await <u>structural specification of the "stress tensor"</u> σ_{ij}. It is in this latter connection that specific fluid models are described/distinguished/classified. General considerations (angular momentum conservation) can be shown to force the *symmetry* of the stress tensor ($\sigma_{ij} = \sigma_{ji}$), but still leave the model-builder with a vast amount of freedom. "Newtonian fluids" arise from the assumption

$$\sigma_{ij} = -p\,\delta_{ij} + \sum_{k,l} \mathcal{D}_{ijkl} V_{kl}$$

where $V_{kl} \equiv \frac{1}{2}(\partial_l v_k + \partial_k v_l)$ are components of the so-called "rate of deformation tensor," where the $\mathcal{D}_{ijkl}$ are the so-called "viscosity coefficients" and where p is the "static pressure." Isotropy (the rotational invariance of $\mathcal{D}_{ijkl}$) can be shown to entail $\mathcal{D}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ and thus to reduce the number of independently specifiable $\mathcal{D}$-coefficients from 36 to 2, giving

$$\sigma_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\sum_k V_{kk} + 2\mu V_{ij}$$

Then $\sum_k \sigma_{kk} = -3p + (3\lambda + 2\mu)\sum_k V_{kk}$ and in the case $3\lambda + 2\mu = 0$ we obtain the stress tensor characteristic of a "Stokes fluid"

$$\sigma_{ij} = -p\delta_{ij} + 2\mu V_{ij} - \tfrac{2}{3}\mu\delta_{ij}\sum_k V_{kk}$$

For an "incompressible Stokes fluid" this simplifies

$$\sigma_{ij} = -p\delta_{ij} + 2\mu V_{ij}$$

and in the absence of viscosity simplifies still further

$$\sigma_{ij} = -p\delta_{ij}$$

At zero pressure we obtain what is technically called **dust**:

$$\sigma_{ij} = 0 \tag{316}$$

We will have need of (313), (315) and (316). Other remarks on this page have been included simply to place what we will be doing in its larger context, to stress that we will be concerned only with the simplest instance of a vast range of structured possibilities—the number of which is increased still further when one endows the fluid with "non-Newtonian," or thermodynamic, or (say) magnetohydrodynamic properties. END OF DIGRESSION

The charges which comprise the "sources" of an electromagnetic field must, for fundamental reasons, satisfy *Lorentz-covariant equations of motion*. We propose to consider the sources to comprise collectively a kind of "fluid." We stand in need, therefore, of a relativistic fluid dynamics. To that end we observe that equations $c\cdot(313)\oplus(315)$ comprise a quartet of equations that can be written

$$\partial_\mu s^{\mu\nu} = f^\nu \tag{316}$$

with

$$\|s^{\mu\nu}\| \equiv \begin{pmatrix} \rho c^2 & \rho c v_1 & \rho c v_2 & \rho c v_3 \\ \rho c v_1 & \rho v_1 v_1 - \sigma_{11} & \rho v_1 v_2 - \sigma_{12} & \rho v_1 v_3 - \sigma_{13} \\ \rho c v_2 & \rho v_2 v_1 - \sigma_{21} & \rho v_2 v_2 - \sigma_{22} & \rho v_2 v_3 - \sigma_{23} \\ \rho c v_3 & \rho v_3 v_1 - \sigma_{31} & \rho v_3 v_2 - \sigma_{32} & \rho v_3 v_3 - \sigma_{33} \end{pmatrix}, \quad \|f^\nu\| \equiv \begin{pmatrix} 0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

In the instantaneous rest frame of a designated fluid element

$$\|s^{\mu\nu}\| \downarrow = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ 0 & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ 0 & -\sigma_{31} & -\sigma_{32} & -\sigma_{33} \end{pmatrix}$$

and for a "non-viscous Newtonian fluid"—a model that is, as will emerge, adequate to our intended application—we obtain

$$\|s^{\mu\nu}\| \downarrow = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{317}$$

Equation (316) looks a lot more relativistic than (at the moment) it is, but *becomes fully relativistic if it is assumed that*

i) $s^{\mu\nu}$ and f^ν are prescribed in the local rest frame and

ii) respond tensorially to Lorentz transformations.

Thus

$$\begin{aligned} \partial_\mu s^{\mu\nu} &= f^\nu \quad \text{in the rest frame of a fluid element} \\ &\downarrow \\ \partial_\mu s^{\mu\nu} &= k^\nu \quad \text{in the lab frame} \end{aligned} \tag{318.1}$$

where

$$k^\nu \equiv \Lambda^\nu{}_\beta(\boldsymbol{\beta}) f^\beta \equiv \text{"Minkowski force density"} \tag{318.2}$$

$$\boldsymbol{\beta} \equiv \tfrac{1}{c}\cdot\begin{pmatrix}\text{velocity with which we in the lab frame}\\ \text{see the fluid element to be moving}\end{pmatrix}$$

$$s^{\mu\nu} \equiv \Lambda^\mu{}_\alpha(\boldsymbol{\beta})\Lambda^\nu{}_\beta s^{\alpha\beta} \tag{318.3}$$

Details relating to the construction (318.2) of k^ν have been described already at (290/291) on page 197. We look now to details implicit in the construction (318.3) of the "stress-energy tensor $s^{\mu\nu}$ of the relativistic fluid." Notice first that $s^{\mu\nu}$ shares the physical dimensionality of $S^{\mu\nu}$:

$$[s^{\mu\nu}] = \frac{\text{force}}{(\text{length})^2} = \frac{\text{energy}}{\text{3-volume}} = \text{pressure}$$

If we take $s^{\mu\nu}$ to be given by (317) and $\mathbb{\Lambda}(\boldsymbol{\beta})$ to possess the general boost design (209) then a straightforward computation[177] supplies

$$s^{\mu\nu} = (\rho + \tfrac{1}{c^2}p)u^\mu u^\nu - pg^{\mu\nu} \tag{319}$$

where

$$\rho \equiv \text{mass density in the local rest frame}$$
$$u^\mu \equiv \gamma \begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix} \equiv \text{4-velocity of the fluid element}$$

At (319) we encounter the stress-energy tensor of a "relativistic non-viscous Newtonian fluid" which plays a major role in relativistic cosmology, where theorists speak of a "fluid" the elements of which are galaxies![178] If in (319) we set $p = 0$ we obtain the **stress-energy tensor of relativistic dust**

$$s^{\mu\nu} = \rho u^\mu u^\nu \tag{320}$$

where $u^\mu(x)$ is the 4-velocity field characteristic of the moving dust, and $\rho(x)$ is the rest mass density. The simplicity of (320) reflects *the absence (in dust) of any direct interparticle interaction*, and has the consequence that (for dust) the fluid dynamical equations

$$\partial_\mu s^{\mu\nu} = k^\nu$$

are but thinly disguised variants of the equations of particulate motion:

$$\begin{aligned}\text{expression on the left} &= u^\nu \cdot \underbrace{\partial_\mu(\rho u^\mu)}_{0 \text{ by mass conservation}} + \rho(u^\mu \partial_\mu)u^\nu \\ &= \rho(\tfrac{d}{d\tau})u^\nu \\ &= k^\nu \quad \text{by Minkowski's equation (275), adapted here to mass/force \underline{densities}}\end{aligned}$$

For <u>a "dust cloud" which contains but a single particle</u> we expect $s^{\mu\nu}(x)$ to *vanish except on the worldline of the particle*, and are led from (320) to the odd-looking construction

$$s^{\mu\nu}(x) = mc \int_{-\infty}^{+\infty} u^\mu(\tau)u^\nu(\tau)\delta(x - x(\tau))\, d\tau \tag{321}$$

where $x(\tau)$ is the solution of $m\frac{d}{d\tau}u^\nu = K^\nu$

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[177] PROBLEM 55.

[178] See, for example, C. W. Misner, K. S. Thorne & J. A. Wheeler, *Gravitation* (1973), pages 153–154.

where the c-factor arises from dimensional considerations.[179] Equation (321) describes the *stress-energy tensor of a relativistic mass point*,[180] and if, in particular, it is the Lorentz force

$$K^\mu = (q/c)F^\mu{}_\alpha u^\alpha \tag{295}$$

that "steers" the particle then (321) becomes the **stress-energy tensor of a relativistic charged particle**—a concept introduced by Minkowski himself in 1908.

If all the constituent particles in a *charged dust cloud* are of then same species (*i.e.*, if the value of q/m is invariable within the cloud) then

$$\begin{aligned}\rho u^\mu &\equiv \text{mass-current 4-vector field}\\ &= (m/q)\cdot\text{charge-current 4-vector field}\\ &= (m/q)\cdot J^\mu\end{aligned}$$

and (320) becomes

$$s^{\mu\nu}(x) = (m/q)\cdot J^\mu(x)u^\nu(x) \tag{322}$$

This is the stress-energy tensor of a single-species charged dust cloud. For a single charged particle—looked upon as a "degenerate charged dust cloud"—we have

$$J^\mu(x) = qc\int_{-\infty}^{+\infty} u^\mu(\tau)\delta(x - x(\tau))\,d\tau \tag{323}$$

which when introduced into (320) gives back (321).

From (295)—written

$$K^\mu/(\text{unit 3-volume}) = \tfrac{1}{c}F^\mu{}_\alpha(q/\text{unit 3-volume})\,u^\alpha$$

—we infer that the *Lorentz force density experienced by a charged dust cloud* can be described

$$k^\mu = \tfrac{1}{c}F^\mu{}_\alpha J^\alpha \tag{324}$$

which positions us to address the main point of this discussion: I show now how (324) can be used to motivate the definition (309) of the stress-energy tensor $S^{\mu\nu}$ of the electromagnetic field. Most of the work has, in fact, already been

[179] [4-dimensional δ-function]= (4-volume)$^{-1}$ so

$$[c\delta(x - x(x))] = (\text{3-volume})^{-1}$$

[180] PROBLEM 56.

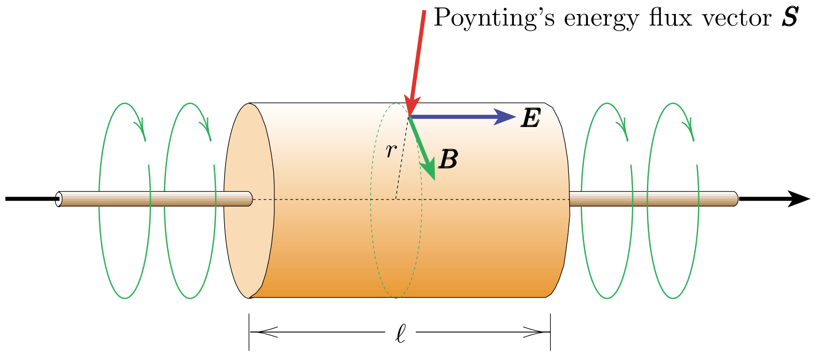


FIGURE 78: *Current I passes through a cylindrical resistor with resistance $R = \rho\ell/\pi r^2$. The potential $V = IR$ implies the existence of an axial electric field $\boldsymbol{E}$ of magnitude $E = V/\ell$, while at the surface of the resistor the magnetic field is solenoidal, of strength $B = I/c2\pi r$. The Poynting vector $\boldsymbol{S} = c(\boldsymbol{E}\times\boldsymbol{B})$ is therefore centrally directed, with magnitude $S = cEB$, which is to say: the field dumps energy into the resistor at the rate given by*

$$\begin{aligned}\text{rate of energy influx} &= S\cdot 2\pi r\ell\\ &= c(IR/\ell)(I/c2\pi r)2\pi r\ell\\ &= I^2R\end{aligned}$$

The steady field can, from this point of view, be considered to act as a conduit for energy that flows from battery to resistor. The resistor, by this account, heats up not because copper atoms are jostled by conduction electrons, but because it drinks energy dumped on it by the field.

done: we have (drawing only upon Maxwell's equations and the antisymmetry of $F^{\mu\nu}$) at (312) already established that

$$\tfrac{1}{c}F^{\nu}{}_{\alpha}J^{\alpha} \quad \text{can be expressed} \quad -\partial_\mu S^{\mu\nu}$$
$$\text{with}\quad S^{\mu\nu} \equiv F^{\mu}{}_{\alpha}F^{\alpha\nu} - \tfrac{1}{4}(F^{\alpha\beta}F_{\beta\alpha})g^{\mu\nu}$$

So we have

$$\partial_\mu s^{\mu\nu} = k^\nu = -\partial_\mu S^{\mu\nu}$$

giving

$$\partial_\mu \underbrace{(s^{\mu\nu} + S^{\mu\nu})}_{\text{stress-energy tensor of } \textbf{total system: sources + field}} = 0 \tag{325}$$

This equation provides (compare page 218) a detailed local description of energy/momentum traffic back and forth between the field and its sources, and does so in a way that conforms manifestly to the principle of relativity. We speak with intuitive confidence about the energy and momentum of particulate systems, and of their continuous limits (*e.g.*, fluids), and can on the basis of (325) speak with that same confidence about the "energy & momentum of the electromagnetic field."

The language employed by Maxwell (quoted on page 216) has by this point lost much of its quaintness, for the electromagnetic field has begun to acquire the status of a physical "object"—a sloshy object, but as real as any fluid. The emerging image of "field as dynamical object" acquires even greater plausibility from illustrative applications—such as that presented here as Figure 78—and from the discussion to which we now turn:

3. Electromagnetic angular momentum. If $\boldsymbol{E}$ and $\boldsymbol{B}$ describe the electric and magnetic fields at a point $\boldsymbol{x}$ then (see again page 216) $\boldsymbol{\mathfrak{P}} = \frac{1}{c}(\boldsymbol{E}\times\boldsymbol{B})$ describes the momentum density at $\boldsymbol{x}$, and it becomes natural to suppose that

$$\boldsymbol{\mathfrak{L}} \equiv \boldsymbol{x}\times\boldsymbol{\mathfrak{P}} = \tfrac{1}{c}\boldsymbol{x}\times(\boldsymbol{E}\times\boldsymbol{B}) \tag{326}$$

describes—relative to the origin—the *angular momentum density* of the field at $\boldsymbol{x}$. From the "triple cross product identity" we infer that

$$\boldsymbol{\mathfrak{L}} = \tfrac{1}{c}\big\{(\boldsymbol{x}\cdot\boldsymbol{B})\boldsymbol{E} - (\boldsymbol{x}\cdot\boldsymbol{E})\boldsymbol{B}\big\} \quad \text{lies in the local } (\boldsymbol{E}\cdot\boldsymbol{B})\text{-plane}$$

We expect that the *total angular momentum* resident in the field will be given by an equation of the form

$$\boldsymbol{L} = \int_{\text{all space}} \boldsymbol{\mathfrak{L}}\, d^3x$$

. . . that *angular momentum flux vectors* will be associated with each of the components of $\boldsymbol{\mathfrak{L}}$ . . . and that there will, in general be *angular momentum exchange between the field and its sources*. All these expectations—modulo some surprises—will be supported by subsequent events. We begin, however, by looking not to formal fundamentals but to the particulars of a tractable special case:

ELECTROMAGNETIC GYROSCOPE WITH NO MOVING PARTS

Suppose—with J. J. Thompson (1904)—that an electric charge e has been glued to one end of a stick of length a, and that a "magnetic charge" g has been glued to the other end. It is immediately evident (see Figure 79) that the superimposed $\boldsymbol{E}$ and $\boldsymbol{B}$-fields that result from such a static charge configuration give rise to a momentum field $\boldsymbol{\mathfrak{P}} = \frac{1}{c}(\boldsymbol{E}\times\boldsymbol{B})$ that circulates about the axis defined by the stick, so that if you held such a construction in your hand it would feel and act like a gyroscope . . . though it contains *no moving parts*! We wish to quantify that intuitive insight, to calculate the total angular momentum resident within the static electromagnetic field. Taking our notation from the figure, we have

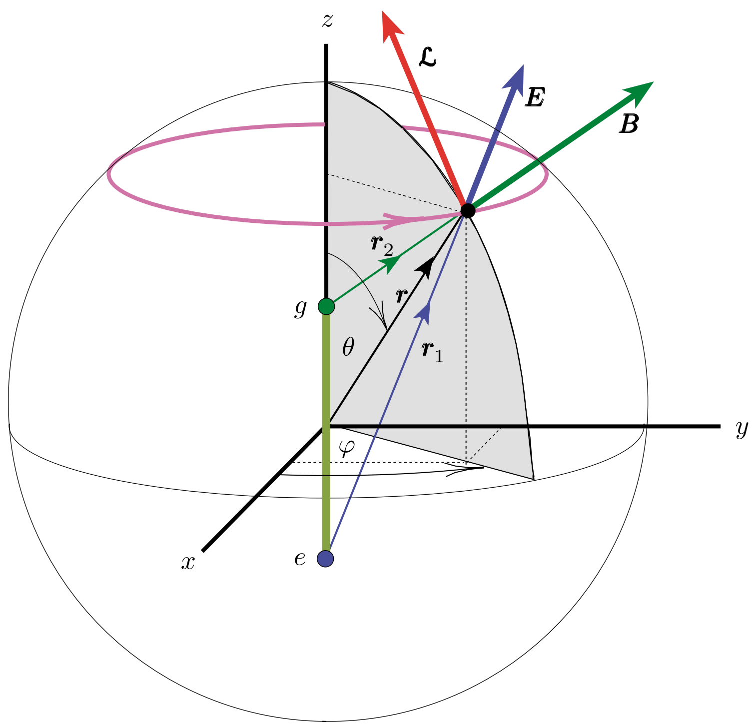


FIGURE 79: *Notations used in analysis of the "Thompson monopole" (or "mixed dipole").* Momentum circulation *is represented by the purple ellipse, and is right-handed with respect to the axis defined by the vector* $\boldsymbol{a}$ *directed from e to g* : (• → •)*. Momentum circulation gives rise to a local angular momentum density that lies in the local* $(\boldsymbol{E}, \boldsymbol{B})$*-plane. Only the axial component of* $\boldsymbol{L} = \int \boldsymbol{\mathfrak{L}}\, d^3x$ *survives the integration process.*

$$\boldsymbol{E} = \frac{e}{4\pi r_1^3}\boldsymbol{r}_1 \quad \text{with} \quad \boldsymbol{r}_1 = \boldsymbol{r} + \tfrac{1}{2}\boldsymbol{a}$$
$$r_1^2 = r^2 + \boldsymbol{r}\cdot\boldsymbol{a} + \tfrac{1}{4}a^2$$
$$\boldsymbol{B} = \frac{g}{4\pi r_1^3}\boldsymbol{r}_2 \quad \text{with} \quad \boldsymbol{r}_2 = \boldsymbol{r} - \tfrac{1}{2}\boldsymbol{a}$$
$$r_2^2 = r^2 - \boldsymbol{r}\cdot\boldsymbol{a} + \tfrac{1}{4}a^2$$

giving

$$\boldsymbol{\mathfrak{P}} = \frac{eg/c}{(4\pi)^2}\frac{1}{r_1^3 r_2^3}\,\boldsymbol{a}\times\boldsymbol{r}$$
$$\boldsymbol{\mathfrak{L}} = \frac{eg/c}{(4\pi)^2}\frac{1}{r_1^3 r_2^3}\,\boldsymbol{r}\times(\boldsymbol{a}\times\boldsymbol{r})$$

But

$$\boldsymbol{r}\times(\boldsymbol{a}\times\boldsymbol{r}) = r^2\boldsymbol{a} - (\boldsymbol{r}\cdot\boldsymbol{a})\boldsymbol{r} = r^2 a\begin{pmatrix} -\cos\theta\cdot\sin\theta\cos\varphi \\ -\cos\theta\cdot\sin\theta\sin\varphi \\ 1-\cos\theta\cdot\cos\theta \end{pmatrix}$$

The x and y-components are killed by the process $\int_0^{2\pi} d\varphi$, so (as already anticipated) we have

$$\boldsymbol{L} = \begin{pmatrix} 0 \\ 0 \\ L \end{pmatrix}$$

with

$$\begin{aligned} L &= \frac{eg/c}{(4\pi)^2} \iiint \frac{1}{r_1^3 r_2^3}\, r^2 a \sin^2\theta \cdot r^2 \sin\theta\, dr d\theta d\varphi \\ &= 2\pi \frac{eg/c}{(4\pi)^2} \iint \frac{1}{r_1^2 r_2^2} \frac{ra\sin\theta}{r_1 r_2} (r\sin\theta)^2 \cdot r dr d\theta \end{aligned}$$

Write $r = \frac{1}{2}sa$ and obtain

$$= 4\pi \frac{eg/c}{(4\pi)^2} \iint \frac{1}{s_1^2 s_2^2} \frac{s\sin\theta}{s_1 s_2} (s\sin\theta)^2 \cdot s\, ds d\theta \tag{327}$$

$$\begin{aligned} s_1^2 &\equiv s^2 + 1 + 2s\cos\theta \\ s_1^2 &\equiv s^2 + 1 - 2s\cos\theta \end{aligned}$$

from which all reference to the stick-length—the only "natural length" which Thompson's system provides—has disappeared:

> The angular momentum in the field of Thompson's mixed dipole is *independent of stick-length.*

Evaluation of the $\iint$ poses a non-trivial but purely technical problem which has been discussed in detail—from at least six points of view!—by I.Adawi.[181] The argument which follows—due in outline to Adawi—illustrates the power of what might be called "symmetry-adapted integration" and the sometimes indispensable utility of "exotic coordinate systems."

Let (327) be written

$$L = \frac{eg/c}{4\pi} \iint \Big(\frac{w}{s_1 s_2}\Big)^3 d(\text{area}) \tag{328}$$

with $w = s\sin\theta$ and $d(\text{area}) = s\, ds d\theta$. The dimensionless variables s_1, s_2 and w admit readily of geometric interpretation (see Figure 80). Everyone familiar with the "string construction" knows that

$$s_1 + s_2 = 2u \quad \text{describes an ellipse with pinned foci}$$

and will be readily convinced that

$$s_1 - s_2 = 2v \quad \text{describes (one branch of) a hyperbola}$$

[181] "Thompson's monopoles," AJP **44**, 762 (1976). Adawi learned of this problem—as did I—when we were both graduate students of Philip Morrison at Cornell University (1955/56). Adawi was famous among his classmates for his exceptional analytical skill.

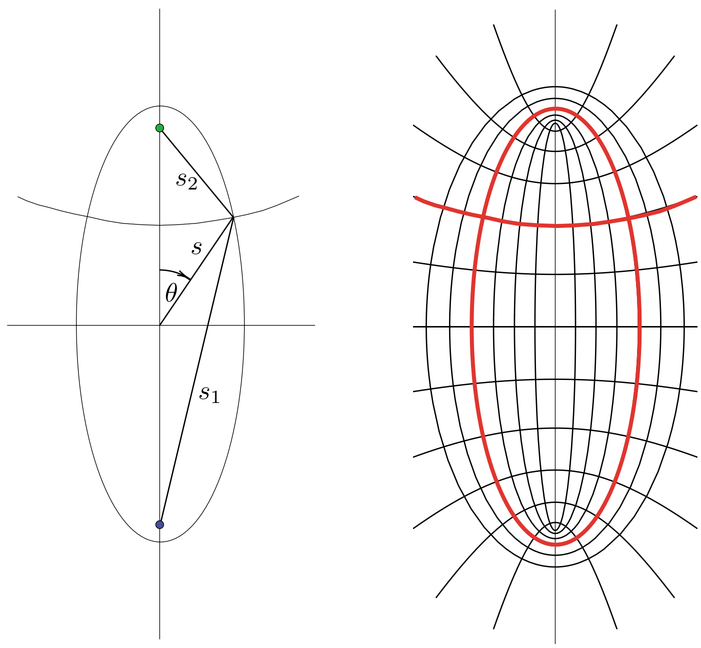


FIGURE 80: *In dimensionless variables*

$$\zeta \equiv s\cos\theta = 2z/a \quad \textit{and} \quad w \equiv s\sin\theta = (2r/a)\sin\theta$$

the electric charge • sits on the ζ-axis at $\zeta = -1$, the magnetic charge • at $\zeta = +1$. The "confocal conic coordinate system," shown at right, simplifies the analysis because it conforms optimally to the symmetry of the system.

It is equally evident on geometrical grounds that the parameters u and v are subject to the constraints indicated in Figure 81 below, and that the (u, v)-parameterized ellipses/hyperbolas are *confocal*. Some tedious but straightforward analytical geometry shows moreover that

$$\frac{\zeta^2}{u^2} + \frac{w^2}{u^2 - 1} = 1 \quad \text{describes the } u\text{-ellipse}$$
$$\frac{\zeta^2}{v^2} - \frac{w^2}{1 - v^2} = 1 \quad \text{describes the } v\text{-hyperbola}$$

Equivalently

$$\frac{\zeta^2}{\cosh^2\alpha} + \frac{w^2}{\sinh^2\alpha} = 1 \quad \text{with} \quad u \equiv \cosh\alpha$$
$$\frac{\zeta^2}{\cos^2\beta} - \frac{w^2}{\sin^2\beta} = 1 \quad \text{with} \quad v \equiv \cos\beta$$

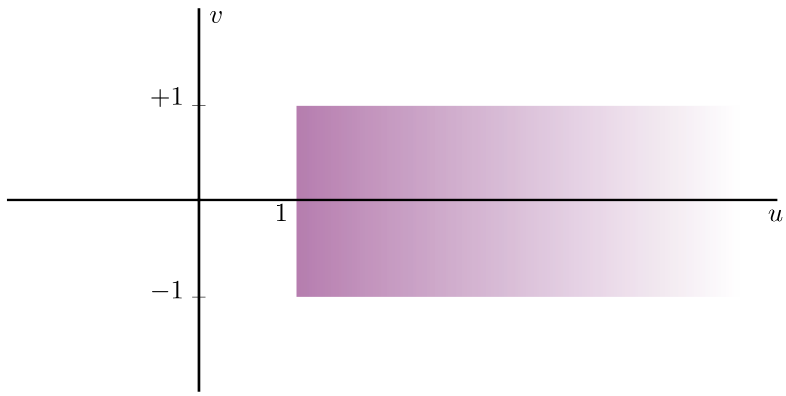


FIGURE 81: *The parameters u and v are subject to the constraints*

$$1 < u < \infty$$
$$-1 < v < +1$$

which is to say: they range on the purple strip.

from which it follows readily that

$$\zeta = \cosh\alpha\cos\beta = u\,v$$
$$w = \sinh\alpha\sin\beta = \sqrt{(u^2-1)(1-v^2)}$$

The last pair of equations describe a coordinate transformation

$$(\zeta, w) \longmapsto (u, v)$$

and it is in the confocal coordinates (u, v) that we propose to evaluate the $\iint$. To that end, we observe that

$$s_1 s_2 = \Big(\frac{s_1+s_2}{2}\Big)^2 - \Big(\frac{s_1-s_2}{2}\Big)^2 = u^2 - v^2$$

and

$$d\zeta dw = J\,dudv$$
$$J = \det\begin{pmatrix} \frac{\partial\zeta}{\partial u} & \frac{\partial\zeta}{\partial v} \\ \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v}\end{pmatrix} = \frac{u^2-v^2}{\sqrt{(u^2-1)(1-v^2)}} = \frac{s_1 s_2}{w}$$

Returning with this information to (328) we obtain

$$L = 2\cdot\frac{eg/c}{4\pi}\int_0^1 dv\int_1^\infty \frac{(u^2-1)(1-v^2)}{(u^2-v^2)^2}\,du$$

where the leading 2-factor comes from $\int_{-1}^{+1} = 2\int_0^{+1}$ (because the integrand is an even function of v). Finally write $u = 1/t$ and use $du = -(1/t^2)dt$ to obtain the remarkably symmetric result

$$= 2\cdot\frac{eg/c}{4\pi}\int_0^1\!\!\int_0^1 \frac{(1-t^2)(1-v^2)}{(1-t^2v^2)^2}\,dtdv$$

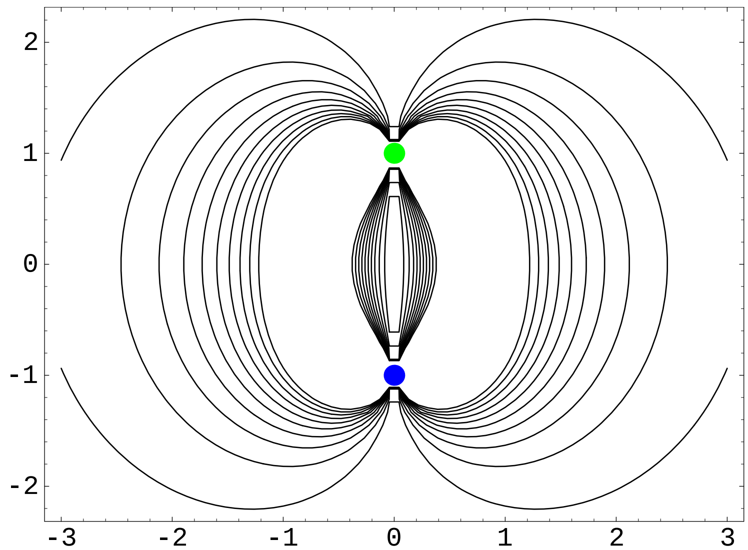


FIGURE 82: *Only the axial component (the component parallel to* • → •*) of* $\boldsymbol{\mathfrak{L}}$ *survives the integration process. From results developed in the text we discover the density of that component to be given in Cartesian coordinates by* $\mathfrak{L}_{\text{axial}} = \frac{1}{4\pi}(eg/c)\cdot f(w,\zeta)$ *with*

$$f(w,\zeta) = \frac{w^2}{[w^2+(\zeta-1)^2]^{\frac{3}{2}}[w^2+(\zeta+1)^2]^{\frac{3}{2}}}$$

of which the figure provides a contour plot. The angular momentum of Thompson's mixed dipole is seen to reside mainly in the "meat of the apple," exclusive of its core.

The double integral yields to a rather pretty direct analysis,[182] but I will on this occasion be content simply to ask *Mathematica*, who supplies

$$\int_0^1\!\!\int_0^1 \frac{(1-t^2)(1-v^2)}{(1-t^2v^2)^2}\,dtdv = \int_0^1 \frac{v^3-v+(1-v^4)\tanh^{-1}v}{2v^3}\,dv = \tfrac{1}{2}$$

So we have Thomson's relation

$$L = \frac{eg}{4\pi c}$$

which in rationalized units $\tilde{e}\equiv e/\sqrt{4\pi}$ and $\tilde{g}\equiv g/\sqrt{4\pi}$ assumes the still simpler

[182] See CLASSICAL ELECTRODYNAMICS (1980), page 319.

form

$$L = \frac{\tilde{e}\tilde{g}}{c} \quad : \quad \text{independently of the “stick length” } a \tag{328}$$

We know (which Thompson did not) that the intrinsic angular momentum (“spin”) of an elementary particle is always an integral multiple of $\frac{1}{2}\hbar$. It becomes attractive therefore to set

$$= n \cdot \tfrac{1}{2}\hbar$$

giving

$$\tilde{e}\tilde{g} = n\tfrac{1}{2}\hbar c$$

But

$$\hbar c = 137\,\tilde{e}^2$$

so on these grounds

$$\tilde{g} = n\tfrac{137}{2}\,\tilde{e} \tag{329}$$

which suggests that *if the universe contained even a single magnetic monopole then we could on this basis understand the observed* *quantization of electric charge*. Magnetic monopoles are, according to (329) “strongly” charged, and therefore should be conspicuous. On the other hand, they should be relatively hard to isolate, for they are bound by forces $(n\frac{137}{2})^2 = 4692n^2$ times stronger than the forces which bind electric monopoles. This line of thought originates in a paper of classic beauty by P. A. M. Dirac (1931), and after seventy years continues to haunt/taunt the imagination of physicists (J. Schwinger, A. O. Barut and many others). For a good review (and basic references) see §6.12 in J. D. Jackson's *Classical Electrodynamics* (3rd edition 1999).

We return now—with our relativistic goggles on—to the more general issues posed on page 226. I ask: *How does* $\boldsymbol{\mathcal{L}}$ *transform*? . . . my double intent being

1) to achieve manifest conformity with the principle of relativity, and
2) to develop formulæ which describe the angular momentum flux vectors.

Here as so often, index play provides the essential clue. If we bring to (326) the recollection (page 216) that

$$\mathcal{P}^i = \tfrac{1}{c}S^{0i} \quad : \quad i = 1, 2, 3$$

we obtain

$$\begin{aligned}
\mathcal{L}_1 &= x^2\mathcal{P}^3 - x^3\mathcal{P}^2 = \tfrac{1}{c}(x^2S^{03} - x^3S^{02}) \equiv \mathcal{L}^{023}\\
\mathcal{L}_2 &= x^3\mathcal{P}^1 - x^1\mathcal{P}^3 = \tfrac{1}{c}(x^3S^{01} - x^1S^{03}) \equiv \mathcal{L}^{031}\\
\mathcal{L}_3 &= x^1\mathcal{P}^2 - x^2\mathcal{P}^1 = \tfrac{1}{c}(x^1S^{02} - x^2S^{01}) \equiv \mathcal{L}^{012}
\end{aligned}$$

From $\mathcal{L}_1 \equiv \mathcal{L}^{023}$ and the experience of pages 212–216 we infer that the equations $\mathcal{L}^{i23} \equiv \frac{1}{c}(x^2S^{i3} - x^3S^{i2})$ may very well describe the components $(i = 1, 2, 3)$ of the $\mathcal{L}_1$-*flux vector*. This is a conjecture which can be confirmed by direct calculation:

$$\begin{aligned}
\partial_\alpha\mathcal{L}^{\alpha 23} &= \tfrac{1}{c}\partial_\alpha(x^2S^{\alpha 3} - x^3S^{\alpha 2})\\
&= \tfrac{1}{c}(\delta^2{}_\alpha S^{\alpha 3} + x^2\partial_\alpha S^{\alpha 3} - \delta^3{}_\alpha S^{\alpha 2} - x^3\partial_\alpha S^{\alpha 2})
\end{aligned}$$

The 2$^{\text{nd}}$ and 4$^{\text{th}}$ terms on the right vanish individually (in source-free regions) as instances of momentum conservation ($\partial_\alpha S^{\alpha i} = 0$), so

$$\begin{aligned} &= \tfrac{1}{c}(S^{23} - S^{32}) \\ &= 0 \quad \text{by the symmetry of } S^{\mu\nu} \end{aligned}$$

Similar remarks pertain to $\mathcal{L}_2$ and $\mathcal{L}_3$. Indeed, the same argument supplies

$$\partial_\alpha \mathcal{L}^{\alpha\mu\nu} = 0 \quad \text{in source-free regions} \ : \ \mu,\nu = 0,1,2,3 \tag{330}$$

where

$$\mathcal{L}^{\alpha\mu\nu} \equiv \tfrac{1}{c}(x^\mu S^{\alpha\nu} - x^\nu S^{\alpha\mu}) \tag{331}$$

has obviously the following *antisymmetry* property:

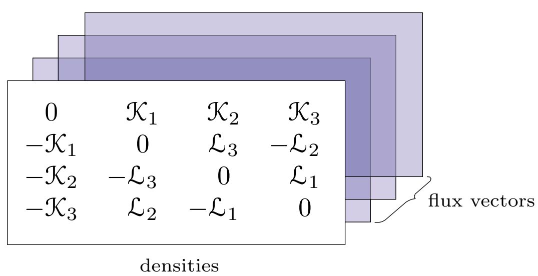
$$\mathcal{L}^{\alpha\mu\nu} = -\mathcal{L}^{\alpha\nu\mu} \tag{332}$$

Starting from the construction (326) of the three components of the angular momentum density vector $\boldsymbol{\mathcal{L}}$, and drawing upon a little bit of relativity . . . we have been led

- to explicit descriptions of the associated angular momentum fluxes, and
- to three unanticipated conservation laws:

$$\partial_\alpha \mathcal{K}^{\alpha 1} = \partial_\alpha \mathcal{K}^{\alpha 2} = \partial_\alpha \mathcal{K}^{\alpha 2} = 0 \quad \text{with} \quad \mathcal{K}^{\alpha i} \equiv \mathcal{L}^{\alpha 0 i} \tag{333}$$

We have been led, in short, from an initial trio of field functions to a final total of 24—the components of a $\mu\nu$-antisymmetric third-rank tensor $\mathcal{L}^{\alpha\mu\nu}$



. . . all of which become intricately (but linearly) intermixed when Lorentz transformed. And an anticipated trio of conservation laws (conservation of angular momentum) have—by force of Lorentz covariance—been joined by an unanticipated second trio. We confront, therefore, this unanticipated question: *What is the physical significance of the conserved vector*

$$\boldsymbol{K} \equiv \int_{\text{all space}} \boldsymbol{\mathcal{K}}\, d^3x \tag{334.1}$$

$$\boldsymbol{\mathcal{K}} \equiv \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix} \quad \text{with} \quad \mathcal{K}_i \equiv \mathcal{K}^{0i} \equiv \mathcal{L}^{00i} \tag{334.2}$$

4. Motion of the "center of mass" of a free field. Bringing to (334.2) the definition (331) we have

$$\mathcal{K}_i = \mathcal{L}^{00i} = \tfrac{1}{c}(x^0 S^{0i} - x^i S^{00})$$

which in the notations introduced at the bottom of page 215 becomes

$$\begin{aligned} &= \tfrac{1}{c}\Big\{(ct)(c\mathcal{P}^i) - x^i \mathcal{E}\Big\} \\ &= c(t\mathcal{P}^i - \mathcal{M}x^i) \end{aligned}$$

$$\mathcal{M} \equiv \mathcal{E}/c^2 \equiv \text{local "mass density" of the field} \tag{335}$$

giving

$$\boldsymbol{\mathcal{K}} = c(t\boldsymbol{\mathcal{P}} - \mathcal{M}\boldsymbol{x}) \tag{336}$$

For free fields

$$\boldsymbol{P} \equiv \int \boldsymbol{\mathcal{P}}\, d^3x = \text{total linear momentum}$$

and

$$\begin{aligned} M \equiv \int \mathcal{M}\, d^3x &= \text{total effective mass} \\ &= \frac{\text{total energy}}{c^2} \end{aligned}$$

are known to be constants of the motion. So writing

$$\begin{aligned} \boldsymbol{K} &\equiv \int \boldsymbol{\mathcal{K}}\, d^3x \\ &= c(t\boldsymbol{P} - \underbrace{\int \mathcal{M}\boldsymbol{x}\, d^3x}_{=\,M\boldsymbol{X}(t)}) \end{aligned}$$

$$\begin{aligned} \boldsymbol{X}(t) &\equiv \tfrac{1}{M}\int \boldsymbol{x}\mathcal{M} d^3x = \tfrac{1}{E}\int \boldsymbol{x}\,\mathcal{E} d^3x \\ &= \textbf{center of mass/energy} \text{ of the free field} \end{aligned} \tag{337}$$

we see that $\boldsymbol{K}$-conservation

$$\tfrac{d}{dt}\boldsymbol{K} = \mathbf{0}\,, \quad \text{the upshot of the local conservations laws (333)}$$

amounts simply to the satisfying statement that *the center of mass/energy of a free electromagnetic field* *moves uniformly/rectilinearly*:

$$\tfrac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{P}/M = \text{constant} \tag{338}$$

In this respect a free electromagnetic field is very like a Newtonian free particle! Or more precisely: like an *isolated system* of Newtonian particles.

> IMPORTANT REMARK: Such frequently-encountered (because frequently useful) abstractions as "electromagnetic plane waves" are utterly non-localized. Their total mass/energy/momentum are defined by non-convergent integrals so the definition (337) becomes meaningless: **no center of mass can be assigned** to such idealized solutions of Maxwell's equations. We are led to regard as "physical" only those free fields to which the center of mass concept *does* pertain—fields which (because of the manner in which they "vanish at infinity") can be considered to be "isolated." Fourier analysis is in this respect strange (though no stranger here than in quantum mechanics), for it invites us to display semi-localized physical free fields as wavepacket-like superpositions of idealized *non*-physical free fields.

Distributed quantities—wherever in pure/applied mathematics they may be encountered—are often most usefully described in terms of their *moments* of ascending order. If, for example, $\rho(\boldsymbol{x})$ describes a mass distribution in 3-space then we standardly define

$$\begin{aligned}
0^{\text{th}}\text{ moment } M &\equiv \int \rho(\boldsymbol{x})\,d^3x \equiv \langle 1\rangle = \text{total mass}\\
1^{\text{st}}\text{ moments } M^i &\equiv \int x^i\rho(\boldsymbol{x})\,d^3x \equiv \langle x^i\rangle\\
2^{\text{nd}}\text{ moments } M^{ij} &\equiv \int x^ix^j\rho(\boldsymbol{x})\,d^3x \equiv \langle x^ix^j\rangle\\
&\vdots
\end{aligned}$$

and from those construct such objects as[183]

$$\begin{aligned}
\text{center of mass vector} &\quad:\quad X^i \equiv \frac{\langle x^i\rangle}{\langle 1\rangle}\\
\text{matrix of centered } 2^{\text{nd}}\text{ moments} &\quad:\quad C^{ij} \equiv \langle (x^i - X^i)(x^j - X^j)\rangle\\
\text{moment of inertia matrix} &\quad:\quad I^{ij} \equiv (C^{11}+C^{22}+C^{33})\delta^{ij} - C^{ij}\\
&\quad\vdots
\end{aligned}$$

where C^{ij} provides leading-order information about *how the mass is distributed about* the center of mass, I^{ij} is a construction natural to the dynamics of rigid bodies, etc. The point is that *such objects—defined in reference to a variety of density functions—can be associated with isolated electromagnetic fields*. This is not commonly done, but is an analytical device that has been exploited to good effect by Schwinger.[184] Following (except notationally) in Schwinger's

---

[183] See CLASSICAL GYRODYNAMICS (1976), pages 9–11.

[184] See J. Schwinger *et al*, *Classical Electrodynamics* (1998), Chapter 3.

footsteps, let us agree to write

$$\begin{aligned}
\langle \boldsymbol{x}\rangle^0 &\equiv \frac{1}{E}\int_{\text{pulse}} \boldsymbol{x}\,\mathcal{E}\;d^3x \quad : \quad \mathcal{E}\text{-weighted mean position}\\
\langle \boldsymbol{x}\rangle^i &\equiv \frac{1}{P^i}\int_{\text{pulse}} \boldsymbol{x}\,\mathcal{P}^i\,d^3x \quad : \quad \mathcal{P}^i\text{-weighted mean position}\\
&\vdots \text{ and more generally}\\
\langle \bullet\rangle^\nu &\equiv \left[\int_{\text{pulse}} S^{0\nu}\,d^3x\right]^{-1}\int_{\text{pulse}} \bullet\, S^{0\nu}\,d^3x \quad : \quad S^{0\nu}\text{-weighted mean }\bullet
\end{aligned}$$

where "pulse" is the term used by Schwinger to emphasize that his results—all of which refer to the motion of moments—pertain only to *isolated* electromagnetic fields. In this notation (338) reads

$$M\tfrac{d}{dt}\langle \boldsymbol{x}\rangle^0 = \boldsymbol{P}$$

which when integrated becomes

$$\begin{aligned}
&\langle \boldsymbol{x}\rangle^0_t = \boldsymbol{v}\,t + \langle \boldsymbol{x}\rangle^0_0\\
&\qquad \boldsymbol{v} \equiv \tfrac{1}{M}\boldsymbol{P} \equiv \text{constant velocity of the center of energy} \qquad (339.1)
\end{aligned}$$

A natural companion to the preceding statement arises from Schwinger's (characteristically clever) observation that

$$\begin{aligned}
\tfrac{d}{dt}\int_{\text{pulse}} \boldsymbol{x}\cdot\boldsymbol{\mathcal{P}}\,d^3x &= \tfrac{d}{dt}\Big\{P^1\langle x^1\rangle^1 + P^2\langle x^2\rangle^2 + P^3\langle x^3\rangle^3\Big\}\\
&= -\int_{\text{pulse}} x_i\,\underbrace{\partial_0 c\mathcal{P}^i}_{=\,\partial_0 S^{0i}\,=\,-\partial_k S^{ki}\text{ by }\partial_\mu S^{\mu i}=0}\,d^3x\\
&= +\int_{\text{pulse}} \big[\underbrace{\partial_k(x_i S^{ki})}_{\text{contributes a surface term, which vanishes}} - S^{ki}g_{ki}\big]\,d^3x\\
&= E \quad \text{because } S^k{}_k = -S^0{}_0 = -\mathcal{E} \text{ by (311)}
\end{aligned}$$

The implication is that if we define

$$\boldsymbol{u} \equiv \tfrac{d}{dt}\boldsymbol{\xi} \quad\text{with}\quad \boldsymbol{\xi} \equiv \begin{pmatrix}\langle x^1\rangle^1\\ \langle x^2\rangle^2\\ \langle x^3\rangle^3\end{pmatrix}$$

↑ an object curiouser than Schwinger would have us believe!

then

$$E = \boldsymbol{P}\cdot\boldsymbol{u} \qquad (340)$$

. . . which tells us nothing about $\boldsymbol{u}_\perp$ but informs us that

$$\boldsymbol{u}_\parallel = u_\parallel \hat{\boldsymbol{P}} \quad \text{is a constant vector, with} \quad u_\parallel = E/P \tag{339.2}$$

From equations (339) it follows that

$$\boldsymbol{v}\cdot\boldsymbol{u}_\parallel = \tfrac{1}{M}(E/P)\boldsymbol{P}\cdot\hat{\boldsymbol{P}} = c^2$$

Schwinger observes that *if*

i) $\boldsymbol{v}$ refers to the velocity of energy transport

ii) $\boldsymbol{u}_\parallel$ refers to the velocity of momentum transport

iii) and if, moreover, (as would then seem plausible) those are identical

then

$$v = c \quad : \quad \begin{cases} \text{for isolated free fields ("pulses") with} \\ \text{identical energy/momentum transport} \\ \text{velocites } (\boldsymbol{v} = \boldsymbol{u}_\parallel) \text{ the transport speed} \\ \text{is necessarily the speed of light} \end{cases} \tag{341}$$

and (340) becomes

$$E = cP \tag{342}$$

which—interestingly—is of the design assumed by (282) in the massless limit:

$$\begin{aligned} E &= c\sqrt{\boldsymbol{p}\cdot\boldsymbol{p} + (mc)^2} \\ &\downarrow \\ &= cp \quad \text{as} \quad m \downarrow 0 \end{aligned}$$

But this line of argument provides no insight into the (seemingly plausible, but in fact highly specialized) conditions under which Schwinger's hypotheses hold.

Sharpened results can be obtained by looking to motion of the energetic *second* moment $\langle g_{\mu\nu}x^\mu x^\nu\rangle^0$: from local energy conservation $\partial_\alpha S^{\alpha 0} = 0$ it follows trivially that

$$(g_{\mu\nu}x^\mu x^\nu)\partial_\alpha S^{0\alpha} = \partial_\alpha\big[(g_{\mu\nu}x^\mu x^\nu)S^{0\alpha}\big] - 2S^{0\alpha}x_\alpha = 0$$

which can be spelled out

$$\begin{aligned} \tfrac{1}{c}\partial_t\big[(c^2t^2 - \boldsymbol{x}\cdot\boldsymbol{x})\mathcal{E}\big] + \boldsymbol{\nabla}\cdot(\textbf{etc.}) - 2c(\mathcal{E}\,t - \boldsymbol{\mathfrak{P}}\cdot\boldsymbol{x}) &= 0 \\ &\Downarrow \\ \tfrac{d}{dt}\int_{\text{pulse}}(c^2t^2 - \boldsymbol{x}\cdot\boldsymbol{x})\mathcal{E}\,d^3x + \text{vanishing surface term} - 2c^2\Big\{Et - \int_{\text{pulse}}\boldsymbol{\mathfrak{P}}\cdot\boldsymbol{x}\,d^3x\Big\} &= 0 \end{aligned}$$

But it was established on the preceding page that $\frac{d}{dt}\{\text{etc.}\} = 0$; *i.e.*, that $\{\text{etc.}\}$ is a constant of the motion:

$$Et - \int_{\text{pulse}}\boldsymbol{\mathfrak{P}}\cdot\boldsymbol{x}\,d^3x = Et - \boldsymbol{P}\cdot\boldsymbol{\xi}_t = \text{constant} = -\boldsymbol{P}\cdot\boldsymbol{\xi}_0$$

from which we could recover $E = \boldsymbol{P}\cdot\boldsymbol{u}$ by t-differentiation. So we have

$$\tfrac{d}{dt}(c^2t^2E - E\langle\boldsymbol{x}\cdot\boldsymbol{x}\rangle^0\,) + 2c^2\boldsymbol{P}\cdot\boldsymbol{\xi}_0 = 0$$

giving $E\frac{d}{dt}\langle\boldsymbol{x}\cdot\boldsymbol{x}\rangle^0 = 2c^2(Et + \boldsymbol{P}\cdot\boldsymbol{\xi}_0)$ whence (divide by $E = Mc^2$ and integrate)

$$\langle\boldsymbol{x}\cdot\boldsymbol{x}\rangle^0_t = c^2t^2 + 2\tfrac{1}{M}\boldsymbol{P}\cdot\boldsymbol{\xi}_0 t + \langle\boldsymbol{x}\cdot\boldsymbol{x}\rangle^0_0 \tag{343}$$

To gain leading-order information about the *evolving spatial distribution* of the field we introduce the centered second moment with respect to $\mathcal{E}$:

$$\begin{aligned}\sigma^2 &\equiv \tfrac{1}{E}\int_{\text{pulse}}(\boldsymbol{x} - \langle\boldsymbol{x}\rangle^0)\cdot(\boldsymbol{x} - \langle\boldsymbol{x}\rangle^0)\mathcal{E}\,d^3x\\ &= \langle\boldsymbol{x}\cdot\boldsymbol{x}\rangle^0 - \langle\boldsymbol{x}\rangle^0\cdot\langle\boldsymbol{x}\rangle^0\end{aligned}$$

Necessarily $\sigma^2 \geqslant 0$, with equality if and only if the pulse is "point-like." Results in hand now supply

$$\begin{aligned}\sigma^2_t &= \Big[c^2t^2 + 2\tfrac{1}{M}\boldsymbol{P}\cdot\boldsymbol{\xi}_0 t + \langle\boldsymbol{x}\cdot\boldsymbol{x}\rangle^0_0\Big] - \Big[\tfrac{1}{M}\boldsymbol{P}t + \langle\boldsymbol{x}\rangle^0_0\Big]\cdot\Big[\tfrac{1}{M}\boldsymbol{P}t + \langle\boldsymbol{x}\rangle^0_0\Big]\\ &= \Big[1 - \frac{P^2}{M^2c^2}\Big](ct)^2 + 2\frac{1}{Mc}\boldsymbol{P}\cdot\big[\boldsymbol{\xi}_0 - \langle\boldsymbol{x}\rangle^0_0\big]ct + \sigma^2_0\\ &\equiv A\,(ct)^2 + 2B\,(ct)^1 + C\,(ct)^0\end{aligned} \tag{344}$$

. . . which pertains to *all* isolated fields, and is plotted in Figure 83. The roots of $\sigma^2_t = 0$ are evidently both *complex*, which entails

$$0 \leqslant B^2 \leqslant AC \tag{345}$$

But $C \equiv \sigma^2_0 \geqslant 0$ so necessarily

$$A \equiv \Big[1 - \Big(\frac{cP}{E}\Big)^2\,\Big] \geqslant 0$$

Evidently (342) identifies the exceptional condition $A = 0$, which by (345) entails $B = 0$. And this, by (344), entails $\boldsymbol{P}\cdot\boldsymbol{\xi}_0 = \boldsymbol{P}\cdot\langle\boldsymbol{x}\rangle^0_0$. But we have already established that

$$\begin{aligned}\boldsymbol{P}\cdot\boldsymbol{\xi}_0 &= \boldsymbol{P}\cdot\boldsymbol{\xi}_t - Et\\ &= \int\boldsymbol{x}\cdot\boldsymbol{\mathcal{P}}\,d^3x - Et\\ \boldsymbol{P}\cdot\langle\boldsymbol{x}\rangle^0_0 &= \boldsymbol{P}\cdot\Big\{\langle\boldsymbol{x}\rangle^0_t - \tfrac{1}{M}\boldsymbol{P}t\Big\}\\ &= \Big[\int\boldsymbol{\mathcal{P}}\,d^3x\Big]\cdot\Big[\tfrac{1}{E}\int\boldsymbol{x}\,\mathcal{E}\,d^3x\Big] - \frac{P^2}{E/c^2}\,t\end{aligned}$$

and the t-terms are rendered equal by the condition $cP/E = 1$ which is now in force. The implication is that

$$B = 0 \quad\Longleftrightarrow \Big[\int\mathcal{E}\,d^3x\Big]\Big[\int\boldsymbol{x}\cdot\boldsymbol{\mathcal{P}}\,d^3x\Big] = \Big[\int\boldsymbol{\mathcal{P}}\,d^3x\Big]\cdot\Big[\int\boldsymbol{x}\,\mathcal{E}\,d^3x\Big]$$

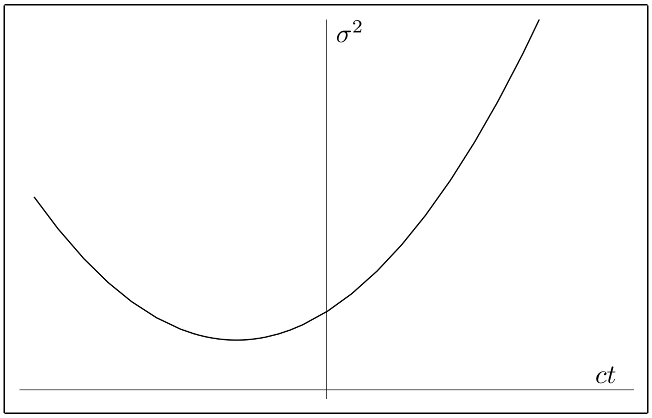


FIGURE 83: *Graph—computed from the right side of* (344)*— of the function* σ_t^2 *that describes (in leading approximation) how the energy in an isolated free field becomes spatially dispersed. This is what would happen to (for example) the Coulomb field of a charge if the charge were suddenly "turned off." It follows immediately from* (344) *that*

$$\tfrac{d}{dt}\sigma_t \to c \quad \text{as} \quad t\uparrow\infty$$

In the text the fact that the curve cannot cross the time-axis is shown to have important general implications.

which is readily seen to be satisfied if (but *only* if?) it is everywhere and always the case that

$$\boldsymbol{\mathcal{P}}E = \boldsymbol{P}\mathcal{E}$$

This is a very strong condition, for it forces the momentum density $\boldsymbol{\mathcal{P}}$ to be everywhere and always proportional to the constant vector $\hat{\boldsymbol{P}}$:

$$\boldsymbol{\mathcal{P}} = \tfrac{1}{c}\mathcal{E}\hat{\boldsymbol{P}} \tag{346.1}$$

Integration over the isolated free field gives

$$\boldsymbol{P} = \tfrac{1}{c}E\hat{\boldsymbol{P}} \tag{346.2}$$

What can one say about the structure of the electric/magnetic fields which is forced by (what we now recognize to be) the strong condition

$$E = c|\boldsymbol{P}|, \quad \text{equivalently} \quad E = cP \tag{347}$$

On the one hand we have[185]

[185] I hope it will be clear from context when, in the following discussion, E means "total energy" and when it means "magnitude of $\boldsymbol{E}$."

$$
\begin{aligned}
E &= c|\boldsymbol{P}| \\
&\downarrow \\
\int \frac{E^2+B^2}{2}\,d^3x &= \left|\int \boldsymbol{E}\times\boldsymbol{B}\,d^3x\right| \\
&\leqslant \int \big|\boldsymbol{E}\times\boldsymbol{B}\big|\,d^3x
\end{aligned}
\tag{348.1}
$$

with equality if and only if $\boldsymbol{E}\times\boldsymbol{B}$ is *unidirectional*. On the other hand

$$
\begin{aligned}
\big|\boldsymbol{E}\times\boldsymbol{B}\big|^2 &= E^2B^2 - (\boldsymbol{E}\cdot\boldsymbol{B})^2 \\
&= \Big[\frac{E^2+B^2}{2}\Big]^2 - \Big\{\Big[\frac{E^2-B^2}{2}\Big]^2 + (\boldsymbol{E}\cdot\boldsymbol{B})^2\Big\} \\
&\leqslant \Big[\frac{E^2+B^2}{2}\Big]^2 \quad : \quad \text{equality if and only if } E^2 = B^2 \text{ and } \boldsymbol{E}\cdot\boldsymbol{B} = 0
\end{aligned}
$$

so

$$
\int \big|\boldsymbol{E}\times\boldsymbol{B}\big|\,d^3x \leqslant \int \frac{E^2+B^2}{2}\,d^3x
\tag{348.2}
$$

which is (348.1) but with the inequality reversed. If we are to achieve (347) then *both* inequalities must hold: both, in other words, must reduce to equalities. The relation $E = cP$ is seen thus to require that it be everywhere and always the case that

1) $\boldsymbol{E}\times\boldsymbol{B}$ is unidirectional
2) $E^2 = B^2$
3) $\boldsymbol{E} \perp \boldsymbol{B}$

These same conditions will assume major importance when we come to consider plane wave solutions of the free field equations . . . which is curious, since (as was remarked already on page 235) plane waves cannot be "isolated," cannot be considered to comprise "pulses."

The discussion of recent paragraphs illustrates the power of the "momental mode of argument" (and illustrates also the deft genius of Schwinger!), but by no means exhausts the resources of the method: much fruit awaits the picking. More to the immediate point, it shows that basic mechanical properties of electromagnetic fields can be exposed *without direct appeal to Maxwell's equations*. Collectively, those properties encourage us to think of the (free) field as a mechanical object . . . even as a mechanical object which is—to a remarkable degree—"particle-like."

5. Zilch, spin & other exotic constructs. However "particle-like" we may consider the electromagnetic field to be, it does—because a field—possess many more degrees of freedom than a particle (infinitely many!), and can be expected to possess correspondingly many more constants of the motion. That one can actually write some of these down was discovered—by accident, and to everyone's surprise—by D. M. Lipkin in 1964.[186] Lipkin happened somehow to notice that if he defined

$$\left.\begin{aligned} Z^0 &\equiv \boldsymbol{E}\cdot\text{curl}\boldsymbol{E} + \boldsymbol{B}\cdot\text{curl}\boldsymbol{B} \\ \boldsymbol{Z} &\equiv \tfrac{1}{c}\Big[\boldsymbol{E}\times\tfrac{\partial}{\partial t}\boldsymbol{E} + \boldsymbol{B}\times\tfrac{\partial}{\partial t}\boldsymbol{B}\Big] \end{aligned}\right\} \tag{349}$$

then[187] it follows from the free-field Maxwell equations[188] that

$$\partial_0 Z^0 + \boldsymbol{\nabla}\cdot\boldsymbol{Z} = 0 \tag{350}$$

This he interpreted to provide local expression of the fact that

$$\text{total "zilch"} \equiv \int Z^0\, d^3x$$

is a constant of the free-field motion. The name he gave his discovery reflects the fact that he had (nor, to this day, does anyone have, so far as I am aware) no sense of what the *physical* significance of "zilch" might be. He drew attention to the fact that field *derivatives*—so conspicuously absent from the stress-energy and angular momentum tensors—enter into the definitions (349).

One is tempted at (350) to write $\partial_\alpha Z^\alpha = 0$, but such an equation would make relativistic good sense only if the Z^α transform as components of a 4-vector ...which, as it turns out, they do not. One confronts therefore the question: *How to bring Lipkin's discovery into manifest compliance with the principle of relativity*? Persuit of this issue led Lipkin to the identification of nine <u>*additional*</u> *new conservation laws*. More specifically, he was led to write

$$Z^\alpha = V^{00\alpha}$$

where—as T. A. Morgan[189] was quick to discover—the tensor components of $V^{\mu\nu\alpha}$ can be described quite simply as follows:

$$V^{\mu\nu\alpha} \equiv (\partial^\alpha G^\mu{}_\lambda)F^{\lambda\nu} - (\partial^\alpha F^\mu{}_\lambda)G^{\lambda\nu} \tag{351}$$

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[186] "Existence of a new conservation law in electromagnetic theory," J. Math. Phys. **5**, 696 (1964).

[187] PROBLEM 57.

[188] In (65) set $\rho = 0$ and $\boldsymbol{j} = \boldsymbol{0}$.

[189] "Two classes of new conservation laws for the electromagnetic field and other massless fields," J. Math. Phys. **5**, 1659 (1964). See also T. A. Morgan & D. W. Joseph, "Tensor lagrangians and generalized conservation laws for free fields," Nuovo Cimento **39**, 494 (1965) and R. F. O'Connell & D. R. Tompkins, "Generalized solutions for massless free fields and consequent generalized conservation laws," J. Math. Phys. **6**, 1952 (1965). It follows easily from (351) that

$$V^{00\alpha} = -(\boldsymbol{E}\cdot\partial^\alpha\boldsymbol{B} - \boldsymbol{B}\cdot\partial^\alpha\boldsymbol{E})$$

One achieves conformity with (349) by drawing upon the free field equations

This discovery motivated Morgan to write

$$V^{\mu\nu\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q} \equiv \begin{aligned}[t] &(\partial^{\alpha_1}\cdots\partial^{\alpha_p}G^\mu{}_\lambda)(\partial^{\beta_1}\cdots\partial^{\beta_q}F^{\lambda\nu}) \\ -&(\partial^{\alpha_1}\cdots\partial^{\alpha_p}F^\mu{}_\lambda)(\partial^{\beta_1}\cdots\partial^{\beta_q}G^{\lambda\nu})\end{aligned}$$

$$T^{\mu\nu\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q} \equiv \tfrac{1}{2}\Big[\begin{aligned}[t] &(\partial^{\alpha_1}\cdots\partial^{\alpha_p}F^\mu{}_\lambda)(\partial^{\beta_1}\cdots\partial^{\beta_q}F^{\lambda\nu}) \\ +&(\partial^{\alpha_1}\cdots\partial^{\alpha_p}G^\mu{}_\lambda)(\partial^{\beta_1}\cdots\partial^{\beta_q}G^{\lambda\nu})\Big]\end{aligned}$$

and to observe that—in consequence of the free field equations and certain fundamental "dualization identities"[190]—each of the above quantities is

1) $\mu\nu$-symmetric: $V^{\mu\nu\cdots} = V^{\nu\mu\cdots}$ and $T^{\mu\nu\cdots} = T^{\nu\mu\cdots}$
2) traceless: $V^\mu{}_\mu{}^{\cdots} = T^\mu{}_\mu{}^{\cdots} = 0$, and
3) locally conserved: $\partial_\mu V^{\mu\nu\cdots} = \partial_\mu T^{\mu\nu\cdots} = 0$.

In the absence of "spectator indices" (*i.e.*, in the case $p = q = 0$) $T^{\mu\nu\cdots}$ reduces to the familiar stress-energy tensor (309), so at least that member of Morgan's infinite population of functionally-independent conservation laws has a strong claim to physical significance. Lipkin's tensor $V^{\mu\nu\alpha}$ has moreover the property (which recommended it to his attention in the first place—namely) that

$$\partial_\alpha V^{\mu\nu\alpha} = 0 \quad : \quad \text{These are } \underline{\text{Lipkin's 10 conservation laws}}$$

. . . but the proof of that fact (see the papers cited above) is intricate, and will be omitted.

The solitary conservation law (350) discovered by Lipkin is seen in retrospect to have been but the tip of an iceberg. Of methodological interest is the observation that it was *relativity* that led from the tip to a perception of the iceberg as a whole. On page 233 we were led from the three components of angular momentum density to the 24 elements of $\mathcal{L}^{\alpha\mu\nu}$. Here the relativistic payoff has been infinitely richer . . . but to what effect? Although the theoretical placement of zilch-like conservation laws has been somewhat clarified,[191] the subject has passed into almost total obscurity: "zilch" is indexed in none of the standard texts, and appears to be on nobody's mind. I know of no argument

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(continued from the preceding page) and upon (compare (5)) the following uncommon but quite elementary identity:

$$\sum_{k=1}^{3}(A_k\boldsymbol{\nabla}B_k - B_k\boldsymbol{\nabla}A_k) = \boldsymbol{A}\times\text{curl}\boldsymbol{B} - \boldsymbol{B}\times\text{curl}\boldsymbol{A} + \boldsymbol{A}\,\text{div}\boldsymbol{B} - \boldsymbol{B}\,\text{div}\boldsymbol{A} - \text{curl}(\boldsymbol{A}\times\boldsymbol{B})$$

Note that the curl$(\boldsymbol{A}\times\boldsymbol{B})$-term makes no contribution to $\boldsymbol{\nabla}\cdot\boldsymbol{Z}$, so can be omitted (Lipkin's option) from the definition of $\boldsymbol{Z}$.

[190] See page 16 in ELEMENTS OF RELATIVITY (1966).

[191] See especially T. W. B. Kibble, "Conservation laws for free fields," J. Math. Phys. **6**, 1022 (1965).

to the effect that zilch is a concept too fundamentally trivial to support useful physics, but the effort to expose that physics appears to lie in the distant future. A place to start might be to describe the zilch-like features of some specific solutions of the free field equations, the objective being to gain a sharper *intuitive* sense of what those infinitely many conservation laws are trying to tell us. "Infinitely many conservation laws" seems a treasure too rich to ignore.

Classical mechanics came into the world as the theory of a particular system—the gravitational two-body system—and it was Newton's descriptive success in that special case that lent credibility to the concepts and methods he had created. But Newton's $\boldsymbol{F} = \frac{d}{dt}\boldsymbol{p}$ was by itself insufficient to support a theory of mechanical-systems-in-general, for it assumed $\boldsymbol{F}$ to be known/given in advance, and had nothing to say about how the forces (most conspicuously: the forces of constraint) internal to multiparticle systems *come* to be known. The *general theory of mechanical systems* had to await the cultivation of ideas that radiate from the work of Lagrange,[192] and only when such a theory was in place could the deepest and most subtle aspects of the original two-body problem be exposed. So it was also in the history of classical field theory: Maxwell gave us the theory of a particular classical field system—a theory which Einstein showed to be "naturally relativistic"—but motivation to create a *general theory of relativistic classical fields* had to await the development of interest a "relativistic theory of gravitation," the theory which by the time it had become ripe enough to fall from the tree had metamorphosed into "general relativity." It emerged that Lagrangian methods provide—ready made—the language of choice for the description of relativistic classical fields, and that the "mechanical properties of fields" are brought into focus (Noether's insight) by <u>conservation laws that reflect symmetries of the dynamical action</u>:[193]

$$S_{\mathcal{R}}[\varphi] \equiv \iiiint_{\mathcal{R}} \mathcal{L}(\varphi, \partial\varphi)\, d^4x$$

Here φ is any *solution* of the field equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi_{a,\mu}} - \frac{\partial \mathcal{L}}{\partial \varphi_a} = 0$$

[192] Lagrange's *Mechanique analytique* was published in 1788—101 years after the publication of Newton's *Philosophiae Naturalis Principia Mathematica*. Another near-half-century was to elapse before Hamilton—who took his inspiration directly from what he called Lagrange's "scientific poem"—completed his own contributions to mechanics ("On a general method in dynamics" appeared in 1834, and his "Second essay on a general method in dynamics" in 1835) and it was not until 1918 that Emmy Noether placed the elegant capstone on Lagrangian dynamics.

[193] For more detailed discussion see, for example, CLASSICAL FIELD THEORY (1999), Chapter 1, pages 15–32 or Herbert Goldstein, *Classical Mechanics* (2nd edition 1980), Chapter 12.

$\mathcal{R}$ is any "bubble" in spacetime, and a indexes the individual components of the multi-component field system. When one returns with such general principles to the electrodynamic birthplace of relativistic field theory one acquires deepened insight into the meaning—and a greater respect for the "naturalness"—of constructions that in §§1–3 were introduced in a somewhat improvisatory *ad hoc* manner. Specifically, one finds that (see again (304) and (309)) the ${}^{\nu}$-indexed quartet of conservation laws

$$\partial_\alpha S^{\alpha\nu} = 0 \qquad (352.1)$$
$$S^{\mu\nu} \equiv F^\mu{}_\alpha F^{\alpha\nu} - \tfrac{1}{4}(F^{\alpha\beta}F_{\beta\alpha})g^{\mu\nu}$$

reflects the **translational** symmetry of the electromagnetic free-field action function, and that (see again (330) and (331)) the antisymmetrically ${}^{\mu\nu}$-indexed sextet of conservation laws

$$\partial_\alpha \mathcal{L}^{\alpha\mu\nu} = 0 \qquad (352.2)$$
$$\mathcal{L}^{\alpha\mu\nu} \equiv \tfrac{1}{c}(x^\mu S^{\alpha\nu} - x^\nu S^{\alpha\mu})$$

reflects the **Lorentz** symmetry of the action. Three of the latter (those that arise from the *rotational* component of the Lorentz group) refer to the conservation of angular momentum $\boldsymbol{L}$, while the other three (those that arise from boosts) refer to the conservation of $\boldsymbol{K}$. We know, however, that Maxwellian electrodynamics is conformally covariant, and that the 4-dimensional conformal group is a 15-parameter group that—in addition to translations, rotations and boosts—contains "dilations" (one parameter) and "Möbius transformations" (four parameters). What are the associated conservation laws? This question was studied by E. Bessel-Hagen (1921), whose work is reviewed in a very accessible paper by B. F. Plybon.[194] It develops that **dilational** symmetry of the action entails

$$\partial_\alpha(S^\alpha{}_\beta x^\beta) = 0 \qquad (352.3)$$

while **Möbius** symmetry supplies a ${}^{\mu}$-indexed quartet of conservation laws

$$\partial_\alpha(2S^\alpha{}_\beta x^\beta x^\mu - S^{\alpha\mu}\cdot x^\beta x_\beta) = 0 \qquad (352.4)$$

Recalling from (310) & (311) that $S^{\mu\nu}$ is symmetric and traceless, we observe (with Plybon) that[195]

- (352.2) follows from (352.1) and the symmetry of $S^{\mu\nu}$
- (352.3) follows from (352.1) and the tracelessness of $S^{\mu\nu}$
- (352.4) follows from (352.1) and the traceless symmetry of $S^{\mu\nu}$

So (352.4) provides no information additional to that conveyed already by the conservation laws (352.1/2/3) and it is therefore pointless to inquire after the

[194] "Observations on the Bessel-Hagen conservation laws for electromagnetic fields," AJP **42**, 998 (1974).
[195] PROBLEM 58.

"independent physical meaning" of the Möbius invariants.[196] The physical meanings of the translational and Lorentz invariants has already been established, while (352.3) supplies the dilational invariant

$$\begin{aligned} D &\equiv \int (S^0{}_\beta x^\beta)\, d^3x \\ &= c\int (\mathcal{E}\,t - \boldsymbol{\mathcal{P}}\boldsymbol{\cdot}\boldsymbol{x})\, d^3x \\ &= c(E\,t - \boldsymbol{P}\boldsymbol{\cdot}\boldsymbol{\xi}_t) \end{aligned}$$

. . . the invariance of which was encountered/exploited already at the bottom of page 237.

This elegant train of thought lends new interest to the zilch-like free-field conservation laws discussed previously, for it is easily demonstrated that those are of a design to which standard "Noetherian analysis" can never lead. This observation led Morgan & Joseph[189] to construct a highly non-standard theory of "tensor Lagrangians"

$$\mathcal{L} \quad\longrightarrow\quad \mathcal{L}_{\text{population of tensor indices}}$$

in which all of the infinitely many "conservation of zilch" statements can be attributed to the translational invariance of the associated tensor Lagrangians. They note, however, that free fields are *unobservable in principle*: that <u>it is by their interactions that systems announce themselves</u> . . . and that it appears to be impossible to build interactions into a tensor Lagrangian theory. It is, in their view, this circumstance that robs "conservation of zilch" of any claim to physical significance, and that explains why only scalar Lagrangians are encountered in theories of the observable real world.

To approach the subject of "spin," as it is (but only rarely!) encountered in classical electrodynamics I must back up a bit. In 1936 A. Proca undertook to apply orthodox Lagrangian methods to the construction of what might be called a "relativistic electrodynamics of massive photons," his hope being that such objects might be identified with Yukawa's conjectured "mesons" (1934: see again page 18). Proca was led[197] to a system of field equations which in

[196] This, however, is not to say that (352.4) is useless. Used in conjunction with (352.1) and the traceless symmetry of $S^{\mu\nu}$ it supplies

$$\partial_\alpha\big[(x^\beta x_\beta)S^{\mu\alpha}\big] - 2S^{\mu\alpha}x_\alpha = 0$$

which in the case $\mu = 0$ was used (at the middle of page 237) to good effect by Schwinger.

[197] Details are developed in CLASSICAL FIELD THEORY (1999), Chapter 2, pages 16–22 and 51–56.

manifestly Lorentz covariant notation read

$$
\begin{gathered}
G^{\mu\nu} = \partial^\mu U^\nu - \partial^\nu U^\mu \\
\partial^\lambda G^{\mu\nu} + \partial^\mu G^{\nu\lambda} + \partial^\nu G^{\lambda\mu} = 0 \\
\partial_\mu G^{\mu\nu} + \varkappa^2 U^\nu = 0 \\
\partial_\nu U^\nu = 0
\end{gathered}
$$

and in this electrodynamically-inspired notation

$$
\|G^{\mu\nu}\| = \begin{pmatrix} 0 & -\mathfrak{E}_1 & -\mathfrak{E}_2 & -\mathfrak{E}_3 \\ \mathfrak{E}_1 & 0 & -\mathfrak{B}_3 & \mathfrak{B}_2 \\ \mathfrak{E}_2 & \mathfrak{B}_3 & 0 & -\mathfrak{B}_1 \\ \mathfrak{E}_3 & -\mathfrak{B}_2 & \mathfrak{B}_1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} U^0 \\ U^1 \\ U^2 \\ U^3 \end{pmatrix} = \begin{pmatrix} \phi \\ \boldsymbol{\mathfrak{A}} \end{pmatrix}
$$

become

$$
\begin{gathered}
\boldsymbol{\mathfrak{E}} = -\boldsymbol{\nabla}\phi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\mathfrak{A}} \qquad \text{and} \qquad \boldsymbol{\mathfrak{B}} = \boldsymbol{\nabla}\times\boldsymbol{\mathfrak{A}} \\
\boldsymbol{\nabla}\times\boldsymbol{\mathfrak{E}} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\mathfrak{B}} = \mathbf{0} \qquad \text{and} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{B}} = 0 \\
\boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{E}} = -\varkappa^2\phi \qquad \text{and} \qquad \boldsymbol{\nabla}\times\boldsymbol{\mathfrak{B}} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\mathfrak{E}} = -\varkappa^2\boldsymbol{\mathfrak{A}} \\
\tfrac{1}{c}\tfrac{\partial}{\partial t}\phi + \boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{A}} = 0
\end{gathered}
$$

Here

$$
\varkappa \equiv mc/\hbar \quad \text{with physical dimension} \quad [\varkappa] = (\text{length})^{-1}
$$

is Proca's "mass parameter"—the reciprocal of the λ encountered already on page 18. The formal success of Proca's program resides in the observation that in the limit $\varkappa \downarrow 0$ these equations assume precisely the form of the free-field Maxwell equations (61) in what we will later learn to call the "Lorentz gauge." Noether's argument leads from the translational invariance of Proca's Lagrangian to a stress-energy tensor which is *not symmetric*, but which after "Belinfante symmetrization" becomes[198]

$$
\begin{aligned}
T^{\mu\nu} &= G^\mu{}_\sigma G^{\sigma\nu} + \mathcal{L}g^{\mu\nu} + \varkappa^2 U^\mu U^\nu \\
&\qquad \mathcal{L} \equiv \tfrac{1}{2}\Big\{G^{\sigma\rho}(U_{\rho,\sigma} - U_{\sigma,\rho}) - \tfrac{1}{2}G^{\rho\sigma}G_{\rho\sigma}\Big\} - \tfrac{1}{2}\varkappa^2 U^\rho U_\rho
\end{aligned}
$$

which is manifestly symmetric, but traceless only in the limit $\varkappa \downarrow 0$, and which supplies

$$
\begin{aligned}
\text{energy density} &= \tfrac{1}{2}\Big[\big(G_{01}^2 + G_{02}^2 + G_{03}^2 + G_{12}^2 + G_{23}^2 + G_{31}^2\big) \\
&\qquad\qquad + \varkappa^2\big(U_0^2 + U_1^2 + U_2^2 + U_3^2\big)\Big] \\
&= \tfrac{1}{2}\big\{\mathfrak{E}^2 + \mathfrak{B}^2 + \varkappa^2(\phi^2 + \mathfrak{A}^2)\big\} \geqslant 0 \\
\text{momentum density vector} &= \tfrac{1}{c}\big\{\boldsymbol{\mathfrak{E}}\times\boldsymbol{\mathfrak{B}} + \varkappa^2\phi\,\boldsymbol{\mathfrak{A}}\big\}
\end{aligned}
$$

[198] We write $T^{\mu\nu}$ instead of $S^{\mu\nu}$ because S has been preempted by Spin.

These formulæ give back their electromagnetic counterparts in the limit $\varkappa \downarrow 0$ ... and bring us at last to the main point of this discussion: Noether's argument leads from the Lorentz invariance of Proca's Lagrangian to an angular momentum tensor of the form

$$\begin{aligned} \mathcal{J}^{\mu\alpha\beta} &= \tfrac{1}{c}\big(x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}\big) \\ &= \mathcal{L}^{\mu\alpha\beta} + \mathcal{S}^{\mu\alpha\beta} \\ &= \text{orbital component} + \text{intrinsic or ``spin'' component} \end{aligned}$$

with

$$\mathcal{S}^{\mu\alpha\beta} = \tfrac{1}{c}\big(G^{\alpha\mu}U^\beta - G^{\beta\mu}U^\alpha\big)$$

Both $\partial_\mu \mathcal{L}^{\mu\alpha\beta}$ and $\partial_\mu \mathcal{S}^{\mu\alpha\beta}$ fail to vanish, but they do so in such a concerted way that $\partial_\mu \mathcal{J}^{\mu\alpha\beta} = 0$ (which arise from the familiar pair of circumstances: $\partial_\mu T^{\mu\nu} = 0$ and $T^{\mu\nu} = T^{\nu\mu}$). Straightforward extension of (see again page 233) the definition

$$\text{angular momentum density vector} = \begin{pmatrix} \mathcal{L}^{023} \\ \mathcal{L}^{031} \\ \mathcal{L}^{012} \end{pmatrix}$$

supplies

$$\begin{aligned} \text{spin density vector} = \begin{pmatrix} \mathcal{S}^{023} \\ \mathcal{S}^{031} \\ \mathcal{S}^{012} \end{pmatrix} &= \tfrac{1}{c}\begin{pmatrix} G^{20}U^3 - G^{30}U^2 \\ G^{30}U^1 - G^{10}U^3 \\ G^{10}U^2 - G^{20}U^1 \end{pmatrix} \\ &= \tfrac{1}{c}\,\boldsymbol{\mathfrak{E}}\times\boldsymbol{\mathfrak{A}} \end{aligned} \tag{353}$$

which does check out dimensionally: from

$$[\boldsymbol{\mathfrak{E}}] = \sqrt{\text{energy density}} \quad \text{and} \quad [\boldsymbol{\mathfrak{A}}] = \text{length}\cdot\sqrt{\text{energy density}}$$

we have

$$\begin{aligned} [\text{spin density}] &= \text{time}\cdot\text{energy density} \\ &= \text{action density} \\ &= \text{angular momentum density} \end{aligned}$$

Remarkably, (353) contains no reference to $\varkappa$, therefore no reference to either $\hbar$ or m. We expect it therefore to retain its meaning even in the classical electromagnetic limit ... or would but for this awkward detail: in Proca theory ($\varkappa \neq 0$) $\partial_\nu U^\nu = 0$ enjoys the status of a field equation, but in the Maxwellian limit ($\varkappa = 0$) it acquires the status of an arbitrarily imposed side condition (the "Lorentz gauge condition," which will acquire major importance later in our work). In electrodynamics we expect therefore to have

$$\text{spin density } \boldsymbol{\mathcal{S}} = \tfrac{1}{c}\,\boldsymbol{E}\times\boldsymbol{A}, \quad \textit{but only in the Lorentz gauge!} \tag{354}$$

But if (354) requires us to nail down the gauge, it does not require us to nail down the coordinate system, to specify a "reference point": since the expression on the right lacks the "momental structure" of $\boldsymbol{x}\times\boldsymbol{p}$ it is insensitive to where we have elected to place of the origin of the $\boldsymbol{x}$-coordinate system. If $\boldsymbol{\mathfrak{S}}$ has anything at all to do with "angular momentum" it must have to do with "*intrinsic* angular momentum" (or "spin").

Equation (354) appears on page 115 of Davison Soper's *Classical Field Theory* (1976) but nowhere else in the pedagogical literature, so far as I have been able to discover. That the construction $\frac{1}{c}\boldsymbol{E}\times\boldsymbol{A}$ does indeed have "something to do with angular momentum" Soper argues as follows: Look to the case

$$\|A^\mu\| = \begin{pmatrix} \varphi \\ \boldsymbol{A} \end{pmatrix} = \begin{pmatrix} 0 \\ -A\sin[k(ct-z)] \\ \pm A\cos[k(ct-z)] \\ 0 \end{pmatrix}$$

Then the Lorentz gauge condition $\partial_\mu A^\mu = 0$ becomes trivial, and

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} = \begin{pmatrix} Ak\cos[k(ct-z)] \\ \pm Ak\sin[k(ct-z)] \\ 0 \end{pmatrix}$$

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} = \begin{pmatrix} \mp Ak\sin[k(ct-z)] \\ +Ak\cos[k(z-ct)] \\ 0 \end{pmatrix} = \hat{\boldsymbol{z}}\times\boldsymbol{E}$$

describe ↺ / ↻ circularly polarized plane waves of frequency $\omega = kc$, advancing up the z-axis with speed c. We compute

$$\boldsymbol{\mathfrak{S}} = \tfrac{1}{c}\boldsymbol{E}\times\boldsymbol{A} = \pm A^2(k/c)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Noting that the energy density is $\mathcal{E} = \frac{1}{2}(E^2+B^2) = A^2k^2$ we have

$$\boldsymbol{\mathfrak{S}} = (\mathcal{E}/\omega)\,\hat{\boldsymbol{z}} \tag{355}$$

To interpret this result, Soper draws upon a "photonic" conception of the electromagnetic field: he imagines it to contain N photons per unit volume, each carrying energy $\hbar\omega$. Then $\mathcal{E} = N\hbar\omega$ gives

$$\boldsymbol{\mathfrak{S}} = (N\hbar)\,\hat{\boldsymbol{z}}$$

which Soper interprets to state that

Each photon carries $\hbar$ units of intrinsic angular momentum (356)

In the *quantum* theory of angular momentum one is brought to the conclusion that the

"allowed values" of **orbital** angular momentum are $0, \hbar, 2\hbar, 3\hbar, \ldots$

"allowed values" of **spin** angular momentum are $0, \frac{1}{2}\hbar, \hbar, \frac{3}{2}\hbar, 2\hbar, \ldots$

and is led to assert that "electrons are spin $\frac{1}{2}$ particles," that "fermions carry half-integral spin, bosons carry integral spin," etc. We are (in view of what happens to those statements in the limit $\hbar \downarrow 0$) not surprised to encounter the frequently-repeated to claim—not quite accurate, as it turns out[199]—that "spin is an intrinsically quantum mechanical phenomenon, and at the deepest level a *relativistic* quantum mechanical phenomenon." It becomes in this light interesting to notice that (355) is *classically meaningful as it stands, that the introduction of "photonic language"—though possible—is inessential*. And indeed: the first direct *experimental* support of (355) was reported by authors who, while they allowed themselves to make casual use of "photonic language," employed methods that were in fact entirely classical.[200]

Yet subtleties lurk within the preceding account of the angular momentum of electromagnetic free fields, and literature bearing on the subject remains to this day often confused/misleading. Some authors fall into paradox when they talk about orbital angular momentum but imagine themselves to be talking about spin angular momentum,[201] though by the present account the two could hardly be more different: an unbounded circularly polarized plane wave carries

- infinite spin angular momentum but (by a symmetry argument)
- zero orbital angular momentum.

Richly detailed accounts of orbital angular momentum can be found in §2.7 and Chapter 9 of J. W. Simmons & M. J. Guttmann's *States, Waves and Photons: A Modern Introduction to Light* (1970) and in a recent paper by L. Allen, M. J. Padgett & M. Babiker,[202] but those authors do not share my interest in probing the outer limits of *classical* electrodynamics: they have other fish to fry, and at critical moments reveal themselves to have photons on the brain. Nor are things quite so simple as I have represented them to be: in §3 of the last of the papers mentioned above we encounter the observation that

> "*. . . there is a considerable literature which warns against such a separation* [as is conveyed by the equation $\boldsymbol{\mathfrak{J}} = \boldsymbol{\mathfrak{L}} + \boldsymbol{\mathfrak{S}}$: they cite sources, and continue . . .] *Biedenharn & Louck write 'It is, indeed,*

[199] See the material collected in H. C. Corben, *Classical & Quantum Theories of Spinning Particles* (1968).

[200] R. A. Beth, "Mechanical detection and measurement of the angular momentum of light," Phys. Rev. **50**, 115 (1936). Beth used a torsion balance to measure the change in the angular momentum of a circularly polarized light beam on passage through a doubly refracting crystal plate. He worked at Princeton, and was in correspondence with A. H. S. Holbourn (at Cambridge) who obtained similar results at the same time (Nature **137**, 31 (1936)). Beth reports that an equation equivalent to (355) can be found in J. H. Poynting, Proc. Roy. Soc. **A82**, 560 (1909).

[201] See, for example, R. I. Khrapko, "Question #79. Does plane wave not carry a spin?" AJP **69**, 405 (2001).

[202] "The orbital angular momentum of light," Progress in Optics (1999), pages 294–372.

> *not possible to separate the total angular momentum of the photon field into and "orbital" and a "spin" part (this would contradict gauge invariance); the best that can be done is to define the helicity operators ... which is an observable (Beth).'*"[203]

The problems to which these authors allude do *not* arise in the Proca theory. They are parts of an interrelated nest of problems that arise in the Maxwellian limit $\varkappa \downarrow 0$—problems to which we will have occasion to revisit after we have acquired some sharper tools. Appeals to the Proca theory will often prove of assistance in those endeavors.

6. Conclusion. The work of this chapter has shown the electromagnetic field to be richly endowed with "mechanical properties"... to possess, indeed, all the properties that we standardly/intuitively associate with "particles" *except spatial localization*. The results developed are of a practical importance that should by now be obvious. On the philosophical side ... while they do not of themselves "resolve" the question "Is the electromagnetic field 'real'"? they do have clear relevance to any attempt to assess the status of that question. It is my personal opinion that any attempt to dismiss the electromagnetic field as "a computational convenience ... but a physical fiction"

- has overwhelmingly much to answer for
- is therefore quite unlikely to succeed, and
- would almost entail more cost than benefit.

Readers should, however, be aware that some very able physicists—the young Feynman, among others!—have from time to time been motivated to adopt—tentatively, and without compelling success—the opposite view, and that this minor tradition has exposed isolated points of great interest.[204]

Our work has also served—an many points—to illustrate the remarkable *theory-shaping power of special relativity*.

It is for theory-shaping reasons that I have raised (and will raise again) the "reality question." We do not expect the methods of physics ever to part the final veil and reveal "the stark beauty of naked Reality": we are, after all, decendents of Newton, the great natural philosopher who, though he yearned to know what gravity *is*, recognized that he/we must be content to *describe* what gravity *does* ("I do not philosophize ..."). But when we look to the history of physics we find that major developments have often entailed shifts in the points at which we imagine the "reality" in our theories to be invested. So it is for pragmatic reasons that we must pay attention to the "reality question"

[203] Their reference here is to L. C. Biedenharn & J. D. Luock, *Angular Momentum in Quantum Physics: Vol. VIII of the Encyclopaedia of Mathematics & its Applications* (1980).

[204] See the papers reprinted in E. H. Kerner (editor), *The Theory of Action-at-a-Distance in Relativistic Particle Physics* (1972). Also relevant is F. Hoyle & J. V. Narlikar's *Action at a Distance in Physics and Cosmology* (1974).

if it is our ambition to contribute to the next such development. My claims regarding the "reality of the electromagnetic field" draw only weak support from electro/magnetostatics, stronger support from electro*dynamics* ... but in the end we must admit that in fact we never *observe* $\boldsymbol{E}$-fields or $\boldsymbol{B}$-fields their naked selves: what we observe are (ramifications of) their mechanical properties, the results of their interaction with other mechanical systems (which themselves remain similarly unobservable in isolation!). It might therefore be argued that we should assign tentative "reality" not to $F^{\mu\nu}$ but to objects like $S^{\mu\nu}$. But even then the situation is not entirely clear cut ... for suppose were were to form

$${\color{red}S}^{\mu\nu} \equiv S^{\mu\nu} + \partial_\alpha W^{\alpha\mu\nu}$$

$$W^{\alpha\mu\nu} \text{ assumed to be } \begin{cases} \alpha\mu\text{-antisymmetric} \\ \mu\nu\text{-symmetric} \end{cases}$$

Then

$$\begin{aligned} \partial_\mu S^{\mu\nu} = 0 \quad &\Longleftrightarrow \quad \partial_\mu {\color{red}S}^{\mu\nu} = 0 \\ S^{\mu\nu} = S^{\nu\mu} \quad &\Longleftrightarrow \quad {\color{red}S}^{\mu\nu} = {\color{red}S}^{\nu\mu} \end{aligned}$$

and (because $\alpha\mu$-antisymmetry entails $\partial_\alpha W^{\alpha 0\nu} = \sum_k \partial_k W^{k0\nu} \equiv \boldsymbol{\nabla}\cdot\boldsymbol{W}^\nu$)

$$\begin{aligned} \int {\color{red}S}^{0\nu}\, d^3x &= \int S^{0\nu}\, d^3x + \int \boldsymbol{\nabla}\cdot\boldsymbol{W}^\nu\, d^3x \\ &= \text{ditto} + \int \boldsymbol{W}^\nu\cdot \boldsymbol{d\sigma} \\ &= \int S^{0\nu}\, d^3x \quad \text{if the surface term is assumed to vanish} \end{aligned}$$

show that, while ${\color{red}S}^{\mu\nu}$ assigns

- different energy/momentum densities but
- the same *total* energy/momentum

to the field, it satisfies all formal requirements (symmetry, local conservation) just as well as $S^{\mu\nu}$. We possess therefore as many viable candidate stress-energy tensors as there are ways to assign value to $W^{\alpha\mu\nu}$ *and no principle of choice*. It becomes difficult in such a circumstance to argue that one member of the population has a stronger claim to "reality" than another.[205]

[205] For related discussion see ELECTRODYNAMICAL APPLICATIONS OF THE EXTERIOR CALCULUS (1996) pages 57–61 and "ELECTRODYNAMICS" IN 2-DIMENSIONAL SPACETIME (1997) pages 13–14.