ABNORMAL MODES ‡

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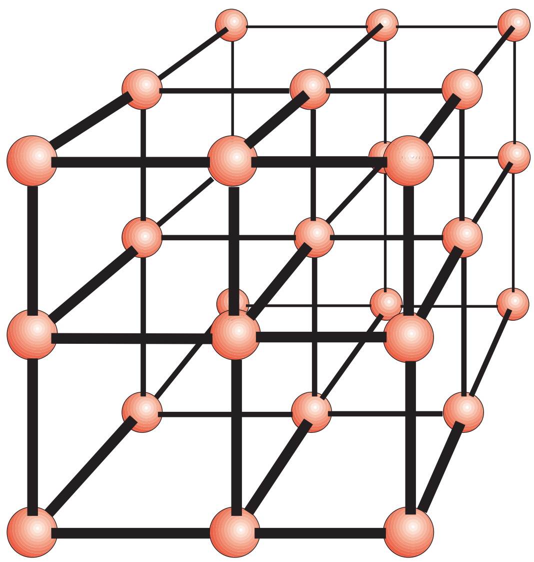

FIGURE 1: *A system of masses coupled by springs, intended to call to mind an enormous and highly varied population of "classical molecules and crystals."*

Introduction. When presented with physical systems such as those illustrated in FIGURES 1,2 & 3 a physicist thinks "normal modes." My purpose here today will be to explore some aspects of this well-studied theory that remain relatively little known. The points that will emerge seem to me to be of interest both mathematically and physically. Some of them will call into question the very *meaning* of the normal mode concept.

‡ Notes for a Reed College Physics Seminar presented 2 November 2005.

1. Setting up the problem. Look to the system illustrated below. From the Lagrangian

$$L = \tfrac{1}{2}\{m_1\dot{x}_1^2 + m_2\dot{x}_2^2\} - \tfrac{1}{2}\{k_1x_1^2 + K(x_2 - x_1)^2 + k_2x_2^2\}$$

we obtain the coupled equations of motion

$$\begin{aligned} m_1\ddot{x}_1 + k_1x_1 - K(x_2 - x_1) &= 0 \\ m_2\ddot{x}_2 + k_2x_2 + K(x_2 - x_1) &= 0 \end{aligned}$$

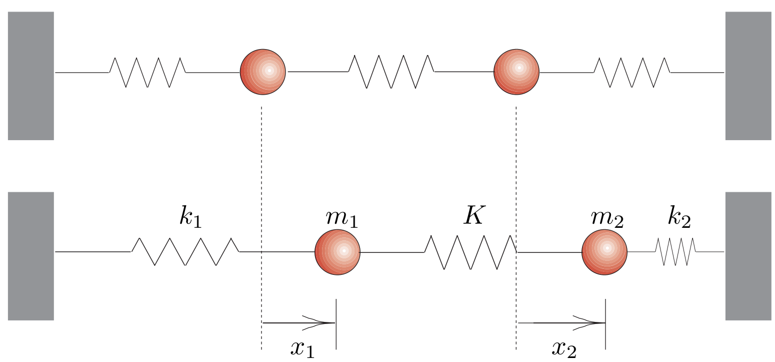

FIGURE 2: *A simple system of coupled oscillators*

In matrix notation

$$L = \tfrac{1}{2}\dot{\boldsymbol{x}}^{\mathsf{T}}\mathbb{M}\,\dot{\boldsymbol{x}} - \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\mathbb{K}\,\boldsymbol{x} \tag{1}$$

where $\mathbb{M}$ and $\mathbb{K}$ are the real symmetric matrices

$$\mathbb{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad \text{and} \quad \mathbb{K} = \begin{pmatrix} k_1 + K & -K \\ -K & k_2 + K \end{pmatrix}$$

In this notation the equations of motion become

$$\mathbb{M}\,\ddot{\boldsymbol{x}} + \mathbb{K}\boldsymbol{x} = \boldsymbol{0} \tag{2}$$

Here it is the off-diagonal elements of $\mathbb{K}$ that serve to couple the equations of motion. Had we been working from FIGURE 3 we would have had

$$\begin{aligned} L_1\ddot{q}_1 + M\ddot{q}_2 + C_1^{-1}q_1 &= 0 \\ L_2\ddot{q}_2 + M\ddot{q}_1 + C_2^{-1}q_2 &= 0 \end{aligned}$$

which can be cast in the form (2) with

$$\mathbb{M} = \begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix} \quad \text{and} \quad \mathbb{K} = \begin{pmatrix} C_1^{-1} & 0 \\ 0 & C_2^{-1} \end{pmatrix}$$

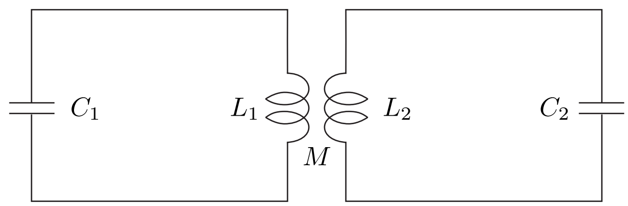

FIGURE 3: *Electrical analog of the mechanical system shown in the first figure. It is established in the text that the analogy is not quite perfect.*

Notice that it is the off-diagonal elements not of $\mathbb{K}$ but of $\mathbb{M}$—the coefficients of mutual inductance—that serve to couple the circuit equations. So (small point!) the so-called "electro-mechanical analogy" is in this respect (also in another, as will emerge) not quite perfect.

Thus is our initial attention directed to n-variable systems of the type

$$L = \tfrac{1}{2}\dot{\boldsymbol{x}}^{\mathsf{T}}\mathbb{M}\,\dot{\boldsymbol{x}} - \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\mathbb{K}\,\boldsymbol{x} \tag{1}$$

and to differential equations of motion of the coupled linear form

$$\mathbb{M}\,\ddot{\boldsymbol{x}} + \mathbb{K}\boldsymbol{x} = \boldsymbol{0} \tag{2}$$

Note the equations themselves provide no indication of whether (in—say—the case $n = 6$) we have in mind

- the motion of one particle in 6-space;
- the motion of two particles in 3-space;
- the motion of three particles in 2-space;
- the motion of six particles in 1-space.

Note also that if we take (1) as our point of departure then the symmetery of $\mathbb{M}$ and $\mathbb{K}$ can be assumed witout loss of generality, but if we proceed from (2) then symmetry must be explicitly stipulated (in its absence we are dealing with a system that lies beyond the reach of the Lagrangian formalism). Symmetry forces the eigenvalues $\{m_1, m_2, \ldots, m_n\}$ and $\{k_1, k_2, \ldots, k_n\}$ of the real matrices $\mathbb{M}$ and $\mathbb{K}$ to be real. The energy of such a system is given by

$$E = \tfrac{1}{2}\dot{\boldsymbol{x}}^{\mathsf{T}}\mathbb{M}\,\dot{\boldsymbol{x}} + \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\mathbb{K}\,\boldsymbol{x}$$

and from the invariable non-negativity of energy[1] it follows that all eigenvalues are positive; *i.e.*, that the matrices $\mathbb{M}$ and $\mathbb{K}$ are positive-definite.

The problem of solving equations (2) can be approached in several ways:

[1] Obvious for mechanical systems, this holds also for their electrical analogs.

2. Solution of the equations of motion "by Ansztz." Let us exercise our option—made available by the facts that (i) $\mathbb{M}$ and $\mathbb{K}$ are real and (ii) the equations of motion are linear—to construe $\boldsymbol{x}$ to be the real part of a complex variable $\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y}$. *Assume* that the variables z_i move in harmonic synchrony:

$$\text{ANSATZ:}\quad \boldsymbol{z}(t) = \boldsymbol{Z}e^{i\omega t}$$

Equation (2) then supplies

$$\big(\mathbb{K} - \omega^2\mathbb{M}\big)\boldsymbol{Z} = \boldsymbol{0}$$

From $\det\big(\mathbb{K} - \omega^2\mathbb{M}\big) = 0$ we obtain roots $\{\omega_1^2, \omega_2^2, \ldots, \omega_n^2\}$ and by command

$$\texttt{NullSpace[}\,\mathbb{K} - \omega_j^2\mathbb{M}\,\texttt{]}$$

we obtain linearly independent vectors $\{\boldsymbol{Z}_1, \boldsymbol{Z}_2, \ldots, \boldsymbol{Z}_n\}$, each of which is (by the reality of $\mathbb{K} - \omega_j^2\mathbb{M}$) proportional to a *real* vector:

$$\boldsymbol{Z}_j = A_j e^{i\delta_j}\cdot \boldsymbol{X}_j$$

Thus are we led to solutions

$$\begin{aligned}\boldsymbol{x}(t) &= \text{real part of }\Big\{e^{i\omega_1 t}\boldsymbol{Z}_1 + e^{i\omega_2 t}\boldsymbol{Z}_2 + \cdots + e^{i\omega_n t}\boldsymbol{Z}_n\Big\}\\ &= A_1\cos(\omega_1 t + \delta_1)\boldsymbol{X}_1 + A_2\cos(\omega_2 t + \delta_2)\boldsymbol{X}_2 + \cdots + A_n\cos(\omega_n t + \delta_n)\boldsymbol{X}_n\end{aligned}$$

where the $\{A_1, \delta_1, A_2, \delta_2, \ldots, A_n, \delta_n\}$ might be chosen to achieve conformity with prescribed initial data $\{\boldsymbol{x}(0), \dot{\boldsymbol{x}}(0)\}$.

EXAMPLE: Look to the case

$$\mathbb{M} = \begin{pmatrix} 2 & 1\\ 1 & 3\end{pmatrix},\quad \mathbb{K} = \begin{pmatrix} 3 & 2\\ 2 & 7\end{pmatrix}$$

—both of which are obviously real/symmetric and demonstrably positive-definite. From $\det\big(\mathbb{K} - \omega^2\mathbb{M}\big) = 0$ we obtain

$$\begin{aligned}\omega_1 &= \pm 1.53566\\ \omega_2 &= \pm 1.20073\end{aligned}$$

`NullSpace` commands now give

$$\boldsymbol{Z}_1 = \begin{pmatrix} -0.20871\\ 1\end{pmatrix}\quad\text{and}\quad \boldsymbol{Z}_2 = \begin{pmatrix} -4.79129\\ 1\end{pmatrix}$$

which are obviously real and demonstrably $\mathbb{M}$-orthogonal. After $\mathbb{M}$-normalization those vectors become

$$\boldsymbol{X}_1 = \begin{pmatrix} -0.127737\\ 0.612025\end{pmatrix},\quad \boldsymbol{X}_2 = \begin{pmatrix} -0.763992\\ 0.159454\end{pmatrix}$$

which are $\mathbb{M}$-orthonormal:

$$\boldsymbol{X}_i^{\mathsf{T}}\mathbb{M}\,\boldsymbol{X}_j = \delta_{ij}$$

By calculation we confirm that

$$\boldsymbol{z}_1(t) = e^{i\omega_1 t}\boldsymbol{X}_1 \quad \text{and} \quad \boldsymbol{z}_2(t) = e^{i\omega_2 t}\boldsymbol{X}_2$$

are indeed solutions of $\mathbb{M}\,\ddot{\boldsymbol{z}} + \mathbb{K}\,\boldsymbol{z} = \boldsymbol{0}$. Writing

$$\boldsymbol{x}(t) = (a_1 \cos\omega_1 t + b_1 \sin\omega_1 t)\boldsymbol{X}_1 + (a_2 \cos\omega_2 t + b_2 \sin\omega_2 t)\boldsymbol{X}_2$$

we find that imposition of the initial conditions

$$\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \dot{\boldsymbol{x}}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

requires that we set

$$a_1 = 0.35655,\ \ b_1 = 1.11244,\ \ a_2 = -1.36853,\ \ b_2 = -0.23788$$

Graphs derived from the resulting $\boldsymbol{x}(t)$ are shown below:

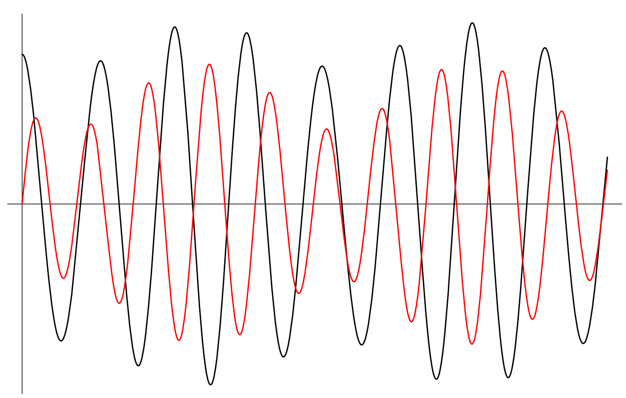

FIGURE 4: *Graph of $\boldsymbol{x}(t)$, interpreted as a reference to the motion $x_1(t)$ and $x_2(t)$ of two points in one dimension.*

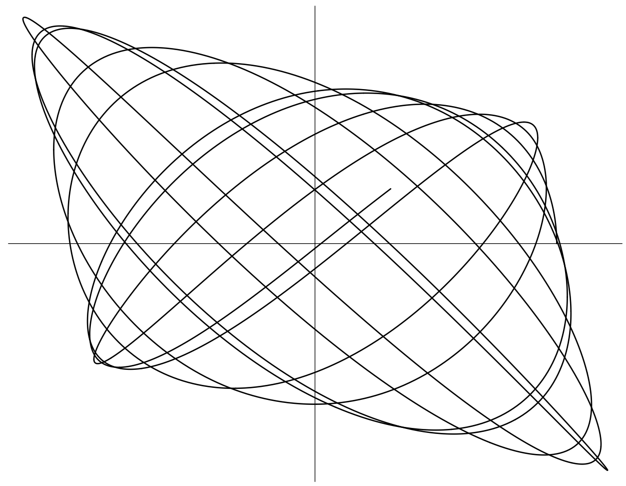

FIGURE 5: *Exactly the same information as was presented in the preceding figure, displayed here as though it referred to the motion of a single point in 2-space.*

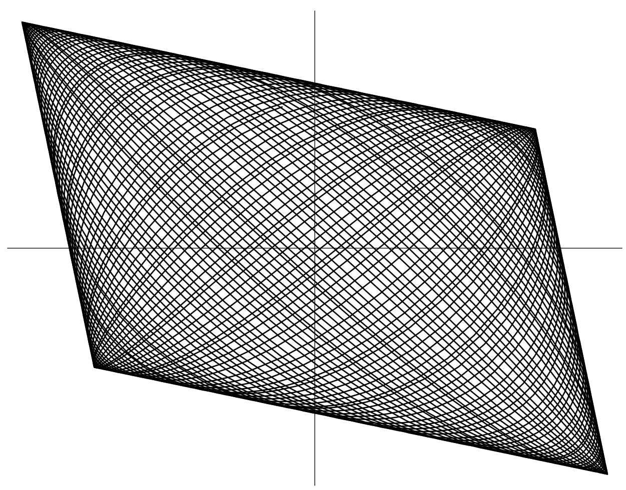

FIGURE 6: *In the preceding pair of figures t ran from 0 to 41. Here it runs from 0 to 250. Closure requires that ω_1/ω_2 be rational.*

3. Solution by simultaneous diagonalization. Established properties of $\mathbb{M}$ assure that it can always be developed

$$\mathbb{M} = \mathbb{R}^{\intercal}\begin{pmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & m_n \end{pmatrix}\mathbb{R}$$

where $\mathbb{R}$ is a rotation matrix and $\{m_1, m_2, \dots, m_n\}$ are the eigenvalues of $\mathbb{M}$ (which, as previously remarked, are positive for a physical reason). It follows that we can write

$$\mathbb{M} = \mathbb{N}^{\mathsf{T}}\mathbb{N} \quad \text{with} \quad \mathbb{N} = \begin{pmatrix} \sqrt{m_1} & 0 & \dots & 0 \\ 0 & \sqrt{m_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{m_n} \end{pmatrix} \mathbb{R} \tag{3}$$

The Lagrangian (1) can therefore be rendered

$$\begin{aligned} L &= \tfrac{1}{2}\dot{\boldsymbol{x}}^{\mathsf{T}}\mathbb{N}^{\mathsf{T}}\mathbb{N}\,\dot{\boldsymbol{x}} - \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\mathbb{N}^{\mathsf{T}}\Omega^2\,\mathbb{N}\,\boldsymbol{x} \\ &= \tfrac{1}{2}\dot{\boldsymbol{y}}^{\mathsf{T}}\mathbb{I}\,\dot{\boldsymbol{y}} - \tfrac{1}{2}\boldsymbol{y}^{\mathsf{T}}\Omega^2\,\boldsymbol{y} \end{aligned} \tag{4}$$

where $\boldsymbol{y} = \mathbb{N}\boldsymbol{x}$ and Ω^2 is the positive-definite real symmetric matrix defined

$$\Omega^2 = (\mathbb{N}^{\mathsf{T}})^{-1}\,\mathbb{K}\,(\mathbb{N})^{-1} \tag{5}$$

The equations of motion have now become

$$\ddot{\boldsymbol{y}} + \Omega^2\boldsymbol{y} = \mathbf{0} \tag{6}$$

Proceeding now as before, we look for solutions of the harmonically synchronized form

$$\boldsymbol{y}(t) = \text{real part of } \boldsymbol{n}e^{i\omega t}$$

Equation (6) then assumes the classic form

$$\big(\,\Omega^2 - \omega^2\mathbb{I}\,\big)\,\boldsymbol{n} = \mathbf{0} \tag{7}$$

We write $\{\omega_1^2, \omega_2^2, \dots, \omega_n^2\}$ to describe the spectrum of Ω^2, the eigenvectors of which are (if the spectrum is nondegenerate) automatically orthogonal, and can be assumed to have been normalized:

$$\boldsymbol{n}_i^{\mathsf{T}}\boldsymbol{n}_j = \delta_{ij} \tag{8}$$

It is this circumstance that permits one to speak of **normal modes** of harmonic motion. "Modal coordinates" $\boldsymbol{y}$ stand to "physical coordinates" $\boldsymbol{x}$ in the relation

$$\boldsymbol{y} = \mathbb{N}\boldsymbol{x}$$

Introducing $\boldsymbol{n}_i = \mathbb{N}\boldsymbol{X}_i$ into the orthogonormality statement (8) we recover the $\mathbb{M}$-orthonormality relations $\boldsymbol{X}_i^{\mathsf{T}}\mathbb{M}\,\boldsymbol{X}_j = \delta_{ij}$ familar form page 5.

EXAMPLE REVISITED: `SingularValueDecomposition[` $\mathbb{M}$ `]` supplies

$$\mathbb{M} = \mathbb{R}^{\mathsf{T}}\begin{pmatrix} 3.618 & 0 \\ 0 & 1.382 \end{pmatrix}\mathbb{R} \quad : \quad \mathbb{R} = \begin{pmatrix} 0.526 & 0.851 \\ -0.851 & 0.526 \end{pmatrix}$$

whence

$$\mathbb{N} = \begin{pmatrix} 1 & 1.618 \\ -1 & 0.618 \end{pmatrix}$$

We use this information to compute

$$\mho^2 = (\mathbb{N}^{\mathsf{T}})^{-1}\,\mathbb{K}\,(\mathbb{N})^{-1} = \begin{pmatrix} 2.124 & 0.400 \\ 0.400 & 1.676 \end{pmatrix}$$

the eigenvalues of which are found to be precisely the squares of the ω_i to which wewere led already on page 4:

$$\mho^2 \text{ spectrum is } \{\omega_1^2, \omega_2^2\}, \text{ with } \begin{cases} \omega_1 = \pm 1.53566 \\ \omega_2 = \pm 1.20073 \end{cases}$$

and (normalized) eigenvectors

$$\boldsymbol{n}_1 = \begin{pmatrix} 0.86254 \\ 0.50599 \end{pmatrix} = \mathbb{N}\boldsymbol{X}_1, \quad \boldsymbol{n}_2 = \begin{pmatrix} -0.50599 \\ 0.86254 \end{pmatrix} = \mathbb{N}\boldsymbol{X}_2$$

So we know exactly what we mean when we write

$$\boldsymbol{y}(t) = (a_1\cos\omega_1 t + b_1\sin\omega_1 t)\,\boldsymbol{n}_1 + (a_2\cos\omega_2 t + b_2\sin\omega_2 t)\,\boldsymbol{n}_2$$

Which is exactly the result to which we would have been led had we multiplied $\mathbb{N}$ into

$$\boldsymbol{x}(t) = (a_1\cos\omega_1 t + b_1\sin\omega_1 t)\,\boldsymbol{X}_1 + (a_2\cos\omega_2 t + b_2\sin\omega_2 t)\,\boldsymbol{X}_2$$

The following figure presents a graph of the $\boldsymbol{y}(t)$ that results from imposition of the transformed initial conditions

$$\boldsymbol{y}(0) = \mathbb{N}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \dot{\boldsymbol{y}}(0) = \mathbb{N}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It is instructive/useful to carry the preceding discussion one step further. The SVD of the positive-definite real symmetric matrix $\mho^2$ supplies information that can be displayed this way:

$$\mathbb{S}^{\mathsf{T}}\mho^2\mathbb{S} = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$$

where $\mathbb{S}$ is a rotation matrix assembled from the eigenvectors of $\mho^2$:

$$\mathbb{S} = \begin{pmatrix} 0.86254 & -0.50599 \\ 0.50599 & 0.86254 \end{pmatrix}$$

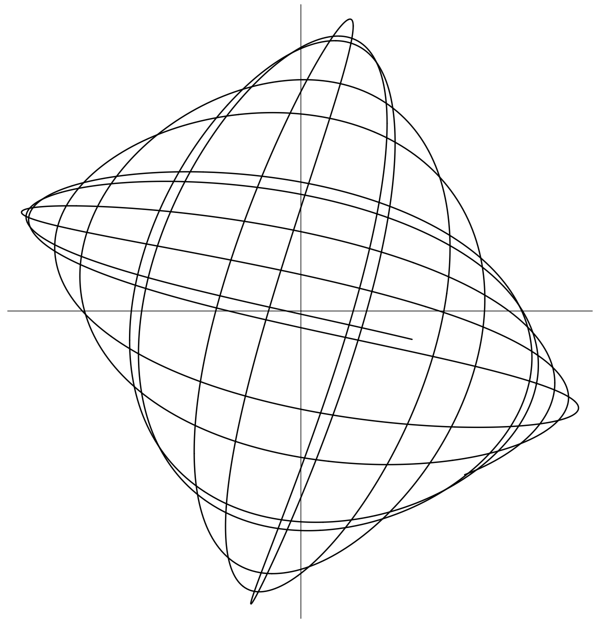

FIGURE 7: *Same motion as was shown in* FIGURE 5*, but displayed here in reference not to the "natural"* $\boldsymbol{x}$*-coordinate system but in reference to the "normal"* $\boldsymbol{y}$*-coordinate system. Note that the graph has been "rectified" but remains rotated with respect to the coordinate axes.*

The implication is that if we write

$$\boldsymbol{y} = \mathbb{S}\,\hat{\boldsymbol{y}}$$

then the Lagrangian (4) assumes the form

$$L = \tfrac{1}{2}\,\hat{\boldsymbol{y}}^{\mathsf{T}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\hat{\boldsymbol{y}} - \tfrac{1}{2}\,\hat{\boldsymbol{y}}^{\mathsf{T}}\begin{pmatrix}\omega_1^2 & 0\\ 0 & \omega_2^2\end{pmatrix}\hat{\boldsymbol{y}}$$

and **the equations of motion have become decoupled**

$$\ddot{\hat{y}}_i + \omega_i^2\hat{y}_i = 0 \quad : \quad i = 1, 2$$

with consequences that are illustrated in FIGURE 8.

I understand decoupled motion to be the signature attribute of the modal concept. To recapitulate how it was achieved in the example: the initial Lagrangian was assembled from a pair of quadratic forms:

$$L = \tfrac{1}{2}\dot{\boldsymbol{x}}^{\mathsf{T}}\mathbb{M}\,\dot{\boldsymbol{x}} - \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\mathbb{K}\,\boldsymbol{x}$$

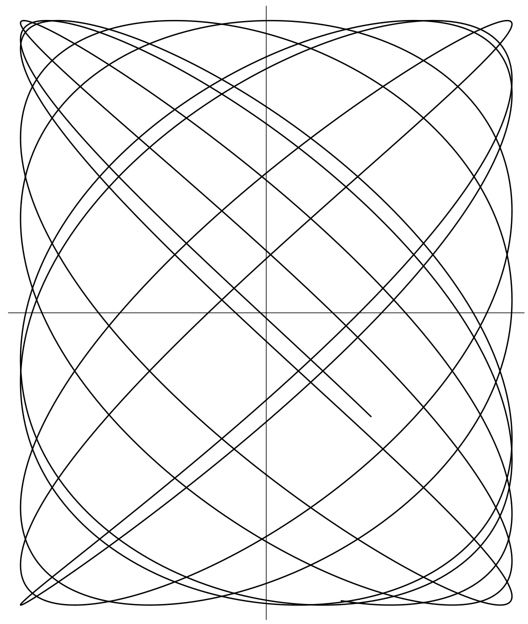

FIGURE 8: *Same motion as was shown in* FIGURE 7, *but displayed here in reference* $\hat{\boldsymbol{y}}$*-coordinate system. The design reflects the fact that* $\hat{y}_1$ *and* $\hat{y}_2$ **oscillate independently**.

The linear transformation $\boldsymbol{x} \longmapsto \boldsymbol{y} = \mathbb{N}\boldsymbol{x}$ (rotation, followed by a dilation) brought L to the form

$$L = \tfrac{1}{2}\,\boldsymbol{y}^{\mathsf{T}}\mathbb{I}\,\boldsymbol{y} - \tfrac{1}{2}\,\boldsymbol{y}^{\mathsf{T}}\Omega^2\,\boldsymbol{y}$$

A second rotation $\boldsymbol{y} \longmapsto \hat{\boldsymbol{y}} = \mathbb{S}\,\boldsymbol{y}$ served to diagonalize the potential energy form while preserving the diagonality of the kinetic energy form ($\boldsymbol{y}^{\mathsf{T}}\mathbb{I}\,\boldsymbol{y} = \hat{\boldsymbol{y}}^{\mathsf{T}}\,\mathbb{I}\,\hat{\boldsymbol{y}}$). The elementary logic of the procedure is illustrated in FIGURE 9.

4. Introduction & management of "gyroscopic" terms. In quest of the most general linear system (we have $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$ in mind) we might look to the most general quadratic Lagrangian

$$L = \tfrac{1}{2}\dot{\boldsymbol{x}}^{\mathsf{T}}\mathbb{M}\,\dot{\boldsymbol{x}} + \dot{\boldsymbol{x}}^{\mathsf{T}}\mathbb{G}\,\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\mathbb{K}\,\boldsymbol{x}$$

The resulting equations of motion are

$$\mathbb{M}\,\ddot{\boldsymbol{x}} + (\mathbb{G} - \mathbb{G}^{\mathsf{T}})\dot{\boldsymbol{x}} + \mathbb{K}\boldsymbol{x} = 0$$

to which, interestingly, the symmetric part of $\mathbb{G} = \mathbb{S} + \mathbb{A}$ makes no contribution . . . for this reason:

$$\tfrac{d}{dt}\big(\tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\mathbb{S}\,\boldsymbol{x}\big) = \dot{\boldsymbol{x}}^{\mathsf{T}}\mathbb{G}\boldsymbol{x} \quad \text{is a gauge term}$$

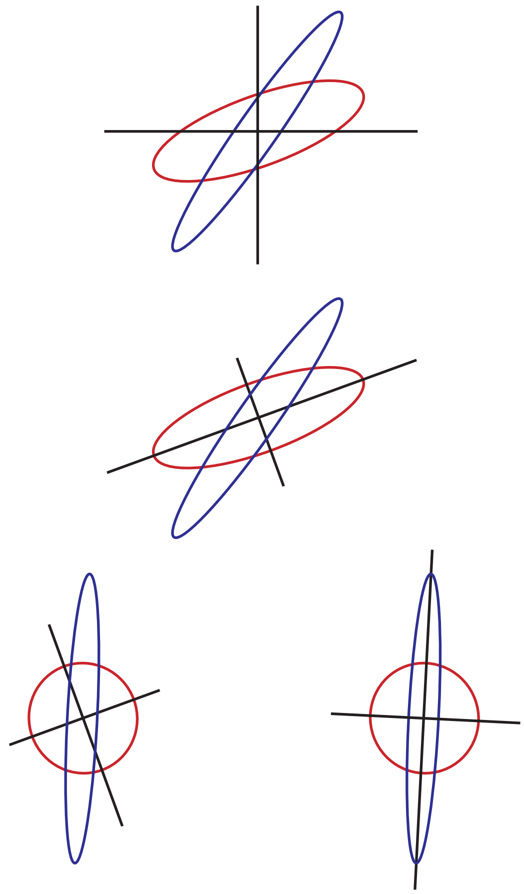

FIGURE 9: *The ellipses arise from* $\boldsymbol{x}^{\mathsf{T}}\mathbb{M}\,\boldsymbol{x} = 1$ *and* $\boldsymbol{x}^{\mathsf{T}}\mathbb{K}\,\boldsymbol{x} = 1$*, where* $\mathbb{M}$ *and* $\mathbb{K}$ *are arbitrary real/symmetric/positive-definite matrices, and would in the n-dimensional case be hyper-ellipsoids. The* **simultaneous diagonalization** *procedure runs*

- *rotate to the principal axes of the* $\mathbb{M}$*-ellipse;*
- *dilate so as to render the* $\mathbb{M}$*-ellipse spherical;*
- *rotate to the principal axes of the (now deformed)* $\mathbb{K}$*-ellipse; the* $\mathbb{M}$*-sphere is invariant under that adjustment.*

It is evident that **the process cannot be continued** *for the simple reason that it is (except in special circumstances) possible to make only* one sphere at a time.

Discarding the gauge term, we are left with

$$L = \tfrac{1}{2}\dot{\boldsymbol{x}}^{\mathsf{T}}\mathbb{M}\,\dot{\boldsymbol{x}} + \dot{\boldsymbol{x}}^{\mathsf{T}}\mathbb{A}\,\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\mathbb{K}\,\boldsymbol{x} \quad : \quad \mathbb{A}\text{ antisymmetric} \tag{9}$$

which gives

$$\mathbb{M}\,\ddot{\boldsymbol{x}} + 2\mathbb{A}\,\dot{\boldsymbol{x}} + \mathbb{K}\,\boldsymbol{x} = \mathbf{0} \tag{10}$$

The new term *looks* like a damping term, but isn't. For t-differentiation of the energy $E = \frac{1}{2}\dot{\boldsymbol{x}}^\mathsf{T}\mathbb{M}\,\dot{\boldsymbol{x}} + \frac{1}{2}\boldsymbol{x}^\mathsf{T}\mathbb{K}\,\boldsymbol{x}$ gives

$$\begin{aligned}\dot{E} &= \big[\tfrac{1}{2}\dot{\boldsymbol{x}}^\mathsf{T}\mathbb{M}\,\ddot{\boldsymbol{x}} + \tfrac{1}{2}\dot{\boldsymbol{x}}^\mathsf{T}\mathbb{K}\,\boldsymbol{x}\big] + \text{transpose}\\ &= \big[\tfrac{1}{2}\dot{\boldsymbol{x}}^\mathsf{T}\big\{-2\mathbb{A}\,\dot{\boldsymbol{x}} - \mathbb{K}\,\boldsymbol{x}\big\} + \tfrac{1}{2}\dot{\boldsymbol{x}}^\mathsf{T}\mathbb{K}\,\boldsymbol{x}\big] + \text{transpose}\\ &= 0\end{aligned}$$

from which the $\dot{\boldsymbol{x}}^\mathsf{T}\mathbb{A}\,\dot{\boldsymbol{x}}$-term has disappeared by antisymmetry.

The "simultaneous diagonalization of $\mathbb{M}$, $\mathbb{A}$ and $\mathbb{K}$" was just shown to be generally impossible, and anyway: a diagonalized $\mathbb{A}$ would—in consequence of its antisymmetry—disappear altogether!

I propose to discuss the typical dynamical consequences of the $\mathbb{A}$-term in the equation of motion (10). By way of preparation, write $\mathbb{M} = \mathbb{N}^\mathsf{T}\mathbb{N}$, multiply $(\mathbb{N}^\mathsf{T})^{-1} = (\mathbb{N}^{-1})^\mathsf{T}$ into (10) and obtain

$$\mathbb{N}\,\ddot{\boldsymbol{x}} + 2(\mathbb{N}^{-1})^\mathsf{T}\mathbb{A}\mathbb{N}^{-1}\cdot\mathbb{N}\,\dot{\boldsymbol{x}} + (\mathbb{N}^{-1})^\mathsf{T}\mathbb{K}\mathbb{N}^{-1}\cdot\mathbb{N}\,\boldsymbol{x} = \mathbf{0}$$

which we agree to notate

$$\ddot{\boldsymbol{y}} + 2\mathbb{B}\,\dot{\boldsymbol{y}} + \Omega_0^2\,\boldsymbol{y} = \mathbf{0} \tag{11}$$

(the $\boldsymbol{y}$-variables have abandoned their former hats) and which possess the structure

$$\ddot{\boldsymbol{y}} + 2\mathbb{B}\,\dot{\boldsymbol{y}} + \begin{pmatrix}\omega_{10}^2 & 0 & \dots & 0\\ 0 & \omega_{20}^2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \omega_{n0}^2\end{pmatrix}\boldsymbol{y} = \mathbf{0} \;\;:\;\; \mathbb{B}\text{ antisymmetric}$$

We have by this point diagonalized what we could, but are left with a system of equations that <u>cannot be uncoupled</u>. The motion of such a system cannot be resolved into superimposed normal modes of harmonic vibration. Indeed, such a system **cannot properly be said to possess "normal modes"** in the familiar sense. Nevertheless, and as will emerge: such systems are endowed with *natural frequencies*, and do display what might be called *abnormal modes* of harmonic motion. I find it convenient to develop these points in reference to a specific

EXAMPLE: Look to $\ddot{\boldsymbol{y}} + 2\mathbb{B}\,\dot{\boldsymbol{y}} + \Omega_0^2\,\boldsymbol{y} = \mathbf{0}$ in the case

$$\mathbb{B} = \begin{pmatrix}0 & -\beta\\ \beta & 0\end{pmatrix}, \quad \Omega_0^2 = \begin{pmatrix}2^2 & 0\\ 0 & 3^2\end{pmatrix}$$

I will sketch three distinct approaches to the solution of this pair of equations, all of which extend straightforwardly to systems with more than two degrees of freedom.

First approach: We could simply ask *Mathematica* to do our work for us, bringing its `DSolve` command to the system

$$\begin{aligned} \ddot{y}_1 - 2\beta\dot{y}_2 + 4y_1 &= 0 \\ \ddot{y}_2 + 2\beta\dot{y}_1 + 9y_2 &= 0 \\ y_1(0) &= 1 \\ y_2(0) &= 0 \\ \dot{y}_1(0) &= 0 \\ \dot{y}_2(0) &= 1 \end{aligned}$$

In the trivial case $\beta = 0$ we get

$$\begin{aligned} y_1(t) &= \phantom{\tfrac{1}{3}}\cos 2t \\ y_2(t) &= \tfrac{1}{3}\sin 3t \end{aligned}$$

which is plotted below:

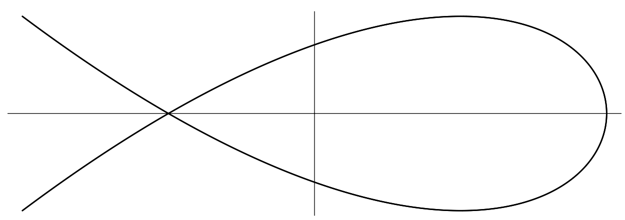

FIGURE 10: *Motion in the case under study when the gyroscopic coupling is turned off*: $\beta = 0$.

In the case $\beta = \frac{1}{10}$ (on which I will concentrate) *Mathematica* is no less accommodating: it supplies a result that, when all the complicated surds have been reduced to numbers, can be rendered

$$\left.\begin{aligned} y_1(t) &= 1.0330\cos(1.9921\,t) - 0.0330\cos(3.0119\,t) \\ y_2(t) &= 0.0818\,\sin(1.9921\,t) + 0.2778\,\sin(3.0119\,t) \end{aligned}\right\} \tag{12}$$

and is plotted in FIGURE 11.

Second approach: Assume $\boldsymbol{y}$ to be the real part of $\boldsymbol{Z}e^{i\omega t}$. Define

$$\mathbb{F}(\omega) = \begin{pmatrix} 4-\omega^2 & -\frac{1}{5}i\omega \\ -\frac{1}{5}i\omega & 9-\omega^2 \end{pmatrix}$$

and from $\det\mathbb{F}(\omega) = 0$ recover precisely the ω-values implicit in (12):

$$\begin{aligned} \omega_1 &= \pm 3.0119 \\ \omega_2 &= \pm 1.9921 \end{aligned}$$

Command `Transpose[NullSpace[N[`$\mathbb{F}(\omega_k)$`]]]` and get complex

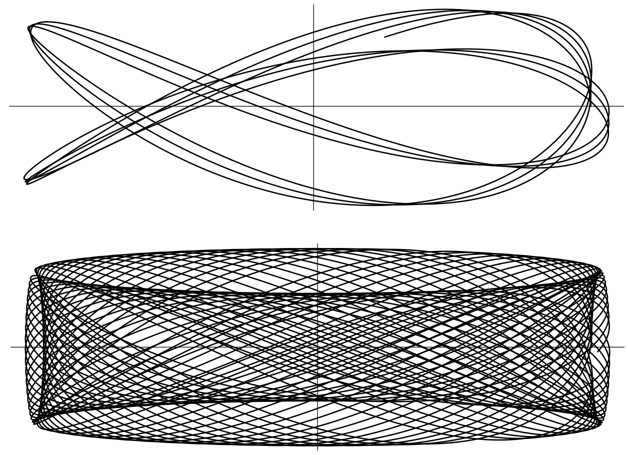

FIGURE 11: *Figures derived from* (12). *In the top figure* t *runs from* 0 *to* 19.6. *In the bottom figure* t *runs to* 120.

vectors

$$\boldsymbol{Z}_1 = \boldsymbol{X}_1 + i\boldsymbol{Y}_1 = \begin{pmatrix} -0.1179 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ -0.9930 \end{pmatrix}$$
$$\boldsymbol{Z}_2 = \boldsymbol{X}_2 + i\boldsymbol{Y}_2 = \begin{pmatrix} -0.9969 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ +0.0789 \end{pmatrix}$$

From those construct

$$\boldsymbol{Z}_k e^{i\omega_k t} = \underbrace{\big[\boldsymbol{X}_k \cos\omega_k t - \boldsymbol{Y}_k \sin\omega_k t\big]}_{\boldsymbol{u}_k(t)} + i\underbrace{\big[\boldsymbol{X}_k \sin\omega_k t + \boldsymbol{Y}_k \cos\omega_k t\big]}_{\boldsymbol{v}_k(t)}$$

which in the case at hand supply

$$\boldsymbol{u}_1(t) = \begin{pmatrix} -0.1179\cos\omega_1 t \\ +0.9930\,\sin\omega_1 t \end{pmatrix}$$
$$\boldsymbol{v}_1(t) = \begin{pmatrix} -0.1179\,\sin\omega_1 t \\ -0.9930\cos\omega_1 t \end{pmatrix}$$
$$\boldsymbol{u}_2(t) = \begin{pmatrix} -0.9969\cos\omega_2 t \\ -0.0789\,\sin\omega_2 t \end{pmatrix}$$
$$\boldsymbol{v}_2(t) = \begin{pmatrix} -0.9969\,\sin\omega_2 t \\ +0.0789\cos\omega_2 t \end{pmatrix}$$

We verify by calculation that each of those satisfies

$$\ddot{\boldsymbol{y}} + \tfrac{1}{5}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\dot{\boldsymbol{y}} + \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix}\boldsymbol{y} = \boldsymbol{0}$$

and expect the general solution to be describable

$$\boldsymbol{y}(t) = a_1\boldsymbol{u}_1(t) + b_1\boldsymbol{v}_1(t) + a_2\boldsymbol{u}_2(t) + b_2\boldsymbol{v}_2(t)$$

Realization of the previously-stated initial conditions is found to entail that we set $b_1 = b_2 = 0$ and

$$\begin{aligned} a_1 &= 0.2799 \\ a_2 &= -1.0362 \end{aligned}$$

When this is done we *recover precisely the equation* (12) *that gave us* FIGURE 11.

The preceding discussion shows that the presence of gyroscopic terms in the equations of motion **serves not to destroy the modal concept, but to make it richer**: the vectors

$$(a\cos\omega t + b\sin\omega t)\boldsymbol{n}$$

that describe modal motion in the conventional sense all lie in a 1-space, while the vectors

$$a\big[\boldsymbol{X}\cos\omega t - \boldsymbol{Y}\sin\omega t\big] + b\big[\boldsymbol{X}\sin\omega t + \boldsymbol{Y}\cos\omega t\big]$$

trace ellipses in a 2-space. One recovers conventional modality in the limit that $\boldsymbol{Y}$ (else $\boldsymbol{X}$) vanishes, which must evidently happen when $\mathbb{B} \to \mathbb{O}$. At work here is the circumstance that in the absence of gyroscopic terms the Ansatz gives rise to vectors $\boldsymbol{Z}$-vectors that are invariably *real to within a numerical complex factor*, whnle in the presence of such terms $\boldsymbol{Z}$ becomes *essentially* complex.

It has been remarked that if the $\mathbb{B}$ in

$$\ddot{\boldsymbol{y}} + 2\mathbb{B}\,\dot{\boldsymbol{y}} + \Omega_0^2\boldsymbol{y} = \boldsymbol{0} \tag{11}$$

could be diagonalized it would—by antisymmetry—disappear altogether. In point of fact it *can* be made to disappear, but at cost. To that end, introduce "spinning variables" $\boldsymbol{s}$ by

$$\boldsymbol{y} = \mathbb{W}\boldsymbol{s} \quad \text{with} \quad \mathbb{W}(t) = e^{-\mathbb{B}t}$$

Then

$$\begin{aligned} \dot{\boldsymbol{y}} &= \mathbb{W}(\dot{\boldsymbol{s}} - \mathbb{B}\,\boldsymbol{s}) &&= \mathbb{W}(\tfrac{d}{dt} - \mathbb{B})\boldsymbol{s} \\ \ddot{\boldsymbol{y}} &= \mathbb{W}(\ddot{\boldsymbol{s}} - 2\mathbb{B}\,\dot{\boldsymbol{s}} + \mathbb{B}^2\boldsymbol{s}) &&= \mathbb{W}(\tfrac{d}{dt} - \mathbb{B})^2\boldsymbol{s} \end{aligned}$$

and (11) becomes

$$\mathbb{W}(\ddot{\boldsymbol{s}} - 2\mathbb{B}\,\dot{\boldsymbol{s}} + \mathbb{B}^2\boldsymbol{s}) + 2\mathbb{B}\mathbb{W}(\dot{\boldsymbol{s}} - \mathbb{B}\,\boldsymbol{s}) + \Omega_0^2\,\mathbb{W}\boldsymbol{s} = \boldsymbol{0}$$

$\mathbb{B}$ and $\mathbb{W}$ commute, so the $\dot{\boldsymbol{s}}$ terms drop away, leaving

$$\mathbb{W}\ddot{\boldsymbol{s}} + (\Omega_0^2 - \mathbb{B}^2)\mathbb{W}\boldsymbol{s} = \boldsymbol{0}$$

Multiplication by $\mathbb{W}^{-1} = \mathbb{W}^{\mathsf{T}}$ yields finally

$$\ddot{\boldsymbol{s}} + \Omega^2\boldsymbol{s} = \boldsymbol{0} \tag{13.1}$$

where Ω^2 is the t-dependent real/symmetric matrix defined

$$\Omega^2(t) = e^{+\mathbb{B}t}\,\Omega_0^2\,e^{-\mathbb{B}t} - \mathbb{B}^2 \tag{13.2}$$

Note that the motion of $\Omega_0^2(t)$ is such that its spectrum remains constant.

EXAMPLE REVISITED: Look again to $\ddot{\boldsymbol{y}} + 2\mathbb{B}\,\dot{\boldsymbol{y}} + \Omega_0^2\boldsymbol{y} = \boldsymbol{0}$ in the case

$$\mathbb{B} = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}, \quad \Omega_0^2 = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix}$$

The `MatrixExp` command supplies

$$\mathbb{W} = e^{-\mathbb{B}t} = \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}$$

whence

$$\Omega^2(t) = \begin{pmatrix} 4\cos^2\beta t + 9\sin^2\beta t + \beta^2 & -5\cos\beta t\sin\beta t \\ -5\cos\beta t\sin\beta t & 9\cos^2\beta t + 4\sin^2\beta t + \beta^2 \end{pmatrix}$$

which, we note in passing, has eigenvalues

$$2^2 + \beta^2, \quad 3^2 + \beta^2 \quad : \quad \text{all } t$$

The system motion is described therefore by the coupled differential equations

$$\left.\begin{aligned} \ddot{s}_1 + (4\cos^2\beta t + 9\sin^2\beta t + \beta^2)s_1 - (5\cos\beta t\sin\beta t)s_2 = 0 \\ \ddot{s}_2 + (9\cos^2\beta t + 4\sin^2\beta t + \beta^2)s_2 - (5\cos\beta t\sin\beta t)s_1 = 0 \end{aligned}\right\} \tag{14.1}$$

which, of course, decouple at $\beta^2 = 0$ (where the spinning stops and the $\boldsymbol{s}/\boldsymbol{y}$ distinction disappears) to give back again

$$\begin{aligned} \ddot{y}_1 + 4y_1 = 0 \\ \ddot{y}_2 + 9y_2 = 0 \end{aligned}$$

To translate initial data into conditions of spinning variable we use

$$\begin{aligned} \boldsymbol{s} &= \mathbb{W}^{-1}\,\boldsymbol{y} \\ \dot{\boldsymbol{s}} &= \mathbb{W}^{-1}\,\dot{\boldsymbol{y}} + \mathbb{B}\,\mathbb{W}^{-1}\,\boldsymbol{y} \end{aligned}$$

which in the case of immediate interest ($\beta = \frac{1}{10}$) becomes

$$\left.\begin{aligned} \boldsymbol{s}(0) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \dot{\boldsymbol{s}}(0) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \tfrac{1}{10}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{11}{10} \end{pmatrix} \end{aligned}\right\} \tag{14.2}$$

We are in position now to use `NDSolve` to obtain descriptions of $s_1(t)$ and $s_2(t)$ as a pair of `InterpolatingFunction`s, which when plotted yield FIGURE 12. If, on the other hand, we multiply $\mathbb{W}^{-1}(t)$

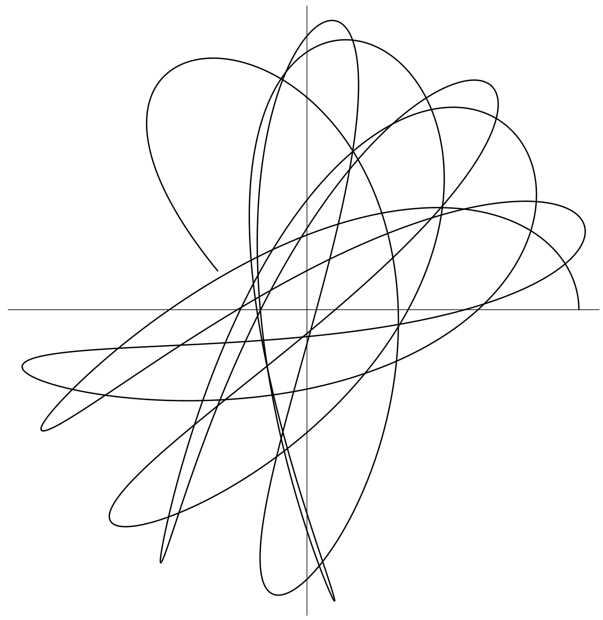

FIGURE 12: *Graph of the* $\boldsymbol{s}(t)$ *that results at* $\beta = \frac{1}{10}$ *from* (14) *as* t *runs from* 0 *to* 19.6.

into the $\boldsymbol{y}(t)$ of (12)—which produced FIGURES 11—we obtain

$$\begin{pmatrix} s_1(t) \\ s_2(t) \end{pmatrix} = \begin{pmatrix} \cos(t/10) & -\sin(t/10) \\ \sin(t/10) & \cos(t/10) \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

which when graphed *exactly reproduces the preceding figure*. Figures referred to the rotating $\boldsymbol{s}$-frame become much prettier/less chaotic when $\beta \ll \{\omega_1, \omega_2\}$. If, for example, we set $\beta = \frac{1}{100}$ then the ω_k acquire new values $\{2.9999, 2.0001\} \approx \{3, 2\}$—so also does $\dot{\boldsymbol{s}}(0)$—and in place of FIGURE 12 we obtain FIGURE 13.

Anyone who has played with an oscilloscope has had fun with the Lissajous figures

$$\begin{aligned} x_1(t) &= a_1 \sin(\omega_1 t) \\ x_2(t) &= a_2 \sin(\omega_2 t + \delta) \end{aligned}$$

that appear to "lock" when ω_1/ω_2 is a small rational number, and that appear to "roll" when one of the signals is not quite in tune. It should be appreciated that "Lissajous rolling" and "gyro precession" are distinct phenomena, with distinct causes, though they may on casual inspection look quite similar.

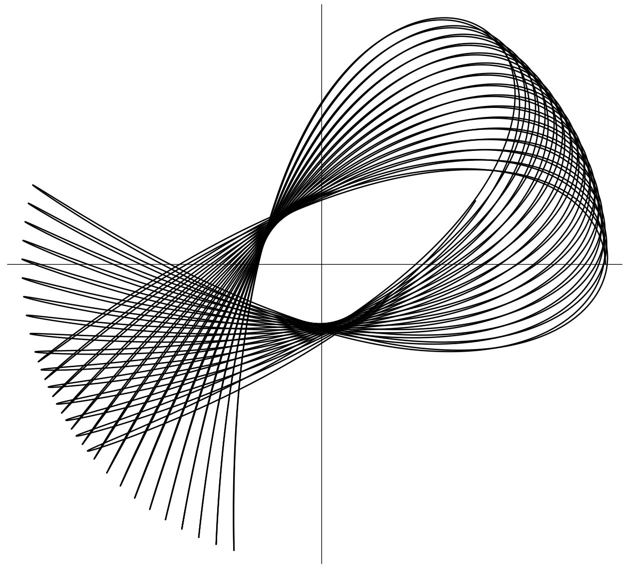

FIGURE 13: *Graph of the* $\boldsymbol{s}(t)$ *that results at* $\beta = \frac{1}{100}$ *from* (14) *as* t *runs now from* 0 *to* 100*. The adjustment requires that we adjust also the value of* $\dot{\boldsymbol{s}}(0)$:

$$\dot{\boldsymbol{s}}(0) = \begin{pmatrix} 0 \\ \frac{101}{100} \end{pmatrix}$$

We recover FIGURE 10 *in the limit* $\beta \to 0$.

5. Damping. To describe charge flow in a pair of inductively coupled RLC circuits (FIGURE 14) we would write

$$\begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix} \ddot{\boldsymbol{q}} + \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \dot{\boldsymbol{q}} + \begin{pmatrix} C_1^{-1} & 0 \\ 0 & C_2^{-1} \end{pmatrix} \boldsymbol{q} = \mathbf{0}$$

of which we will take

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \ddot{\boldsymbol{q}} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \dot{\boldsymbol{q}} + \begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix} \boldsymbol{q} = \mathbf{0} \tag{15}$$

to provide a representative instance. Proceeding in the now-familiar way to the simultaneous diagonalization of the inductance and capacitance matrices, we are led to the construction of a matrix

$$\mathbb{Q} = \begin{pmatrix} -0.53891 & -0.55640 \\ 0.59435 & -0.21621 \end{pmatrix}$$

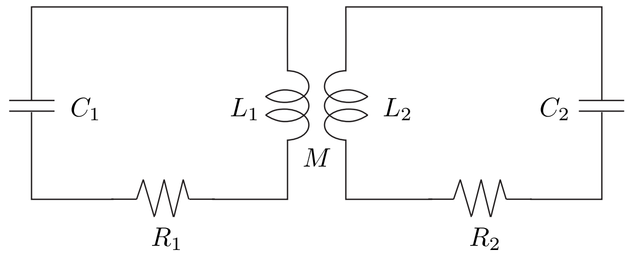

FIGURE 14: *Inductively coupled RLC circuits, produced by inserting resistors into the circuits shown in* FIGURE 3.

with the property that

$$\mathbb{Q}^{\mathsf{T}}\begin{pmatrix}2&1\\1&3\end{pmatrix}\mathbb{Q}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$
$$\mathbb{Q}^{\mathsf{T}}\begin{pmatrix}3&0\\0&7\end{pmatrix}\mathbb{Q}=\begin{pmatrix}\omega_{10}^2&0\\0&\omega_{20}^2\end{pmatrix}$$

where

$$\omega_{10}^2 = 3.34403$$
$$\omega_{20}^2 = 1.25597$$

are recognized to be squares of the roots of

$$\det\left[\begin{pmatrix}3&0\\0&7\end{pmatrix}-\begin{pmatrix}2&1\\1&3\end{pmatrix}\omega^2\right]=0$$

We are led therefore to introduce new variables $\boldsymbol{y}$ (who have again lost their hats) by

$$\boldsymbol{q}=\mathbb{Q}\,\boldsymbol{y}$$

in terms of which the circuit equations (after multiplication on the left by $\mathbb{Q}^{\mathsf{T}}$) become

$$\ddot{\boldsymbol{y}}+\mathbb{R}\,\dot{\boldsymbol{y}}+\begin{pmatrix}\omega_{10}^2&0\\0&\omega_{20}^2\end{pmatrix}\boldsymbol{y}=\boldsymbol{0} \tag{16.1}$$

where the "resistance matrix"

$$\mathbb{R}=\mathbb{Q}^{\mathsf{T}}\begin{pmatrix}1&0\\0&2\end{pmatrix}\mathbb{Q}=\begin{pmatrix}0.99693&0.04284\\0.04284&0.40307\end{pmatrix} \tag{16.2}$$

is real/symmetric/positive-definite. In (16) we encounter once again a situation in which the diagonalization process has run out of steam before complete decoupling has been accomplished. A theorist, anxious to preserve the relevance of the "normal mode" concept, might be tempted to introduce "modal damping"

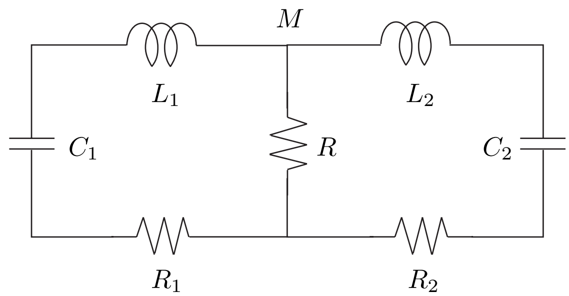

FIGURE 15: *Circuit redesigned to make it possible to achieve "modal damping."*

by returning to (16) and *by hand* setting

$$\mathbb{R} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

But that would entail introducing precisely calibrated off-diagonal elements into the resistance matrix in (15), and to accomplish that one would have to redesign the circuit (FIGURE 15), setting

$$\begin{aligned} R_1 &= 0.23374\,r_1 + 1.76626\,r_2 \\ R \ &= -0.60150\,r_1 + 1.60150\,r_2 \\ R_2 &= 1.54789\,r_1 + 1.45211\,r_2 \end{aligned}$$

Equations (16)—into which I now find it useful to introduce an adjustable constant k, writing

$$\ddot{\boldsymbol{y}} + k\cdot\mathbb{R}\,\dot{\boldsymbol{y}} + \begin{pmatrix} \omega_{10}^2 & 0 \\ 0 & \omega_{20}^2 \end{pmatrix}\boldsymbol{y} = \boldsymbol{0} \tag{17}$$

—do, however, yield to solution by the now-familiar variety of means. At $k = 1$ `DSolve` supplies an expression containing four constants of integration, each multiplied into an oddly weighted linear combination of terms of the types

$$e^{-0.49844\,t}\begin{cases}\cos(1.75858\,t)\\ \sin(1.75858\,t)\end{cases} \quad\text{and}\quad e^{-0.20156\,t}\begin{cases}\cos(1.10293\,t)\\ \sin(1.10293\,t)\end{cases} \tag{18}$$

that contains four constants of integration. If we retain the initial conditions familiar from previous examples then `NDSolve` supplies interpolating functions that when plotted yield figures like those presented on the next two pages.

If k becomes sufficiently large the oscillatory factors disappear and tyhe system enters into an **overdamped regime**. Experiments with `DSolve` indicate that for the system at hand oscillations are still present at $k = 5.6$, but they

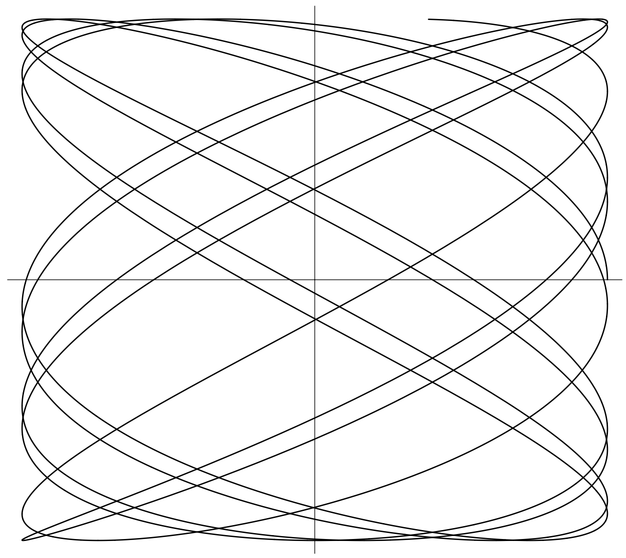

FIGURE 16: *Curve derived from* (17) *in the case* $k = 0$ (*resistences turned off*)*. The initial conditions are again those that have become standard in this work* $(y_1(0) = \dot{y}_2(0) = 1,\ y_2(0) = \dot{y}_1(0) = 0)$ *and* t *runs from* 0 *to* 35*.*

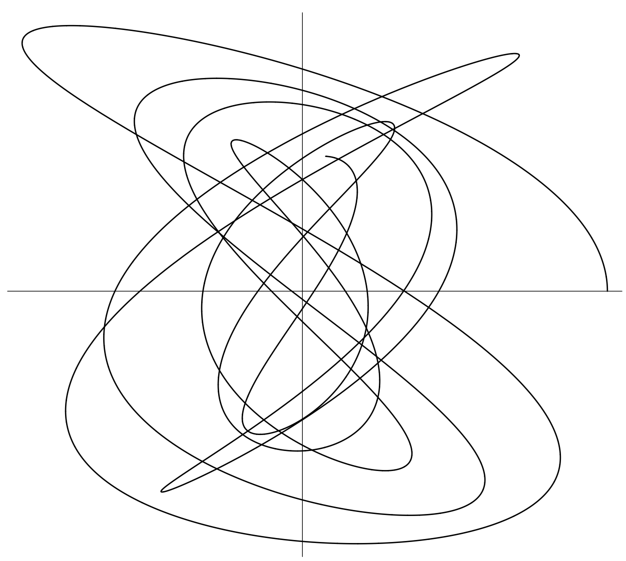

FIGURE 17: *Same as above, except that now* $k = 0.1$*.*

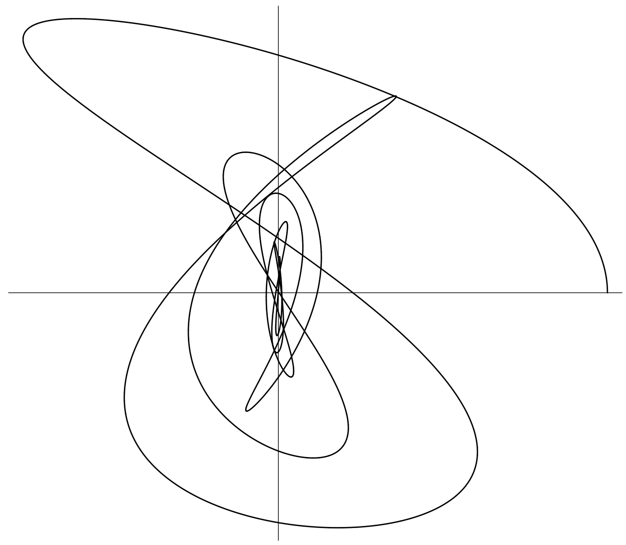

FIGURE 18: *Same as above, except that now $k = 0.3$.*

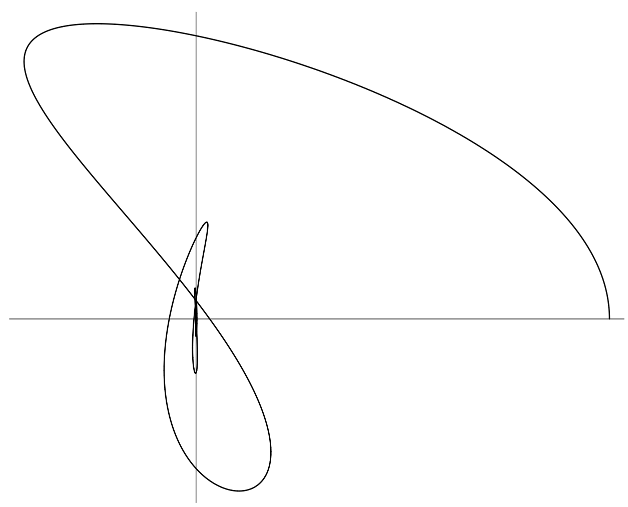

FIGURE 19: *Same as above, except that now $k = 1.0$.*

have disappeared by the time resistance has grown to the level indicated by $k = 5.7$: $\boldsymbol{y}(t)$ is then assembled from terms proportional to

$$e^{-5.03875\,t}, \quad e^{-1.29343\,t}, \quad e^{-1.00934\,t}, \quad \text{and} \quad e^{-0.63848\,t}$$

For a graph of $\boldsymbol{y}(t)$ in that case see FIGURE 20.

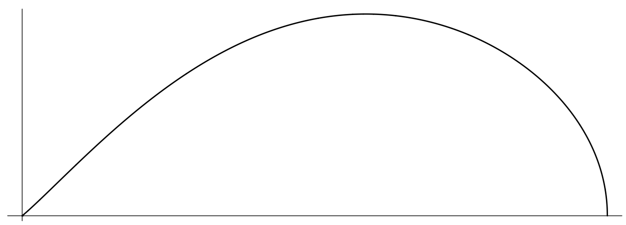

FIGURE 20: *Curve obtained from* (17) *in the overdamped case* $k = 5.7$.

If one looks for solutions of the form $\boldsymbol{Z}e^{i\omega t}$ then one has first to solve

$$\det(\mathbb{F}(\omega;k)) = 0$$

with

$$\mathbb{F}(\omega;k) = \begin{pmatrix} \omega_{10}^2 & 0 \\ 0 & \omega_{20}^2 \end{pmatrix} + ik\cdot\mathbb{R}\,\omega - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\omega^2$$

This (in the case $k = 1$) gives

$$\begin{aligned} \omega_1 &= \pm 1.75858 + 0.49844\,i \\ \omega_2 &= \pm 1.10293 + 0.20156\,i \end{aligned}$$

—in precise agreement with the results reported at (18). The commands `NullSpace[`$\mathbb{F}(\omega_j)$`]` now supply $\boldsymbol{Z}_j = \boldsymbol{X}_j + i\,\boldsymbol{Y}_j$ with

$$\begin{aligned} \boldsymbol{X}_1 &= \begin{pmatrix} 0.92571 \\ -0.00505 \end{pmatrix} & \boldsymbol{Y}_1 &= \begin{pmatrix} 0.37635 \\ 0.03743 \end{pmatrix} \\ \boldsymbol{X}_2 &= \begin{pmatrix} -0.02199 \\ -0.17753 \end{pmatrix} & \boldsymbol{Y}_1 &= \begin{pmatrix} 0.00727 \\ -0.98384 \end{pmatrix} \end{aligned}$$

Interest here attaches not to the digits but to the simple fact that the $\boldsymbol{Z}_j$-vectors are *essentially complex*,[2] for here as in the gyroscopic case

> It is the complexity of the $\boldsymbol{Z}$s that announces the breakdown of the normal mode concept, the intrusion of abnormal modes.

Recent figures all show *extinction of motion*, with which we associate energy loss. Looking now specifically to the energetics: the line of argument encountered at the top of page 12 serves to establish that if

$$\mathbb{M}\ddot{\boldsymbol{x}} + 2\mathbb{G}\dot{\boldsymbol{x}} + \mathbb{K}\boldsymbol{x} = \boldsymbol{0} \quad \text{and} \quad E = \tfrac{1}{2}\dot{\boldsymbol{x}}^{\mathsf{T}}\mathbb{M}\dot{\boldsymbol{x}} + \tfrac{1}{2}\boldsymbol{x}^{\mathsf{T}}\mathbb{K}\boldsymbol{x}$$

[2] We say that a complex vector $\boldsymbol{Z}$ "essentially complex" if it is not possible to write $\boldsymbol{Z} = (\text{complex number})\cdot(\text{real vector})$, and in the contrary case that it is "inessentially complex." In one dimension the distinction does not arise.

then

$$\dot{E} = -\tfrac{1}{2}\dot{\boldsymbol{x}}^{\mathsf{T}}(\mathbb{G} + \mathbb{G}^{\mathsf{T}})\,\dot{\boldsymbol{x}}$$

It was remarked previously that the antisymmetric part of $\mathbb{G} = \mathbb{S} + \mathbb{A}$ does not contribute to energy dissipation. What we have just established is that the symmetric part, on the other hand, does: we have

$$\dot{E} = -\dot{\boldsymbol{x}}^{\mathsf{T}}\mathbb{S}\,\dot{\boldsymbol{x}}$$

which in electrical applications reads $\dot{E} = -\dot{\boldsymbol{q}}^{\mathsf{T}}\mathbb{R}\,\dot{\boldsymbol{q}}$ and is called "I^2R loss."

6. Introduction of inflating coordinates. We have the shift rule

$$e^{-\mathbb{G}t}\,\mathbb{D}\,e^{\mathbb{G}t} = (\mathbb{D} + \mathbb{G}) \quad : \quad \mathbb{D} \equiv \tfrac{d}{dt}\mathbb{I}$$

from which it follows in particular that $\mathbb{D}^2 e^{\mathbb{G}t}\boldsymbol{y} = e^{\mathbb{G}t}\big\{\ddot{\boldsymbol{y}} + 2\mathbb{G}\,\dot{\boldsymbol{y}} + \mathbb{G}^2\boldsymbol{y}\big\}$. So if vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ stand in the relation $\boldsymbol{z} = e^{\mathbb{G}t}\boldsymbol{y}$ then

$$\ddot{\boldsymbol{z}} = e^{\mathbb{G}t}\big\{\ddot{\boldsymbol{y}} + 2\mathbb{G}\,\dot{\boldsymbol{y}} + \mathbb{G}^2\boldsymbol{y}\big\}$$

Adding a term $e^{\mathbb{G}t}\big\{\Omega_0^2\,\boldsymbol{y} - \mathbb{G}^2\boldsymbol{y}\big\}$ amounts to adding a term $\big\{e^{\mathbb{G}t}\,\Omega_0^2\,e^{-\mathbb{G}t}\boldsymbol{z} - \mathbb{G}^2\boldsymbol{z}\big\}$ on the left. We conclude that

$$\ddot{\boldsymbol{y}} + 2\mathbb{G}\,\dot{\boldsymbol{y}} + \Omega_0^2\boldsymbol{y} = \boldsymbol{0} \quad \Longleftrightarrow \quad e^{\mathbb{G}t}\big\{\ddot{\boldsymbol{y}} + 2\mathbb{G}\,\dot{\boldsymbol{y}} + \Omega_0^2\boldsymbol{y}\big\} = \boldsymbol{0}$$

which is in turn equivalent to

$$\ddot{\boldsymbol{z}} + \Omega^2(t)\boldsymbol{z} = \boldsymbol{0} \quad \text{with} \quad \Omega^2(t) = e^{\mathbb{G}t}\,\Omega_0^2\,e^{-\mathbb{G}t} - \mathbb{G}^2 \tag{19}$$

No use has been made here of the symmetry properties of $\mathbb{G}$: if we understand $\mathbb{G}$ to be antisymmetric we recover (13), from which a gyroscopic term had been removed. But if $\mathbb{G}$ is symmetric then it is a damping term that has been eliminated from (19).

To see how this works in a concrete case I return to (17). Looking to the case $k = \frac{1}{10}$, we have $\mathbb{G} = \frac{1}{20}\mathbb{R}$ (which has eigenvalues $\{0.05, 0.02\}$), command `MatrixExp[`$\mathbb{G}t$`]` and obtain

$$e^{\mathbb{G}t} = \begin{pmatrix} 0.99488 & 0.07139 \\ 0.07139 & 0.00512 \end{pmatrix} e^{0.05\,t} + \begin{pmatrix} 0.00512 & -0.07139 \\ -0.07139 & 0.99488 \end{pmatrix} e^{0.02\,t}$$

We use `DSolve` to produce $\boldsymbol{y}(t)$ and then construct

$$\boldsymbol{z}(t) = e^{\mathbb{G}t}\boldsymbol{y}(t)$$

FIGURE 21 shows a typical result of such a procedure. In the one-dimensional theory of damped oscillators it is my experience that inflating coordinates can be used to good advantage, but in the present context I can't claim them to be good for much beyond the production of pretty pictures.

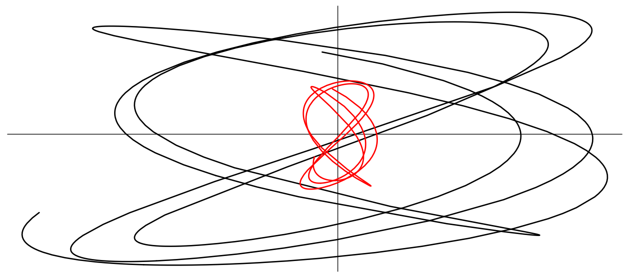

FIGURE 21: *The red curve depicts the solution* $\boldsymbol{y}(t)$ *of* (17) *in the case* $k = \frac{1}{10}$ (*in which connection see again* FIGURE 17) *as* t *ranges from* 30 *to* 50. *The black curve derives from* $\boldsymbol{z}(t) = e^{\mathbb{G}t}\boldsymbol{y}(t)$ *and provides a distortedly magnified representation of the same data.*

7. Conclusions & prospects. I have stressed that the presence of a "3$^{\text{rd}}$ matrix" in the linear differential equations that describe the oscillations of complex mechanical/systems brings one into conflict with the elementary fact that one can (in general) diagonalize not more than two symmetric matrices at once. This circumstance spells the demise of the familiar "normal mode" concept, which is replaced by a richer conception: the "abnormal mode."

I have described several approaches to the *solution* of the equations that describe abnormal modes, and shown that problem to be scarcely more difficult that the problem posed by normal mode systems, even though the motion executed by abnormal systems is relatively more complicated.

Does the normal/abnormal distinction matter? Hardly at all. For only rarely does one have physical interest in the specific movement of the component parts of a multi-component vibrating system: one is most commonly interested in the **spectroscopy** of such systems, in the natural frequencies associated with that motion, and the theoretical determination of those is wholely independent of the distinction in question.[3]

Spectroscopic techniques refer to the behavior of *stimulated* vibrational systems. It would be interesting—and possibly useful—to inquire therefore into the response of abnormal systems to stimulation.

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[3] Here as always, it is the inverse problem that is most interesting: given a measured spectrum, how much damping, how much gyration should one introduce into one's model to *account* for the data?