THE ANISOTROPIC STRING

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Introduction. To describe the motion of waves on a stretched string we write

$$\big\{\partial_x^2 - \tfrac{1}{uu}\partial_t^2\big\}\,\varphi(x,t) = 0 \tag{1}$$

which (since the predominant waves on a string are transverse, and their description requires therefore that we monitor *two* fields) relates more properly to compressional waves on a stretched spring. Factorization of the wave operator

$$\partial_x^2 - \tfrac{1}{uu}\partial_t^2 = \big(\partial_x + \tfrac{1}{u}\partial_t\big)\big(\partial_x - \tfrac{1}{u}\partial_t\big)$$

leads directly to the conclusion that φ can be developed

$$\varphi(x,t) = f(x-ut) + g(x+ut)$$

where the right-running wave $f(x-ut)$ is killed by $\big(\partial_x + \frac{1}{u}\partial_t\big)$, the left-running wave $g(x+ut)$ is killed by $\big(\partial_x - \frac{1}{u}\partial_t\big)$.

Which brings us to the simple point of this note: Suppose it were the case that waves run right and left with *distinct speeds* u and v. We would then be led to write

$$\varphi(x,t) = f(x-ut) + g(x+vt)$$

which is a solution of

$$\big(\partial_x + \tfrac{1}{u}\partial_t\big)\big(\partial_x - \tfrac{1}{v}\partial_t\big)\,\varphi = \big\{\partial_x^2 + \big(\tfrac{1}{u} - \tfrac{1}{v}\big)\partial_x\partial_t - \tfrac{1}{uv}\partial_t^2\big\}\varphi = 0 \tag{2}$$

We observe that (2) gives back (1) in the case $u = v$. My intention is to examine some of the formal/physical properties of the anisotropic system (2).

1. Lagrangian formalism. Introduce

$$\mathcal{L}(\varphi,\partial\varphi) = \tfrac{1}{2uv}\varphi_t^2 - \tfrac{1}{2}\big(\tfrac{1}{u} - \tfrac{1}{v}\big)\varphi_x\varphi_t - \tfrac{1}{2}\varphi_x^2 \tag{3}$$

and from

$$\Big\{\partial_t\frac{\partial}{\partial\varphi_t} + \partial_x\frac{\partial}{\partial\varphi_x} - \frac{\partial}{\partial\varphi}\Big\}\,\mathcal{L} = 0$$

obtain

$$\tfrac{1}{uv}\varphi_{tt} - \big(\tfrac{1}{u} - \tfrac{1}{v}\big)\varphi_{tx} - \varphi_{xx} = 0$$

—which is (2). The Lagrangian (3) gives rise to a stress-energy tensor the components of which can be described[1]

$$\left.\begin{aligned}
S^t{}_t &= \frac{\partial\mathcal{L}}{\partial\varphi_t}\varphi_t - \mathcal{L}\\
&= \tfrac{1}{2uv}\varphi_t^2 + \tfrac{1}{2}\varphi_x^2\\
&\equiv \text{energy density } \mathcal{E}\\[2ex]
S^x{}_t &= \frac{\partial\mathcal{L}}{\partial\varphi_x}\varphi_t\\
&= -\tfrac{1}{2}\big(\tfrac{1}{u}-\tfrac{1}{v}\big)\varphi_t^2 - \varphi_x\varphi_t\\
&\equiv \text{energy flux } \mathcal{F}\\[2ex]
S^t{}_x &= \frac{\partial\mathcal{L}}{\partial\varphi_t}\varphi_x\\
&= \tfrac{1}{uv}\varphi_t\varphi_x - \tfrac{1}{2}\big(\tfrac{1}{u}-\tfrac{1}{v}\big)\varphi_x^2\\
&\equiv \text{momentum density } \mathcal{P}\\[2ex]
S^x{}_x &= \frac{\partial\mathcal{L}}{\partial\varphi_x}\varphi_x - \mathcal{L}\\
&= -\tfrac{1}{2uv}\varphi_t^2 - \tfrac{1}{2}\varphi_x^2\\
&\equiv \text{momentum flux } \mathcal{G}
\end{aligned}\right\} \tag{4}$$

By calculation we verify that

$$\begin{aligned}
\partial_t S^t{}_t + \partial_x S^x{}_t &= \varphi_t\big[\tfrac{1}{uv}\varphi_{tt} - \big(\tfrac{1}{u}-\tfrac{1}{v}\big)\varphi_{tx} - \varphi_{xx}\big] = 0\\
\partial_t S^t{}_x + \partial_x S^x{}_x &= \varphi_x\big[\tfrac{1}{uv}\varphi_{tt} - \big(\tfrac{1}{u}-\tfrac{1}{v}\big)\varphi_{tx} - \varphi_{xx}\big] = 0
\end{aligned}$$

which are statements of local energy/momentum conservation.

2. The equivalent lattice. Consider the familiar "one-dimensional crystal" that has been assembled from identical particles—each of mass m, each coupled to its nearest neighbors by identical springs of elasticity k. To describe the instantaneous position of the n^{th} particle we write

$$\begin{aligned}
x_n(t) &= na + \varphi_n(t)\\
&= \text{equilibrium position} + \text{displacement}
\end{aligned}$$

where a is the "lattice constant." To describe the motion of the n^{th} particle (unless it is an end-particle) we write

$$\begin{aligned}
m\ddot{\varphi}_n &= k(\varphi_{n+1}-\varphi_n) - k(\varphi_n - \varphi_{n-1})\\
&= -k(-\varphi_{n-1} + 2\varphi_n - \varphi_{n+1})
\end{aligned} \tag{4}$$

[1] See CLASSICAL FIELD THEORY (1999), Chapter 1, page 63.

This coupled system of equations can be notated

$$\mathbb{M}\ddot{\boldsymbol{\varphi}} + \mathbb{K}\boldsymbol{\varphi} = \mathbf{0} \tag{6}$$

with $\mathbb{M} \equiv m\mathbb{I}$ and

$$\mathbb{K} \equiv k\begin{pmatrix} * & * & & & & & * \\ \alpha & \beta & \alpha & & & & \\ & \alpha & \beta & \alpha & & & \\ & & \alpha & \beta & \alpha & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \alpha & \beta & \alpha \\ * & & & & & * & * \end{pmatrix}$$

where $\alpha = -1$, $\beta = +2$, the values assigned to the $*$s depends upon how we elect to manage the end-particles (coupled to fixed walls? coupled to each other?), and all other elements are zero. We observe that $\mathbb{M}$ and $\mathbb{K}$ are both symmetric, and that (6) can be obtained from a Lagrangian of the design

$$L_0 = \tfrac{1}{2}\dot{\boldsymbol{\varphi}}\cdot\mathbb{M}\dot{\boldsymbol{\varphi}} + \tfrac{1}{2}\boldsymbol{\varphi}\cdot\mathbb{K}\boldsymbol{\varphi}$$

Standard stuff. But we take now a non-standard step: we introduce into the Lagrangian a "gyroscopic term," writing

$$L = \tfrac{1}{2}\dot{\boldsymbol{\varphi}}\cdot\mathbb{M}\dot{\boldsymbol{\varphi}} + \tfrac{1}{2}\dot{\boldsymbol{\varphi}}\cdot\mathbb{G}\boldsymbol{\varphi} + \tfrac{1}{2}\boldsymbol{\varphi}\cdot\mathbb{K}\boldsymbol{\varphi} \tag{7}$$

where $\mathbb{G}$ is an *anti*symmetric matrix of the design

$$\mathbb{G} \equiv g\begin{pmatrix} 0 & -1 & & & \\ +1 & 0 & -1 & & \\ & +1 & 0 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & & \end{pmatrix}$$

The equations of motion (6)—thus adjusted—become

$$\mathbb{M}\ddot{\boldsymbol{\varphi}} + \mathbb{G}\dot{\boldsymbol{\varphi}} + \mathbb{K}\boldsymbol{\varphi} = \mathbf{0}$$

which in fine detail (compare (4)) read

$$m\ddot{\varphi}_n = g(\dot{\varphi}_{n+1} - \dot{\varphi}_{n-1}) - k(-\varphi_{n-1} + 2\varphi_n - \varphi_{n+1}) \tag{8}$$

We want now to "refine the lattice"—to make the particles progressively more numerous, individually (though not collectively) less massive, more closely spaced and to regulate the values of g and k in such a way as to obtain a meaningful continuous (or "field-theoretic") limit of the discrete system presently in hand. The program[2] proceeds from a notational adjustment: in place of $\varphi_n(t)$ we write $\varphi(na,t)$, which in the continuous limit becomes $\varphi(x,t)$ and will be abbreviated $\varphi(x)$ when t is not a participant but merely a spectator. In that notation (8) becomes

$$\begin{aligned}\varphi_{tt}(x) = \frac{2g(a)\,a}{m(a)}\cdot\frac{\varphi_t(x+a)-\varphi_t(x+a)}{2a}\\ +\frac{k(a)\,a^2}{m(a)}\cdot\frac{\frac{\varphi(x+a)-\varphi(x)}{a}-\frac{\varphi(x)-\varphi(x-a)}{a}}{a}\end{aligned}$$

To recover (2)—*i.e.*, to recover

$$\varphi_{tt}-(v-u)\varphi_{tx}-uv\,\varphi_{xx}=0$$

—in the limit $a\downarrow 0$ we have only to stipulate that

$$\lim_{a\downarrow 0}\frac{2g(a)\,a}{m(a)} = v-u \qquad \text{and} \qquad \lim_{a\downarrow 0}\frac{k(a)\,a^2}{m(a)} = uv \tag{9}$$

Since

$$\frac{m(a)}{a}\text{ models linear mass density }\mu$$

we are at (9) requiring in effect that

$$g(a) = \tfrac{1}{2}\mu\cdot(v-u)$$

remains constant as a diminishes, while the springs get stiffer as they get shorter, and become infinitely stiff in the limit[3]

$$a\,k(a) = \mu uv$$

The Lagrangian of the discrete system (which for us has become an a-parameterized *sequence* of discrete systems)—which at (7) is presented as a sum of terms—is readily seen to go over in the continuous limit into an integral:

$$\begin{aligned}L &= \int \mathcal{L}\,dx\\ \mathcal{L} &= \tfrac{1}{2}\mu\varphi_t^2-\tfrac{1}{2}\mu(v-u)\varphi_t\varphi_x-\tfrac{1}{2}\mu uv\,\varphi_x^2\end{aligned}$$

Division by μuv gives back the Lagrange density encountered at (3).

[2] See pages 5–8 in the class notes just cited, or §13.1 in H. Goldstein *et al*, *Classical Mechanics* (3$^{\text{rd}}$ edition 2002).

[3] This development is not at all "strange:" it is basic that springs get softer when connected in series, stiffer when chopped into fragments.

In his Reed College thesis[4] Mark Galassi demonstrated that lattices of the design

$$ABCABCABCABC\cdots$$

do *not* support anisotropic wave physics, even though they present distinct faces to right-moving and left-moving tourists. What we have learned is that anisotropy is a "gyroscopic" effect,[5] that it arises from a velocity-dependent "gyroscopic coupling" of particles or string-elements to their (near) neighbors.

It was established on page 8 of the material just cited that gyroscopic terms have no effect upon the energetics of oscillatory systems. At (4) we found that such terms have no effect either upon the energy density of anisotropic (or as we now understand them to be: gyroscopic) strings. But they do show up in the formulæ that describe energy *flux* and *momentum* density.

3. Transformational aspects of anisotropy. If we, in the rest frame of the string, see waves to run $\rightarrow$ with speed u, $\leftarrow$ with speed v, then we expect an observer who is himself running $\rightarrow$ with speed c to see

- waves to run $\rightarrow$ with speed $u-c$
- waves to run $\leftarrow$ with speed $v+c$

and to <u>see apparent isotropy</u> in the case $c=\frac{1}{2}(u-v)$, the $\leftrightarrow$ wave speed in that case being

$$w=\tfrac{1}{2}(u+v)$$

Our moving observer has *transformationally eliminated the anisotropy*, which is consonant with the upshot of §3 in the material just cited, where it is established that "the gyroscopic term can always be rotated away."

It is easy to show[6] that under Galilean transformation

$$\begin{aligned}\boldsymbol{t}&=t\\ \boldsymbol{x}&=x+ct\end{aligned}$$

the familiar isotropic wave equation

$$\Big\{\big(\tfrac{\partial}{\partial\boldsymbol{x}}\big)^2-\tfrac{1}{w^2}\big(\tfrac{\partial}{\partial\boldsymbol{t}}\big)^2\Big\}\boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{t})=0$$

goes over into

$$\Big\{\big(\tfrac{\partial}{\partial x}\big)^2+\tfrac{2c}{w^2-c^2}\tfrac{\partial}{\partial x}\tfrac{\partial}{\partial t}-\tfrac{1}{w^2-c^2}\big(\tfrac{\partial}{\partial t}\big)^2\Big\}\varphi(x,t)=0$$

But $w+c=u$, $w-c=v$ and $2c=u-v$ so we recover the equation

$$\varphi_{xx}-\big(\tfrac{1}{u}-\tfrac{1}{v}\big)\varphi_{tx}-\tfrac{1}{uv}\varphi_{tt}=0$$

that appears at the bottom of page 1. From this point of view the Lorentz transformations can be said to have been invented "to kill the gyroscopic term," and thus to turn isotropy into a frame-independent (observer-independent) concept.

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[4] "Lagrangian field theory of anisotropic systems" (1986).

[5] See §§2&3 in Chapter 3 of ADVANCED CLASSICAL MECHANICS (2004).

[6] See "Electrodynamics in 2-dimensional spacetime" (1997), page 20.

4. Running planewaves. At the end of the preceding section a fresh trail led us back again to our starting point. So does this short trail:

Require of the function $e^{i(kx-\omega t)}$ that it be a solution of (2), written

$$uv\varphi_{xx} + (v-u)\varphi_{xt} - \varphi_{tt} = 0 \tag{10}$$

Immediately $-uvk^2 + (v-u)k\omega + \omega^2 = 0$ which gives

$$\omega = \tfrac{1}{2}[(u-v) \pm (u+v)]k = \begin{cases} +ku, & \text{else} \\ -kv \end{cases}$$

So we have right/left-running planewaves

$$e^{ik(x-ut)} \quad \text{and} \quad e^{ik(x+vt)}$$

All waves within each population run with the same phase velocity (u else $-v$), so superpositions

$$f(x-ut) = \int F(k)e^{ik(x-ut)}\,dk$$
$$g(x+vt) = \int G(k)e^{ik(x+vt)}\,dk$$

are non-dispersive.

A simple model of a <u>dispersive anisotropic string</u> results if into (10) we introduce a "Klein-Gordon term," writing

$$uv\varphi_{xx} + (v-u)\varphi_{xt} - \varphi_{tt} - uv\kappa^2\varphi = 0 \tag{10}$$

We then find

$$\omega = \frac{(u-v)k \pm \sqrt{(u+v)^2k^2 + 4uv\kappa^2}}{2} \tag{11}$$

In the case $u = v = c$ the field equation (10) assumes standard Klein-Gordon form

$$\varphi_{xx} - \tfrac{1}{c^2}\varphi_{tt} - \kappa^2\varphi = 0$$

and the dispersion equation (11) becomes

$$\omega/c = \pm\sqrt{k^2 + \kappa^2}$$

We conclude that anisotropy and dispersion are in any event not mutually exclusive. Indeed, we may expect their simultaneous presence to the rule rather than the exception, for—on the general grounds that very little in physics is *exactly* so—we cannot expect material media or even the vacuum to be exactly isotropic, however exquisite may be the approximation, nor can we expect even the photon to be exactly massless.