KEPLER PROBLEM BY DESCENT FROM THE EULER PROBLEM ‡

Nicholas Wheeler, Reed College Physics Department
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Introduction. The exactly soluble problems of physics relate typically to systems that dwell not in Nature, but in the mind. It is by gross over-simplification—joined sometimes by lucky accident—that they acquire their tractability, and it is on account of their exceptional tractability that they are disproportionately prominent in the classroom, where exactly soluble systems serve usefully to illustrate points of principle and technique. But this they do at cost: they tend to obscure the force of the frequently-heard allegation that "Mother Nature is a cunning bitch."

The latter circumstance is, of course, not news to mature physicists, who collect and treasure exactly soluble systems for their own good reasons. Such systems are valuable as "laboratories" in which to test the worth of fresh ideas. Perhaps even more importantly, they provide the points of departure for the perturbation theories which are the stuff of "real physics." The "catalog of exactly soluable systems" can be looked upon as a catalog of (latent) zeroth approximations.

Exactly soluble physical systems see service also in other—less familiar—ways. Reflecting on the work which will concern me today, I was surprised to realize that my own career has been marked by a recurrence of problems of the type

$$\text{Exactly soluble problem A} \longrightarrow \text{Exactly soluble problem B}$$

where my objective has been to "morph" A into B, to find a point of view from which A and B can be seen to be particular instances of some over-arching (soluble) problem C. One's objective in such a context is to blaze a trail, to move as gracefully and informatively as possible through the analytical terrain—typically a jungle—that separates A from B, and it has been my experience that the value of such exercise lies most typically not so much in the completed

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trail as in the bushwhacking that created it, which tends to be analytically to be quite informative. I recall, for example, an occasion when I had interest in the rolling motion of a loaded massless disk. The disk has radius R, and the load (a point mass m) is attached at a distance r from the center of the disk. The path traced by the load m is cycloidal—"curtate" if $r < R$, "prolate" if $r > R$, and "common" if $r = R$.[1] The small-amplitude motion of such a system is harmonic in the curtate and prolate cases, but m "bounces" anharmonically in the critical case $r = R$. The question, therefore, was this: how does the familiar physics of an oscillator "morph" continuously into the (also familiar but) quite different physics of a bouncing ball? The answer was found[2] to hinge on a remarkable property of the complete elliptic integrals

$$\begin{aligned}\mathbf{K}(p) &\equiv \int_0^{\pi/2} \frac{1}{\sqrt{1-p^2\sin^2\theta}}\,d\theta\\ \mathbf{E}(p) &\equiv \int_0^{\pi/2} \sqrt{1-p^2\sin^2\theta}\;d\theta\end{aligned}$$

These familiar functions become abruptly complex (and $\mathbf{K}(p)$ becomes singular) at $p = 1$, but at points just shy of that critical point (i.e., when $q \equiv \sqrt{1-p^2}$ is real but small) one has these wonderful "one-sided expansions"[3]

$$\begin{aligned}\mathbf{K} &= \Lambda + \tfrac{1}{4}(\Lambda - 1)q^2 + \tfrac{9}{64}(\Lambda - \tfrac{7}{6})q^4 + \cdots\\ \mathbf{E} &= 1 + \tfrac{1}{2}(\Lambda - \tfrac{1}{2})q^2 + \tfrac{3}{16}(\Lambda - \tfrac{13}{12})q^4 + \cdots\\ &\qquad\Lambda \equiv \log\tfrac{4}{q}\end{aligned}$$

On another occasion I was motivated to study how the (quantum) physics of a confined bounding ball goes over into the physics of a free "particle in a box" as the "ceiling height" is reduced. On yet another occasion I had reason to construct a unified quantum theory of systems of the type

$$H(x,p) = A(x)p^2 + B(x)p + C(x)$$

from which the quantum theories of free particles, particles in free fall and harmonically bound particles could all be obtained as special cases. These projects entailed heavy use of the theory of higher functions, but contributed

[1] I have borrowed my terminology from §19 of F. L. Griffin's *Introduction to Mathematical Analysis: Higher Course* (1927).

[2] See GYRODYNAMICS (1976), pp. 191–205. The problem was discussed subsequently, from a different point of view, by Jean-Marc Lévy-Leblond; see "Rock and roll: Non-isochronous small oscillations (an example)," AJP **46**, 106 (1978).

[3] See E. Jahnke & F. Emde, *Tables of Functions* (1938), p. 73 or J. Spanier & K. Oldham, *An Atlas of Functions* (1987), p. 612.

by virtue both of their over-arching generality and their analytical novelty to my understanding of some aspects of Feynman's sum-over-paths formalism. The list of examples which might be drawn from my own work could be very greatly extended, and when we look to the published literature it can be extended *ad infinitum*, for my personal experience has been not at all atypical; when we come upon a particular result our instinct is to look to its generalizability, and conversely: when we come upon a general result it is our invariable instinct to particularize, to test it against special cases. Usually the latter activity poses no problem, but in cases of the sort to which I have drawn attention it entails some analytical effort, some "bushwhacking," which most typically assumes the guise of "asymptotic analysis."

The problem of interest to me today—which might be described

$$\text{Kepler Problem} \longleftarrow\!\!\!\text{———}\ \text{Euler Problem}$$

—has much in common with problems of the type just described. It sprang, in fact, from the exercise of nothing more profound than our just-noted instinct to simplify, to particularize, to see what intricate novelties have to say when applied to familiar simplicities. But the results I will be reporting gain interest also from another (quite unanticipated) circumstance. Standardly we use simple problems (whether by perturbation theory or otherwise) to illuminate hard ones. Today, however, I will proceed in the opposite direction; I will describe some of the ways in which "Euler's Problem"—a hard problem—can be used to illuminate some pretty but little-known aspects of the relatively simple (but inexhaustibly rich) "Kepler Problem."

The Kepler Problem is a "soluble problem with a difference" in the sense that it stands in convincingly close approximation to some *physical systems that matter*—systems that lie at the heart both of celestial mechanics and of atomic physics. It was observational data that—by inspired analysis—led Kepler to the laws which now bear his name, but Kepler himself was never in position to formulate what we now call "the Kepler Problem," for he inhabited a pre-dynamical world. It was Newton who was first in position to pose—and to solve—the dynamical "Kepler Problem," and it was his success in this regard which most strongly recommended Newton's Laws of Motion to the attention of the world. And it was a variant of this same physical model that in our own century inspired the work of, and lent credibility to, the successive accomplishments of Bohr, Heisenberg and Schrödinger. The Kepler Problem has served as a primary stimulus to mathematical/physical invention for now more than 300 years, and still today retains many of its secrets.

The "Euler Problem" came into the world in response to no such physical imperative, but as the fruit of a formal act of straightforward generalization, a mathematical curiosity.[4] And a curiosity it has remained—made the curiouser

[4] Lagrange, in his *Méchanique Analytique*, begins his own discussion of the problem by apologizing for treating a system "which has nothing corresponding to it in the system of the world."

by the fact that at several points Euler made use of ideas that "hadn't been invented yet" (and wouldn't be for decades). Though the quantum version of the problem can be argued to stand near the heart of theoretical chemistry (as was first appreciated by Øyvind Burrau[5] in 1928), the classical problem has, so far as I am aware, been discussed by no modern author on celestial mechanics. Brief mention does appear in §53 of E. T. Whittaker's *Analytical Mechanics* (1937); Whittaker calls Euler's Problem "the most famous of the known soluble problems [of its class]," and cites (in addition to papers by Euler (1760–1764) and Lagrange (1766–1769)) one early 20$^{\text{th}}$-Century paper by the Princeton mathematician A. M. Hiltebeitel,[6] who in turn makes reference to 19$^{\text{th}}$-Century contributions by Liouville and Darbeaux.[7] Mathematicians all. I hope, however, to demonstrate that the Euler Problem has much of importance to teach us not only about the Kepler Problem, and the physics of Kepler-like systems, but about the general principles of mechanics.

1. Statement of the Kepler Problem. Two particles move in 3-space, subject only to a (rotationally invariant) central interaction. The center of mass of such a system moves uniformly/rectilinearly—which is to say, uninterestingly. Plant an inertial Cartesian frame on the center of mass and observe that the position/motion of either particle is implicit in the position/motion of the other. One arrives thus at the "reduced central force problem"

$$L(\dot{\mathbf{x}}, \mathbf{x}) = \tfrac{1}{2}\mu\big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big) - U(r) \tag{1}$$

where

$$\mu \equiv \big(\tfrac{1}{m_1} + \tfrac{1}{m_2}\big)^{-1}$$

defines the so-called "reduced mass," and where $r \equiv \sqrt{x^2 + y^2 + z^2}$. If one of the masses is much larger than the other, then $\mu \sim$ lesser of the masses; I shall, as a notational convenience, assume this to be the case, writing m in place of μ. The central force problem (1) yields generally to analysis in spherical coordinates

$$\begin{aligned} x &= r\,\sin\theta\cos\phi \\ y &= r\,\sin\theta\,\sin\phi \\ z &= r\cos\theta \end{aligned}$$

[5] See L. Pauling & E. B. Wilson, *Introduction to Quantum Mechanics, with Applications to Chemistry* (1935) §42c and N. F. Mott & I. N. Sneddon, *Wave Mechanics & Its Applications* (1948) §33.4.

[6] "On the problem of two fixed centers and certain of its generalizations," Amer. J. of Math. **33**, 337 (1911).

[7] A usefully annotated discussion of Euler's Problem and related material—which Whittaker, though famous for his exhaustive command of the literature, elected not to cite—can be found also at the beginning of Chapter VIII in E. J. Routh's *Dynamics of a Particle*, which had appeared in 1898, six years prior to the publication of the first edition of Whittaker's own text.

but since the motion is necessarily confined to a plane (the plane defined by the initial values of $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$) one can without loss of generality assume that $\theta = \frac{1}{2}\pi$ initially and at all subsequent times: the motion is confined to the equatorial plane. In the original (physical) problem configuration space is 6-dimensional (and phase space 12-dimensional), but in the "twice reduced problem"

$$L(\dot{x},\dot{y},x,y) = \tfrac{1}{2}m\big(\dot{x}^2+\dot{y}^2\big) - U(r) \qquad r = \sqrt{x^2+y^2} \tag{2}$$

it is only 2-dimensional. In two cases and two only, the problem thus posed has the property[8] that *all (bound) orbits are reentrant* (and therefore periodic). Those are the cases

$U(r) = +kr^{+2}$: 2-dimensional harmonic oscillator

$U(r) = -kr^{-1}$: **Kepler Problem**

Remarkably (and, as it turns out, relatedly), in those cases—and those only—the equations of motion are *"soluble by separation of variables"*[9] *in more than one coordinate system*. In the former case, separability can be achieved not only in polar coordinates but also—trivially—in Cartesian coordinates; one has

$$L = \tfrac{1}{2}m\big(\dot{x}^2+\dot{y}^2\big) - k\big(x^2+y^2\big) \tag{3}$$

with obvious consequences. Less obviously, the Kepler Problem separates not only in polar coordinates but also[10] in the *confocal parabolic coordinate system* defined[11]

$$\left.\begin{aligned} x &= \tfrac{1}{2}(\mu^2-\nu^2)\\ y &= \mu\nu \end{aligned}\right\} \tag{4}$$

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[8] This is the upshot of "Bertrand's theorem;" see, for example, Appendix A in H. Goldstein's *Classical Mechanics* (Second edition, 1980).

[9] We will be concerned here with the separation of variables technique only as it relates to the ordinary differential equations of motion; its application to the associated partial differential equations of motion (Hamilton-Jacobi equation, Schrödinger equation) will be taken up elsewhere.

[10] We touch here on an aspect of the Kepler Problem which has, I guess, to be considered "well-known." It is interesting in this connection to recall that, in his very first quantum mechanical publication (i.e., in the paper which announced to the world the equation which now bears his name), Schrödinger solved the hydrogen problem not only in spherical coordinates—in the manner now standard to the textbooks—but also in confocal parabolic coordinates. See (for example) L. I. Schiff, *Quantum Mechanics* (3rd Edition 1968), pp. 95, 98, 139 & 265 or E. Merzbacher, *Quantum Mechanics* (1961), pp. 192, 243 & 385. Parabolic separation acquires special utility in connection with the *unbound* states (scattering theory) of the hydrogen atom.

[11] See P. Moon & D. E. Spencer, *Field Theory Handbook* (1961), pp. 21–24, 34–36, 52 & 57)

or, more compactly,

$$x + iy = \tfrac{1}{2}(\mu + i\nu)^2 \tag{5}$$

Since (4) entails

$$(ds)^2 = (dx)^2 + (dy)^2 = (\mu^2 + \nu^2)[(d\mu)^2 + (d\nu)^2] \tag{6}$$

and

$$x^2 + y^2 = \tfrac{1}{4}(\mu^2 + \nu^2)^2 \tag{7}$$

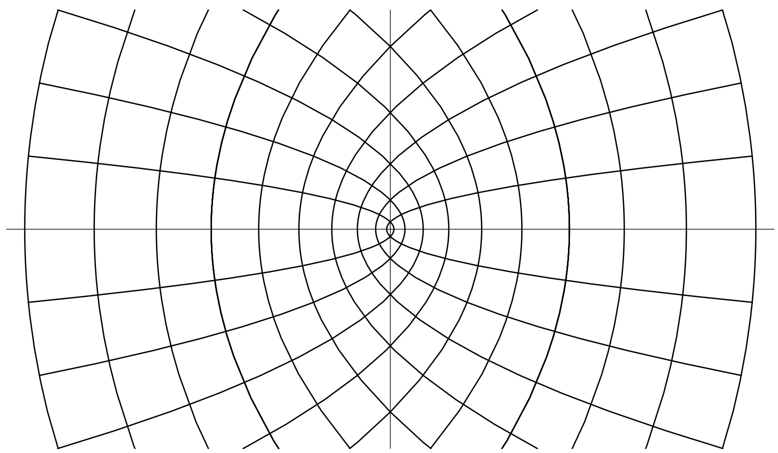

FIGURE 1: *Confocal parabolic coordinate system (4). Curves of constant μ open to the right, curves of constant ν to the left. Note that $\{\mu,\nu\}$ and $\{-\mu,-\nu\}$ map to the same point.*

we have

$$L_{\text{Kepler}}(\dot\mu,\dot\nu,\mu,\nu) = \tfrac{1}{2}m(\mu^2+\nu^2)(\dot\mu^2+\dot\nu^2) + \frac{2k}{\mu^2+\nu^2} \tag{8}$$

to which we will have occasion soon to return. Before we are done, it will be forced upon our attention that the Kepler Problem separates also in an infinitude of yet other coordinate systems.

2. Statement of the Euler Problem. The Kepler Problem can be considered to refer to the motion of a mass m in the gravitational field of a mass m_1 pinned to the origin of the $\{x,y\}$-plane:

$$L(\dot x,\dot y,x,y) = \tfrac{1}{2}m\big(\dot x^2+\dot y^2\big) + \frac{M}{\sqrt{x^2+y^2}}$$

where $M \equiv Gmm_1$. The Euler Problem refers similarly to the planar (!) motion of m in the gravitational field of a *pair* of masses, m_1 and m_2, which we can without loss of generality assume to have been pinned to the points $\{+a,0\}$ and $\{-a,0\}$, respectively:

$$L(\dot{x},\dot{y},x,y)=\tfrac{1}{2}m\big(\dot{x}^2+\dot{y}^2\big)+\frac{M_1}{\sqrt{(x-a)^2+y^2}}+\frac{M_2}{\sqrt{(x+a)^2+y^2}} \tag{9}$$

Note that the motion of m in such a field will, in general, *not* be planar; to obtain Euler's Problem one must explicitly assume that $\dot{\boldsymbol{x}}$ initially (whence also at all subsequent times) had no component normal to the plane defined by $\{m,\ m_1, m_2\}$.

Amongst the planar orbits possible in the presence only of m_1 (i.e., *in the absence* of m_2) would be a family of ellipses whose "other focus" lies at the empty position $x=-a$. And amongst those possible in the presence only of m_2 would be a family (the same family) of ellipses whose "other focus" lies at the empty position $x=+a$. "Bonnet's theorem"—due actually to Legendre (1817), and presumably not available in 1760 to Euler—asserts[12] that those orbits will remain members of the family of possible orbits even when m_1 and m_2 are simultaneously present. On these grounds, Euler's central idea seems (in retrospect) almost natural. That idea was this: introduce **confocal conic coordinates**[13] ξ and η as follows: write

$$\begin{aligned} x &= a\cosh\xi\cos\eta \\ y &= a\ \sinh\xi\sin\eta \end{aligned} \tag{10}$$

or, more compactly,

$$x+iy=a\cosh(\xi+i\eta) \tag{11}$$

where $0\le\xi<\infty$, $0\le\eta<2\pi$. Elimination first of η, then of ξ, gives

$$\begin{aligned} \left(\frac{x}{a\cosh\xi}\right)^2+\left(\frac{y}{a\sinh\xi}\right)^2&=1 \\ \left(\frac{x}{a\cos\eta}\right)^2-\left(\frac{y}{a\sin\eta}\right)^2&=1 \end{aligned} \tag{12}$$

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[12] See Whittaker's §51 or Routh's §§271–275. Note that Bonnet's theorem refers to the geometry of orbits, not to motion along those orbits. Bonnet did, however, observe that if P is a point on a mutual orbit $\mathcal{C}$, and if m moves past P with speed v_1 when subject only to the influence of m_1 (speed v_2 when subject only to the influence of m_2) then it will move with speed $v=\sqrt{v_1^2+v_2^2}$ when subject to the simultaneous influences of m_1 and m_2.

[13] See Moon & Spencer, pp. 17–20. Routh reports, on the authority of Serret, that this was the first application of confocal conic coordinates (also called "elliptical coordinates") to a physical problem. P. Serret (1819–1885), who was a contemporary of Bonnet and Bertrand and editor of the collected works of Lagrange, in 1861 became professor of celestial mechanics at the Collège de France and reportedly wrote a history of the Euler Problem.

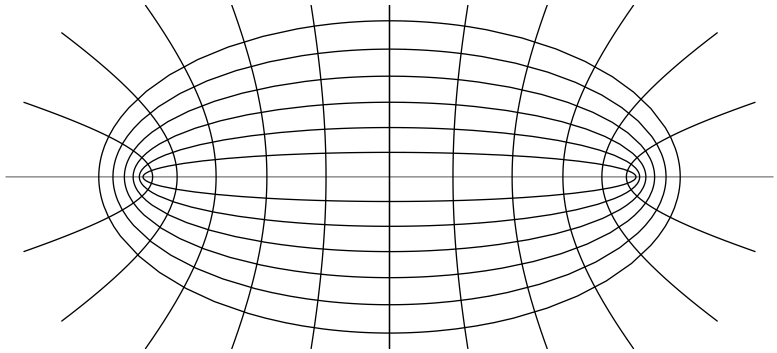

Figure 2: *Confocal conic (or elliptic) coordinate system (10). The ellipses are curves of constant* ξ, *the hyperbolæ curves of constant* η. *A line of coordinate singularity links the foci.*

according to which the curves of constant ξ are ellipses with foci at $x = \pm a$, and the curves of constant η are confocal hyperbolæ. The Euclidean interval acquires the description

$$(ds)^2 = (dx)^2 + (dy)^2 = a^2(\cosh^2\xi - \cos^2\eta)[(d\xi)^2 + (d\eta)^2] \tag{13}$$

while

$$(x \pm a)^2 + y^2 = a^2(\cosh\xi \pm \cos\eta)^2 \tag{14}$$

So we have

$$L = \tfrac{1}{2}ma^2(\cosh^2\xi - \cos^2\eta)(\dot{\xi}^2 + \dot{\eta}^2) + \frac{M_1}{a(\cosh\xi - \cos\eta)} + \frac{M_2}{a(\cosh\xi + \cos\eta)} \tag{15}$$

Our problem is to solve the resulting equations of motion. As a preparatory step, we notice that (15) can be written

$$\left.\begin{aligned} L &= \tfrac{1}{2}ma^2(\cosh^2\xi - \cos^2\eta)(\dot{\xi}^2 + \dot{\eta}^2) \\ &\qquad + \frac{(M_1 + M_2)ma\cosh\xi + (M_1 - M_2)ma\cos\eta}{ma^2(\cosh^2\xi - \cos^2\eta)} \\ &= \tfrac{1}{2}\big\{u_1(\xi) + u_2(\eta)\big\}(\dot{\xi}^2 + \dot{\eta}^2) + \frac{w_1(\xi) + w_2(\eta)}{u_1(\xi) + u_2(\eta)} \\ &= \tfrac{1}{2}u\cdot(\dot{\xi}^2 + \dot{\eta}^2) + \frac{w_1(\xi) + w_2(\eta)}{u} \end{aligned}\right\} \tag{16}$$

with

$$\left.\begin{aligned} u_1(\xi) &= +ma^2\cosh^2\xi \\ u_2(\eta) &= -ma^2\cos^2\eta \\ w_1(\xi) &= (M_1 + M_2)\,ma\cosh\xi \\ w_2(\eta) &= (M_1 - M_2)\,ma\cos\eta \\ u &= u_1(\xi) + u_2(\eta) \end{aligned}\right\} \tag{17}$$

I turn now to discussion of why (16)—as unpromising as on its face it appears to be—is welcome news.

3. Degrees of separation: Liouville's method. "Separation of variables" is such a primitive notion that, though the point is seldom remarked, it can—depending on circumstances—be realized in several distinct ways. Separability comes in several flavors. We have remarked already in connection with (3) that Lagrangians of the type

$$L(\dot{x}, \dot{y}, x, y) = \dot{x}^2 + \dot{y}^2 + w_1(x) + w_2(y) \tag{18}$$

separate spontaneously/trivially in this sense: they yield uncoupled equations of motion. Equally familiar, but distinct in its detailed mechanism, is the separability achieved when (for example) the central-force Lagrangian (2) is expressed in polar coordinates. Writing

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}\right\} \tag{19}$$

we obtain

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r) \tag{20}$$

giving equations of motion

$$\begin{aligned} m\ddot{r} - mr\dot{\varphi}^2 + U'(r) &= 0 \\ \frac{d}{dt}\{mr^2\dot{\varphi}\} &= 0 \end{aligned}$$

which are, as they stand, still coupled. But the latter equation yields an immediate first integral

$$mr^2\dot{\varphi} = \ell$$

which when fed back into the radial equation gives an equation

$$m\ddot{r} + \left\{U'(r) - \frac{\ell^2}{mr^3}\right\} = 0$$

from which all reference to φ has disappeared.[14] Notice that ℓ has actually the status not of a "separation constant" but of a constant of integration (physically interpretable as angular momentum). The separation technique just outlined owes its success to the fortuitous φ-independence of the Lagrangian (20), and

[14] Immediately

$$\dot{r} = \sqrt{\tfrac{2}{m}[E - U_{\text{eff}}(r)]} \quad \text{where} \quad U_{\text{eff}}(r) \equiv U(r) + \frac{\ell^2}{2mr^2}$$

which is soluble by quadrature. But to pursue the argument further would be to digress from my main theme; see (for example) §3 of Goldstein's Chapter 3.

for that reason (i.e., because no coordinate is cyclic) fails when brought to bear either upon the $L_{\text{Kepler}}(\dot{\mu},\dot{\nu},\mu,\nu)$ of (8) or upon any of the previously encountered variants of L_{Euler}.

Now another anachronism. In 1849—when Euler's contribution to our topic lay a full eighty-five years in the past—Liouville observed that Lagrangians of the type (16)—a class which includes among its members also $L_{\text{Kepler}}(\dot{\mu},\dot{\nu},\mu,\nu)$ —yield equations of motion which admit of solution by a clever modification of the separation technique. Liouville's line of argument, as it pertains to (16), runs as follows: multiply

$$\left\{\frac{d}{dt}\frac{\partial}{\partial\dot{\xi}}-\frac{\partial}{\partial\xi}\right\}L=\frac{d}{dt}(u\cdot\dot{\xi})-\tfrac{1}{2}\frac{\partial u}{\partial\xi}(\dot{\xi}^2+\dot{\eta}^2)-\frac{\partial}{\partial\xi}\left(\frac{w_1}{u}\right)=0$$

by $2u\cdot\dot{\xi}$ to obtain

$$\frac{d}{dt}(u^2\cdot\dot{\xi}^2)-u\cdot\dot{\xi}\,\frac{\partial u}{\partial\xi}(\dot{\xi}^2+\dot{\eta}^2)-2u\cdot\dot{\xi}\,\frac{\partial}{\partial\xi}\left(\frac{w_1}{u}\right)=0 \tag{21}$$

Now use energy conservation

$$\tfrac{1}{2}u\cdot(\dot{\xi}^2+\dot{\eta}^2)-\frac{w_1(\xi)+w_2(\eta)}{u}=E \tag{22}$$

to obtain

$$u\cdot(\dot{\xi}^2+\dot{\eta}^2)=2\left[E+\frac{w_1(\xi)+w_2(\eta)}{u}\right]$$

which when fed back into (21) gives

$$\begin{aligned}\frac{d}{dt}(u^2\cdot\dot{\xi}^2)&=2\left[E+\frac{w_1(\xi)+w_2(\eta)}{u}\right]\dot{\xi}\,\frac{\partial u}{\partial\xi}+2u\cdot\dot{\xi}\,\frac{\partial}{\partial\xi}\left(\frac{w_1}{u}\right)\\&=2\,\dot{\xi}\frac{\partial}{\partial\xi}\left[\left(E+\frac{w_1(\xi)+w_2(\eta)}{u}\right)u\right]\\&=2\,\dot{\xi}\frac{\partial}{\partial\xi}[E\cdot u_1(\xi)+w_1(\xi)]\\&=2\frac{d}{dt}[E\cdot u_1(\xi)+w_1(\xi)]\end{aligned} \tag{23}$$

The last two steps are brilliancies—the point of the whole argument—and (when taken in conjunction with the companion η-argument) lead Liouville at last to these two not-yet-quite-separated 1^{st}-order differential equations of motion:

$$\left.\begin{aligned}\tfrac{1}{2}u^2\cdot\dot{\xi}^2&=E\cdot u_1(\xi)+w_1(\xi)+\epsilon_1\\\tfrac{1}{2}u^2\cdot\dot{\eta}^2&=E\cdot u_2(\eta)+w_2(\eta)+\epsilon_2\end{aligned}\right\} \tag{24}$$

Here ϵ_1 and ϵ_2 are separation constants—actually constants of integration—subject (if we are to achieve consistency with the energy relation (22)) to the constraint

$$\epsilon_1 + \epsilon_2 = 0 \tag{25}$$

Bringing (25) to (24) we obtain finally

$$\left.\begin{aligned} \tfrac{1}{2}u^2 \cdot \dot{\xi}^2 &= E \cdot u_1(\xi) + w_1(\xi) + \epsilon \\ \tfrac{1}{2}u^2 \cdot \dot{\eta}^2 &= E \cdot u_2(\eta) + w_2(\eta) - \epsilon \end{aligned}\right\} \tag{26}$$

It is on account of the u^2-term on the left that equations (26) remain "not-quite-separated." Were it the case that $u = 1$ then (26) would as they stand be fully separated equations of motion, soluble by quadrature to yield

$$\begin{aligned} \xi &= \xi(t; \xi_0, E, +\epsilon) \\ \eta &= \eta(t; \eta_0, E, -\epsilon) \end{aligned}$$

But in such a case the Lagrangian (16) gives rise to equations of motion which separate spontaneously, and we have no need of Liouville's clever argument. Liouville was concerned with the generality of cases in which

$$u = u(\xi, \eta) = u_1(\xi) + u_2(\eta)$$

and Euler—nearly a century before him—had been concerned (as are we now) with a particular such case. One obvious way to eliminate the offending u^2-term from (26) would be to divide it out, writing

$$\frac{d\xi}{d\eta} = \sqrt{\frac{E \cdot u_1(\xi) + w_1(\xi) + \epsilon}{E \cdot u_2(\eta) + w_2(\eta) - \epsilon}} \tag{27.1}$$

This equation—which is equivalent to the following pair of equations

$$\begin{aligned} \frac{d\xi}{d\tau} &= \sqrt{2[E \cdot u_1(\xi) + w_1(\xi) + \epsilon]} \\ \frac{d\eta}{d\tau} &= \sqrt{2[E \cdot u_2(\eta) + w_2(\eta) - \epsilon]} \end{aligned} \tag{27.2}$$

—serves (note that all reference to t has been eliminated) to describe not the motion of m but the *geometry of its orbit*. Integration of (27.2) gives

$$\int_{\xi_0}^{\xi} \frac{1}{\sqrt{2[E \cdot u_1(\xi) + w_1(\xi) + \epsilon]}}\, d\xi = \tau = \int_{\eta_0}^{\eta} \frac{1}{\sqrt{2[E \cdot u_2(\eta) + w_2(\eta) - \epsilon]}}\, d\eta \tag{28}$$

whence

$$\tau = F(\xi; \xi_0, E, \epsilon) \quad \Longrightarrow \quad \xi = \xi(\tau; \xi_0, E, \epsilon) \tag{29.1}$$

$$\tau = G(\eta; \eta_0, E, \epsilon) \quad \Longrightarrow \quad \eta = \eta(\tau; \eta_0, E, \epsilon) \tag{29.2}$$

where $\Longrightarrow$ signifies functional inversion. Equations (29) provide in principle a τ-parameterized description of the orbit $\mathcal{C}\{\xi_0, \eta_0, E, \epsilon\}$, and introduction of the latter into the former gives this explicit description:

$$\xi = \xi(\tau(\eta; \eta_0, E, \epsilon); \xi_0, E, \epsilon) = X(\eta; \xi_0, \eta_0, E, \epsilon) \tag{30}$$

I say "in principle" because, in practical applications, one might find oneself unable to perform the quadratures or unable to accomplish the functional inversion.

To proceed from (29) to a description of the *motion* of the particle one must discover the t-dependence of the parameter τ. Such information can be obtained from considerations which I must on this occasion be content simply to sketch. We proceed from the observation that to write (see again (16))

$$L = T - U = \underbrace{\tfrac{1}{2}u\cdot(\dot{\xi}^2 + \dot{\eta}^2)}_{=\,\frac{1}{2}m\dot{s}^2} + \frac{w_1(\xi) + w_2(\eta)}{u}$$

is not only to ascribe to $U(\xi, \eta)$ a specialized structure, but to ascribe specialized structure also to the metric, i.e., to the (ξ, η)-description of differential arc length (see again, in this light, (6) and (13)):

$$(ds)^2 = \tfrac{1}{m}u\cdot[(d\xi)^2 + (d\eta)^2] \tag{31}$$

By energy conservation

$$\dot{s}^2 = \tfrac{2}{m}[E - U]$$

On the other hand, (31) gives $\dot{s}^2 = \frac{1}{m}u\cdot[\dot{\xi}^2 + \dot{\eta}^2]$ so by (27.2) we have

$$\begin{aligned}\dot{s}^2 &= \tfrac{1}{m}u\cdot[2E\cdot(u_1 + u_2) + 2(w_1 + w_2)]\cdot\dot{\tau}^2\\ &= \tfrac{2}{m}[E - U]\cdot u^2\dot{\tau}^2\end{aligned}$$

from which, it is interesting to note, ϵ has dropped away. The implication is that $u\cdot\dot{\tau} = 1$, and it was to enhance the simplicity of this result that a seemingly unmotivated $\sqrt{2}$ was introduced at (27.2) into the definition of τ. The t-dependence of τ can in principle be obtained now by functional inversion of the integral

$$t(\tau) = \int_0^\tau \frac{1}{u_1(\xi(\tau'; \xi_0, E, \epsilon)) + u_2(\eta(\tau'; \eta_0, E, \epsilon))}\,d\tau' \tag{32}$$

$$= \tau \quad \text{in the trivial case } u \equiv u_1 + u_2 = 1$$

In the normal course of expository events one would at this point cast about for a simple case which might serve to illustrate the practical application of Liouville's method, and would come soon to an appreciation of the fact that the system (8)—the Kepler Lagrangian in confocal parabolic coordinates—is for such purposes particularly attractive. I have, however, elected to forego such an exercise, for several reasons: my objective in this essay is a limited one, and in pursuit of that objective I have been led to adopt a line of argument

$$\text{Kepler Problem} \longleftarrow\!\!\!-\!\!\!-\!\!\!- \text{Euler Problem}$$

which will have the unintended effect of giving rise to a *unified* account of the Kepler Problem. It will, in particular, emerge that the polar version

$$L_{\text{Kepler}}(r,\varphi) = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{k}{r} \tag{33}$$

of the Kepler Problem—which on its face is not even susceptible to analysis by Liouville's method—can by slight and well-motivated adjustment be made to yield a Liouville system of exceptional simplicity. It is, therefore, to avoid fragmenting a discussion which stands at the verge of striking unification that I turn directly to the topic which served initially to motivate the preceding remarks.

4. Application of Liouville's method to the Euler Problem. Liouville described "Liouville's method" in 1849, but his primiary contributions to the literature concerning the Euler Problem were published in 1846 and 1848. I suspect, therefore, that it was the latter which led him (by generalization/abstraction) to the former, and that in using Liouville's method to attack Euler's Problem I am, in fact, reversing the historical sequence of events.

Bringing (17) to (28) we find that in the particular case of interest to Euler

$$\tau = \int_{\xi_0}^{\xi} \frac{1}{\sqrt{2[+Ema^2\cosh^2\xi + (M_1+M_2)ma\cosh\xi + \epsilon]}}\,d\xi \tag{34.1}$$

$$= \int_{\eta_0}^{\eta} \frac{1}{\sqrt{2[-Ema^2\cos^2\eta + (M_1-M_2)\,ma\cos\eta\, - \epsilon]}}\,d\eta \tag{34.2}$$

The former of the preceding integrals can by a simple change of variables (set $\cosh\xi = v$, which gives $d\xi = dv/\sqrt{v^2-1}$) be written

$$\tau = +\int_{\cosh\xi_0}^{\cosh\xi} \frac{1}{\sqrt{A(v^2+2\hat{B}v+C)(v^2-1)}}\,dv$$

while the latter integral (set $\cos\eta = v$, which gives $d\eta = dv/\sqrt{1-v^2}$) becomes

$$\tau = -\int_{\cos\eta_0}^{\cos\eta} \frac{1}{\sqrt{A(v^2-2\check{B}v+C)(v^2-1)}}\,dv$$

where $A \equiv 2Ema^2$, $\hat{B} \equiv (M_1+M_2)ma/A$, $\check{B} \equiv (M_1-M_2)ma/A$ and $C \equiv 2\epsilon/A$. In either case we confront an integral of the form

$$\int \frac{1}{\sqrt{\text{quartic}}}\,dv = \int \frac{1}{(v-v_1)(v-v_2)(v-v_3)(v-v_4)}\,dv$$

and are informed by *Mathematica*[15] that

$$= \frac{2}{\sqrt{(v_1-v_3)(v_2-v_4)}}F(\phi;p)$$

where

$$\phi(v) \equiv \arcsin\left[\frac{(v_2-v_4)(v-v_1)}{(v_1-v_4)(v-v_2)}\right] \tag{35.1}$$

$$p \equiv \left[\frac{(v_1-v_4)(v_2-v_3)}{(v_1-v_3)(v_2-v_4)}\right]^{\frac{1}{2}} \tag{35.2}$$

and where $F(\phi;p)$ is the incomplete elliptic integral of the first kind:

$$F(\phi;p) \equiv \int_0^\phi \frac{1}{\sqrt{1-p^2\sin^2\theta}}\,d\theta = \int_0^{\sin\phi} \frac{1}{\sqrt{(1-p^2t^2)(1-t^2)}}\,dt$$
$$\downarrow$$
$$= \text{complete elliptic integral } \mathbf{K}(p) \text{ at } \phi = \tfrac{1}{2}\pi$$

The functional inversions contemplated at (29) are accomplished by appeal to the theory of Jacobian elliptic functions,[16] which was invented—by Gauss, Abel and Jacobi[17] —for precisely this purpose. Inversion of the function $x(y)$ defined

$$x(y) \equiv \int_0^y \frac{1}{\sqrt{(1-p^2t^2)(1-t^2)}}\,dt$$

gives

$$x = \int_0^{y(x)\equiv\text{sn}(x)} \frac{1}{\sqrt{(1-p^2t^2)(1-t^2)}}\,dt$$

and it is with the wonderful properties of the function $\text{sn}(x;p)$ and its close relatives that the theory of Jacobian elliptic functions is concerned. In the case $p = 0$ we have $x(y) = \sin y$, which shows $\text{sn}(x;p)$ to be a generalization of the function $\arcsin x$. To establish more immediate contact with the results in hand, we set $y = \sin\phi$ and writing

$$X(\phi) \equiv \int_0^{\sin\phi} \frac{1}{\sqrt{(1-p^2t^2)(1-t^2)}}\,dt$$

obtain $\phi = \arcsin(\text{sn}(X))$.

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[15] See also I. S. Gradshteyn & I. M. Ryzhik, *Table of Integrals, Series, and Products* (1965), **3.147.8**.

[16] See L. M. Milne-Thomson, *Jacobian Elliptic Function Tables* (1950) or Chapter 63 of J. Spanier & K. B. Oldham, *An Atlas of Functions* (1987).

[17] For accounts of this development of the late 1820s (Euler himself had died nearly a half century previously, in 1783), which was central to the 19th Century history of mathematics, see C. B. Boyer, *A History of Mathematics* (1968), pp. 554–557 or pp. 395–399 of E. T. Bell, *Development of Mathematics* (1940).

Preceding remarks establish the technical sense in which the Euler Problem is "soluble," but as they stand contribute essentially nothing to our direct intuitive understanding of the physics of the matter. The task of extracting the physics from the formalism is a task of intimidating complexity, but has been undertaken by Eli Snyder in his Reed College thesis.[18] Snyder has spelled out many of the the analytical details (which I have been content merely to sketch), has identified a number of relatively tractable special cases, and has produced a very instructive collection of computer-generated graphical representations of Eulerian orbits. My own objective on this occasion is much more modest; it is to cast light on this question:

What is the physical interpretation of the separation constant ϵ ?

But before I take up that question I pause to draw attention to the importance in other connections—and latently in the present connection—of the ratios which occur in equations (35).

If $\{x_1, x_2, x_3, x_4\}$ is an ordered set of four real numbers, then

$$\begin{aligned}
[x_1 x_2, x_3 x_4] &\equiv \frac{\text{ratio in which } x_3 \text{ divides the interval } [x_1, x_2]}{\text{ratio in which } x_4 \text{ divides the interval } [x_1, x_2]} \\
&= \frac{x_3 - x_1}{x_2 - x_3}\Big/\frac{x_4 - x_1}{x_2 - x_4} \\
&= \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)} \\
&= [x_3 x_4, x_1 x_2] = [x_2 x_1, x_4 x_3] = [x_4 x_3, x_2 x_1]
\end{aligned}$$

defines the so-called "cross ratio" of $\{x_1, x_2, x_3, x_4\}$. Cross ratio is a concept derived from projective geometry,[19] into which it enters as the fundamental invariant; indeed, one can (in the spirit of Klein's Erlanger Program) define projective geometry to be "the study of propositions which are invariant with respect to transformations which preserve cross ratio," and can be confident that when—as in (35)—cross ratio makes a natural appearance, projective geometry cannot be far away. Cross ratio has the property that if $\{x_1, x_2, x_3, x_4\}$ is sent into $\{x_1', x_2', x_3', x_4'\}$ by invertible fractional linear transformation

$$x \longmapsto x' = \frac{ax + b}{cx + d}$$

then

$$[x_1' x_2', x_3' x_4'] = [x_1 x_2, x_3 x_4]$$

To encounter cross ratio within a function-theoretic context is to be put therefore in mind of the rich theory of automorphic functions, which is itself

[18] "Euler's Problem: the Problem of Two Centers" (1966).

[19] See Chapter VI of W. C. Graustein, *Introduction to Higher Geometry* (1930), J. A. Todd, *Projective and Analytical Geometry* (1947) or Chapter IV §3 in the 2nd edition of R. Courant & H. Robbins, *What is Mathematics?* (1996).

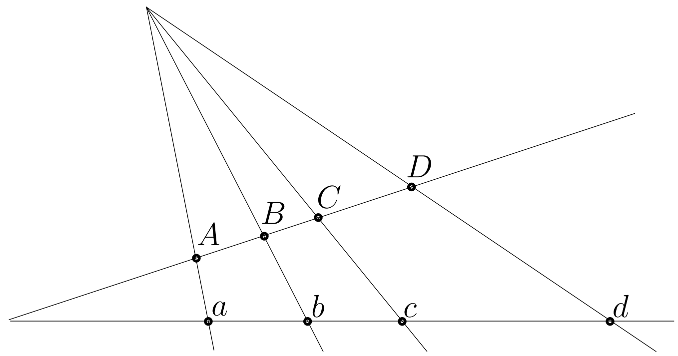

FIGURE 3: *Let AB denote the Euclidean length of the line segment linking $A \to B$, etc. It was known already to Pappus of Alexandria (first half of the* 4^{th} *Century)—and, on evidence of his text, probably known already to Euclid—that cross ratio is a projective invariant*

$$[AB, CD] = [ab, cd]$$

but the deep significance of this fact was first appreciated only 2250 years later, by Chasles in the mid-19^{th} *Century. Pappus was aware also that*

$$AD^2 \cdot BC + BD^2 \cdot CA + CD^2 \cdot AB + BC \cdot CA \cdot AB = 0$$

*which is much easier to prove, but is usually attributed to Mathew Stewart (*1746*).*

an outgrowth from the theory of elliptic functions, and a generalization of the theory of periodic functions.[20] Just as a function $f(z)$ is said to be "periodic" if it is invariant

$$f(x + na) = f(x) \quad : \quad n = 0, \pm 1, \pm 2, \ldots$$

with respect to a discrete subgroup of the translation group $x \longmapsto x' = x + t$, so is $f(z)$ said to be "automorphic" if it is invariant

$$f\Big(\frac{az + b}{cz + d}\Big) = f(z)$$

[20] See L. R. Ford, *Automorphic Functions* (1929) or Chapters XIII and XIV of A. Erdélyi *et al Higher Transcendental Functions* (1955). Also interesting in this connection is F. Klein, *The Icosahedron* (1884).

with respect to a subgroup of the linear fractional group. It is interesting to notice that the derivative of an automorphic function is, in general, not automorphic.[21]

5. Hamiltonian formalism. Equation (30) speaks of an orbit $\mathcal{C}$ which passes through the point (ξ_0, η_0), along which m moves with energy E (and therefore with a speed which is, in principle, known). The separation parameter ϵ must evidently refer—however indirectly—to the *direction* of its passage. Insofar as ϵ enters into (30) as an "orbit identifier," as an embodiment of some of the information resident in $(\dot{\xi}_0, \dot{\eta}_0)$, the numerical value of ϵ must be constant on $\mathcal{C}$; it follows—quite independently of any statement about how $\mathcal{C}$ is pursued in time—that ϵ is a constant of the motion, and it is to lend sharpened emphasis to that fact that I pass now over to the Hamiltonian formulation of the Euler Problem.[22]

From the description (16) of $L_{\text{Liouville}}(\dot{\xi}, \dot{\eta}, \xi, \eta)$ it follows that

$$\begin{aligned} p_\xi &= u\,\dot{\xi} \quad \text{whence} \quad \dot{\xi} = \frac{1}{u}\,p_\xi \\ p_\eta &= u\,\dot{\eta} \quad \text{whence} \quad \dot{\eta} = \frac{1}{u}\,p_\eta \end{aligned}$$

and therefore that

$$\begin{aligned} H_{\text{Liouville}}(p_\xi, p_\eta, \xi, \eta) &= p_\xi \cdot \dot{\xi} + p_\eta \cdot \dot{\eta} - L_{\text{Liouville}} \\ &= \frac{1}{2u}\big(\,p_\xi^2 + p_\eta^2\,\big) - \frac{w_1(\xi) + w_2(\eta)}{u} \end{aligned} \tag{36}$$

$$\downarrow$$

$$\begin{aligned} H_{\text{Euler}}(p_\xi, p_\eta, \xi, \eta) &= \tfrac{1}{2ma^2}\tfrac{1}{(\cosh^2\xi - \cos^2\eta)}\big(\,p_\xi^2 + p_\eta^2\,\big) \\ &\qquad - \tfrac{M_1}{a(\cosh\xi - \cos\eta)} - \tfrac{M_2}{a(\cosh\xi + \cos\eta)} \\ &= \tfrac{1}{a(\cosh^2\xi - \cos^2\eta)}\Big\{\tfrac{1}{2ma}\big(\,p_\xi^2 + p_\eta^2\,\big) \\ &\qquad - (M_1 + M_2)\cosh\xi - (M_1 - M_2)\cos\eta\Big\} \end{aligned} \tag{37}$$

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[21] Pursuit of this remark leads directly (see Ford §44 or Klein, p. 80) to the construction of the so-called

$$\text{Schwarz derivative} \quad : \quad \{f, z\} \equiv \frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2$$

and to the discovery that $\{f, z\}/f'^2$ is automorphic. For discussion of the main properties and diverse mathematical/physical occurances of the Schwarz derivative see my "Theory and Physical Applications of the Schwarz Derivative" (Notes for the Reed College Physics Seminar of 23 May 1968) in COLLECTED SEMINARS 1963–1970.

[22] We are informed by Routh (on authority of Serret) that the Hamiltonian formulation of Euler's Problem was first explored by Liouville himself, in 1847.

while from (22) and (26) we obtain

$$\begin{aligned}
\epsilon &= \tfrac{1}{2}u\cdot(u_2\dot{\xi}^2 - u_1\dot{\eta}^2) - \frac{u_2w_1 - u_1w_2}{u}\\
&\downarrow\\
G_{\text{Liouville}}(p_\xi, p_\eta, \xi, \eta) &\equiv \frac{1}{2u}\Big\{\big(u_2p_\xi^2 - u_1p_\eta^2\big) - 2\big(u_2w_1 - u_1w_2\big)\Big\} \qquad (38)\\
&\downarrow\\
G_{\text{Euler}}(p_\xi, p_\eta, \xi, \eta) &= -\tfrac{1}{2(\cosh^2\xi - \cos^2\eta)}\Big\{\big(p_\xi^2\cos^2\eta + p_\eta^2\cosh^2\xi\big)\\
&\qquad - 2ma\cosh\xi\cos\eta\big[(M_1+M_2)\cos\eta\\
&\qquad\qquad + (M_1-M_2)\cosh\xi\big]\Big\} \qquad (39)
\end{aligned}$$

where it is by particularization—i.e., by application $\downarrow$ of (17)—that Liouville's generic expressions (36) and (38) have become expressions specific to the Euler Problem. Within the Hamiltonian formalism, the elementary statement $\dot{\epsilon} = 0$ assumes the form

$$[G, H] \equiv \frac{\partial G}{\partial \xi}\frac{\partial H}{\partial p_\xi} + \frac{\partial G}{\partial \eta}\frac{\partial H}{\partial p_\eta} - \frac{\partial H}{\partial \xi}\frac{\partial G}{\partial p_\xi} - \frac{\partial H}{\partial \eta}\frac{\partial G}{\partial p_\eta} = 0 \qquad (40)$$

I have, mainly as a check on the accuracy of my expressions (and with the indispensible assistance of *Mathematica*), confirmed (40), both generically and after Eulerian particularization.

But how does one make sense of, how does one get a handle on the physical interpretation of, an object so complicated as G_{Euler}? It would be nice to trace (39) to a "symmetry," in the sense of Noether's theorem. But of this there is, in the present instance, no hope, for the conservation laws which emerge from Noether's theorem are of the general form

$$\dot{J}_\alpha = 0 \quad \text{with} \quad J_\alpha \equiv \sum_k p_k A^k_\alpha(q,t) + H(p,q)B_\alpha(q,t) + C_\alpha(q,t)$$

The expressions $J_\alpha(p,q,t)$ are, in other words, *linear in the momenta except for such quadratic or higher-order momentum-dependence as may reside within the Hamiltonian*, and the quantities $G(p,q)$ are inconsistent with that principle; they exhibit a quadratic momentum-dependence which is distinct from that of $H(q,p)$.

I have drawn attention already to the commonplace fact that one standard aid to the digestion of complex results involves the examination of relatively simple/familiar special cases, limiting cases (asymptotics). It is in that spirit that we are led now to ask: What becomes of G_{Euler} in the Keplerian limit? To ask the question is to realize that one can recover the Kepler Problem from the Euler Problem in three distinct ways:

- One can allow m_1 and m_2 to coalesce: $a \downarrow 0$. This is, in effect, to restrict one's attention to orbits which remain in regions remote from the duplex force center (high energy regime). The Euler Problem looks Keplerian when viewed from far away.

- One can place the origin at the location of (say) m_1 and remove m_2 to infinity: $a \uparrow \infty$. This is, in effect to restrict one's attention to orbits which remain in in close proximity to one of the force centers (low energy regime). The Euler Problem looks Keplerian when viewed from very close up.

- One can simply extinguish one of the fixed masses: $m_2 \downarrow 0$.

Three opening moves, leading to three quite different end-games—each of which is of interest in its own right... as will emerge. But before we can play the game we must set up the pieces. I begin by looking from a Lagrangian point of view to the implications of the preceeding • remarks.

6. Final preparations: three approaches to the Keplerian limit. In terms of the complex variable $z = x + iy$ one has (see again (9))

$$\begin{aligned} L &= \tfrac{1}{2}m\dot{z}^*\dot{z} + \frac{M_1}{\sqrt{(z-a)^*(z-a)}} + \frac{M_2}{\sqrt{(z+a)^*(z+a)}} \qquad (41)\\ &= \tfrac{1}{2}ma^2\dot{\zeta}^*\dot{\zeta}\sinh\zeta^*\sinh\zeta \\ &\qquad + \frac{M_1}{a\sqrt{(\cosh\zeta^*-1)(\cosh\zeta-1)}} + \frac{M_2}{a\sqrt{(\cosh\zeta^*+1)(\cosh\zeta+1)}} \qquad (42)\end{aligned}$$

where (see again (11)) I have written

$$z = a\cosh\zeta \quad \text{with} \quad \zeta = \xi + i\eta \tag{43}$$

and made use of the facts that $(\cosh\zeta)^* = \cosh\zeta^*$, $(\sinh\zeta)^* = \sinh\zeta^*$ and $\dot{z} = a\dot{\zeta}\sinh\zeta$. Now it is clear from

$$z = a(\cosh\xi\cos\eta + i\sinh\xi\sin\eta)$$

—and more particularly from

$$|z|^2 = a^2(\cosh^2\xi - \sin^2\eta)$$

—that to achieve $|z| \gg a$ (which for any non-zero z becomes sooner or later the case as $a \downarrow 0$) we must assign to ξ a large value. But as ξ becomes large we have

$$\left.\begin{aligned}\cosh\zeta &= \tfrac{1}{2}(e^{\zeta} + e^{-\zeta})\\ \sinh\zeta &= \tfrac{1}{2}(e^{\zeta} - e^{-\zeta})\end{aligned}\right\} \longrightarrow \tfrac{1}{2}e^{\zeta} = \tfrac{1}{2}e^{\xi}(\cos\eta + i\sin\eta) \tag{44}$$

One implication of this result is that

$$\begin{aligned} z = a\cosh\zeta \longrightarrow z &= \tfrac{1}{2}ae^{\zeta} = \tfrac{1}{2}ae^{\xi}e^{i\eta} \quad \text{as } a\downarrow 0 \\ &= re^{i\eta} \qquad \text{with } r \equiv \tfrac{1}{2}ae^{\xi} \end{aligned} \tag{45}$$

i.e., that *confocal conic coordinates become polar in the limit* $a \downarrow 0$. Returning with this information to (42) we obtain

$$\begin{aligned} L &= \tfrac{1}{2}ma^2(\dot\xi^2+\dot\eta^2)(\tfrac{1}{2}e^{\xi})^2 + \frac{M_1}{a\sqrt{(\frac{1}{2}e^{\xi})^2 - 2(\frac{1}{2}e^{\xi})\cos\eta + 1}} \\ &\qquad\qquad + \frac{M_2}{a\sqrt{(\frac{1}{2}e^{\xi})^2 + 2(\frac{1}{2}e^{\xi})\cos\eta + 1}} \\ &= \tfrac{1}{2}m(\dot r^2 + r^2\dot\eta^2) + \frac{M_1}{\sqrt{r^2 - 2ar\cos\eta + a^2}} + \frac{M_2}{\sqrt{r^2 + 2ar\cos\eta + a^2}} \qquad (46) \\ &\Downarrow \\ &= \tfrac{1}{2}m(\dot r^2 + r^2\dot\eta^2) + \frac{M_1+M_2}{r} \qquad \text{as } a\downarrow 0 \qquad (47) \end{aligned}$$

In (46) we have obtained precisely the Lagrangian of the Euler Problem in polar coordinates, while (47) is the polar Lagrangian appropriate to the case in which the two source masses have been lumped: (47) is, to within a trivial notational adjustment, precisely the $L_{\text{Kepler}}(r,\varphi)$ of (33).[23]

[23] We are in position now to address a curious formal point: (47) was obtained here from a Lagrangian of Liouville's type, but (as was remarked near the end of §2) is itself not of that type. We got off the track when, at (45), we allowed ourselves to write $r = \frac{1}{2}ae^{\xi}$. If in place of (19) one writes

$$x = ae^s\cos\varphi \quad ; \quad y = ae^s\sin\varphi$$

then in place of (33) one obtains

$$L_{\text{Kepler}}(s,\varphi) = \tfrac{1}{2}ma^2e^{2s}(\dot s^2 + \dot\varphi^2) + \frac{k}{ae^s} = \tfrac{1}{2}u\cdot(\dot s^2+\dot\varphi^2) + \frac{w_1(s)+w_2(\varphi)}{u}$$

with $u = u_1(s) + u_2(\varphi)$ and

$$\begin{aligned} u_1(s) &= ma^2e^{2s} \\ u_2(\varphi) &= 0 \\ w_1(s) &= kmae^s \\ w_2(\varphi) &= 0 \end{aligned}$$

The "log-polar" description of the Kepler Problem provides an especially simple context—simpler even than its parabolic companion (8)—within which to explore detailed implications of Liouville's method.

As was noted already at (12), $z = a\cosh\zeta$ gives rise to conics which are confocal at $z = \pm a$. We are (when we shift our attention from the high energy regime to the low energy regime) interested in physics in the vicinity of (say) m_1, which lives at $z = +a$, so we write

$$z = a + Z \tag{48}$$

and require that $|Z| \ll a$; the latter condition becomes inevitable as m_2 is "removed to infinity," i.e., as $a \uparrow \infty$. We have

$$\begin{aligned}\cosh\zeta &= 1 + \tfrac{1}{2!}\zeta^2 + \tfrac{1}{4!}\zeta^4\cdots\\ &= 1 + (Z/a)\end{aligned}$$

which in the operative limit entails

$$\begin{aligned} Z &= \tfrac{1}{2}a\zeta^2\\ &\Downarrow\\ &= \tfrac{1}{2}\Omega^2 \qquad \text{where } \Omega = \mu + i\nu \qquad (49)\\ X + iY &= \tfrac{1}{2}(\mu^2-\nu^2) + i\mu\nu \qquad (50)\end{aligned}$$

Thus do confocal conic coordinates become spontaneously *parabolic in the near neighborhood of a focal point*, and as $a \uparrow 0$ the "near neighborhood" becomes coextensive with the entire Z-plane. Returning with (49) and (50) to (41) we find

$$\begin{aligned} L &= \tfrac{1}{2}m\dot{Z}^*\dot{Z} + \frac{M_1}{\sqrt{Z^*Z}} + \frac{M_2}{\sqrt{(Z+2a)^*(Z+2a)}}\\ &\Downarrow\\ &= \tfrac{1}{2}m\dot{Z}^*\dot{Z} + \frac{M_1}{\sqrt{Z^*Z}} \qquad \text{as } a\uparrow\infty\end{aligned}$$

giving

$$\begin{aligned} L &= \tfrac{1}{2}m|\Omega|^2|\dot{\Omega}|^2 + \frac{M_1}{|\Omega|}\\ &= \tfrac{1}{2}m(\mu^2+\nu^2)(\dot{\mu}^2+\dot{\nu}^2) + \frac{2M_1}{\mu^2+\nu^2}\end{aligned} \tag{44}$$

—in precise agreement with (8).

The third line of attack seems on its face almost too simple to merit discussion, but is, in my view, actually the most deeply informative; one returns to (41) or (42)—or equivalently but even more simply to (16)—and sets $M_2 = 0$ to obtain

$$L = \tfrac{1}{2}ma^2(\cosh^2\xi - \cos^2\eta)(\dot{\xi}^2 + \dot{\eta}^2) + \frac{M_1 ma(\cosh\xi + \cos\eta)}{ma^2(\cosh^2\xi - \cos^2\eta)} \tag{52}$$

But (52) shows the Kepler Problem to be (in the sense of Liouville) separable in confocal conic coordinates when the force center is situated at one focus and the other focus is *anywhere*! The Kepler Problem is, in other words, *separable in an infinitude of coordinate systems additional to the familiar polar and confocal parabolic systems* (which, as we have seen, can be recovered as limiting cases of conic systems). In this simple fact lies an important lesson. We tend to think of spherical coordinates as "natural" to the Kepler Problem (and to central force problems generally) because coordinate surfaces conform to the geometry of the equipotentials. This is a false conception. Spherical coordinates are natural for the same reason that parabolic and conic coordinates are natural: they conform to the geometry of subpopulations within the population of orbits. None of the coordinate systems in question conform to the diploid equipotentials encountered in connection with the Euler Problem, but the conic coordinates (uniquely) do conform to the geometry of an orbital subpopulation.

It is in terms of the Lagrangian formalism that the Kepler Problem is usually set up and solved, but it is only from within the Hamiltonian formalism that the deeper reaches of the problem begin to come into view. In Cartesian coordinates the 3-dimensional Keplerian Hamiltonian reads

$$H = \frac{1}{2m}\mathbf{p}\cdot\mathbf{p} - \frac{M}{r}$$

The rotational invariance of the problem leads familiarly to three conservation laws

$$[\mathbf{L}, H] = \mathbf{0} \quad \text{where} \quad \mathbf{L} \equiv \mathbf{r}\times\mathbf{p} = \begin{pmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{pmatrix}$$

Conservation of angular momentum $\dot{\mathbf{L}} = \mathbf{0}$ is a property shared by all central force systems. Special to the Kepler Problem (though a related construct occurs in connection with the oscillator) is the additional triple of conservation laws

$$\begin{aligned} [\mathbf{K}, H] = \mathbf{0} \quad \text{where} \quad \mathbf{K} &\equiv \frac{1}{m}(\mathbf{p}\times\mathbf{L}) - \frac{M}{r}\mathbf{r} \\ &= \frac{1}{m}[(\mathbf{p}\cdot\mathbf{p})\,\mathbf{r} - (\mathbf{p}\cdot\mathbf{r})\,\mathbf{p}\,] - \frac{M}{r}\mathbf{r} \end{aligned}$$

The Lenz vector $\mathbf{K}$ lies in the plane (while $\mathbf{L}$ stands normal to the plane) of the orbit. Assuming (as heretofore we have) the orbit to lie in the (x, y)-plane, we have

$$\left.\begin{aligned} H &= \tfrac{1}{2m}(p_x^2 + p_y^2) - \tfrac{M}{r} \\ \\ L_x &= 0 \\ L_y &= 0 \\ L_z &= xp_y - yp_x \\ \\ K_x &= \tfrac{1}{m}p_y(xp_y - yp_x) - \tfrac{M}{r}x \\ K_y &= \tfrac{1}{m}p_x(yp_x - xp_y) - \tfrac{M}{r}y \\ K_z &= 0 \end{aligned}\right\} \tag{53}$$

We note that **K** is quadratic in the momenta, and that it is moreover quadratic in such a way as not to permit derivation from a symmetry *via* Noether's theorem. In this respect **K** resembles G, and is quite unlike **L**. Computation (I omit the details) shows that in polar coordinates (r,φ) we have

$$\left.\begin{aligned} H\; &= \tfrac{1}{2m}(p_r^2 + \tfrac{1}{r^2}p_\varphi^2) - \tfrac{M}{r} \\ L_z &= p_\varphi \\ K_x &= \tfrac{1}{m}[\tfrac{1}{r}p_\varphi^2 \cos\varphi + p_r p_\varphi \sin\varphi] - M\cos\varphi \\ K_y &= \tfrac{1}{m}[\tfrac{1}{r}p_\varphi^2 \sin\varphi - p_r p_\varphi \cos\varphi] - M\sin\varphi \end{aligned}\right\} \tag{54.1}$$

in confocal parabolic coordinates we have

$$\left.\begin{aligned} H\; &= \tfrac{1}{(\mu^2+\nu^2)}\left\{\tfrac{1}{2m}(p_\mu^2 + p_\nu^2) - 2M\right\} \\ L_z &= \tfrac{1}{2}(\mu p_\nu - \nu p_\mu) \\ K_x &= \tfrac{1}{\mu^2+\nu^2}\left\{\tfrac{1}{2m}(\mu^2 p_\nu^2 - \nu^2 p_\mu^2) - M(\mu^2-\nu^2)\right\} \\ K_y &= \tfrac{1}{\mu^2+\nu^2}\left\{\tfrac{1}{2m}[\mu\nu(p_\mu^2 + p_\nu^2) - p_\mu p_\nu(\mu^2+\nu^2)] - 2M\mu\nu\right\} \end{aligned}\right\} \tag{54.2}$$

while if we relocate the force center (place it at $x = a$) and transform to confocal conic coordinates we obtain

$$H\; = \tfrac{1}{2ma^2}\tfrac{1}{(\cosh^2\xi - \cos^2\eta)}(p_\xi^2 + p_\eta^2) - \tfrac{M}{a(\cosh\xi - \cos\eta)} \tag{55.1}$$

$$L_z = -\left\{p_\xi \tfrac{\sin\eta}{\cosh\xi+\cos\eta} - p_\eta \tfrac{\sinh\xi}{\cosh\xi+\cos\eta}\right\} \tag{55.2}$$

$$\begin{aligned} K_x = -\tfrac{1}{m}&\left\{\tfrac{1}{a(\cosh^2\xi-\cos^2\eta)}(p_\xi\cosh\xi\sin\eta + p_\eta\sinh\xi\cos\eta)\right\} \\ &\cdot\left\{p_\xi \tfrac{\sin\eta}{\cosh\xi+\cos\eta} - p_\eta \tfrac{\sinh\xi}{\cosh\xi+\cos\eta}\right\} - M\tfrac{\cosh\xi\cos\eta-1}{\cosh\xi-\cos\eta} \end{aligned} \tag{55.3}$$

$$\begin{aligned} K_y = +\tfrac{1}{m}&\left\{\tfrac{1}{a(\cosh^2\xi-\cos^2\eta)}(p_\xi\sinh\xi\cos\eta - p_\eta\cosh\xi\sin\eta)\right\} \\ &\cdot\left\{p_\xi \tfrac{\sin\eta}{\cosh\xi+\cos\eta} - p_\eta \tfrac{\sinh\xi}{\cosh\xi+\cos\eta}\right\} - M\tfrac{\sinh\xi\sin\eta}{\cosh\xi-\cos\eta} \end{aligned} \tag{55.4}$$

These (highly patterned) results are off-puttingly complicated, but I have high confidence in their accuracy, for *Mathematica* has labored to assure me that in all cases

$$[L_z, H] = [K_x, H] = [K_y, H] = 0 \tag{56}$$

7. End game: many conservation laws from one. We have now all the pieces on the board, and are ready to play the game. We return to (39), set $M_2 = 0$ and, drawing upon (55), at length simply *observe* that

$$G = -ma^2 H - maK_x - \tfrac{1}{2}L_z^2 \tag{57}$$

I initially undertook this exercise on the hunch that G would turn out to refer to some kind of "generalized Lenz vector." But the fact of the matter is rather more interesting than I had guessed. In Cartesian coordinates (57) reads

$$
\begin{aligned}
G = -ma^2 &\left\{ \tfrac{1}{2m}(p_x^2 + p_y^2) - \tfrac{M}{\sqrt{(x-a)^2+y^2}} \right\} \\
&- ma \left\{ \tfrac{1}{m} p_y \, L_z - \tfrac{M}{\sqrt{(x-a)^2+y^2}}(x-a) \right\} - \tfrac{1}{2}\Big\{ (x-a)p_y - yp_x \Big\}^2
\end{aligned}
$$

Repositioning the origin of the (x, y)-coordinate system (identifying it, that is to say, with the sole surviving force center—located formerly at $x = a$), we have

$$
\begin{aligned}
G = -ma^2 &\left\{ \tfrac{1}{2m}(p_x^2 + p_y^2) - \tfrac{M}{\sqrt{x^2+y^2}} \right\} \\
&- ma \left\{ \tfrac{1}{m} p_y \, (xp_y - yp_x) - \tfrac{M}{\sqrt{x^2+y^2}} x \right\} - \left\{ \tfrac{1}{2}(xp_y - yp_x)^2 \right\}
\end{aligned}
$$

where all allusions to a have been expunged from the observables interior to the braces. The implication—since the surviving a's can be assigned arbitrary value—is that those observables must *separately* be constants of the motion. Thus does Euler's Problem—which supplies a single conservation law

$$[G, H] = 0$$

—give rise in the Keplerian limit to a trio of conservation laws

$$[H, H] = [K_x, H] = [L_z, H] = 0$$

And these, by commutator closure (i.e., by Jacobi's identity and the easily established statement $[L_z, K_x] = K_y$), supply also a fourth:

$$[\,\underbrace{[K_x, H]}_{0}, L_z] + [\,\underbrace{[H, L_z]}_{0}, K_x] + [\,\underbrace{[L_z, K_x]}_{K_y}, H] = 0 \tag{58}$$

Thus do we recover (56) as a *set* of statements which the solitary Eulerian statement $[G, H] = 0$ calls suddenly into being at (but not before) the Keplerian limit. Moreover, L_x is, by axial symmetry, a constant of 3-dimensional Eulerian motion, and is therefore a constant also of 3-dimensional Keplerian motion. But

$$[L_z, L_x] = L_y$$

Therefore (arguing as in (58)) so also is L_y a constant of Keplerian motion. And so also is $K_z = [L_x, K_y]$. In short, *all* of the Keplerian statements $\dot{\mathbf{L}} = \dot{\mathbf{K}} = \mathbf{0}$ are latent in the solitary Eulerian statement $[G, H] = 0$.

Within the Hamiltonian formulation of mechanics, constants of the motion (conserved observables) assume importance partly from the circumstance that when pressed into service as the Lie generators of canonical transformations they generate transformations which (in phase space) carry dynamical flowlines into dynamical flowlines. From examination[24] of the commutation relations satisfied by the generators

$$\begin{matrix} 0 & K_x & K_y & K_z \\ & 0 & L_z & L_y \\ & & 0 & L_x \\ & & & 0 \end{matrix}$$

one discovers that they act on dynamical flow lines with $E < 0$ (elliptical case) to produce a representation of 0(4), and that they act on flowlines with $E > 0$ (hyperbolic case) to produce a representation of the Lorentz group. This action, when "projected" from 4-dimensional phase space onto 2-dimensional configuration space, becomes visible as an isoenergetic adjustment of the figure (orientation/eccentricity) of the orbit in question.

Conclusions and prospects. We have been listening to crosstalk between the Euler Problem (a hard problem) and the Kepler Problem (a relatively easy one). Rather surprisingly, the hard problem has served to sharpen our understanding of the easy one. But so far as the hard problem itself is concerned, we have satisfied ourselves that the Eulerian constant of the motion G is not utterly alien, yet have still no clear sense of what it is trying to tell us about Eulerian motion (or about the theory of elliptic functions). Looking to

$$G_{\text{Kepler}} \quad \longleftarrow \quad G_{\text{Euler}}$$

we have seen that G_{Kepler} spontaneously explodes into a richly structured constellation of conservation laws; we infer that familiarly rich structure to be but a palid reflection of structure which is present in $G \equiv G_{\text{Euler}}$ but which remains still largely hidden from view. We cannot, as yet, claim to possess an answer to our motivating question: "What is the physical interpretation and significance of the Liouvillian separation constant ϵ?"

Further insight might be gained from study of the projections back into configuration space of the infinitesimal canonical transformations generated by G. Or we might form $A \equiv [G, L_x]$, $B \equiv [A, G]$, $C \equiv [A, L_x]$, etc. until we have achieved algebraic closure, and then look for group structure.

Retreating to water less deep. . . I can report that during the course of this work I have frequently been struck by the elegance of the relations which come to light when one uses confocal parabolic coordinates in connection with the Kepler Problem. I hope one day to understand why so many of those relations are reminiscent of relations encountered in the theory of 2-spinors; the discussion I have presently in mind would interrelate the following facts: (*i*) the (bound) orbits encountered in the Kepler Problem are ellipses, (*ii*) the natural descriptors of ellipses (at least of centrally placed ellipses; some interesting adjustments would be required to describe focally placed ellipses)

[24] For details, see my CLASSICAL DYNAMICS 1964–65, Chapter IX, pp. 61–74.

are the so-called "Stokes parameters," and (*iii*) Stokes' parameters derive from the association $SU(2) \longleftrightarrow O(3)$.

Two general points which came only incidentally to my attention during the course of this work are among the points which stick most vividly in my mind, and which have already informed my work in quite other connections. The first has to do with the fact that it is sometimes futile to try to find Noether's theorem at work when one encounters a conservation law, and that an elementary "futility test" is available to us. The second has to do with the fact that it is not the geometry of equipotentials but the more subtile geometry of orbits that lies nearer the heart of the separation method. The question arises: what is the quantum mechanical meaning of the latter insight? The answer must, of course, be formulated in terms which do not make literal reference to "orbits."

In some unrelated recent work I have pointed out that it is always possible —and sometimes quite useful—to separate "the problem of motion" from "the problem of orbital design." There I show more particularly that it is the proper business of the Jacobi-Hertz Principle of Least Action to provide us with a system of "trajectory equations." It would be of interest to recover (30) by application of that theory.

One would like to be in position to relax the artificial restriction that Eulerian orbits be confined to the plane. Such a restriction can (it is physically evident) play no role in the quantum chemistry to which I allude in FOOTNOTE 5, so established quantum theory should provide in this connection some guidance, the curious pattern of the argument being

$$\text{classical theory} \quad \longleftarrow \quad \text{quantum theory}$$

Here the practitioners of the "old quantum theory," who by the mid-1920s had carried classical Hamilton-Jacobi theory to its highest point of development, speak most directly to our needs. Consulting M. Born's *Atommechanik* (1922, republished in English translation as *The Mechanics of the Atom* in 1960), we find that §39 is given over to a discussion—Hamilton-Jacobi theory in elliptic coordinates!—of "the problem of two centers." For discussion of the non-planar aspects of the Euler Problem, Born cites a paper by none other than the young W. Pauli (1922). Born points out that in the local vicinity of one source point the field due to sufficiently remote "other source point" is uniform, so that by going to the parabolic limit of the elliptical system one obtains what is in effect a theory of the Stark effect. Chapter IX of G. Birtwistle's *The Quantum Theory of the Atom* (1926)—with attribution to P. Epstein (1916) and K. Schwartzschild (1926) but no mention of Schrödinger, whose work appeared just in time to render Birtwistle's beautiful book instantly obsolete!—provides an elaborately detailed account of that pretty idea, which Burrau and others cited in FOOTNOTE 5 were quick to appropriate and to re-stage in modern dress. It is my sense that the ideas just touched upon would figure prominently in any truly deep account of the Euler Problem.

ADDENDA: It has come belatedly to my attention that V. I. Arnold shares my affection for confocal conic coordinates: Appendix 15 to the second edition (1989) of his *Mathematical Methods of Classical Mechanics* bears the title "On elliptic coordinates," and treats several modern aspects of that subject with his usual elegance and sophistication. Arnold attributes the foundations of the subject to Jacobi; he identifies several open questions, and cites references through the mid-1980's.

My work in quite another problem area[25] has led me to peruse the wonderfully complete and detailed bibliography printed in C. Grosche's recent monograph, *Path Integrals, Hyperbolic Spaces, and Selberg Trace Formulae*. There, at entry [108], one encounters reference to C. A. Coulson & A. Joseph, "A constant of motion for the two-center Kepler problem," Int. J. Quantum Chem. **1**, 337 (1967) and to "Spheroidal wave functions for the hydrogen atom," Proc. Phys. Soc. **90**, 887 (1967), a companion article by the same authors. Examination of those papers reveals that the thrust of the argument put forward by Coulson & Joseph anticipates that of mine, though they elect to work the quantum mechanical side of the street, and I the classical: they—as I—attempt to obtain insight into the Kepler problem "by descent" from the less well known physics of the "two-center problem." At the algebraic heart of their work lies precisely the conserved observable G_{Euler} encountered at (39). This they use to cast some welcome light upon the famously opaque theory of spheroidal wave functions,[26] and to "account for the observed apparent breakdown in the noncrossing rule for the potential energy curves [of the hydrogen ion]." But their "derivation of the [new] constant of motion" presented in §4 of their first paper amounts really only to a demonstration that the fruit of some improvisatory guesswork does in fact commute with the two-center Hamiltonian. Ultimately they acknowledge that their new conserved observable can be found already in H. A. Erikson & E. L. Hill, "A note on the one-electron states of diatomic molecules," Phys. Rev. **75**, 29 (1949). But the authors of that two-page paper are content simply to pull G_{Euler} and the associated operator $\mathbb{G}_{\text{Euler}}$ out of some unidentified hat; they claim only to have made explicit a constant of the motion which has been implicit in the work of all who have considered the two-center problem, from Euler down through Ø. Burrau (1927), A. H. Wilson (1928), E. Teller (1930), E. Hylleraas (1931) and others.[5] From this historical perspective my own principal accomplishment has been to show that G_{Euler} emerges as a natural by-product of Liouville's separation strategy. I am led to conclude, therefore, with this question: Can a *quantum analog of Liouville's method* be devised which leads with similar naturalness to $\mathbb{G}_{\text{Euler}}$?

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[25] "2-dimensional particle-in-a-box problems in quantum mechanics" (1997) and "Applied theta functions of one or several variables" (1997).

[26] See Chapter 21 in M. Abramowitz & I. Stegun, *Handbook of Mathematical Functions* (1964).