3

CLASSICAL GAUGE FIELDS

Introduction. The theory of "gauge fields" (sometimes called "compensating fields"[1]) is today universally recognized to constitute one of the supporting pillars of fundamental physics, but it came into the world not with a revolutionary bang but with a sickly whimper, and took a long time to find suitable employment. It sprang from the brow of the youthful Hermann Weyl (1885–1955), who is generally thought of as a mathematician, but for the seminal importance of his contributions to general relativity and quantum mechanics—and, more generally, to the "geometrization of physics"—must be counted among the greatest physicists of the 20th Century. Weyl's initial motivation (1918) was to loosen up the mathematical apparatus of general relativity[2] just enough to find a natural dwelling place for electromagnetism. In 1927 Fritz London suggested that Weyl's idea rested more naturally upon quantum mechanics (then fresh out of the egg!) than upon general relativity, and in 1929 Weyl published a revised elaboration of his original paper—the classic "Elektron und Gravitation" to which I have already referred.[3] The influential Wolfgang Pauli became an ardent champion of the ideas put forward by Weyl, and it was *via* Pauli (whose "Wellenmechanik" article in the *Handbuch der Physik* (1933) had made a profound impression upon him) that those ideas

---

[1] See Section 21 in F. A. Kaempffer's charmingly eccentric *Concepts in Quantum Mechanics* (1965).

[2] Recall that Einstein's theory of gravitation had been completed only in 1915, and that its first observationial support was not forthcoming until 1919.

[3] The text, in English translation, can be found (together with historical commentary) in Lochlainn O'Raufeartaugh's splendid *The Dawning of Gauge Theory* (1997), which should be consulted for a much more balanced account of events than I can present here.

came to the attention of C. N. Yang, in the early 1950's. The attempt by Yang & Mills (1954) to construct a "gauge theory of nuclear forces" failed, for reasons (it became clear in retrospect) having to do with the fact that the nuclear force is too densely phenomenological—too far removed from fundamentals—to admit of any elegantly simple theory. The Yang–Mills theory did serve to bring gauge theory to the general attention of theorists, but several developments had to transpire...

- attention had to shift from the interaction of nucleons to the physics *interior* to nucleons (this development hinged upon the invention of the quark, by Gell-Mann and Zweig in 1964)
- the ideas had to come into place which made possible the development (by Weinberg and Salam in 1967) of a unified theory of electromagnetic and weak interactions[4]

...before it became evident (by the early 1970's) how gauge field theory fit within the Grand Scheme of Things.

The developments to which I have alluded, insofar as they refer to particle physics, are profoundly quantum mechanical. But the associated gauge field theory is, to a remarkable degree, susceptible to description in the language of *classical* field theory, and it is to that language—to the physics of "classical gauge fields"—that I here confine myself; $\hbar$'s will intrude, but they will always be "soft $\hbar$'s," inserted for dimensional reasons but stripped of their quantum mechanical burden.

Basic objective of the theory, as standardly conceived. It is a familiar fact that the physical output of quantum theory is phase insensitive—invariant, that is to say, under

$$\psi \longrightarrow \psi' \equiv e^{i\,\omega}\psi \tag{1}$$

We may attribute this circumstance to the reality of the Schrödinger Lagrangian

$$\mathcal{L} = \tfrac{1}{2}i\hbar(\psi_t^*\psi - \psi^*\psi_t) + \tfrac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^*\!\cdot\!\boldsymbol{\nabla}\psi + \psi^*U\psi \tag{2}$$

from which at (1–91) we extracted the Noetherean conservation law

$$\tfrac{\partial}{\partial t}(\psi^*\psi) + \boldsymbol{\nabla}\!\cdot(\text{probability current}) = 0$$

If we adopt the polar representation $\psi = R\exp\left\{\frac{i}{\hbar}S\right\}$ then becomes[5]

$$\mathcal{L} = R^2\left[S_t + \tfrac{1}{2m}\boldsymbol{\nabla}S\!\cdot\!\boldsymbol{\nabla}S + U\right] + \tfrac{\hbar^2}{2m}\boldsymbol{\nabla}R\!\cdot\!\boldsymbol{\nabla}R$$

which is manifestly invariant under this reformulation of (1):

$$S \longrightarrow S + \text{constant} \tag{3}$$

---

[4] For a good brief account of the developments to which I refer, see Chapter 1 in David Griffiths' *Introduction to Elementary Particles* (1987).

[5] See again (1–78).

Dropping the final term in the preceding Lagrangian, we obtain precisely the Lagrangian

$$\mathcal{L} = R^2 \left[S_t + \tfrac{1}{2m}\boldsymbol{\nabla} S\boldsymbol{\cdot}\boldsymbol{\nabla} S + U\right] \tag{4}$$

which was seen at (1–108) to give rise to classical Hamilton-Jacobi theory; the evident invariance of that theory under (3) can by

$$S = \int L\,dt$$

be attributed to the well-known fact that the physical output from Lagrangian mechanics is insensitive to gauge transformations

$$L \longrightarrow L + \tfrac{d}{dt}(\text{anything})$$

Pauli, in a paper[6] which was influential in the history of thus subject, called

- physics-preserving transformations-by-*multiplicative*-adjustment (such as our (1)) "gauge transformations of the 1st type," and
- physics-preserving transformations-by-*additive*-adjustment (such as (3)) "gauge transformations of the 2nd type."

In gauge field theory the two tend to be joined at the hip, and in casual usage the term "gauge transformation" may refer to either.

Write (1) more carefully

$$\psi(x,t) \longrightarrow \psi'(x,t) \equiv e^{i\,\omega}\psi(x,t) \tag{5}$$

to emphasize the presumed $\{x,t\}$-independence of the phase factor, which we have in mind when we refer to the "global" character of the gauge transformation. The "local" analog of (5) reads

$$\psi(x,t) \longrightarrow \psi'(x,t) \equiv e^{ig\Omega(x,t)}\psi(x,t) \tag{6}$$

—the point being that *the phase factor is allowed now to vary from point to point*. Evidently $\psi^*\psi$ is invariant under (6), but from

$$\partial\psi(x,t) \longrightarrow \partial\psi'(x,t) \equiv e^{ig\Omega(x,t)}\big\{\partial\psi(x,t) + ig[\partial\Omega(x,t)]\,\psi\big\} \tag{7}$$

we see that the adjustment (5)$\longrightarrow$(6) serves to disrupt the invariance of expressions assembled from derivatives (unless $\partial\Omega = 0$, which would take us back to the global theory). Gauge field theory presents a general mechanism for restoring gauge invariance to theories which the adjustment

$$\text{global} \longrightarrow \text{local}$$

[6] "Relativistic theories of elementary particles," Rev. Mod. Phys. **13**, 203 (1941). See the text subsequent to equations (23) in Part I, Section 2.

has served to disrupt. That mechanism, in its most frequently encountered (but, as will emerge, not its simplest) manifestation, can be described as follows:[7]

STEP ONE Make everywhere the substitutional replacement

$$\begin{array}{c}\partial_\mu\\ \downarrow\\ \partial_\mu - igA_\mu\end{array}$$

where $A_\mu(x)$ is a "gauge field" ("compensating field"), endowed with properties soon to be specified. Consider (7) to have, in consequence, become

$$(\partial_\mu - igA_\mu)\psi \longrightarrow (\partial_\mu - igA'_\mu)\psi' = e^{ig\Omega}\big\{(\partial_\mu - igA'_\mu)\psi + ig\tfrac{\partial\Omega}{\partial x^\mu}\psi\big\}$$

and STEP TWO assign to the "local gauge transformation" concept this enlarged meaning

$$\left.\begin{aligned}\psi &\longrightarrow \psi' = e^{ig\Omega}\cdot\psi\\ A_\mu &\longrightarrow A'_\mu = A_\mu + \tfrac{\partial\Omega}{\partial x^\mu}\end{aligned}\right\}\tag{8}$$

so as to achieve

$$\mathcal{D}_\mu\psi \longrightarrow \mathcal{D}'_\mu\psi' = e^{ig\Omega}\cdot\mathcal{D}_\mu\psi\tag{9}$$

$$\mathcal{D}_\mu \equiv \partial_\mu - igA_\mu\tag{10}$$

which mimics the structure of the first of equations (8).

Given interest in a system $\mathcal{L}_0(\varphi,\partial\varphi)$, STEP THREE look to the modified system

$$\mathcal{L}_1(\varphi,\partial\varphi,A) \equiv \mathcal{L}_0(\varphi,\mathcal{D}\varphi)\tag{11}$$

which will be locally gauge invariant if the initial system was globally so.

To see how this works in a particular case, let us look to the relativistic complex scalar field system (2–19)

$$\mathcal{L}_0(\psi,\psi^*,\partial\psi,\partial\psi^*) = \tfrac{\hbar^2}{2m}\big\{g^{\alpha\beta}\psi^*_{,\alpha}\psi_{,\beta} - \varkappa^2\psi^*\psi\big\}\tag{12}$$

where I have set $K = mc^2/\varkappa^2 = \hbar^2/2m$ in order to achieve

$$[\psi^*\psi] = 1/(\text{length})^3$$

The invariance of L_0 under the *global* gauge transformation (1) is manifest, and was shown at (2–21) to entail conservation of the real-valued Noetherean current

$$Q^\mu \equiv \tfrac{\hbar}{m}g^{\mu\alpha}\left\{\frac{\psi^*_{,\alpha}\psi - \psi^*\psi_{,\alpha}}{2i}\right\}\tag{13}$$

[7] Gauge field theory is not intrinsically/essentially relativistic, but it is (like field theory generally) "relativistically predisposed," and I find that it serves expository simplicity to make use here of the notational conventions of relativity.

where the $\hbar/m$ was introduced in order to achieve $[Q] = 1/(\text{area}\cdot\text{time})$. To achieve *local* phase invariance we look to the modified system

$$\begin{aligned}\mathcal{L}_1(\psi,\psi^*,\partial\psi,\partial\psi^*,A) &= \tfrac{\hbar^2}{2m}\big\{g^{\alpha\beta}(\psi^*_{,\alpha}+igA_\alpha\psi^*)(\psi_{,\beta}-igA_\beta\psi)-\varkappa^2\psi^*\psi\big\}\\ &= \mathcal{L}_0(\psi,\psi^*,\partial\psi,\partial\psi^*)+\underbrace{ig\tfrac{\hbar^2}{2m}(\psi^*\psi_{,\alpha}-\psi^*_{,\alpha}\psi)A^\alpha}_{g\hbar Q_\alpha A^\alpha}+g^2\tfrac{\hbar^2}{2m}(\psi^*\psi)A_\alpha A^\alpha\end{aligned}$$

Looking to the equations of motion, we find by calculation that

$$\Big\{\partial_\nu\frac{\partial}{\partial\psi^*_{,\nu}}-\frac{\partial}{\partial\psi^*}\Big\}\mathcal{L}=0\quad\text{becomes}\quad(g^{\alpha\beta}\mathcal{D}_\alpha\mathcal{D}_\beta+\varkappa^2)\psi=0\tag{14.1}$$

$$\Big\{\partial_\nu\frac{\partial}{\partial\psi_{,\nu}}-\frac{\partial}{\partial\psi}\Big\}\mathcal{L}=0\quad\text{gives the conjugated equation}\tag{14.2}$$

Finally—in what is perhaps the most amazingly productive step in the entire procedure—we $\boxed{\text{STEP FOUR}}$ *launch the gauge field into motion* by introducing some
- quadratic
- gauge-invariant, and (in relativistic field theory also)
- Lorentz-invariant

∂A_μ-dependence into the Lagrangian. To that end, we note that

$$F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu\quad\text{is transparently gauge-invariant}\tag{15}$$

and that $F_{\alpha\beta}F^{\alpha\beta}$ answers to our other requirements; we look, therefore, to the twice-modified system

$$\begin{aligned}\mathcal{L}_2(\psi,\psi^*,\partial\psi,\partial\psi^*,A,\partial A) &= \mathcal{L}_0(\psi,\psi^*,\partial\psi,\partial\psi^*)\\ &\quad+\mathcal{L}_{\text{int}}(\psi,\psi^*,\partial\psi,\partial\psi^*,A)+\mathcal{L}(A,\partial A)\end{aligned}\tag{16.0}$$

where the "interaction term"

$$\mathcal{L}_{\text{int}}(\psi,\psi^*,\partial\psi,\partial\psi^*,A)=ig\tfrac{\hbar^2}{2m}(\psi^*\psi_{,\alpha}-\psi^*_{,\alpha}\psi)A^\alpha+g^2\tfrac{\hbar^2}{2m}(\psi^*\psi)A_\alpha A^\alpha\tag{16.1}$$

was developed already at the top of the page, and where the "free gauge field" will be governed by

$$\begin{aligned}\mathcal{L}(A,\partial A) &= \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\\ &= \tfrac{1}{4}g^{\alpha\rho}g^{\beta\sigma}(A_{\beta,\alpha}-A_{\alpha,\beta})(A_{\sigma,\rho}-A_{\rho,\sigma})\\ &= \tfrac{1}{2}(g^{\alpha\rho}g^{\beta\sigma}-g^{\alpha\sigma}g^{\beta\rho})A_{\alpha,\beta}A_{\rho,\sigma}\end{aligned}\tag{16.2}$$

Notice also that $\mathcal{L}(A,\partial A)$ is in fact A_μ-*independent*; we have been forced to omit an anticipated $(\varkappa^2 A_\alpha A^\alpha)$-term for the simple but deeply consequential reason that

$$A_\alpha A^\alpha\quad\text{is not gauge-invariant}$$

Notice also that (16.2) requires $[F_{\alpha\beta}F^{\alpha\beta}]$ = (energy density), which entails $[A_\mu] = \sqrt{\text{energy/length}}$. And this—if we are to achieve $[gA_\mu] = 1/(\text{length})$, as required by the definition of $\mathcal{D}_\mu$—in turn entails

$$[g^2] = 1/(\text{energy}\cdot\text{length}) = [1/\hbar c] = 1/(\text{electric charge})^2$$

giving $[g\hbar c]$ = (electric charge). It is on this basis that we will adopt $e \equiv g\hbar c$ as a suggestive notational device.

Working now from the twice-modified Lagrangian (16), we in place of (14.2) obtain

$$\begin{aligned}\partial_\mu F^{\mu\nu} &= \frac{\partial}{\partial A_\mu}\mathcal{L}_{\text{int}}(\psi,\psi^*,\partial\psi,\partial\psi^*,A)\\ &= \tfrac{1}{c}J^\nu \end{aligned}\tag{17}$$

with

$$\begin{aligned} J^\nu &\equiv g\hbar c\big\{Q^\nu + g\tfrac{\hbar}{m}(\psi^*\psi)A^\nu\big\}\\ &= j^\nu + \tfrac{e^2}{mc}\psi^*A^\nu\psi \end{aligned}\tag{18.1}$$

$$j^\nu \equiv eQ^\nu = J\Big|_{A\to 0}\tag{18.2}$$

The gauge-invariance of J^ν—required for the self-consistency of (17)—is not obvious (certainly not "manifest"), but is readily established.

The field equation (14.1) can be written

$$g^{\alpha\beta}\big(\tfrac{\hbar}{i}\partial_\alpha - \tfrac{e}{c}A_\alpha\big)\big(\tfrac{\hbar}{i}\partial_\beta - \tfrac{e}{c}A_\beta\big)\psi = (mc)^2\psi\tag{19}$$

and in this form can be considered to have resulted by ordinary Schrödinger quantization from a classical process of the form

$$\left.\begin{aligned} g^{\alpha\beta}p_\alpha p_\beta &= (mc)^2\\ &\downarrow\\ g^{\alpha\beta}(p_\alpha - \tfrac{e}{c}A_\alpha)(p_\beta - \tfrac{e}{c}A_\beta) &= (mc)^2 \end{aligned}\right\}\tag{20}$$

Note also that

$$[j^\nu] = [J^\nu] = \frac{\text{electrical charge}}{\text{area}\cdot\text{time}} = \text{electrical current density}$$

and that we are now in position to write

$$\begin{aligned}\mathcal{L}_{\text{int}} &= \tfrac{1}{c}j^\alpha A_\alpha + \underbrace{\tfrac{e^2}{2mc^2}(\psi^*A^\alpha\psi)}_{=\frac{1}{2}(J^\alpha - j^\alpha)}A_\alpha\\ &= \tfrac{1}{c}\tfrac{1}{2}(J^\alpha + j^\alpha)A_\alpha\end{aligned}\tag{21}$$

Assuredly,

$$\text{local gauge invariance} \quad\Longrightarrow\quad \text{global gauge invariance}$$

and from the manifest invariance of the twice-modified Lagrangian (16) under the global instance

$$\begin{aligned} \psi &\longrightarrow \psi' = e^{+i\,\omega}\cdot\psi \\ A_\mu &\longrightarrow A'_\mu = A_\mu \end{aligned}$$

of (8)—which infinitesimally becomes

$$\begin{aligned} \psi \;\; &\longrightarrow \psi \;\; + \delta\psi \quad &&\text{with} \quad \delta\psi \;\; = +i\psi\cdot\delta\omega \\ \psi^* &\longrightarrow \psi^* + \delta\psi^* \quad &&\text{with} \quad \delta\psi^* = -i\psi\cdot\delta\omega \\ A_\mu &\longrightarrow A_\mu + \delta A_\mu \quad &&\text{with} \quad \delta A_\mu = 0 \end{aligned}$$

—we are, by Noether's theorem, led to the conservation of

$$\begin{aligned} -(e/\hbar)\Big\{\frac{\partial\mathcal{L}}{\partial\psi_{,\nu}}(+i\psi)+\frac{\partial\mathcal{L}}{\partial\psi^*_{,\nu}}(-i\psi^*)\Big\} &= j^\nu-(e/\hbar)\Big\{\frac{\partial\mathcal{L}_{\text{int}}}{\partial\psi_{,\nu}}(i\psi)-\frac{\partial\mathcal{L}_{\text{int}}}{\partial\psi^*_{,\nu}}(i\psi^*)\Big\} \\ &= j^\nu+\tfrac{e^2}{mc}\psi^*A^\nu\psi \\ &= J^\nu \end{aligned}$$

(i.e., to $\partial_\mu J^\mu = 0$) which at (17) was presented as an automatic consequence of the antisymmetry of $F^{\mu\nu}$. Prior to introduction of the gauge field A_μ we had $\partial_\mu j^\mu = 0$. We can in this light understand the adjustment

$$j^\nu(\psi,\psi^*,\partial\psi,\partial\psi^*) \quad\longrightarrow\quad J^\nu(\psi,\psi^*,\partial\psi,\partial\psi^*,A)$$

as a price paid in our effort

$$\mathcal{L}_0(\psi,\psi^*,\partial\psi,\partial\psi^*) \quad\longrightarrow\quad \mathcal{L}_2(\psi,\psi^*,\partial\psi,\partial\psi^*,A,\partial A)$$

to achieve local gauge invariance.

The effort to which I have just referred has yield up (amongst others) the equations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{and} \quad \partial_\mu F^{\mu\nu} = \tfrac{1}{c}J^\nu$$

It has, in short, delivered Maxwellian electrodynamics to us on a platter. If we were to "turn off" the ψ-field (or set the coupling constant $e = 0$) we would retain free-field electrodynamics as a kind of residue—a gift, for having shopped in the Gauge Store. The "compensating field" has been found in this instance to lead a busy physical life of its own, even when it has nothing to compensate. It was this development which first suggested that gauge field theory might, in fact, be good for something.

We have been supplied, moreover, with a detailed account of how the electromagnetic field $F^{\mu\nu}$ and the ψ-field are coupled—a "theory of field interactions." It was, in fact, a quest for a general theory of field interactions which led Ronald Shaw—in 1953/54 a graduate student of Abdus Salam at Cambridge, working under the influence of Schwinger to the invention of gauge field theory, independently of (and almost simultaneously with) Yang & Mills.[8]

The electromagnetic aspects of the theory to which we have been led do, however, present one problematic (or at least surprising) aspect: the current term J^ν which "stimulates" the electromagnetic field was found at (18) to *itself depend upon* the field (through the 4-potential A_μ). As we move farther into our subject we will remain on the alert for developments which may serve to clarify that circumstance.

Gauge theory of a non-relativistic classical particle. Gauge field theory was born of general/special relativistic parents, and has spent its adult life married to quantum mechanics. It may be well, therefore, to be reminded that the central idea is so robust that it can flourish even when deprived of either or both of those controlling influences. To illustrate the point, I look to the classical Hamilton-Jacobi theory of a non-relativistic particle:

Assume the Hamiltonian to have the form $H(\boldsymbol{p}, \boldsymbol{x}) = \frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} + U(\boldsymbol{x})$. The Hamilton-Jacobi equation then reads

$$\tfrac{1}{2m}\boldsymbol{\nabla}S\cdot\boldsymbol{\nabla}S + U(\boldsymbol{x}) + S_t = 0 \tag{22.1}$$

and[9] when joined by its companion

$$R_t + \boldsymbol{\nabla}\cdot(\tfrac{1}{m}R\boldsymbol{\nabla}S) = 0 \tag{22.2}$$

can be consider to derive from the Lagrangian

$$\mathcal{L}_0(S, \partial S, R) = R\cdot\Big\{\tfrac{1}{2m}\big[(\partial_x S)^2 + (\partial_y S)^2 + (\partial_z S)^2\big] + U + (\partial_t S)\Big\} \tag{23}$$

This Lagrangian is manifestly invariant under the global gauge transformations

$$\left.\begin{aligned} S &\longrightarrow S' = S + \hbar\omega \\ R &\longrightarrow R' = R \end{aligned}\right\} \tag{24}$$

which—compare (5)—are, in Pauli's terminology, "gauge transformations of the 2$^{\text{nd}}$ kind," into which $\hbar$ has been introduced as a "soft constant of the action" in order to preserve the dimensionlessness of ω. To achieve the

$$\text{global gauge} \quad\Longrightarrow\quad \text{local gauge}$$

[8] Relevant sections of Shaw's thesis (1955) are reprinted in O'Raifeartaigh.[3]

[9] See again the discussion which culminated in (1–108).

symmetry enhancement we adjust the design of the system

$$
\begin{aligned}
&\mathcal{L}_0(S,\partial S,R)\\
&\qquad\downarrow\\
&\mathcal{L}_0(S,\mathcal{D}S,R) = R\cdot\Big\{\tfrac{1}{2m}\big[(\partial_x S+g\hbar C_x)^2+(\partial_y S+g\hbar C_y)^2+(\partial_z S+g\hbar C_z)^2\big]\\
&\qquad\qquad\qquad\qquad\qquad + U + (\partial_t S + g\hbar C_t)\Big\}\\
&\qquad\qquad\equiv \mathcal{L}_1(S,\partial S,R,C_x,C_y,C_z,C_t)
\end{aligned}
\tag{25}
$$

and—so as to achieve

$$\partial_x S' + g\hbar C'_x = \partial_x S + g\hbar C_x \ \ ,\ \text{etc.}$$

—assign to the notion of a "local gauge transformation" an enlarged meaning

$$
\left.
\begin{aligned}
S &\longrightarrow S' = S + g\hbar\Omega(\boldsymbol{x},t)\\
R &\longrightarrow R' = R\\
C_x &\longrightarrow C'_x = C_x - \partial_x\Omega(\boldsymbol{x},t)\\
C_y &\longrightarrow C'_y = C_y - \partial_y\Omega(\boldsymbol{x},t)\\
C_z &\longrightarrow C'_z = C_z - \partial_y\Omega(\boldsymbol{x},t)\\
C_t &\longrightarrow C'_t = C_t - \partial_t\,\Omega(\boldsymbol{x},t)
\end{aligned}
\right\}
\tag{26}
$$

which in the global case $g\Omega(\boldsymbol{x},t)=\omega$ gives back essentially (24).

I turn now to remarks intended to help us mix some gauge-invariant ∂C-dependence into (25), and thus to lauch the gauge fields C into dynamical motion. We proceed from the observation that the expressions[10]

$$
\begin{array}{ccc}
\frac{1}{c}(\partial_t C_x - \partial_x C_t) & \frac{1}{c}(\partial_t C_y - \partial_y C_t) & \frac{1}{c}(\partial_t C_z - \partial_z C_t)\\
 & (\partial_x C_y - \partial_y C_x) & (\partial_x C_z - \partial_z C_x)\\
 & & (\partial_y C_z - \partial_z C_y)
\end{array}
$$

are individually gauge-invariant (because the cross partials of Ω are equal). To keep our theory from coming *rotationally* unstuck, we must require that

$$\begin{pmatrix} C_x\\ C_y\\ C_z\end{pmatrix}\ \text{transforms like } \boldsymbol{\nabla} S\text{; i.e., as a vector}$$

Let us agree to write

$$\begin{pmatrix} C_x\\ C_y\\ C_z\end{pmatrix}\equiv -\boldsymbol{A}\quad\text{and}\quad C_t\equiv c\phi$$

[10] Some "soft c" factors have been introduced so as to render all entries co-dimensional.

where a sign has been introduced to establish contact with pre-established convention. The preceding tableau then becomes

$$\begin{array}{ccc} (-\frac{1}{c}\partial_t\boldsymbol{A}-\boldsymbol{\nabla}\phi)_x & (-\frac{1}{c}\partial_t\boldsymbol{A}-\boldsymbol{\nabla}\phi)_y & (-\frac{1}{c}\partial_t\boldsymbol{A}-\boldsymbol{\nabla}\phi)_z \\ & +(\boldsymbol{\nabla}\times\boldsymbol{A})_z & -(\boldsymbol{\nabla}\times\boldsymbol{A})_y \\ & & +(\boldsymbol{\nabla}\times\boldsymbol{A})_x \end{array}$$

which we will agree to abbreviate

$$\begin{array}{ccc} (\boldsymbol{E})_x & (\boldsymbol{E})_y & (\boldsymbol{E})_z \\ & -(\boldsymbol{B})_z & +(\boldsymbol{B})_y \\ & & -(\boldsymbol{B})_x \end{array}$$

The expressions $\boldsymbol{E}\cdot\boldsymbol{E}$, $\boldsymbol{E}\cdot\boldsymbol{B}$ and $\boldsymbol{B}\cdot\boldsymbol{B}$ are
- quadratic in ∂C
- gauge-invariant, and
- rotationally invariant

and candidates, therefore, for independent inclusion into the design of a modified Lagrangian. Our most recent Lagrangian (25) can, in present notation, be written[11]

$$\mathcal{L}_1 = R\cdot\Big\{\tfrac{1}{2m}\big(\boldsymbol{\nabla}S-\tfrac{e}{c}\boldsymbol{A}\big)\cdot\big(\boldsymbol{\nabla}S-\tfrac{e}{c}\boldsymbol{A}\big)+U+(\partial_tS+e\phi)\Big\} \tag{27}$$

and we are led by the preceding remarks to consider Lagrangians of the modified form

$$\begin{aligned}\mathcal{L}_2(S,\partial S,R,\boldsymbol{A},\phi,\partial\boldsymbol{A},\partial\phi) &= \mathcal{L}_1+\tfrac{1}{2}p\boldsymbol{E}\cdot\boldsymbol{E}+q\boldsymbol{E}\cdot\boldsymbol{B}+\tfrac{1}{2}r\boldsymbol{B}\cdot\boldsymbol{B} \qquad (28)\\ &= \mathcal{L}_1+\tfrac{1}{2}p\,(\tfrac{1}{c}\partial_t\boldsymbol{A}+\boldsymbol{\nabla}\phi)\cdot(\tfrac{1}{c}\partial_t\boldsymbol{A}+\boldsymbol{\nabla}\phi)\\ &\qquad -\;\; q\,(\tfrac{1}{c}\partial_t\boldsymbol{A}+\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})\\ &\qquad +\tfrac{1}{2}r\,(\boldsymbol{\nabla}\times\boldsymbol{A})\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})\end{aligned}$$

Look to the associated field equations and conservation laws. From

$$\Big\{\partial_t\frac{\partial}{\partial R_t}+\boldsymbol{\nabla}\cdot\frac{\partial}{\partial\boldsymbol{\nabla}R}-\frac{\partial}{\partial R}\Big\}\mathcal{L}_2=0$$

we obtain[12]

$$\tfrac{1}{2m}\big(\boldsymbol{\nabla}S-\tfrac{e}{c}\boldsymbol{A}\big)\cdot\big(\boldsymbol{\nabla}S-\tfrac{e}{c}\boldsymbol{A}\big)+U+(\partial_tS+e\phi)=0 \tag{28.1}$$

---

[11] One needs to notice that the compensating fields C have—by contrivance—the same physical dimension as the gauge fields formerly notated A_μ (to which they are really identical), and that consequently it still makes sense to write $g\hbar = e/c$.

[12] Compare this with the $H=\frac{1}{2m}(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})+e\phi$ which appears, for example, as (8–27) in Goldstein's *Classical Mechanics* (2nd edition 1980). There it arises from $L=\frac{1}{2}m\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}+\frac{e}{c}\dot{\boldsymbol{x}}\cdot\boldsymbol{A}-e\phi$, which hinges on the observation that the Lorentz force law $\boldsymbol{F}=e(\boldsymbol{E}+\frac{1}{c}\dot{\boldsymbol{x}}\times\boldsymbol{B})=e\big\{-\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}+\frac{1}{c}\dot{\boldsymbol{x}}\times\boldsymbol{\nabla}\times\boldsymbol{A}\big\}$ can be obtained by Lagrange differentiation of $U=e\big(\phi-\frac{1}{c}\dot{\boldsymbol{x}}\cdot\boldsymbol{A}\big)$. See Goldstein's §1–5 for details.

while

$$\Big\{\partial_t \frac{\partial}{\partial S_t} + \boldsymbol{\nabla}\!\cdot \frac{\partial}{\partial \boldsymbol{\nabla} S} - \frac{\partial}{\partial S}\Big\}\mathcal{L}_2 = 0$$

gives

$$\partial_t R + \boldsymbol{\nabla}\!\cdot \tfrac{1}{m}R\big(\boldsymbol{\nabla}S - \tfrac{e}{c}\boldsymbol{A}\big) = 0 \tag{28.2}$$

Interestingly, this last field equation displays the design of a continuity equation, and is in fact *precisely conservation law which by Noether's theorem arises as an expression of the global gauge invariance* of the locally gauge-invariant Lagrangian (27). A simple dimensional argument gives $[R] = 1/(\text{length})^3$, so if we introduce the notations

$$\left.\begin{array}{lcl} \rho \equiv eR & : & \text{charge density} \\ \boldsymbol{J} \equiv \frac{e}{m}R\big(\boldsymbol{\nabla}S - \frac{e}{c}\boldsymbol{A}\big) & : & \text{current density} \end{array}\right\} \tag{29}$$

then (28.2) can be read as a statement

$$\partial_t \rho + \boldsymbol{\nabla}\!\cdot\boldsymbol{J} = 0 \tag{30}$$

of charge conservation. Looking finally to the dynamics of the gauge fields: from

$$\Big\{\partial_t \frac{\partial}{\partial \phi_t} + \boldsymbol{\nabla}\!\cdot \frac{\partial}{\partial \boldsymbol{\nabla} \phi} - \frac{\partial}{\partial \phi}\Big\}\mathcal{L}_2 = 0$$

we obtain $\boldsymbol{\nabla}\!\cdot\big\{p\,(\frac{1}{c}\partial_t\boldsymbol{A} + \boldsymbol{\nabla}\phi) - q\boldsymbol{\nabla}\times\boldsymbol{A}\big\} - eR = 0$ which (recall that div curl always vanishes) can be expressed $-p\boldsymbol{\nabla}\!\cdot\boldsymbol{E} = \rho$, while

$$\Big\{\partial_t \frac{\partial}{\partial A_{x,t}} + \partial_x \frac{\partial}{\partial A_{x,x}} + \partial_y \frac{\partial}{\partial A_{x,y}} + \partial_z \frac{\partial}{\partial A_{x,z}} - \frac{\partial}{\partial A_x}\Big\}\mathcal{L}_2 = 0$$

is found after simplifications to yield the x-component of

$$p\tfrac{1}{c}\partial_t\boldsymbol{E} + q\big(\tfrac{1}{c}\partial_t\boldsymbol{B} + \boldsymbol{\nabla}\times\boldsymbol{E}\big) + r\boldsymbol{\nabla}\times\boldsymbol{B} = \tfrac{1}{c}\boldsymbol{J}$$

But from the standing definitions

$$\boldsymbol{E} \equiv -\tfrac{1}{c}\partial_t\boldsymbol{A} - \boldsymbol{\nabla}\phi \quad \text{and} \quad \boldsymbol{B} \equiv \boldsymbol{\nabla}\times\boldsymbol{A} \tag{31}$$

it follows automatically that

$$\left.\begin{array}{r} \boldsymbol{\nabla}\!\cdot\boldsymbol{B} = 0 \\ \frac{1}{c}\partial_t\boldsymbol{B} + \boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0} \end{array}\right\} \tag{32.1}$$

and from $\mathcal{L}_2$ we have obtained this additional information:

$$\left.\begin{array}{r} -p\boldsymbol{\nabla}\!\cdot\boldsymbol{E} = \rho \\ p\frac{1}{c}\partial_t\boldsymbol{E} + r\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{1}{c}\boldsymbol{J} \end{array}\right\} \tag{32.2}$$

Note that (32.2) renders $\partial_t\rho + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{J} = 0$ automatic in all cases (as it must, since the continuity equation derives from built-in global gauge invariance), and that equations (32.2)

become precisely the sourcey Maxwell equations in the case $r = -p = 1$

But what heretofore neglected physical principle serves to enforce such conditions? An answer emerges from study of the energy/momentum/angular momentum properties of the gauged Hamilton-Jacobi field system.

By way of preparation, we look first to those properties as they refer to the ungauged system $\mathcal{L}_0$ which provided our point of departure. A little exploratory tinkering motivates these definitions:

$$\begin{aligned}\mathcal{E} &\equiv -\Big[\Big\{R_t\frac{\partial}{\partial R_t} + S_t\frac{\partial}{\partial S_t}\Big\}\mathcal{L}_0 - \mathcal{L}_0\Big] &(33.10)\\ &= R\cdot\Big\{\tfrac{1}{2m}\big[(\partial_x S)^2 + (\partial_y S)^2 + (\partial_z S)^2\big] + U\Big\}\\ &= R\cdot H(\boldsymbol{\nabla}S,\boldsymbol{x}) &(33.11)\end{aligned}$$

$$\begin{aligned}\mathcal{F}^x &\equiv -\Big[\Big\{R_t\frac{\partial}{\partial R_x} + S_t\frac{\partial}{\partial S_x}\Big\}\mathcal{L}_0\Big] &(33.20)\\ &= -\big[R\cdot\tfrac{1}{m}(\partial_x S)\big]S_t, \quad \text{with } \mathcal{F}^y \text{ and } \mathcal{F}^z \text{ described similarly}\\ &= +\big[R\cdot\tfrac{1}{m}(\partial_x S)\big]\cdot H(\boldsymbol{\nabla}S,\boldsymbol{x}) \quad \text{by the Hamilton-Jacobi equation}\\ &\downarrow\\ \boldsymbol{\mathcal{F}} &= \big[\tfrac{1}{m}R\,\boldsymbol{\nabla}S\big]\cdot H(\boldsymbol{\nabla}S,\boldsymbol{x}) &(33.21)\end{aligned}$$

$$\begin{aligned}\mathcal{P}_x &\equiv +\Big[\Big\{R_x\frac{\partial}{\partial R_t} + S_x\frac{\partial}{\partial S_t}\Big\}\mathcal{L}_0\Big] &(33.30)\\ &= R\cdot(\partial_x S), \quad \text{with } \mathcal{P}_y \text{ and } \mathcal{P}_z \text{ described similarly}\\ &\downarrow\\ \boldsymbol{\mathcal{P}} &= R\,\boldsymbol{\nabla}S &(33.31)\end{aligned}$$

$$\begin{aligned}\mathcal{T}^u{}_v &\equiv +\Big[\Big\{R_v\frac{\partial}{\partial R_u} + S_v\frac{\partial}{\partial S_u}\Big\}\mathcal{L}_0 - \delta^u{}_v\mathcal{L}_0\Big] \;:\; \{u,v\}\in\{x,y,z\} &(33.40)\\ &= R\cdot\big[\tfrac{1}{m}S_uS_v - H\delta^u{}_v\big] &(33.41)\end{aligned}$$

It is not difficult to establish that

$$\begin{aligned}\tfrac{\partial}{\partial t}\mathcal{E} + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{\mathcal{F}} &= +R\tfrac{\partial}{\partial t}U &(34.1)\\ &= 0 \quad \text{if } U \text{ is } t\text{-independent}\end{aligned}$$

$$\begin{aligned}\tfrac{\partial}{\partial t}\mathcal{P}_x + \tfrac{\partial}{\partial x}\mathcal{T}^x{}_x + \tfrac{\partial}{\partial y}\mathcal{T}^y{}_x + \tfrac{\partial}{\partial z}\mathcal{T}^z{}_x &= -R\tfrac{\partial}{\partial x}U \;:\; \text{ditto with } {}_x \to {}_{y,z} &(34.2)\\ &= 0 \quad \text{if } U \text{ is } x\text{-independent}\end{aligned}$$

and from the manifest *symmetry* of $\mathcal{T}^u{}_v$ it follows that angular momentum is locally conserved at points where the "torque density" $\boldsymbol{x}\times(-\boldsymbol{\nabla}U)$ vanishes. These are physically satisfying results, but my main point has been to identify the contrasting *signs* which enter most sensibly into the preceding definitions.

Preserving those sign conventions, we look now to the energy/momentum desnities/fluxes which arise from

$$\begin{aligned}\mathcal{L}_{\text{free gauge field}} &= \tfrac{1}{2}p\,(\tfrac{1}{c}\partial_t\boldsymbol{A}+\boldsymbol{\nabla}\phi)\boldsymbol{\cdot}(\tfrac{1}{c}\partial_t\boldsymbol{A}+\boldsymbol{\nabla}\phi)+\tfrac{1}{2}r\,(\boldsymbol{\nabla}\times\boldsymbol{A})\boldsymbol{\cdot}(\boldsymbol{\nabla}\times\boldsymbol{A})\\ &= \tfrac{1}{2}p\,\boldsymbol{E}\boldsymbol{\cdot}\boldsymbol{E}+\tfrac{1}{2}r\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{B}\end{aligned}\tag{35}$$

where the originally conjectured q-term, since it made no contribution to the field equations (32.2), has been dropped, and where it is for present computational purposes most efficient to write

$$\mathcal{L}_{\text{free gauge field}} = -p\boldsymbol{E}\boldsymbol{\cdot}\begin{pmatrix}\frac{1}{c}A_{x,t}+\phi_{,x}\\ \frac{1}{c}A_{y,t}+\phi_{,y}\\ \frac{1}{c}A_{z,t}+\phi_{,z}\end{pmatrix}+r\boldsymbol{B}\boldsymbol{\cdot}\begin{pmatrix}A_{z,y}-A_{y,z}\\ A_{x,z}-A_{z,x}\\ A_{y,x}-A_{x,y}\end{pmatrix}$$

Looking first to the *energy density* of the free gauge field system, we find

$$\begin{aligned}\tilde{\mathcal{E}} &\equiv -\Big[\Big\{\phi_{,t}\frac{\partial}{\partial\phi_{,t}}+A_{x,t}\frac{\partial}{\partial A_{x,t}}+A_{y,t}\frac{\partial}{\partial A_{y,t}}+A_{z,t}\frac{\partial}{\partial A_{z,t}}\Big\}-1\Big]\mathcal{L}_{\text{gauge}}\\ &= -\big[-p\tfrac{1}{c}\boldsymbol{A}_t\boldsymbol{\cdot}\boldsymbol{E}-\tfrac{1}{2}p\,\boldsymbol{E}\boldsymbol{\cdot}\boldsymbol{E}-\tfrac{1}{2}r\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{B}\big]\\ &= \underbrace{-\tfrac{1}{2}p\,\boldsymbol{E}\boldsymbol{\cdot}\boldsymbol{E}+\tfrac{1}{2}r\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{B}}_{\mathcal{E}_{\text{gauge}}}-\underbrace{p\,\boldsymbol{E}\boldsymbol{\cdot}\boldsymbol{\nabla}\phi}_{\text{gauge-dependent term, soon discarded}}\quad\text{by}\quad -\tfrac{1}{c}\partial_t\boldsymbol{A}=\boldsymbol{E}+\boldsymbol{\nabla}\phi\end{aligned}\tag{36.1}$$

and notice that $\mathcal{E}\geqslant 0$ requires $p<0$ and $r>0$. Without loss of generality (since the numerical part of p can be absorbed into the definition of e) we

$$\text{Set } p=-1$$

whereupon (32.2) become

$$\left.\begin{aligned}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{E}&=\rho\\ r\boldsymbol{\nabla}\times\boldsymbol{B}&=\tfrac{1}{c}\big\{\boldsymbol{J}+\tfrac{1}{c}\partial_t\boldsymbol{E}\big\}\end{aligned}\right\}\tag{32.2$'$}$$

For the *components of energy flux* we have

$$\begin{aligned}\tilde{\mathcal{F}}^x &\equiv -\Big[\phi_{,t}\frac{\partial}{\partial\phi_{,x}}+A_{x,t}\frac{\partial}{\partial A_{x,x}}+A_{y,t}\frac{\partial}{\partial A_{y,x}}+A_{z,t}\frac{\partial}{\partial A_{z,x}}\Big]\mathcal{L}_{\text{gauge}}\\ &= -\big[\phi_t E_x+rA_{y,t}B_z-rA_{z,t}B_y\big],\quad\text{with } \tilde{\mathcal{F}}^y \text{ and } \tilde{\mathcal{F}}^z \text{ described similarly}\\ &\downarrow\\ \tilde{\boldsymbol{\mathcal{F}}} &= -\big[r(\partial_t\boldsymbol{A})\times\boldsymbol{B}\;+\;\boldsymbol{E}\,\partial_t\phi\big]\\ &= \underbrace{rc(\boldsymbol{E}\times\boldsymbol{B})}_{\boldsymbol{\mathcal{F}}_{\text{gauge}}}+\underbrace{\big\{rc\boldsymbol{\nabla}\phi\times\boldsymbol{B}-\boldsymbol{E}\,\partial_t\phi\big\}}_{\text{gauge-dependent term, soon discarded}}\end{aligned}\tag{36.2}$$

The "gauge-dependent terms" which enter additively into equations (36) cannot participate in the physical output of the theory, but from results already in hand it follows readily that

$$\begin{aligned}\tfrac{\partial}{\partial t}(\boldsymbol{E}\cdot\boldsymbol{\nabla}\phi)+\boldsymbol{\nabla}\cdot\big\{rc\boldsymbol{\nabla}\phi\times\boldsymbol{B}-\boldsymbol{E}\,\partial_t\phi\big\}&=-\big\{\boldsymbol{J}\cdot\boldsymbol{\nabla}+\rho\,\partial_t\big\}\phi \qquad (37)\\ &=0\quad\text{in the absence of sources}\end{aligned}$$

so those terms can be discarded on grounds that they make no contribution to the *total* energy resident in the gauge field system. Notice that r remains still indeterminate.

Looking next to the *components of momentum density* in the free gauge field system, we have

$$\begin{aligned}\tilde{\mathcal{P}}_x&\equiv\Big[\phi_{,x}\frac{\partial}{\partial\phi_{,t}}+A_{x,x}\frac{\partial}{\partial A_{x,t}}+A_{y,x}\frac{\partial}{\partial A_{y,t}}+A_{z,x}\frac{\partial}{\partial A_{z,t}}\Big]\mathcal{L}_{\text{gauge}}\\ &=\tfrac{1}{c}\big[\boldsymbol{E}\cdot(\partial_x\boldsymbol{A})\big],\quad\text{with }\tilde{\mathcal{P}}_y\text{ and }\tilde{\mathcal{P}}_z\text{ described similarly}\end{aligned}$$

which (by a seldom-encountered but easily established identity) yields

$$\begin{aligned}\tilde{\boldsymbol{\mathcal{P}}}=\tfrac{1}{c}\begin{pmatrix}\boldsymbol{E}\cdot(\partial_x\boldsymbol{A})\\ \boldsymbol{E}\cdot(\partial_y\boldsymbol{A})\\ \boldsymbol{E}\cdot(\partial_z\boldsymbol{A})\end{pmatrix}&=\tfrac{1}{c}\boldsymbol{E}\times(\boldsymbol{\nabla}\times\boldsymbol{A})+\tfrac{1}{c}(\boldsymbol{E}\cdot\boldsymbol{\nabla})\boldsymbol{A}\\ &=\underbrace{\tfrac{1}{c}\boldsymbol{E}\times\boldsymbol{B}}_{\boldsymbol{\mathcal{P}}_{\text{gauge}}}+\text{gauge-dependent term}\end{aligned}\tag{38.1}$$

Looking finally to the *components of the* 3×3 *stress tensor* (i.e., of momentum flux), we use

$$\tilde{\mathcal{T}}^u{}_v=\Big[\phi_{,v}\frac{\partial}{\partial\phi_{,u}}+A_{x,v}\frac{\partial}{\partial A_{x,u}}+A_{y,v}\frac{\partial}{\partial A_{y,u}}+A_{z,v}\frac{\partial}{\partial A_{z,u}}-\delta^u{}_v\Big]\mathcal{L}_{\text{gauge}}$$

and writing

$$\|\tilde{\mathcal{T}}^u{}_v\|=\left(\begin{pmatrix}\tilde{\mathcal{T}}^x{}_x\\ \tilde{\mathcal{T}}^y{}_x\\ \tilde{\mathcal{T}}^z{}_x\end{pmatrix}\quad\begin{pmatrix}\tilde{\mathcal{T}}^x{}_y\\ \tilde{\mathcal{T}}^y{}_y\\ \tilde{\mathcal{T}}^z{}_y\end{pmatrix}\quad\begin{pmatrix}\tilde{\mathcal{T}}^x{}_z\\ \tilde{\mathcal{T}}^y{}_z\\ \tilde{\mathcal{T}}^z{}_z\end{pmatrix}\right)$$

compute

$$\begin{pmatrix}\tilde{\mathcal{T}}^x{}_x\\ \tilde{\mathcal{T}}^y{}_x\\ \tilde{\mathcal{T}}^z{}_x\end{pmatrix}=\boldsymbol{E}\,(\partial_x\phi)+r(\partial_x\boldsymbol{A})\times\boldsymbol{B}+\big\{\tfrac{1}{2}\,\boldsymbol{E}\cdot\boldsymbol{E}-\tfrac{1}{2}r\boldsymbol{B}\cdot\boldsymbol{B}\big\}\begin{pmatrix}1\\0\\0\end{pmatrix},\text{etc.}$$

which (by a seldom-encountered and not-so-easily established population of identities—see below) yields

$$
\begin{aligned}
&= -\boldsymbol{E}E_x - r\boldsymbol{B}B_x + \big\{\tfrac{1}{2}\,\boldsymbol{E}\cdot\boldsymbol{E} + \tfrac{1}{2}r\boldsymbol{B}\cdot\boldsymbol{B}\big\}\begin{pmatrix}1\\0\\0\end{pmatrix}\\
&\qquad\qquad - \tfrac{1}{c}\boldsymbol{E}(\partial_t A_x) - r(\boldsymbol{B}\times\boldsymbol{\nabla})A_x\\
&= \begin{pmatrix}\mathcal{T}^x{}_x\\ \mathcal{T}^y{}_x\\ \mathcal{T}^z{}_x\end{pmatrix}_{\text{gauge}} + \underbrace{\big\{-\tfrac{1}{c}\boldsymbol{E}(\partial_t A_x) - r(\boldsymbol{B}\times\boldsymbol{\nabla})A_x\big\}}_{\text{gauge-dependent term}}
\end{aligned}
\tag{38.2}
$$

with

$$
\begin{pmatrix}\mathcal{T}^x{}_x\\ \mathcal{T}^y{}_x\\ \mathcal{T}^z{}_x\end{pmatrix}_{\text{gauge}} \equiv \begin{pmatrix}-E_xE_x - rB_xB_x + \big\{\tfrac{1}{2}\,\boldsymbol{E}\cdot\boldsymbol{E} + \tfrac{1}{2}r\boldsymbol{B}\cdot\boldsymbol{B}\big\}\\ -E_yE_x - rB_yB_x\\ -E_zE_x - rB_zB_x\end{pmatrix}
$$

But—looking now more closely to the gauge-dependent terms—we find

$$
\begin{aligned}
&\tfrac{\partial}{\partial t}\big\{\tfrac{1}{c}(\boldsymbol{E}\cdot\boldsymbol{\nabla})A_x\big\} + \boldsymbol{\nabla}\cdot\big\{-\tfrac{1}{c}\boldsymbol{E}(\partial_t A_x) - r(\boldsymbol{B}\times\boldsymbol{\nabla})A_x\big\}\\
&\quad= \big\{-\tfrac{1}{c}\boldsymbol{j} + r\boldsymbol{\nabla}\times\boldsymbol{B}\big\}\cdot\nabla A_x + \tfrac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\nabla}(\partial_t A_x)\\
&\qquad\qquad - \tfrac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\nabla}(\partial_t A_x) - \tfrac{1}{c}\rho(\partial_t A_x) - r\boldsymbol{\nabla}\cdot(\boldsymbol{B}\times\boldsymbol{\nabla})A_x\\
&\quad= -\tfrac{1}{c}\big\{\boldsymbol{J}\cdot\boldsymbol{\nabla} + \rho\partial_t\big\}A_x + r\underbrace{\big\{\boldsymbol{\nabla}\times\boldsymbol{B}\cdot\nabla - \boldsymbol{\nabla}\cdot(\boldsymbol{B}\times\boldsymbol{\nabla})\big\}}_{0,\ \text{by quick demonstration}}A_x\\
&\quad= 0 \quad \text{in the absence of sources}
\end{aligned}
\tag{39}
$$

This result supplies the familiar grounds on which we will *abandon* the gauge-sensitive terms. Postponing discussion of the results now in hand. . .

I digress now to establish the identity

$$
\begin{pmatrix}\big[(\partial_x\boldsymbol{A})\times\boldsymbol{B}\big]_x\\ \big[(\partial_x\boldsymbol{A})\times\boldsymbol{B}\big]_y\\ \big[(\partial_x\boldsymbol{A})\times\boldsymbol{B}\big]_z\end{pmatrix} = \begin{pmatrix}-B_xB_x + \boldsymbol{B}\cdot\boldsymbol{B}\\ -B_yB_x\\ -B_zB_x\end{pmatrix} - \begin{pmatrix}(\boldsymbol{B}\times\boldsymbol{\nabla})_xA_x\\ (\boldsymbol{B}\times\boldsymbol{\nabla})_yA_x\\ (\boldsymbol{B}\times\boldsymbol{\nabla})_zA_x\end{pmatrix}
\tag{40}
$$

used in the argument which led to (38.2). By way of preparation, we note that

$$
\boldsymbol{B}\times\boldsymbol{B} = \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{0} \quad\Rightarrow\quad \begin{cases} B_yA_{y,x} - B_yA_{y,x} = B_zA_{x,z} - B_zA_{z,x}\\ B_zA_{z,y} - B_zA_{y,z} = B_xA_{y,x} - B_xA_{x,y}\\ B_xA_{x,z} - B_xA_{z,x} = B_yA_{z,y} - B_yA_{y,z}\end{cases}
$$

and that

$$
\boldsymbol{B}\cdot\boldsymbol{B} = B_x(A_{z,y} - A_{z,y}) + B_y(A_{x,z} - A_{z,x}) + B_z(A_{y,x} - A_{x,y})
$$

Drawing without specific comment upon those facts, we have

$$\begin{aligned}
\big[(\partial_x\boldsymbol{A})\times\boldsymbol{B}\big]_x &= B_zA_{y,x}-B_yA_{z,x}\\
&= -B_xB_x+\boldsymbol{B\cdot B}+\big\{B_zA_{y,x}-B_yA_{z,x}-B_yB_y-B_zB_z\big\}\\
\big\{\text{etc.}\big\} &= B_zA_{y,x}-B_yA_{z,x}-B_yA_{x,z}+B_yA_{z,x}-B_zA_{y,x}+B_zA_{x,y}\\
&= -B_yA_{x,z}+B_zA_{x,y}\\
&= -(\boldsymbol{B}\times\boldsymbol{\nabla})_xA_x\\
\big[(\partial_x\boldsymbol{A})\times\boldsymbol{B}\big]_y &= B_xA_{z,x}-B_zA_{x,x}\\
&= -B_yB_x+\big\{B_xA_{z,x}-B_zA_{x,x}+B_y(A_{z,y}-A_{y,z})\big\}\\
\big\{\text{etc.}\big\} &= B_xA_{z,x}-B_zA_{x,x}+B_x(A_{x,z}-A_{z,x})\\
&= -B_zA_{x,x}+B_xA_{x,z}\\
&= -(\boldsymbol{B}\times\boldsymbol{\nabla})_yA_x\\
\big[(\partial_x\boldsymbol{A})\times\boldsymbol{B}\big]_z &= B_yA_{x,x}-B_xA_{y,x}\\
&= -B_zB_x+\big\{B_yA_{x,x}-B_xA_{y,x}+B_z(A_{z,y}-A_{y,z})\big\}\\
\big\{\text{etc.}\big\} &= B_yA_{x,x}-B_xA_{y,x}+B_x(A_{y,x}-A_{x,y})\\
&= -B_xA_{x,y}+B_yA_{x,x}\\
&= -(\boldsymbol{B}\times\boldsymbol{\nabla})_zA_x
\end{aligned}$$

which serve to establish the identity in question. Companion identities are obtained by cyclic permutation on $\{x,y,z\}$.

The results recently acquired are summarized in the following display:

$$\begin{pmatrix} \mathcal{E} & \mathcal{P}_x & \mathcal{P}_y & \mathcal{P}_z\\ \mathcal{F}^x & \mathcal{T}^x{}_x & \mathcal{T}^x{}_y & \mathcal{T}^x{}_z\\ \mathcal{F}^y & \mathcal{T}^y{}_x & \mathcal{T}^y{}_y & \mathcal{T}^y{}_z\\ \mathcal{F}^z & \mathcal{T}^z{}_x & \mathcal{T}^z{}_y & \mathcal{T}^z{}_z \end{pmatrix} = \begin{pmatrix} \mathcal{E} & \frac{1}{c}(\boldsymbol{E}\times\boldsymbol{B})^\intercal\\ \\ rc\boldsymbol{E}\times\boldsymbol{B} & \mathbb{T} \end{pmatrix} \tag{41}$$

where $\mathcal{E}\equiv\frac{1}{2}(E^2+rB^2)$ and

$$\mathbb{T}\equiv\begin{pmatrix} \mathcal{E}-E_xE_x-rB_xB_x & -E_xE_y-rB_xB_y & -E_xE_z-rB_xB_z\\ -E_yE_x-rB_yB_x & \mathcal{E}-E_yE_y-rB_yB_y & -E_yE_z-rB_yB_z\\ -E_zE_x-rB_zB_x & -E_zE_y-rB_zB_y & \mathcal{E}-E_zE_z-rB_zB_z \end{pmatrix}$$

These results (except, perhaps, for the intrusion of the r-factors, concerning which I will have more to say in a moment) are of precisely the design supplied by Maxwellian electrodynamics.[13] The symmetry $\mathbb{T}^\intercal=\mathbb{T}$ of the "stress tensor" is manifest (no Belinfante symmetrization was required after abandonment of the gauge-sensitive terms), and has been shown to assure *angular momentum conservation* in the free gauge field system.

[13] See D. Griffiths' *Introduction to Electrodynamics* (1981) §7.5; CLASSICAL ELECTRODYNAMICS (1980), p. 300.

We have been led from the non-relativistic classical mechanics written into the non-relativistic Hamilton-Jacobi Lagrangian (23) to the two-parameter class of gauge field theories implicit in the locally gauge-invariant Lagrangian (28) (from which we may consider the physically inconsequential q-term to have been dropped). A physical consideration ($\mathcal{E} \geqslant 0$) has led us to require

$$p < 0 < r$$

but has placed no restrictions upon the *numerical values* of p and r; we found it convenient at one point to set $p = -1$, but were certainly under no obligation to do so. Suppose we set $p = -s$ (with $s > 0$) and rewrite (32) as follows:

$$\left.\begin{aligned} \boldsymbol{\nabla}\cdot[\boldsymbol{B}/c] &= 0\\ \boldsymbol{\nabla}\times\boldsymbol{E} &= -\partial_t[\boldsymbol{B}/c]\\ \boldsymbol{\nabla}\cdot\boldsymbol{E} &= \tfrac{1}{\epsilon}[\epsilon\rho/s]\\ \boldsymbol{\nabla}\times[\boldsymbol{B}/c] &= \tfrac{s}{rc^2\epsilon}[\epsilon\boldsymbol{J}/s] + \tfrac{s}{rc^2}\partial_t\boldsymbol{E}\\ &= \mu[\epsilon\boldsymbol{J}/s] + \mu\epsilon\partial_t\boldsymbol{E} \quad\text{with}\quad c^2\mu\epsilon \equiv s/r = -p/r \end{aligned}\right\}\tag{42}$$

Notational adjustments

$$[\boldsymbol{B}/c] \longrightarrow \boldsymbol{B}, \quad [\epsilon\rho/s] \longrightarrow \rho, \quad [\epsilon\boldsymbol{J}/s] \longrightarrow \boldsymbol{J}$$

lead then to "Maxwell equations" identical to those presented by Griffiths[13] at the beginning of his §7.3.3. In short: gauge theory has led us to a *population* of field theories, any one of which we are prepared by our experience to call "Maxwellian electrodynamics in an isotropic homogeneous medium." Each of those theories has a "relativistic look about it," but only one is relativistic in the Einsteinian sense—namely the one which results when (in effect) one sets

$$-p = r = 1 \quad\text{and}\quad c = \text{the observed constant of Nature}$$

In that case (35) reads

$$\begin{aligned} \mathcal{L}_{\text{free gauge field}} &= -\tfrac{1}{2}\,(\tfrac{1}{c}\partial_t\boldsymbol{A} + \boldsymbol{\nabla}\phi)\cdot(\tfrac{1}{c}\partial_t\boldsymbol{A} + \boldsymbol{\nabla}\phi) + \tfrac{1}{2}\,(\boldsymbol{\nabla}\times\boldsymbol{A})\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})\\ &= -\tfrac{1}{2}\,(\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B}) \end{aligned}\tag{43}$$

which is familiar[14] as a Lorentz invariant

$$= -\tfrac{1}{4}F^{\alpha\beta}F_{\beta\alpha}$$

associated with the electromagnetic field *in vacuuo*. The take-home lesson: gauge theory is "relativistically predisposed," but does not *force* relativity upon us.

[14] See p. 256 in the notes just cited.

The discussion subsequent to (35) was off-puttingly dense. I might have made it less so by using sprinkled "it can be shown"s to surpress details (which I included because they frequently take surprising turns), but it seems well to recognize *why* the discussion was so cluttered: it was, in substantial part, because we worked non-relativistically—deprived of the organizing principles and unifying simplifications inherent in special relativity.

The strategy by which we have achieved

$$\text{global gauge} \quad\Longrightarrow\quad \text{local gauge}$$

has "summoned electrodynamics into being," and has at the same time *lent specific structure to the particle–field interaction*. These concluding remarks are intended to expose more clearly some details associated with the latter aspect of our subject. Recall that at (33), working from the ungauged Lagrangian $\mathcal{L}_0$, we extracted definitions

$$\mathcal{E} = R \cdot \tfrac{1}{2m}\boldsymbol{\nabla}S\boldsymbol{\cdot}\boldsymbol{\nabla}S \equiv RH \qquad \text{and} \qquad \boldsymbol{\mathcal{F}} = \tfrac{1}{m}\mathcal{E}\boldsymbol{\nabla}S = \big(\tfrac{1}{m}R\boldsymbol{\nabla}S\big)H$$

and found

$$\begin{aligned}\mathcal{E}_t + \boldsymbol{\nabla\cdot}\boldsymbol{\mathcal{F}} &= \underbrace{\big\{R_t + \boldsymbol{\nabla\cdot}\big(\tfrac{1}{m}R\boldsymbol{\nabla}S\big)\big\}}_{0}H + \tfrac{1}{m}R\underbrace{\big\{\boldsymbol{\nabla}S\boldsymbol{\cdot}\boldsymbol{\nabla}S_t + \boldsymbol{\nabla}S\boldsymbol{\cdot}\boldsymbol{\nabla}H\big\}}_{0}\\ &= 0 \quad \text{by the ungauged Hamilton-Jacobi equations (22)}\end{aligned}$$

To discover the effect of turning on the gauge field we return to (33.10) and (33.20), make the replacement $\mathcal{L}_0 \to \mathcal{L}_1$, and find

$$\left.\begin{aligned}\mathcal{E} &\to \mathcal{E}' = R\cdot H'\\ \boldsymbol{\mathcal{F}} &\to \boldsymbol{\mathcal{F}}' = \big[\tfrac{1}{m}R\big(\boldsymbol{\nabla}S - \tfrac{e}{c}\boldsymbol{A}\big)\big]\cdot H'\end{aligned}\right\} \tag{44}$$

with $H' \equiv \frac{1}{2m}\big(\boldsymbol{\nabla}S - \frac{e}{c}\boldsymbol{A}\big)\boldsymbol{\cdot}\big(\boldsymbol{\nabla}S - \frac{e}{c}\boldsymbol{A}\big) + e\phi$.[15] We obtain

$$\begin{aligned}\mathcal{E}'_t + \boldsymbol{\nabla\cdot}\boldsymbol{\mathcal{F}}' &= \underbrace{\big\{R_t + \boldsymbol{\nabla\cdot}\big[\tfrac{1}{m}R\big(\boldsymbol{\nabla}S - \tfrac{e}{c}\boldsymbol{A}\big)\big]\big\}}_{0\quad\text{by (28.2)}}H' + R\big\{H'_t + \big[\tfrac{1}{m}R\big(\boldsymbol{\nabla}S - \tfrac{e}{c}\boldsymbol{A}\big)\big]\boldsymbol{\cdot}\boldsymbol{\nabla}H'\big\}\\ &= R\big\{\tfrac{1}{m}\big(\boldsymbol{\nabla}S - \tfrac{e}{c}\boldsymbol{A}\big)\boldsymbol{\cdot}\big(\boldsymbol{\nabla}S_t - \tfrac{e}{c}\boldsymbol{A}_t\big) + e\phi_t\big\} - \big[\tfrac{1}{m}R\big(\boldsymbol{\nabla}S - \tfrac{e}{c}\boldsymbol{A}\big)\big]\boldsymbol{\cdot}\boldsymbol{\nabla}S_t\\ &= -\tfrac{e}{m}R\big(\boldsymbol{\nabla}S - \tfrac{e}{c}\boldsymbol{A}\big)\boldsymbol{\cdot}\tfrac{1}{c}\boldsymbol{A}_t + eR\phi_t\\ &= \tfrac{1}{c}\,\boldsymbol{J\cdot E} + (\boldsymbol{J\cdot\nabla} + \rho\partial_t)\phi\end{aligned} \tag{45}$$

On the other hand, for the gauge field we found the energy density and flux to be given by

$$\begin{aligned}\tilde{\mathcal{E}} &= \tfrac{1}{2}(\boldsymbol{E\cdot E} + r\boldsymbol{B\cdot B}) + \text{gauge term}\\ \tilde{\boldsymbol{\mathcal{F}}} &= rc\boldsymbol{E}\times\boldsymbol{B} + \textbf{gauge term}\end{aligned}$$

[15] I have found it convenient, for the purposes of this discussion to turn the impressed potential U off; i.e., to assume that our Hamilton-Jacobi theory refers to the motion of particles which—except for gauge field effects—move freely.

It follows readily from the field equations (32) that

$$\tfrac{\partial}{\partial t}\left[\tfrac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{E}+r\boldsymbol{B}\cdot\boldsymbol{B})\right]+\boldsymbol{\nabla}\cdot\left[rc\boldsymbol{E}\times\boldsymbol{B}\right]=-\tfrac{1}{c}\,\boldsymbol{J}\cdot\boldsymbol{E} \tag{46.1}$$

while we established at (37) that

$$\tfrac{\partial}{\partial t}\left[\text{gauge term}\right]+\boldsymbol{\nabla}\cdot\left[\textbf{gauge term}\right]=-(\boldsymbol{J}\cdot\boldsymbol{\nabla}+\rho\partial_t)\phi \tag{46.2}$$

From (45/46) we obtain

$$\tfrac{\partial}{\partial t}\left[\mathcal{E}'+\tilde{\mathcal{E}}\,\right]+\boldsymbol{\nabla}\cdot\left[\boldsymbol{\mathfrak{F}}'+\tilde{\boldsymbol{\mathfrak{F}}}\right]=0 \tag{47}$$

which attributes detailed local balance to the energy exchange between the matter field and the gauge field. Three similar results, established by similar means, pertain to local momentum balance.[16]

Note finally that our final (locally gauge-invariant) Lagrangian (28) can be developed (compare (16))

$$\mathcal{L}_{\text{free Hamilton-Jacobi}}+\mathcal{L}_{\text{interaction}}+\mathcal{L}_{\text{free gauge field}}$$

where $\mathcal{L}_{\text{free Hamilton-Jacobi}}$ is just the $\mathcal{L}_0$ which at (23) provided our point of departure, $\mathcal{L}_{\text{free gauge field}}=-\frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{E}-r\boldsymbol{B}\cdot\boldsymbol{B})$ is (35) with $p=-1$, and, reading from (27),

$$\begin{aligned}\mathcal{L}_{\text{interaction}}&=-\tfrac{1}{c}\tfrac{e}{m}R\big(\boldsymbol{\nabla}S-\tfrac{e}{c}\boldsymbol{A}\big)\cdot\boldsymbol{A}-R\tfrac{1}{2m}\big(\tfrac{e}{c}\big)^2\boldsymbol{A}\cdot\boldsymbol{A}+eR\phi\\&=\rho\phi-\tfrac{1}{c}\big(\boldsymbol{J}+R\tfrac{e^2}{2mc}\boldsymbol{A}\big)\cdot\boldsymbol{A}\\&=\rho\phi-\tfrac{1}{c}\tfrac{1}{2}\big(\boldsymbol{J}+\boldsymbol{j}\big)\cdot\boldsymbol{A}\end{aligned} \tag{48}$$

$$\boldsymbol{j}\equiv\tfrac{e}{m}R\boldsymbol{\nabla}S=\boldsymbol{J}\Big|_{\boldsymbol{A}\to\boldsymbol{0}} \tag{49}$$

Equation (48) mimics the design of an equation to which we were led when looking to the gauge theory of a *relativistic complex* scalar field. The agreement (48) $\leftrightarrow$ (21) becomes in fact precise when one writes[17]

$$\|J^\mu\|\equiv\begin{pmatrix}c\rho\\\boldsymbol{J}\end{pmatrix}\quad\text{and}\quad\|A^\mu\|=\begin{pmatrix}\phi\\\boldsymbol{A}\end{pmatrix} \tag{50}$$

and uses the Lorentz metric to lower an index. Equation (48) also conforms to the result achieved when one takes the non-relativistic Schrödinger Lagrangian (1–76) as a point of departure.[18] Pretty clearly: *neither relativity, nor quantum*

[16] Compare (327) on p. 312 of CLASSICAL ELECTRODYNAMICS (1980); the argument there is relativistic, and therefore simpler/briefer.
[17] See CLASSICAL ELECTRODYNAMICS (1980), pp. 161 & 373.
[18] See CLASSICAL FIELD THEORY (1979), 188–193.

mechanics, nor complex-valuedness are essential to the success of the gauge field program. Which was the point at issue—now demonstrated by example.[19]

"Minimal coupling" and the physical significance of current. We consider those subjects in reverse order, taking as our point of departure this question: How does electrical current—so "real" it can be measured with an ammeter—come to be represented in theory by an expression which is (on its face) not even gauge-invariant? To expose the points at issue in their simplest and most essential terms I look to the relativistic classical mechanics of a particle.

Let $x^\mu(\tau)$, $u^\mu(\tau) \equiv \frac{d}{d\tau}x^\mu(\tau)$ and $a^\mu(\tau) \equiv \frac{d}{d\tau}u^\mu(\tau)$ refer the position (with respect to an intertial frame), 4-velocity and 4-acceleration of a mass point m. From the definition of proper time τ it follows that $(u,u) \equiv g_{\alpha\beta}u^\alpha u^\beta = c^2$, and therefore that $(a,u) = 0$; i.e., that $a \perp u$ in the Lorentzian sense. Minkowski's equation of motion reads $ma^\mu = K^\mu$. Necessarily, $(K,u) = 0$: Minkowski forces K^μ are necessarily velocity-dependent. In the simplest case K^μ will depend *linearly* upon 4-velocity: $K^\mu \sim F^{\mu\nu}u_\nu$. From $K^\mu u_\mu = 0$ (all u) it follows that necessarily $F^{\mu\nu}$ is *antisymmetric*. We are led thus to consider relativistic systems of the especially simple design

$$ma^\mu = \tfrac{e}{c}F^{\mu\nu}(x)u_\nu \tag{51}$$

where $[eF^{\mu\nu}] = $ (force) and e is a coupling constant. Passing now from the Minkowskian to the Lagrangian side of the street[20]. . .

The simplest way to build velocity-dependence into a Lagrangian is to write

$$L = \tfrac{1}{2}mg_{\alpha\beta}u^\alpha u^\beta + \tfrac{e}{c}A_\alpha(x)u^\alpha \tag{52}$$

---

[19] I find the example to be of some intrinsic interest. The possibility of a "gauged Hamilton-Jacobi theory" has been known to me since the spring of 1984, when I wrote the material which appears on pp. 489–502 of CLASSICAL MECHANICS (1983–1984), but I had forgotten until this minute that I had ever actually written about the subject; my field theory books provide only the rough seminar notes presented under the title "A sense in which classical mechanics quantizes itself" (1980) and bound as an appendix to CLASSICAL FIELD THEORY (1979). Freshly emerged now from such a writing experience, I am filled with a renewed sense of *what a wonderous activity-organizer is the Lagrangian formalism!* It asks you to do some occasionally intricate things. . . which, however, seem preordained always to work out. I am impressed also by how awkward life can become when relativity is removed from one's toolbox.

[20] This is less easily accomplished than might be supposed, for the constraint $(u,u) = c^2$ must be folded into the meaning of the variational process $\delta\int L\,d\tau$. Various techniques for accomplishing that objective are described in the iintroductory sections of RELATIVISTIC DYNAMICS (1975). But the following discussion leads to equations of motion for which compliance with the constraint becomes—as it happens—*automatic*, and it is this lucky circumstance which permits us to set such subtleties aside.

We are led then to equations of the motion of the form

$$\left\{\frac{d}{d\tau}\frac{\partial}{\partial u^\mu} - \frac{\partial}{\partial x^\mu}\right\}L = ma_\mu + \tfrac{e}{c}A_{\mu,\alpha}(x)u^\alpha - \tfrac{e}{c}A_{\alpha,\mu}(x)u^\alpha = 0$$

which can be written

$$ma^\mu = \tfrac{e}{c}F^{\mu\nu}(x)\,u_\nu \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{53}$$

These equations—which we may interpret to refer to the relativistic motion of a charged mass point in the presence of an impressed electromagnetic field—are invariant under

$$A_\mu \longrightarrow A_\mu + \partial_\mu\Omega \tag{54.1}$$

but derive from a Lagrangian which is *not* gauge-invariant:

$$\begin{aligned} L &\longrightarrow L + \text{offending term} \\ &\qquad\qquad \text{offending term} = \tfrac{e}{c}u^\alpha\partial_\alpha\Omega = \tfrac{d}{d\tau}\big[\tfrac{e}{c}\,\Omega(x)\big] \end{aligned}$$

Notice, however, that we have only to assign an expanded meaning

$$A_\mu \longrightarrow A_\mu + \partial_\mu\Omega \qquad \rightsquigarrow \qquad \begin{cases} A_\mu \longrightarrow A_\mu + \partial_\mu\Omega(x) \\ \\ L \longrightarrow L - \tfrac{d}{d\tau}\big[\tfrac{e}{c}\,\Omega(x)\big] \end{cases} \tag{54.2}$$

to the notion of a "gauge transformation" to acquire gauge-invariance of the Lagrangian, whence of all that follows from the Lagrangian. To illustrate the point:

Working from (52), we find the momentum conjugate to x^μ to be given by

$$p_\mu = \frac{\partial L}{\partial u^\mu} = mu_\mu + \tfrac{e}{c}A_\mu \tag{55}$$

which

- is not gauge-invariant under the interpretation (54.1), but
- is gauge-invariant under the expanded interpretation (54.2).

To say the same thing another way: gauge transformations, under the restricted interpretation (54.1), send

$$\left.\begin{aligned} L &\longrightarrow L + \tfrac{e}{c}u^\alpha\partial_\alpha\Omega \\ p_\mu &\longrightarrow p_\mu + \tfrac{e}{c}\partial_\mu\Omega \end{aligned}\right\} \tag{56.1}$$

but under the expanded interpretation (54.2) send

$$\left.\begin{aligned} L &\longrightarrow \big\{L - \tfrac{d}{d\tau}\big[\tfrac{e}{c}\,\Omega(x)\big]\big\} + \tfrac{e}{c}u^\alpha\partial_\alpha\Omega = L \\ p_\mu &\longrightarrow \big\{p_\mu - \tfrac{\partial}{\partial u^\mu}\tfrac{d}{d\tau}\big[\tfrac{e}{c}\,\Omega(x)\big]\big\} + \tfrac{e}{c}\partial_\mu\Omega = p_\mu \end{aligned}\right\} \tag{56.2}$$

Passing now to the Hamiltonian formalism, we find that

$$H(p,x) = p_\alpha u^\alpha - L(x,u) \qquad \text{with} \qquad u \mapsto \tfrac{1}{m}\big[p - \tfrac{e}{c}A\big] \tag{57}$$

gives

$$= \tfrac{1}{2m} g^{\alpha\beta}\big[p_\alpha - \tfrac{e}{c}A_\alpha\big]\big[p_\beta - \tfrac{e}{c}A_\beta\big] \tag{58}$$

The resulting canonical equations

$$\begin{aligned} u^\mu &= +\tfrac{\partial H}{\partial p_\mu} = \tfrac{1}{m} g^{\mu\alpha}\big[p_\alpha - \tfrac{e}{c}A_\alpha\big] \\ \tfrac{d}{d\tau} p_\mu &= -\tfrac{\partial H}{\partial x^\mu} = \tfrac{1}{m} g^{\alpha\beta}\big[p_\alpha - \tfrac{e}{c}A_\alpha\big]\tfrac{e}{c}A_{\beta,\mu} \end{aligned}$$

are readily seen to reproduce (53).[21] The gauge transformation properties of the Hamiltonian are somewhat subtle, and to sort them out I need to distinguish "L-gauge" ($L \longrightarrow L + u^\alpha \partial_\alpha \Lambda(x)$) from "$A$-gauge" ($A_\mu \longrightarrow A_\mu u + \partial_\alpha \Omega(x)$). The former causes the $H(p,x)$ of (57) to go over into

$$[p_\alpha + \partial_\alpha \Lambda]u^\alpha - [L + u^\alpha \partial_\alpha \Lambda] \qquad \text{with} \qquad u \mapsto \tfrac{1}{m}\big([p - \partial\Lambda] - \tfrac{e}{c}A\big)$$

In short (note the cancellation): L-gauge causes

$$H(p,x) \longrightarrow H(p - \partial\Lambda, x)$$

while reading from (58) we see that that A-gauge causes

$$H(p,x) \longrightarrow H(p - \tfrac{e}{c}\partial\Omega, x)$$

These elementary remarks expose in new light the central idea of gauge field theory: *use one gauge type to cancel the effect of the other*, by setting $\Lambda = -\frac{e}{c}\Omega$.

Electrical "current" enters the discussion as a by-product of steps we take in order to promote the A_μ-field—heretofore considered to have been externally impressed/prescribed—to the status of a *dynamical field in its own right*. From the A-gauge-invariant antisymmetric tensor field $F^{\mu\nu}$ on can—using $g_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$ as "glue"—construct a total of three Lorentz invariants:

$$F^\mu{}_\nu F^\nu{}_\mu\,, \quad F^\mu{}_\nu G^\nu{}_\mu\,, \quad \text{and} \quad G^\mu{}_\nu G^\nu{}_\mu \quad \text{with} \quad G_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

which in index-free notation can be described

$$\mathrm{tr}\,\mathbb{F}\mathbb{F}, \quad \mathrm{tr}\,\mathbb{F}\mathbb{G}, \quad \text{and} \quad \mathrm{tr}\,\mathbb{G}\mathbb{G}$$

[21] . . . and can be considered to arise as "meta-Lagrange equations"

$$\Big\{\frac{d}{d\tau}\frac{\partial}{\partial \dot p_\mu} - \frac{\partial}{\partial p_\mu}\Big\}\mathfrak{L} = 0 \quad \text{and} \quad \Big\{\frac{d}{d\tau}\frac{\partial}{\partial u^\mu} - \frac{\partial}{\partial x^\mu}\Big\}\mathfrak{L} = 0$$

from the "meta-Lagrangian"

$$\mathfrak{L} \equiv p_\alpha u^\alpha - H(p,x)$$

But it is not difficult to show[22] that

$$\mathrm{tr}\,\mathbb{G}\mathbb{G} \sim \mathrm{tr}\,\mathbb{F}\mathbb{F} \quad \text{and} \quad \mathrm{tr}\,\mathbb{F}\mathbb{G} \sim \partial_\mu\big[\epsilon^{\mu\nu\rho\sigma}A_\nu(\partial_\rho A_\sigma - \partial_\sigma A_\rho)\big]$$

So $\mathrm{tr}\,\mathbb{F}\mathbb{F}$ and $\mathrm{tr}\,\mathbb{G}\mathbb{G}$ contribute identically (apart from a trivial factor) when introduced into a Lagrangian, while $\mathrm{tr}\,\mathbb{F}\mathbb{G}$ contributes only an inconsequential gauge term. We are led, therefore, to examine

$$\begin{aligned} L_2 &= \tfrac{1}{2}mg_{\alpha\beta}u^\alpha u^\beta + \tfrac{e}{c}A_\alpha(x)u^\alpha + \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \\ &= \tfrac{1}{2}mg_{\alpha\beta}u^\alpha u^\beta + \tfrac{e}{c}A_\alpha(x)u^\alpha + \tfrac{1}{2}(g^{\alpha\rho}g^{\beta\sigma} - g^{\alpha\sigma}g^{\beta\rho})A_{\alpha,\beta}A_{\rho,\sigma} \end{aligned} \tag{59}$$

From

$$\Big\{\partial_\mu\frac{\partial}{\partial A_{\nu,\mu}} - \frac{\partial}{\partial A_\nu}\Big\}L_2 = \partial_\mu(A^{\nu,\mu} - A^{\mu,\nu}) - \frac{\partial L}{\partial A_\nu} = 0$$

we obtain

$$\partial_\mu F^{\mu\nu} = \tfrac{1}{c}J^\nu \tag{60}$$

with

$$J^\nu \equiv \frac{\partial L}{\partial A_\nu} = eu^\nu \tag{61.1}$$

$$= \frac{e}{m}\big[p^\nu - \tfrac{e}{c}A^\nu\big] \tag{61.2}$$

The expression on the right in (61.1) is—by every interpretation—manifestly gauge-invariant, and conforms precisely to what, on physical grounds, we expect of the "4-current of a charged mass point." The expression of the right in (6.12) is, on the other hand, gauge-invariant only under the expanded interpretation (54.2), and it is under the latter interpretation that the Lagrangian of (59) becomes gauge-invariant. The notation (61.1) permits the interaction term present in (59) to be described

$$\mathcal{L}_{\text{interaction}} = \tfrac{1}{c}J^\alpha A_\alpha(x) \tag{62}$$

This is the term which appears, on its face, to mess up gauge-invariance, but which becomes gauge-invariant in the expanded sense; it was precisely the gauge-failure of $\mathcal{L}_{\text{interaction}}$ that the L-gauge of the overall Lagrangian was tailored to correct.

Passing from (59) to the equivalent Hamiltonian formalism, one has

$$\begin{aligned} H(p,x) &= \tfrac{1}{2m}g^{\alpha\beta}\big[p_\alpha - \tfrac{e}{c}A_\alpha\big]\big[p_\beta - \tfrac{e}{c}A_\beta\big] - \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \\ &= \tfrac{1}{2m}g^{\alpha\beta}\big[p_\alpha - \tfrac{e}{c}A_\alpha\big]\big[p_\beta - \tfrac{e}{c}A_\beta\big] - \tfrac{1}{2}(g^{\alpha\rho}g^{\beta\sigma} - g^{\alpha\sigma}g^{\beta\rho})A_{\alpha,\beta}A_{\rho,\sigma} \end{aligned} \tag{63}$$

In this formalism the extended gauge transformation (54.2) lurks behind the scenery, and presents this face:

$$\left.\begin{aligned} A_\mu &\longrightarrow A_\mu + \partial_\mu\Omega(x) \\ p_\mu &\longrightarrow p_\mu - \tfrac{e}{c}\partial_\mu\Omega(x) \end{aligned}\right\} \tag{64}$$

[22] See CLASSICAL ELECTRODYNAMICS (1980), p. 298.

The invariance of (63) under (64) is manifest. As was noted already in the discussion subsequent to (58), the canonical equations implicit in (63) reproduce our initial description (53) of the dynamics of the charged particle. But what of the dynamics of the gauge field? Yielding unthinkingly to entrenched habit, we construct

$$\Big\{\partial_\mu\frac{\partial}{\partial A_{\nu,\mu}}-\frac{\partial}{\partial A_\nu}\Big\}H=-\partial_\mu(A^{\nu,\mu}-A^{\mu,\nu})-\frac{\partial H}{\partial A_\nu}=0$$

which does indeed give back (60/61):

$$\partial_\mu F^{\mu\nu}=-\frac{\partial H}{\partial A_\nu}=\tfrac{e}{mc}\big[p^\nu-\tfrac{e}{c}A^\nu\big]=\tfrac{1}{c}J^\nu$$

I say "unthinkingly" because we have no secure reason to take the Lagrange derivative of a Hamiltonian! It would, I think, be better form to construct the "meta-Lagrangian"

$$\mathfrak{L}(p,x,\bullet,u,A,\partial A)=p_\alpha u^\alpha-\big\{\tfrac{1}{2m}g^{\alpha\beta}\big[p_\alpha-\tfrac{e}{c}A_\alpha\big]\big[p_\beta-\tfrac{e}{c}A_\beta\big]-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\big\}$$

and recover the canonical equations as "meta-Lagrange equations," but we would come out in the same place.[21]

The Hamiltonian (63) can be developed

$$\begin{aligned}H=\tfrac{1}{2m}g^{\alpha\beta}p_\alpha p_\beta-H_{\text{int}}-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}&\\ H_{\text{int}}\equiv\tfrac{e}{c}\tfrac{1}{m}\big[p^\alpha-\tfrac{e}{c}A^\alpha\big]A_\alpha+\tfrac{1}{2m}\big(\tfrac{e}{c}\big)^2A^\alpha A_\alpha&\\ =\tfrac{1}{c}J^\alpha A_\alpha+\tfrac{e^2}{2mc^2}A^\alpha A_\alpha&\end{aligned}\tag{65}$$

though to do so entails a term-by-term sacrifice of manifest gauge-invariance. If we borrow notation from (18.1), writing $J^\nu\equiv j^\nu-\frac{e^2}{mc}A^\nu$, then we have

$$\begin{aligned}&=\tfrac{1}{c}\tfrac{1}{2}(J^\alpha+j^\alpha)A_\alpha\\ &j^\alpha\equiv\tfrac{e}{m}p^\alpha=J^\alpha\Big|_{A\to0}\end{aligned}\tag{66}$$

which mimics (21) and (48). Notice, however, this curious circumstance: the second term on the right side of (65)—which at (21) read $\frac{e^2}{2mc^2}(\psi^*A^\alpha\psi)A_\alpha$, and at (48) read $\frac{e^2}{2mc^2}R\,\boldsymbol{A}\cdot\boldsymbol{A}$—displays now no reference to the particle; only the e^2 reveals the "interactive" nature of the term, which we might otherwise be tempted to classify as a "mass term" present in the design of $H_{\text{free gauge field}}$.

What have we learned?

People sometimes point to (62)—i.e., to terms of the design $\boldsymbol{J}\cdot\boldsymbol{A}$, which in Lagrangian formalism serve to describe the interaction of charged matter with the electromagnetic field—as the defining symptom of "minimal coupling." But more standardly, the term is taken[23] to refer to the characteristic matter-field

[23] See, for example, M. E. Peskin & D. V. Schroeder, *An Introduction to Quantum Field Theory* (1995), p. 78.

interaction which arises from pursuit of the gauge field program; i.e., which springs spontaneously from $p \longrightarrow p - (e/c)A$ (or again: from $\partial \longrightarrow \mathcal{D}$).

Gauge field theory usually has a quantum mechanical objective, and for that reason is strongly Hamiltonian in spirit, though presented as an exercise in Lagrangian field theory.[24] In the examples we have studied we have been led at (21/48/66) to interaction terms which are "of a type," but more complicated than is suggested by the more purely Lagrangian model (62). We have learned, however, that the isolation of "interaction terms" $\mathcal{L}_{\text{int}}$ typically violates the "principle of manifest gauge-invariance," and thus runs counter to the essential spirit of gauge field theory; it is better to allow the interaction to remain implicit, as (for example) it did when at (19) and (17) we wrote

$$g^{\alpha\beta}\left(\tfrac{\hbar}{i}\partial_\alpha - \tfrac{e}{c}A_\alpha\right)\left(\tfrac{\hbar}{i}\partial_\beta - \tfrac{e}{c}A_\beta\right)\psi = (mc)^2\psi$$
$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \tfrac{1}{c}J^\nu$$

with $J^\nu = e\big[-i\frac{\hbar}{2m}g^{\nu\alpha}\big(\psi^*_{,\alpha}\psi - \psi^*\psi_{,\alpha}\big) + \frac{e}{mc}\big(\psi^*\psi\big)A^\nu\big]$.

Gauge field theory standardly takes a "field theory of matter" as its point of departure, but we have learned that the gauge field concept is so primitive that one can *abandon the initial field theory*: we achieved success when we proceeded from the Lagrangian mechanics of a single relativistic particle. . . and might (with some loss of simplicity) have abandoned the relativity; the resulting theory would have captured the simple essence of our "gauged Hamilton-Jacobi theory."

We have learned that the gauge-invariance of the currents that arise from gauge field theory is invariably *present but covert*. Our particulate model supplied

$$J^\mu = \frac{e}{m}\big[p^\mu - \tfrac{e}{c}A^\mu\big] = eu^\mu$$

which suggests that the complexity of the expressions that serve, in various contexts, to define J^μ can be attributed to the familiar complexity of the relationship between "velocity" (a physical observable, at least in particle mechanics) and "conjugate momentum" (a theoretical construct).

In the beginning was a decision—a decision to "allow the phase factor vary from point to point"—which may at the time have seemed willful, arbitrary, justifiable only by the illuminating results to which it could be shown to lead. Our particulate model has allowed that decision to be replaced by a formal act which I find much more natural: require of the formulæ of (meta-)Lagrangian dynamics that they be manifestly invariant under arbitrary gauge-adjustments

$$L \longrightarrow L + \partial_\alpha\Lambda^\alpha$$

I shall on another occasion describe how gauge field theory might procede from such a starting point (that effort will require only rearrangement of what I have

[24] Both remarks, by the way, pertain also to our "gauged Hamilton-Jacobi theory."

already written, and some shifted emphasis), but turn now to more pressing matters.

Gauged Dirac theory. We take now as our point of departure the Lagrangian

$$\mathcal{L}_0(\psi,\tilde{\psi},\partial\psi,\partial\tilde{\psi}) = \hbar c\Big[\tfrac{1}{2}i\big\{\tilde{\psi}\boldsymbol{\gamma}^\alpha\psi_{,\alpha} - \tilde{\psi}_{,\alpha}\boldsymbol{\gamma}^\alpha\psi\big\} - \varkappa\,\tilde{\psi}\psi\Big] \tag{67}$$

which was seen at (2–56) to yield the Dirac equations

$$\big(\boldsymbol{\gamma}^\mu\partial_\mu + i\varkappa\big)\psi = 0 \quad \text{and its adjoint} \tag{68}$$

From the manifest *global* phase-invariance of $\mathcal{L}_0$—infinitesimally: from the invariance of $\mathcal{L}_0$ under

$$\begin{aligned} \psi &\longrightarrow \psi + \delta_\omega\psi \quad \text{with} \quad \delta_\omega\psi = +i\psi\cdot\delta\omega \\ \tilde{\psi} &\longrightarrow \tilde{\psi} + \delta_\omega\tilde{\psi} \quad \text{with} \quad \delta_\omega\tilde{\psi} = -i\tilde{\psi}\cdot\delta\omega \end{aligned}$$

—we obtain the conservation law

$$\partial_\mu Q^\mu = 0 \tag{69.1}$$

$$Q^\mu \equiv -\tfrac{1}{\hbar}\left\{\frac{\partial\mathcal{L}_0}{\partial\psi_{,\mu}}(i\psi) + \frac{\partial\mathcal{L}_0}{\partial\tilde{\psi}_{,\mu}}(-i\tilde{\psi}\,)\right\} = c\,\tilde{\psi}\boldsymbol{\gamma}^\mu\psi \tag{69.2}$$

where an $\hbar$-factor has been introduced so as to achieve[25]

$$[Q^\mu] = \frac{1}{\text{area}\cdot\text{time}} = \text{number flux}$$

Our objective is to achieve *local* phase invariance. Familiar steps lead us, therefore, to construction of the system

$$\begin{aligned} \mathcal{L}_2(\psi,\tilde{\psi},\partial\psi,\partial\tilde{\psi},A,\partial A) &= \mathcal{L}_0(\psi,\tilde{\psi},[\partial - i\tfrac{e}{\hbar c}A]\psi,[\partial + i\tfrac{e}{\hbar c}A]\tilde{\psi}\,) + \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta} \\ &= \hbar c\Big[\tfrac{1}{2}i\Big\{\tilde{\psi}\boldsymbol{\gamma}^\alpha(\psi_{,\alpha} - i\tfrac{e}{\hbar c}A_\alpha\psi) - (\tilde{\psi}_{,\alpha} + i\tfrac{e}{\hbar c}A_\alpha\tilde{\psi}\,)\boldsymbol{\gamma}^\alpha\psi\Big\} - \varkappa\,\tilde{\psi}\psi\Big] + \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta} \\ &= \mathcal{L}_0(\psi,\tilde{\psi},\partial\psi,\partial\tilde{\psi}\,) + \tfrac{1}{c}J^\alpha A_\alpha + \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta} \end{aligned} \tag{70}$$

where the gauge-invariant antisymmetric tensor field $F_{\mu\nu}$ retains its former definition (15) and where

$$J^\mu \equiv eQ^\mu \tag{71}$$

---

[25] Notice now much simpler (69.2) is than were its non-relativistic/relativistic scalar counterparts (1–87) and (3–13). That simplicity can be attributed to the circumstance that $\mathcal{L}_0$ is *linear in the derivatives* of the complex field. It is, therefore, a simplicity not special to the Dirac theory, but shared by all canonically formulated theories. . . of which the Dirac theory provides merely the simplest instance.

Only once before—at (61.1) and (62), in connection with our particulate model—have we encountered formulæ so simple as those which serve here to describe J^μ and $\mathcal{L}_{\text{interaction}}$; we note in particular that the distinction between J^μ and j^μ does not force itself upon our attention in Dirac theory.

The field equations which arise from the twice-modified (i.e., from the "gauged and launched") Lagrangian (70) read

$$\left.\begin{array}{c}\big[\boldsymbol{\gamma}^\mu(\partial_\mu - i\tfrac{e}{\hbar c}A_\mu) + i\varkappa\big]\psi = 0 \quad \text{and its adjoint}\\ \partial_\mu F^{\mu\nu} = \tfrac{1}{c}J^\nu\end{array}\right\} \tag{72.1}$$

with
$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \tag{72.2}$$

The "synchronized local gauge transformation" with respect to which the theory is—by design—invariant can in present notation be described

$$\left.\begin{array}{rl}\psi \longrightarrow \psi' &= e^{ig\Omega(x)}\cdot\psi\\ A_\mu \longrightarrow A'_\mu &= A_\mu + \partial_\mu\Omega(x)\end{array}\right\} \tag{73}$$

with $g = e/\hbar c$. Those statements were contrived to entail

$$(\partial_\mu - igA_\mu)\psi \longrightarrow e^{ig\Omega(x)}\cdot(\partial_\mu - igA_\mu)\psi \tag{74}$$

and it is, to reiterate, that contrivance—first encountered at (9)—which lies at the heart of guage field theory.

Mathematical interlude: non-Abelian gauge groups. The operations "multiply by a phase factor"—which when ψ is an N-component complex field have this explicit meaning:

$$\begin{pmatrix}\psi^1\\ \psi^2\\ \vdots\\ \psi^N\end{pmatrix} \longrightarrow \begin{pmatrix}e^{i\omega} & 0 & \dots & 0\\ 0 & e^{i\omega} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & e^{i\omega}\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\\ \vdots\\ \psi^N\end{pmatrix}$$

—clearly possess the group property. The abstract group in question is $U(1)$, of which the 1×1 unitary matrices $\boldsymbol{U}(\omega) \equiv \|e^{i\omega}\| = e^{i\omega}\boldsymbol{I}$ provide the simplest representation. It becomes natural in this light to write

$$\begin{pmatrix}\psi^1\\ \psi^2\\ \vdots\\ \psi^N\end{pmatrix} \longrightarrow \begin{pmatrix}U^1{}_1 & U^1{}_2 & \dots & U^1{}_N\\ U^2{}_1 & U^2{}_2 & \dots & U^2{}_N\\ \vdots & \vdots & \ddots & \vdots\\ U^N{}_1 & U^N{}_2 & \dots & U^N{}_N\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\\ \vdots\\ \psi^N\end{pmatrix} \equiv \boldsymbol{U}\begin{pmatrix}\psi^1\\ \psi^2\\ \vdots\\ \psi^N\end{pmatrix}$$

and to admit the possibility that $\boldsymbol{U}$ might be an element of the group $U(N)$ of $N\times N$ unitary matrices. This is the idea which Yang & Mills (1954) were, for their own good reasons,[26] the first to explore... with results which first alerted physicists to the possibility that gauge field theory might be put to more informative uses that the "elegant re-invention of electrodynamics." My objective here will to assemble the mathematical material we will need to pursue that idea.

From the unitarity of $\boldsymbol{U}$ it follows that $(\det \boldsymbol{U})^*(\det \boldsymbol{U}) = 1$, and therefore that

$$\det \boldsymbol{U} = e^{i\vartheta}$$

Write $\boldsymbol{U} = e^{i\boldsymbol{H}}$ and observe that $\boldsymbol{U}$ will be unitarity $\boldsymbol{U}^\dagger = \boldsymbol{U}^{-1}$ if and only if $\boldsymbol{H}$ is hermitian. A general identity supplies $\det \boldsymbol{U} = \exp\big\{i\,(\text{tr}\boldsymbol{H})\big\}$, from which we infer that $\vartheta = \text{tr}\boldsymbol{H}$. If $\boldsymbol{U}$ is unitary then so is $\boldsymbol{S} \equiv e^{-i\omega}\boldsymbol{U}$, and $\det \boldsymbol{S} = e^{i(\vartheta - N\omega)}$, where N refers to the *dimension* of $\boldsymbol{U}$. We have only to set $\omega = \vartheta/N$ to render $\boldsymbol{S}$ *unimodular*; i.e., to achieve $\det \boldsymbol{S} = 1$. Unimodularity is preserved under multiplication: the $N \times N$ matrices $\boldsymbol{S}$ are elements of a subgroup (denoted $SU(N)$ and called the "special unitary group") of $U(N)$. One writes

$$U(N) = U(1) \otimes SU(N)$$

to signify that every element of $U(N)$ can be written

$$\boldsymbol{U} = e^{i\omega}\cdot \boldsymbol{S} \quad \text{with} \quad \begin{cases} e^{i\omega} \in U(N) \\ \boldsymbol{S} \in SU(N) \end{cases}$$

Matrices $\boldsymbol{S} \in SU(N)$ can be written

$$\boldsymbol{S} = e^{i\boldsymbol{H}} \quad \text{where } \boldsymbol{H} \text{ is a } \textit{traceless} \text{ hermitian matrix}$$

The most general such matrix $\boldsymbol{H}$ can be displayed

$$\boldsymbol{H} = \begin{pmatrix} d_1 & a_1 + ib_1 & a_2 + ib_2 & \dots \\ a_1 - ib_1 & d_2 & a_N + ib_N & \dots \\ a_2 - ib_2 & a_N - ib_N & d_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{with } d_1 + d_2 + \cdots + d_N = 0$$

and contains $N^2 - 1$ adjustable constants. The set of such matrices is closed under addition and multiplication by real numbers, so can be considered to comprise a real vector space $V_{\mathcal{N}}$ of $\mathcal{N} \equiv N^2 - 1$ dimensions. Select any basis $\big\{\boldsymbol{h}_1, \boldsymbol{h}_2, \dots, \boldsymbol{h}_{\mathcal{N}}\big\}$ in $V_{\mathcal{N}}$. The general element of $V_{\mathcal{N}}$ can then be developed

$$\boldsymbol{H} = H^1\boldsymbol{h}_1 + H^2\boldsymbol{h}_2 + \cdots + H^{\mathcal{N}}\boldsymbol{h}_{\mathcal{N}}$$

Bases can, of course, be selected in limitlessly many ways, but some algebraic "principles of selection" will soon emerge.

[26] Those reasons are evident already in the title of their paper: "Conservation of isotopic spin and isotopic gauge invariance," Phys. Rev. **96**, 191 (1954).

Look to the case $N = 2$. Pauli (see again (2–54)) would in that case have us write

$$\boldsymbol{H} = H^1\boldsymbol{\sigma}_1 + H^2\boldsymbol{\sigma}_2 + H^3\boldsymbol{\sigma}_3 = \begin{pmatrix} H^3 & H^1 - iH^2 \\ H^1 + iH^2 & -H^3 \end{pmatrix} \tag{75}$$

for the reason that the $\boldsymbol{\sigma}$-matrices thus defined are endowed with some especially attractive/useful algebraic properties:

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_3\boldsymbol{\sigma}_3 = \boldsymbol{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{76.1}$$

$$\left.\begin{aligned} \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 &= i\,\boldsymbol{\sigma}_3 = -\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 &= i\,\boldsymbol{\sigma}_1 = -\boldsymbol{\sigma}_3\boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 &= i\,\boldsymbol{\sigma}_2 = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 \end{aligned}\right\} \tag{76.2}$$

$$\left.\begin{aligned} [\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2] &= 2i\,\boldsymbol{\sigma}_3 \\ [\boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3] &= 2i\,\boldsymbol{\sigma}_1 \\ [\boldsymbol{\sigma}_3, \boldsymbol{\sigma}_1] &= 2i\,\boldsymbol{\sigma}_2 \end{aligned}\right\} \tag{76.3}$$

Look similarly to the case $N = 3$. Gell-Mann[27] found it convenient to write

$$\begin{aligned} \boldsymbol{H} &= H^1\boldsymbol{\lambda}_1 + H^2\boldsymbol{\lambda}_2 + H^3\boldsymbol{\lambda}_3 + H^4\boldsymbol{\lambda}_4 + H^5\boldsymbol{\lambda}_5 + H^6\boldsymbol{\lambda}_6 + H^7\boldsymbol{\lambda}_7 + H^8\boldsymbol{\lambda}_8 \\ &= \begin{pmatrix} H^3 + \frac{1}{\sqrt{3}}H^8 & H^1 - iH^2 & H^4 - iH^5 \\ H^1 + iH^2 & -H^3 + \frac{1}{\sqrt{3}}H^8 & H^6 - iH^7 \\ H^4 + iH^5 & H^6 + iH^7 & -\frac{2}{\sqrt{3}}H^8 \end{pmatrix} \end{aligned} \tag{77}$$

with algebraic consequences so relatively complicated that I will postpone discussion of them until it has become clearer what it is that we want to know.[28]

Our interest in the *multiplicative*—as opposed to the merely additive—properties of the basic matrices $\boldsymbol{h}_a$ comes to the fore when they are pressed into service as the "generators" of finite transformations, in the sense

$$\boldsymbol{S} \equiv e^{i\boldsymbol{H}} = \lim_{m\to\infty}\Big[\boldsymbol{I} + \tfrac{1}{m}(i\boldsymbol{H}\,)\Big]^m$$

Certainly we would develop a lively interest in algebra if we set out to obtain a closed-form *evaluation* of $\sum \frac{1}{n!}(i\boldsymbol{H}\,)^n$. But it is from another (simpler) quarter that our algebraic interest actually springs:

---

[27] M. Gell-Mann & Y. Ne'eman, *The Eightfold Way* (1964), p. 49; see also p. 502 in Peskin & Schroeder,[23] and CLASSICAL GAUGE FIELDS (1981), p. 44.

[28] In the meantime, see Appendix A.3 in M. Kaku, *Quantum Field Theory: A Modern Introduction* (1993).

Generally, the elements of $SU(N)$ fail to commute (which is all one means when one says of $SU(N)$ that it is—in contrast to the "Abelian" group $U(1)$ —"non-Abelian"): $\boldsymbol{S}_1\boldsymbol{S}_2 \neq \boldsymbol{S}_2\boldsymbol{S}_1$. One has[29]

$$\begin{aligned}\boldsymbol{S}_1\boldsymbol{S}_2 &= e^{i\boldsymbol{H}_1}e^{i\boldsymbol{H}_2}\\ &= e^{i(\boldsymbol{H}_1+\boldsymbol{H}_2)-\frac{1}{2}[\boldsymbol{H}_1,\boldsymbol{H}_2]+\text{higher order nested commutators}}\\ &= \boldsymbol{S}_2\boldsymbol{S}_1 \quad \text{if and only if } [\boldsymbol{H}_1,\boldsymbol{H}_2]=\boldsymbol{0}\end{aligned}$$

It was Sophus Lie who first appreciated that the "group multiplication table" (which would supply the evaluation of $\boldsymbol{S}_1\boldsymbol{S}_2$ in all instances) is latent in the commutation properties of the generators. And, moreover, that one can in all cases expect to obtain relations of the form

$$[\text{generator}, \text{generator}] = \text{linear combination of generators} \tag{78}$$

which stands as the defining characteristic of the theory of *Lie algebras*. In the present context (78) becomes

$$[\boldsymbol{h}_p, \boldsymbol{h}_q] = i\sum_{r=1}^{\mathcal{N}} c_p{}^r{}_q \boldsymbol{h}_r \tag{79}$$

The i reflects the elementary circumstance that

$$\begin{aligned}[\text{hermitian}, \text{hermitian}] &= \text{traceless antihermitian}\\ &= i\,(\text{traceless hermitian})\end{aligned}$$

and the real numbers $c_p{}^r{}_q$ are the *structure constants* characteristic of the group. At (76.3) we have already encountered a particular instance of (79).

The structure constants are not freely assignable, but subject to certain constraints. From the antisymmetry of the commutator it follows, for example, that

$$c_p{}^r{}_q = -c_q{}^r{}_p \tag{80.1}$$

while from Jacobi's identity, written $[\boldsymbol{h}_p,[\boldsymbol{h}_q,\boldsymbol{h}_n]]-[\boldsymbol{h}_q,[\boldsymbol{h}_p,\boldsymbol{h}_n]] = [[\boldsymbol{h}_p,\boldsymbol{h}_q],\boldsymbol{h}_n]$, we obtain

$$c_p{}^m{}_k\,c_q{}^k{}_n - c_q{}^m{}_k\,c_p{}^k{}_n = c_p{}^r{}_q\cdot c_r{}^m{}_n \tag{80.2}$$

which can be written

$$\mathbb{C}_p\mathbb{C}_q - \mathbb{C}_q\mathbb{C}_p = i\,c_p{}^r{}_q\,\mathbb{C}_r \tag{81}$$

Evidently the imaginary $\mathcal{N}\times\mathcal{N}$ matrices $\mathbb{C}_r \equiv \|i\,c_r{}^m{}_n\|$ $(r=1,2,\ldots,\mathcal{N})$ provide a representation (the so-called "adjoint representation") of the algebra from

[29] I borrow here from what is called "Campbell-Baker-Hausdorff theory." See CLASSICAL MECHANICS (1980), p. 285 and references cited there. But we need not venture into that intricate topic to understand the simple point at issue.

which they sprang.[30] Look, for example, to the group $SU(2)$: we are led from the structure constants implicit in (76.3) to the matrices

$$\mathbb{C}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2i \\ 0 & +2i & 0 \end{pmatrix},\ \mathbb{C}_2 = \begin{pmatrix} 0 & 0 & +2i \\ 0 & 0 & 0 \\ -2i & 0 & 0 \end{pmatrix},\ \mathbb{C}_3 = \begin{pmatrix} 0 & -2i & 0 \\ +2i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Calculation confirms that these matrices do in fact satisfy the commutation relations (76.3), even though the set $\{\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3\}$ is not multiplicatively closed and therefore cannot possibly satisfy (76.1/2).

The $\mathcal{N}\times\mathcal{N}$ matrix

$$\mathbb{k} \equiv \|k_{pq}\| \quad \text{with} \quad k_{pq} \equiv \mathrm{tr}\,\mathbb{C}_p\mathbb{C}_q \tag{82}$$

is transparently real and symmetric. Proceeding in the assumption that $\det\mathbb{k} \neq 0$, I write $\mathbb{k}^{-1} \equiv \|k^{pq}\|$ and will assign to $\mathbb{k}$ the role of "gauge metric;" i.e., I will use k^{pq} and k_{pq} to raise and lower indices. In the case $SU(2)$ we compute

$$\mathbb{k} = \begin{pmatrix} \mathrm{tr}\,\mathbb{C}_1\mathbb{C}_1 & \mathrm{tr}\,\mathbb{C}_1\mathbb{C}_2 & \mathrm{tr}\,\mathbb{C}_1\mathbb{C}_3 \\ \mathrm{tr}\,\mathbb{C}_2\mathbb{C}_1 & \mathrm{tr}\,\mathbb{C}_2\mathbb{C}_2 & \mathrm{tr}\,\mathbb{C}_2\mathbb{C}_3 \\ \mathrm{tr}\,\mathbb{C}_3\mathbb{C}_1 & \mathrm{tr}\,\mathbb{C}_3\mathbb{C}_2 & \mathrm{tr}\,\mathbb{C}_3\mathbb{C}_3 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

The results just obtained can be interpreted to state that the traceless hermitian 3×3 matrices $\mathbb{E}_p \equiv \frac{1}{\sqrt{8}}\mathbb{C}_p$ are *tracewise orthonormal*, and are *special to the Pauli basis*; with respect to that basis one has $k_{pq} = 8\delta_{pq}$.

We will have need of a result which can be stated

$$C_{psq} \text{ is totally antisymmetric} \tag{83.1}$$

and which I digress now to establish. We have

$$\begin{aligned} C_{psq} = k_{sr}\cdot C_p{}^r{}_q = C_s{}^u{}_v\,&\underbrace{C_r{}^v{}_u\cdot C_p{}^r{}_q} \\ &= C_p{}^v{}_w C_q{}^w{}_u - C_q{}^v{}_w C_p{}^w{}_u \quad \text{by (80.2)} \\ &= C_s{}^u{}_v C_p{}^v{}_w C_q{}^w{}_u + C_v{}^u{}_s C_u{}^w{}_p C_w{}^v{}_q \quad \text{by (80.1)} \\ &= \begin{cases} \text{sum or terms each of which is invariant} \\ \text{under cyclic permutation on } \{spq\} \end{cases} \end{aligned}$$

from which we conclude that $C_{psq} = C_{sqp} = C_{qps}$. But $C_{psq} = -C_{qsp}$. This establishes (83.1), from which it follows as a useful corollary that

$$C_{psr} = -C_{prs} \quad \text{which is to say:} \quad (\mathbb{k}\,\mathbb{C}_p)^{\mathsf{T}} = -(\mathbb{k}\,\mathbb{C}_p) \tag{83.2}$$

---

[30] For discussion which digs deeper into the relevant group theory, see §5.6 in H. Bacry, *Lectures on Group Theory and Particle Theory* (1977).

Finally a word about notation: one designs notation so as to be in position to say simple things simply, to highlight essentials while not masking critical distinctions. In classical non-Abelian gauge theory only simple things are going on (some linear algebra, some elementary calculus), but they are going on in potentially confusing constellation. To write gauge field theory in explicit detail would bring into play such blizzard of indices (of diverse ranges and meanings) as to make it very difficult to gain a sense of *what is going on*. But to surpress such detail—to adopt the scrubbed notation standard to publication in the field—is to risk losing a vivid sense of what the marks on the page specifically *mean*. My purpose here is to point out that classical mathematics does supply a tool which in this instance permits one to strike a happy medium; the tool has a name, but it is a name seldom encountered in the gauge field theoretic literature.

Suppose, by way of introduction, that we have interest in a pair of 3-vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, which we propose first to subject independently to linear transformations, and then to rotationally intermix; we might write[31]

$$\left.\begin{aligned}\boldsymbol{x}\longrightarrow \mathbb{A}\,\boldsymbol{x}\\ \boldsymbol{y}\longrightarrow \mathbb{B}\,\boldsymbol{y}\end{aligned}\right\}\longrightarrow\left\{\begin{aligned}\cos\theta\cdot\mathbb{A}\,\boldsymbol{x}-\sin\theta\cdot\mathbb{B}\,\boldsymbol{y}\\ \sin\theta\cdot\mathbb{A}\,\boldsymbol{x}+\cos\theta\cdot\mathbb{B}\,\boldsymbol{y}\end{aligned}\right.$$

But if we "stack" the 3-vectors (forming a 6-vector) we acquire this alternative means of displaying the same information:

$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\longrightarrow\begin{pmatrix}\mathbb{A} & \mathbb{O}\\ \mathbb{O} & \mathbb{B}\end{pmatrix}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\longrightarrow\begin{pmatrix}\cos\theta\cdot\mathbb{I} & -\sin\theta\cdot\mathbb{I}\\ \sin\theta\cdot\mathbb{I} & \cos\theta\cdot\mathbb{I}\end{pmatrix}\begin{pmatrix}\mathbb{A} & \mathbb{O}\\ \mathbb{O} & \mathbb{B}\end{pmatrix}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}$$

The "Kronecker product" (sometimes called the "direct product") of
- an $m\times n$ matrix $\mathbb{A}$ onto
- a $p\times q$ matrix $\mathbb{B}$

is the $mp\times nq$ matrix defined[32]

$$\mathbb{A}\otimes\mathbb{B}\equiv\|a_{ij}\,\mathbb{B}\| \tag{84}$$

In that notation, the "matrices with matrix-valued elements" encountered in my example can be described

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\otimes\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

[31] For the purposes of this discussion I revert to my former practice of writing vectors in boldface, matrices in blackboard doubleface.

[32] The alternative definition $\mathbb{A}\otimes\mathbb{B}\equiv\|\mathbb{A}\,b_{ij}\|$ gives rise to a "mirror image" of the standard theory. Good discussions can be found in E. P. Wigner, *Group Theory and its Application to the Quantum Theory of Atomic Spectra* (1959), Chapter 2; P. Lancaster, *Theory of Matrices* (1969), §8.2; Richard Bellman, *Introduction to Matrix Analysis* (2nd edition 1970), Chapter 12, §§5–13.

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad + \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Manipulation of expressions involving Kronecker products is accomplished by appeal to general statements such as the following:

$$k(\mathbb{A} \otimes \mathbb{B}) = (k\mathbb{A}) \otimes \mathbb{B} = \mathbb{A} \otimes (k\mathbb{B}) \tag{85.1}$$

$$\left.\begin{aligned} (\mathbb{A} + \mathbb{B}) \otimes \mathbb{C} &= \mathbb{A} \otimes \mathbb{C} + \mathbb{B} \otimes \mathbb{C} \\ \mathbb{A} \otimes (\mathbb{B} + \mathbb{C}) &= \mathbb{A} \otimes \mathbb{B} + \mathbb{A} \otimes \mathbb{C} \end{aligned}\right\} \tag{85.2}$$

$$\mathbb{A} \otimes (\mathbb{B} \otimes \mathbb{C}) = (\mathbb{A} \otimes \mathbb{B}) \otimes \mathbb{C} \equiv \mathbb{A} \otimes \mathbb{B} \otimes \mathbb{C} \tag{85.3}$$

$$(\mathbb{A} \otimes \mathbb{B})^{\mathsf{T}} = \mathbb{A}^{\mathsf{T}} \otimes \mathbb{B}^{\mathsf{T}} \tag{85.4}$$

$$\mathrm{tr}(\mathbb{A} \otimes \mathbb{B}) = \mathrm{tr}\mathbb{A} \cdot \mathrm{tr}\,\mathbb{B} \tag{85.5}$$

—all of which are valid except when meaningless.[33] Less obviously (but often very usefully)

$$(\mathbb{A} \otimes \mathbb{B})(\mathbb{C} \otimes \mathbb{D}) = \mathbb{AC} \otimes \mathbb{BD} \quad \text{if} \quad \begin{cases} \mathbb{A} \text{ and } \mathbb{C} \text{ are } m \times m \\ \mathbb{B} \text{ and } \mathbb{D} \text{ are } n \times n \end{cases} \tag{85.6}$$

from which one can extract[34]

$$\mathbb{A} \otimes \mathbb{B} = (\mathbb{A} \otimes \mathbb{I}_n)(\mathbb{I}_m \otimes \mathbb{B}) \tag{85.7}$$

$$\det(\mathbb{A} \otimes \mathbb{B}) = (\det \mathbb{A})^n (\det \mathbb{B})^m \tag{85.8}$$

$$(\mathbb{A} \otimes \mathbb{B})^{-1} = \mathbb{A}^{-1} \otimes \mathbb{B}^{-1} \tag{85.9}$$

Here I have used $\mathbb{I}_m$ to designate the $m \times m$ identity matrix, and below use $\boldsymbol{I}_m$ for that same purpose (though when the dimension is obvious from the context I allow myself to omit the subscript).

Dirac theory with local SU(2) gauge invariance. I look now to the historic first instance of a "non-Abelian gauge field theory"—the theory put forward by Yang & Mills (though presented here as an exercise in *classical* field theory). We select Dirac theory as our starting point because it is, in many respects, "simplest possible." And we select $SU(2)$ as our non-Abelian gauge group for that same reason (Yang & Mills had their own—more pressingly physical—reasons for both selections).

---

[33] Recall that one cannot add matrices unless they are co-dimensional, and does not speak of the trace of a matrix unless it is square.

[34] See Lancaster[32] for the detailed arguments.

Our first assignment $\boxed{\text{STEP ZERO}}$ is to construct a variant of the Dirac theory which exhibits *global* $SU(2)$ invariance. To that end we "stack" two copies of the familiar Dirac theory: we assemble an 8-component complex field

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \quad \text{with} \quad \psi^a = \begin{pmatrix} \psi^a_1 \\ \psi^a_2 \\ \psi^a_3 \\ \psi^a_4 \end{pmatrix} \quad : \quad a = 1, 2 \tag{86}$$

(for lack of standard terminology I will call the superscripts "gauge indices" and the subscripts "Dirac indices") and require that it satisfy the field equations

$$\big(\boldsymbol{\Gamma}^\mu \partial_\mu + i\boldsymbol{\varkappa}\big)\psi = 0 \quad \text{and adjoint} \tag{87}$$

where

$$\boldsymbol{\Gamma}^\mu \equiv \boldsymbol{I}_2 \otimes \boldsymbol{\gamma}^\mu \quad \text{and} \quad \boldsymbol{\varkappa} \equiv \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{pmatrix} \otimes \boldsymbol{I}_4 \tag{88}$$

The field equations arise from

$$\mathcal{L}_0(\psi, \tilde{\psi}, \partial\psi, \partial\tilde{\psi}) = \hbar c\Big[\tfrac{1}{2}i\big\{\tilde{\psi}\boldsymbol{\Gamma}^\alpha\psi_{,\alpha} - \tilde{\psi}_{,\alpha}\boldsymbol{\Gamma}^\alpha\psi\big\} - \tilde{\psi}\boldsymbol{\varkappa}\psi\Big] \tag{89}$$

which—because the matrices $\boldsymbol{\Gamma}^\mu$ and $\boldsymbol{\varkappa}$ share the block structure

$$\begin{pmatrix}
\bullet & \bullet & \bullet & \bullet & 0 & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & 0 & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & 0 & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \bullet & \bullet & \bullet & \bullet \\
0 & 0 & 0 & 0 & \bullet & \bullet & \bullet & \bullet \\
0 & 0 & 0 & 0 & \bullet & \bullet & \bullet & \bullet \\
0 & 0 & 0 & 0 & \bullet & \bullet & \bullet & \bullet
\end{pmatrix}$$

—is just the sum of the Dirac Lagrangians which separately regulate the (presently) uncoupled motion of ψ^1 and ψ^2.

The 8×8 matrices $\boldsymbol{S} \equiv \boldsymbol{s} \otimes \boldsymbol{I}_4$ mimic the multiplicative properties of the 2×2 unimodular unitary matrices $\boldsymbol{s}$

$$\begin{aligned}
\boldsymbol{S}_2\boldsymbol{S}_1 &= (\boldsymbol{s}_2 \otimes \boldsymbol{I}_4)(\boldsymbol{s}_1 \otimes \boldsymbol{I}_4) \\
&= \boldsymbol{s}_2\boldsymbol{s}_1 \otimes \boldsymbol{I}_4 \quad \text{by (85.6)}
\end{aligned}$$

and are readily seen to be themselves unimodular and unitary.[35] Corresponding to the "Pauli decomposition"

$$\boldsymbol{s} = s^\mu \boldsymbol{\sigma}_\mu$$

we have

$$\boldsymbol{S} = s^\mu \boldsymbol{\Sigma}_\mu \quad \text{with} \quad \boldsymbol{\Sigma}_\mu \equiv \boldsymbol{\sigma}_\mu \otimes \boldsymbol{I}_4 \tag{90}$$

---

[35] Use (85.4), (85.8) and (85.9).

Specifically

$$\mathbf{\Sigma}_0 \equiv \begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{pmatrix}, \; \mathbf{\Sigma}_1 \equiv \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ \boldsymbol{I} & \mathbf{0} \end{pmatrix}, \; \mathbf{\Sigma}_2 \equiv \begin{pmatrix} \mathbf{0} & -\boldsymbol{i} \\ \boldsymbol{i} & \mathbf{0} \end{pmatrix}, \; \mathbf{\Sigma}_3 \equiv \begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{I} \end{pmatrix}$$

where the sub-matrices are 4×4. Note particularly that $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ have the reverse of the block structure pictured above.

From the reversed block structure just mentioned it follows that the Lagrangian (89) will be (manifestly) invariant under $\psi \longrightarrow \boldsymbol{S}\psi$ if and only if $\varkappa_1 = \varkappa_2$, which we will assume.[36] $SU(2)$ is a 3-parameter group, so from the *global* $SU(2)$-invariance of $\mathcal{L}_0$—just established— follow three conservation laws. Infinitesimally, we have

$$\begin{aligned} \psi &\longrightarrow \psi + \delta\psi \\ \delta\psi &= i\mathbf{\Sigma}_1\psi \cdot \delta\omega_1 + i\mathbf{\Sigma}_2\psi \cdot \delta\omega_2 + i\mathbf{\Sigma}_3\psi \cdot \delta\omega_3 \end{aligned}$$

and so are led by Noether to the statements (compare (69))

$$\partial_\mu Q_r^\mu = 0 \quad \text{with} \quad Q_r^\mu \equiv \tfrac{1}{2}c\big\{\tilde{\psi}\boldsymbol{\Gamma}^\mu\mathbf{\Sigma}_r\psi + \text{conjugate}\big\} \quad : \quad r = 1, 2, 3$$

Letting the $\mathbf{\Sigma}$-matrices, as described above, act upon $\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$ we obtain

$$\mathbf{\Sigma}_1\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} \psi^2 \\ \psi^1 \end{pmatrix}, \quad \mathbf{\Sigma}_2\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = -i\begin{pmatrix} \psi^2 \\ -\psi^1 \end{pmatrix}, \quad \mathbf{\Sigma}_3\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} \psi^1 \\ -\psi^2 \end{pmatrix}$$

which yield these more explicit descriptions of the conserved currents Q_r^μ:

$$Q_1^\mu = \tfrac{1}{2}c\big(\tilde{\psi}^1\boldsymbol{\gamma}^\mu\psi^2 + \tilde{\psi}^2\boldsymbol{\gamma}^\mu\psi^1\big) \tag{91.1}$$

$$Q_2^\mu = -i\tfrac{1}{2}c\big(\tilde{\psi}^1\boldsymbol{\gamma}^\mu\psi^2 - \tilde{\psi}^2\boldsymbol{\gamma}^\mu\psi^1\big) \tag{91.2}$$

$$Q_3^\mu = \tfrac{1}{2}c\big(\tilde{\psi}^1\boldsymbol{\gamma}^\mu\psi^1 - \tilde{\psi}^2\boldsymbol{\gamma}^\mu\psi^2\big) \tag{91.3}$$

The twinned Dirac Lagrangian $\mathcal{L}_0$ is also (manifestly) U(1)-invariant, which leads to conservation of

$$Q_0^\mu = \tfrac{1}{2}c\big(\tilde{\psi}^1\boldsymbol{\gamma}^\mu\psi^1 + \tilde{\psi}^2\boldsymbol{\gamma}^\mu\psi^2\big) \tag{91.0}$$

which is the anticipated twinned instance of (69.2).[37]

---

[36] At this point Yang & Mills, following in the footsteps of the inventors of the isotopic spin concept (Heisenberg, Wigner and others, in the late 1930's), were content to draw upon the physical circumstance that the proton and neutron masses are *nearly* the same:

$$m_p = 938.280 \text{ MeV}/c^2 \quad \text{while} \quad m_n = 939.573 \text{ MeV}/c^2$$

[37] The quartet of conservation laws (90) are structurally reminiscent of a quartet encountered in connection with the classical mechanics of an isotropic 2-dimensional oscillator. Nor is that formal connection surprising: here $SU(2)$ is an explicitly imposed symmetry; there it is a "hidden symmetry." See the discussion surrounding equation (162) in my "Ellipsometry" (1999).

Our objective is to achieve *local* $SU(2)$ invariance, and we confront at the outset the familiar problem that

$$\psi \longrightarrow \psi' = \boldsymbol{S}(x)\psi \quad \text{induces} \quad \psi_{,\mu} \longrightarrow \psi'_{,\mu} = \boldsymbol{S}(x)\psi_{,\mu} + \boldsymbol{S}_{,\mu}(x)\psi$$

It is to escape the force of the elementary circumstance that ψ and $\psi_{,\mu}$ transform by different rules that $\boxed{\text{STEP ONE}}$ we make what we have learned to call the "minimal coupling substitution"[38]

$$\begin{aligned}&\boldsymbol{\partial}_\mu\\ &\downarrow\\ &\boldsymbol{\mathfrak{D}}_\mu = \boldsymbol{\partial}_\mu - ig\boldsymbol{A}_\mu \quad \text{with} \quad g \equiv e/\hbar c\end{aligned} \tag{92}$$

and $\boxed{\text{STEP TWO}}$ concoct $\boldsymbol{A}_\mu \to \boldsymbol{A}'_\mu$ so as to achieve $(\boldsymbol{\mathfrak{D}}_\mu\psi)' = \boldsymbol{S}(x)(\boldsymbol{\mathfrak{D}}_\mu\psi)$. From

$$\Big[(\boldsymbol{\partial}_\mu - ig\boldsymbol{A}'_\mu)\boldsymbol{S} = \boldsymbol{S}\boldsymbol{\partial}_\mu + \boldsymbol{S}_{,\mu} - ig\boldsymbol{A}'_\mu\boldsymbol{S}\Big] = \boldsymbol{S}(\boldsymbol{\partial}_\mu - ig\boldsymbol{A}_\mu)$$

we are led thus to this enlarged interpretation

$$\left.\begin{aligned}\psi &\longrightarrow \psi' = \boldsymbol{S}\,\psi\\ \boldsymbol{A}_\mu &\longrightarrow \boldsymbol{A}'_\mu = \boldsymbol{S}\boldsymbol{A}_\mu\boldsymbol{S}^{-1} + i\,\tfrac{1}{g}\,\boldsymbol{S}_{,\mu}\boldsymbol{S}^{-1}\end{aligned}\right\} \tag{93}$$

of what we shall understand the phrase "local $SU(2)$ gauge transformation" to *mean*.

Equation (93) describes the non-Abelian counterpart to (73), and the points of similarity/difference stand out even more clearly when we write

$$\boldsymbol{S} = e^{ig\boldsymbol{\Omega}} \quad : \quad \boldsymbol{\Omega} \text{ traceless hermitian}$$

Whereas the $A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu\Omega$ encountered in the Abelian case $U(1)$ had the form of a

derivative-dependent shift

its non-Abelian counterpart in (93) has the form

similarity transformation + derivative-dependent shift

It is non-commutivity ($[\boldsymbol{S}, \boldsymbol{A}_\mu] \neq \boldsymbol{0}$) which prevents the $\boldsymbol{S}$ from slipping past the $\boldsymbol{A}_\mu$ and cancelling the $\boldsymbol{S}^{-1}$; i.e., which accounts for the survival of the similarity transformation as a characteristic component of non-Abelian gauge.

Differentiation of the unitarity condition $\boldsymbol{S}\boldsymbol{S}^\dagger = \boldsymbol{I}$ supplies the information that $i\boldsymbol{S}_{,\mu}\boldsymbol{S}^{-1}$ is invariably hermitian. And, of course, $\boldsymbol{S}\boldsymbol{A}_\mu\boldsymbol{S}^{-1}$ is (traceless) hermitian if $\boldsymbol{A}_\mu$ is. We therefore *assume the gauge matrices* $\boldsymbol{A}_\mu$ *to be hermitian*, and observe it to be an implication of (93) that

$$\boldsymbol{A}_\mu \longrightarrow \boldsymbol{A}'_\mu \text{ preserves hermiticity}$$

The matrices $\boldsymbol{S}$ are, however, not just *any* old unimodular unitary 8×8 matrices; they possess the specialized structure $\boldsymbol{S} \equiv \boldsymbol{s} \otimes \boldsymbol{I}_4$, reflecting the fact that our interest at the moment lies not in $SU(8)$ but in $SU(2)$. We impute that

[38] It is fussy of me to write $\boldsymbol{\partial}_\mu$ (meaning $\boldsymbol{I}\partial_\mu$) in place more simply of ∂_μ, but it offends my eye to "add a scalar to a matrix."

structure also to the hermitian matrices $\boldsymbol{A}_\mu$, writing

$$\boldsymbol{A}_\mu \equiv \boldsymbol{a}_\mu \otimes \boldsymbol{I}_4 \quad \text{with } \boldsymbol{a}_\mu \text{ } 2\times 2 \text{ hermitian}$$

From the unimodularity of $\boldsymbol{S} = e^{ig\boldsymbol{\Omega}}$ we know that (as previously remarked) $\boldsymbol{\Omega}$ is necessarily traceless, and can therefore be developed

$$\boldsymbol{\Omega} = \qquad \Omega^1 \boldsymbol{\Sigma}_1 + \Omega^2 \boldsymbol{\Sigma}_2 + \Omega^3 \boldsymbol{\Sigma}_3$$

But we might expect to have to write

$$\boldsymbol{A}_\mu = A^0_\mu \boldsymbol{\Sigma}_0 + A^1_\mu \boldsymbol{\Sigma}_1 + A^2_\mu \boldsymbol{\Sigma}_2 + A^3_\mu \boldsymbol{\Sigma}_3$$

I will argue that the ^{0} term can be abandoned; i.e., that one can without loss of generality *assume the gauge matrices $\boldsymbol{A}_\mu$ to be traceless*. The argument proceeds in two steps, of which the first, unfortunately, is a bit intricate: we demonstrate that the traceless assumption—if made—is transformationally stable. Certainly it is the case[39] that if $\boldsymbol{A}_\mu$ is traceless then so is $\boldsymbol{S}\boldsymbol{A}_\mu\boldsymbol{S}^{-1}$. But how to show that $\boldsymbol{S}_{,\mu}\boldsymbol{S}^{-1}$ is traceless? I quote two general identities[40]

$$\begin{aligned} e^{ig\boldsymbol{\Omega}}\boldsymbol{A}\,e^{-ig\boldsymbol{\Omega}} &= \Big\{e^{ig\boldsymbol{\Omega}}, \boldsymbol{A}\Big\} \\ &\equiv \boldsymbol{A} + ig[\boldsymbol{\Omega}, \boldsymbol{A}] + \tfrac{1}{2!}(ig)^2[\boldsymbol{\Omega}, [\boldsymbol{\Omega}, \boldsymbol{A}]] + \cdots \end{aligned} \tag{94.1}$$

$$\begin{aligned} \frac{\partial e^{ig\boldsymbol{\Omega}}}{\partial x}\cdot e^{-ig\boldsymbol{\Omega}} &= \Big\{\frac{e^{ig\boldsymbol{\Omega}} - \boldsymbol{I}}{\boldsymbol{\Omega}}, \tfrac{\partial}{\partial x}\boldsymbol{\Omega}\Big\} \\ &= ig\tfrac{\partial}{\partial x}\boldsymbol{\Omega} + \tfrac{1}{2!}(ig)^2[\boldsymbol{\Omega}, \tfrac{\partial}{\partial x}\boldsymbol{\Omega}] + \tfrac{1}{3!}(ig)^3[\boldsymbol{\Omega}, [\boldsymbol{\Omega}, \tfrac{\partial}{\partial x}\boldsymbol{\Omega}]] + \cdots \end{aligned} \tag{94.2}$$

but have immediate need only of the second. Clearly $\frac{\partial}{\partial x}\boldsymbol{\Omega}$ is traceless if $\boldsymbol{\Omega}$ is. But it was remarked already in connection with (79) that

$$[\text{hermitian}, \text{hermitian}] = i\,(\text{traceless hermitian})$$

so each of the nested commutators presented on the right side of (94.2) are in fact traceless. This establishes the point at issue: if $\boldsymbol{A}_\mu$ is traceless then so, according to (93), is $\boldsymbol{A}'_\mu$. Completion of the argument must await one further development:

Given our initial interest in the Dirac Lagrangian (which at (89) we duplicated, in order to get this show on the road), we $\boxed{\text{STEP THREE}}$ look to the modified system

$$\mathcal{L}_1(\psi, \tilde\psi, \partial\psi, \partial\tilde\psi, A^1, A^2, A^3) = \mathcal{L}_0(\psi, \tilde\psi, \mathcal{D}\psi, \tilde{\mathcal{D}}\tilde\psi)$$

---

[39] Use $\text{tr}(\mathbb{A}\mathbb{B}) = \text{tr}(\mathbb{B}\mathbb{A})$.

[40] See, for example, §4 in R. M. Wilcox, "Exponential operators & parameter differentiation in quantum physics," J. Math. Phys. **8**, 962 (1967). The identities in question are attributed by W. Magnus (in "On the exponential solution of differential equations for a linear operator," Comm. Pure & Appl. Math. **7**, 649 (1954)) to F. Hausdorff (1906). In (93.1) one is free to install subscripts on all the $\boldsymbol{A}$'s, while in (93.2) one can assign any meaning to the parameter x.

In more explicit detail we have

$$
\begin{aligned}
\mathcal{L}_1 &= \hbar c\Big[\tfrac{1}{2}i\Big\{\tilde{\psi}\boldsymbol{\gamma}^\alpha(\psi_{,\alpha} - i\tfrac{e}{\hbar c}\boldsymbol{A}_\alpha\psi) - (\tilde{\psi}_{,\alpha} + i\tfrac{e}{\hbar c}\tilde{\psi}\boldsymbol{A}_\alpha)\boldsymbol{\gamma}^\alpha\psi\Big\} - \varkappa\tilde{\psi}\psi\Big] \\
&= \hbar c\Big[\tfrac{1}{2}i\Big\{\tilde{\psi}\boldsymbol{\gamma}^\alpha\big[\psi_{,\alpha} - i\tfrac{e}{\hbar c}(A^0_\alpha\boldsymbol{\Sigma}_0 + A^p_\alpha\boldsymbol{\Sigma}_p)\psi\big] \\
&\qquad\qquad - \big[\tilde{\psi}_{,\alpha} + i\tfrac{e}{\hbar c}\tilde{\psi}(A^0_\alpha\boldsymbol{\Sigma}_0 + A^p_\alpha\boldsymbol{\Sigma}_p)\big]\boldsymbol{\gamma}^\alpha\psi\Big\} - \varkappa\tilde{\psi}\psi\Big]
\end{aligned}
\tag{95}
$$

We observe that an $A^0_\mu\boldsymbol{\Sigma}_0$ term, if assumed to be present in the composition of $\boldsymbol{A}_\mu$, would (since $\boldsymbol{\Sigma}_0 = \boldsymbol{I}_8$ commutes with everything) simply replicate the adjustment we would make—and *did* make at (70)—if we were trying to achieve $U(1)$ gauge invariance. We will agree to abandon the hypothetical ^{0} term on grounds that it is passive with respect to $SU(2)$, and that its discovered predisposition is to talk about something (electrodynamics) other than the subject that presently interests us.

So we have

$$
\boldsymbol{A}_\mu(x) = A^1_\mu(x)\boldsymbol{\Sigma}_1 + A^2_\mu(x)\boldsymbol{\Sigma}_2 + A^3_\mu(x)\boldsymbol{\Sigma}_3 \tag{96}
$$

and at this point make the acquaintance of the *three "gauge fields"—one for each generator—called into being by the imposition of local $SU(3)$ invariance.* It is important to notice that the gauge fields $A^p_\mu(x)$ are necessarily *real-valued vector* fields, and that they arise as "coordinates" of the more fundamental objects $\boldsymbol{A}_\mu(x)$: select a different basis (in the space of traceless hermitian 2×2 matrices) and be led from the same $\boldsymbol{A}_\mu(x)$ to a different trio of gauge fields. The matrix-valued gauge field $\boldsymbol{A}_\mu(x)$ cannot be accorded "physical immediacy" because susceptible to gauge, and its coordinates have an even more tenuous claim to reality.

We have now on-stage a total of twelve real-number-valued fields

$$
\begin{matrix}
A^1_0(x) & A^1_1(x) & A^1_2(x) & A^1_3(x) \\
A^2_0(x) & A^2_1(x) & A^2_2(x) & A^2_3(x) \\
A^3_0(x) & A^3_1(x) & A^3_2(x) & A^3_3(x)
\end{matrix}
\tag{97}
$$

Lorentz transformations linearly recombine the columns; gauge transformations linearly recombine the rows. It is instructive to inquire into details of the latter process. To that end: observe in connection with (94.1), which presents a "Lie series" (i.e., a series of nested commutators) on its right hand side, that

$$
\begin{aligned}
[\boldsymbol{\Omega}, \boldsymbol{A}\,] &= \Omega^p A^q[\,\boldsymbol{\Sigma}_p, \boldsymbol{\Sigma}_q\,] \quad \text{with } \sum_{p,q=1}^{3} \text{ here as henceforth understood} \\
&= \Omega^p A^q\, i c_p{}^r{}_q\, \boldsymbol{\Sigma}_r
\end{aligned}
\tag{98.1}
$$

and that from the $SU(2)$ commutation relations (76.3) one has, in the Pauli basis,

$$
c_p{}^r{}_q = 2\,\mathrm{sgn}\binom{1\,2\,3}{pqr} \tag{98.2}
$$

which, by the way, conforms nicely to (83.1).[41] Returning with (98.2) to (98.1), we obtain

$$\begin{aligned}
[\boldsymbol{\Omega},\boldsymbol{A}\,] &= 2i\,(\vec{\Omega}\times\vec{A}\,)^r\,\boldsymbol{\Sigma}_r\\
&= \Bigg[\underbrace{\begin{pmatrix} 0 & -2i\Omega^3 & 2i\Omega^2\\ 2i\Omega^3 & 0 & -2i\Omega^1\\ -2i\Omega^2 & 2i\Omega^1 & 0\end{pmatrix}}_{\mathbb{Z}}\begin{pmatrix}A^1\\A^2\\A^3\end{pmatrix}\Bigg]\cdot\begin{pmatrix}\boldsymbol{\Sigma}_1\\\boldsymbol{\Sigma}_2\\\boldsymbol{\Sigma}_3\end{pmatrix}\\
&= (\mathbb{Z}\vec{A}\,)^r\,\boldsymbol{\Sigma}_r\\
[\boldsymbol{\Omega},[\boldsymbol{\Omega},\boldsymbol{A}\,]] &= (\mathbb{Z}^2\vec{A}\,)^r\,\boldsymbol{\Sigma}_r\\
&\;\vdots
\end{aligned}$$

Returning with this information to (94) we find that the second half of (93) can be rendered

$$\vec{A}_\mu \longrightarrow \vec{A}'_\mu = e^{ig\mathbb{Z}}\vec{A}_\mu - \frac{e^{ig\mathbb{Z}}-\mathbb{I}}{ig\mathbb{Z}}\,\vec{\Omega}_{,\mu} \tag{99.1}$$

Looking now with one eye to the definition of $\mathbb{Z}$ and with the other to the equations which (just prior to (82)) served to define $\{\mathbb{C}_1,\mathbb{C}_2,\mathbb{C}_3\}$, we see that

$$\mathbb{Z} = \Omega^1\mathbb{C}_1 + \Omega^2\mathbb{C}_2 + \Omega^3\mathbb{C}_3 \tag{100}$$

and notice, moreover, that

$$ig\mathbb{Z} = g\begin{pmatrix} 0 & 2\Omega^3 & -2\Omega^2\\ -2\Omega^3 & 0 & 2\Omega^1\\ 2\Omega^2 & -2\Omega^1 & 0\end{pmatrix}\quad\text{is real antisymmetric}$$

so

$$\mathbb{R}(x) \equiv e^{ig\mathbb{Z}(x)}\quad\text{is a rotation matrix, an element of } O(3)$$

In this notation (99.1) can be rendered

$$\vec{A}_\mu \longrightarrow \vec{A}'_\mu = \mathbb{R}\vec{A}_\mu - \tfrac{\mathbb{R}-\mathbb{I}}{\log\mathbb{R}}\,\vec{\Omega}_{,\mu} \tag{99.2}$$

Equations (99) say the same thing, the latter being a "coordinatized" version of the former. Note the natural occurance of the adjoint representation at (98).

We have now $\boxed{\text{STEP FOUR}}$ to prepare to lauch the gauge matrix $\boldsymbol{A}_\mu$ (equivalently: the gauge fields $A^p_\mu : p = 1,2,3$) into dynamical motion. To that

[41] One should resist the temptation to write $c_p{}^r{}_q = 2\epsilon_{pqr}$, for although the equation is *numerically* correct *in the Pauli basis* it is transformationally screwy: it presents ^{r} on the left but $_r$ on the right, and would come unstuck if one were to abandon the Pauli basis in favor of some arbitrary alternative.

end, taking (15) as our model, we examine the gauge transformation properties of

$$\boldsymbol{f}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu$$

From (93) we obtain

$$\begin{aligned}\partial_\mu \boldsymbol{A}_\nu \longrightarrow \partial_\mu \boldsymbol{A}'_\nu = \boldsymbol{S}_{,\mu}\boldsymbol{A}_\nu \boldsymbol{S}^{-1} + \boldsymbol{S}(\partial_\mu \boldsymbol{A}_\nu)\boldsymbol{S}^{-1} + \boldsymbol{S}\boldsymbol{A}_\nu(\boldsymbol{S}^{-1})_{,\mu}\\ + i\,\tfrac{1}{g}\,\boldsymbol{S}_{,\nu\mu}\boldsymbol{S}^{-1} + i\,\tfrac{1}{g}\,\boldsymbol{S}_{,\nu}(\boldsymbol{S}^{-1})_{,\mu}\end{aligned}$$

But $\partial_\mu(\boldsymbol{S}\boldsymbol{S}^{-1}) = \boldsymbol{0}$ supplies $(\boldsymbol{S}^{-1})_{,\mu} = -\boldsymbol{S}^{-1}\boldsymbol{S}_{,\mu}\boldsymbol{S}^{-1}$ so

$$\begin{aligned}&= \boldsymbol{S}(\partial_\mu \boldsymbol{A}_\nu)\boldsymbol{S}^{-1}\\ &\quad + \boldsymbol{S}_{,\mu}\boldsymbol{A}_\nu \boldsymbol{S}^{-1} - \boldsymbol{S}\boldsymbol{A}_\nu \boldsymbol{S}^{-1}\boldsymbol{S}_{,\mu}\boldsymbol{S}^{-1} + i\tfrac{1}{g}\,\boldsymbol{S}_{,\mu\nu}\boldsymbol{S}^{-1} - i\tfrac{1}{g}\,\boldsymbol{S}_{,\nu}\boldsymbol{S}^{-1}\boldsymbol{S}_{,\mu}\boldsymbol{S}^{-1}\end{aligned}$$

gives

$$\boldsymbol{f}_{\mu\nu} \longrightarrow \boldsymbol{f}'_{\mu\nu} = \boldsymbol{S}\boldsymbol{f}_{\mu\nu}\boldsymbol{S}^{-1} + \big\{\text{unwelcome term}\big\}$$

with

$$\begin{aligned}\big\{\text{unwelcome term}\big\} = (&\boldsymbol{S}_{,\mu}\boldsymbol{A}_\nu \boldsymbol{S}^{-1} - \boldsymbol{S}_{,\nu}\boldsymbol{A}_\mu \boldsymbol{S}^{-1})\\ &+ (\boldsymbol{S}\boldsymbol{A}_\mu \boldsymbol{S}^{-1}\boldsymbol{S}_{,\nu}\boldsymbol{S}^{-1} - \boldsymbol{S}\boldsymbol{A}_\nu \boldsymbol{S}^{-1}\boldsymbol{S}_{,\mu}\boldsymbol{S}^{-1})\\ &+ i\tfrac{1}{g}(\boldsymbol{S}_{,\mu}\boldsymbol{S}^{-1}\boldsymbol{S}_{,\nu}\boldsymbol{S}^{-1} - \boldsymbol{S}_{,\nu}\boldsymbol{S}^{-1}\boldsymbol{S}_{,\mu}\boldsymbol{S}^{-1})\end{aligned}$$

The non-commutivity responsible for the existence of the "unwelcome term" is responsible also for the existence of a second $\mu\nu$-antisymmetric construct—namely the commutator $[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]$, which is found by straightforward calculation to transform

$$[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu] \longrightarrow [\boldsymbol{A}'_\mu, \boldsymbol{A}'_\nu] = \boldsymbol{S}[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]\boldsymbol{S}^{-1} - i\tfrac{1}{g}\big\{\text{same unwelcome term}\big\}$$

The pretty implication is that

$$\boldsymbol{F}_{\mu\nu} \equiv (\partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu) - ig\,(\boldsymbol{A}_\mu \boldsymbol{A}_\nu - \boldsymbol{A}_\nu \boldsymbol{A}_\mu) \tag{101}$$

gauge-transforms by simple similarity transformation (since the "unwelcome terms" cancel):

$$\boldsymbol{F}_{\mu\nu} \longrightarrow \boldsymbol{F}'_{\mu\nu} = \boldsymbol{S}\boldsymbol{F}_{\mu\nu}\boldsymbol{S}^{-1} \tag{102.1}$$

Which (to say the same thing another way) means that if we write $\boldsymbol{F}_{\mu\nu} = F^p_{\mu\nu}\boldsymbol{\Sigma}_p$ and assemble

$$\vec{F}_{\mu\nu} \equiv \begin{pmatrix} F^1_{\mu\nu} \\ F^2_{\mu\nu} \\ F^3_{\mu\nu} \end{pmatrix} = (\vec{A}_{\beta,\alpha} - \vec{A}_{\alpha,\beta}) + 2g\vec{A}_\alpha \times \vec{A}_\beta \tag{103}$$

then we have

$$\vec{F}_{\mu\nu} \longrightarrow \vec{F}'_{\mu\nu} = \mathbb{R}\vec{F}_{\mu\nu} \tag{102.2}$$

Comparison of (102.1) and (102.2) presents an instance of the well-known connection between $SU(2)$ and $O(3)$.

To kill the subscripts we proceed now in imitation of (16.2), constructing

$$\boldsymbol{F}_{\alpha\beta}\boldsymbol{F}^{\alpha\beta}$$

which is Lorentz invariant, and responds to gauge transformations by similarity transformation. From this result it follows that

- the eigenvalues of the 8×8 matrix $\boldsymbol{F}^2 \equiv \boldsymbol{F}_{\alpha\beta}\boldsymbol{F}^{\alpha\beta}$ are gauge-invariant; equivalently
- the coefficients in $\det(\boldsymbol{F}^2 - \lambda\boldsymbol{I})$ are gauge-invariant; equivalently[42]
- the traces of integral powers of $\boldsymbol{F}^2$ are gauge-invariant.

Proceeding in imitation of our experience in simpler situations (but from no higher necessity[43]) we construct

$$\mathcal{L}_{\text{free gauge field}}(\boldsymbol{A},\partial\boldsymbol{A}) = \tfrac{1}{4}\text{tr}\big\{\boldsymbol{F}_{\alpha\beta}\boldsymbol{F}^{\alpha\beta}\big\} = \tfrac{1}{4}g^{\alpha\rho}g^{\beta\sigma}F^p_{\alpha\beta}F^q_{\rho\sigma}\text{tr}\big\{\boldsymbol{\Sigma}_p\boldsymbol{\Sigma}_q\big\}$$

But

$$\begin{aligned}
\text{tr}\big\{\boldsymbol{\Sigma}_p\boldsymbol{\Sigma}_q\big\} &= \text{tr}\big\{(\boldsymbol{\sigma}_p\otimes\boldsymbol{I}_4)(\boldsymbol{\sigma}_q\otimes\boldsymbol{I}_4)\big\} &&\text{by (88)}\\
&= \text{tr}\big\{(\boldsymbol{\sigma}_p\boldsymbol{\sigma}_q\otimes\boldsymbol{I}_4)\big\} &&\text{by (85.6)}\\
&= 4\,\text{tr}\big\{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_q\big\} &&\text{by (85.5)}\\
&= 8\,\delta_{pq} &&\text{by (76)}\\
&= k_{pq} &&\text{by (82)}
\end{aligned}$$

So we have[44]

$$\mathcal{L}_{\text{free gauge field}}(\boldsymbol{A},\partial\boldsymbol{A}) = \tfrac{1}{4}g^{\alpha\rho}g^{\beta\sigma}k_{pq}F^p_{\alpha\beta}F^q_{\rho\sigma} \tag{104}$$

$$\uparrow$$

$$F^p_{\alpha\beta} \equiv A^p_{\beta,\alpha} - A^p_{\alpha,\beta} + gc_u{}^p{}_v A^u_\alpha A^v_\beta \tag{105}$$

and from

$$\left\{\partial_\mu\frac{\partial}{\partial A^r_{\nu,\mu}} - \frac{\partial}{\partial A^r_\nu}\right\}\mathcal{L}_{\text{free gauge field}} = 0$$

---

[42] See p. 13 of "Some applications of an elegant formula due to V. F. Ivanoff" in COLLECTED SEMINARS (1963/70).

[43] It would be interesting on some future occasion to *evaluate* $\det(\boldsymbol{F}^2 - \lambda\boldsymbol{I})$, to see what expressions $\text{tr}(\boldsymbol{F}^{2n})$ actually contribute to the coefficients, and to see whether incorporation of such higher-order terms into the Lagrangian leads to a useful generalization of standard theory. . . else to identify the principle which forces their exclusion.

[44] It should be noticed that (104) presents not only terms of the type $(\partial A)^2$ first encountered at (16.2), but also terms of the types $A^2\partial A$ and A^4; we can anticipate that *the free gauge field equations will be non-linear*. Notice also that expressions of the design $g^{\alpha\beta}\varkappa^2_{pq}A^p_\alpha A^q_\beta$—analogs of the $g^{\alpha\beta}\varkappa^2 A_\alpha A_\beta$ contemplated earlier—are Lorentz-invariant but *not gauge-invariant; it becomes therefore impossible to assign "mass" to the gauge fields* in any straightforward, gauge-symmetric way.

compute

$$\left.\begin{aligned} \partial_\mu F_r^{\mu\nu} &= \tfrac{1}{c} s_r^\nu \\ s_r^\nu &\equiv c\frac{\partial}{\partial A_\nu^r}\mathcal{L}_{\text{free gauge field}} = gcF_p^{\nu\alpha} c_r{}^p{}_q A_\alpha^q \end{aligned}\right\} \tag{106.1}$$

of which

$$\left.\begin{aligned} \partial_\mu \vec{F}^{\mu\nu} &= \tfrac{1}{c}\vec{s}^{\,\nu} \\ \vec{s}^{\,\nu} &\equiv 2gc\vec{A}_\alpha \times \vec{F}^{\,\nu\alpha} \end{aligned}\right\} \tag{106.2}$$

provides a more picturesque account (but an account available only within the $SU(2)$ theory, and then only if we have elected to work in the Pauli basis). As yet a third alternative we have this *basis-independent* representation of free motion of the gauge field system:

$$\left.\begin{aligned} \partial_\mu \boldsymbol{F}^{\mu\nu} &= \tfrac{1}{c}\boldsymbol{s}^\nu \\ \boldsymbol{s}^\nu &\equiv igc[\boldsymbol{F}^{\nu\alpha}, \boldsymbol{A}_\alpha] \end{aligned}\right\} \tag{106.3}$$

Equations (106) become meaningful/informative only after one has imported—"by hand," as it were—from (105/3/1) the definition of $F_r^{\mu\nu}$ else $\vec{F}^{\mu\nu}$ else $\boldsymbol{F}^{\mu\nu}$. There is, however, a way to circumvent this formal blemish (if such it be): borrowing a trick from the theory of Procca fields,[45] let us, in place of (104), write

$$\mathcal{L}_{\text{free gauge}} = \tfrac{1}{2}\big\{F_p^{\alpha\beta}\big[A^p_{\beta,\alpha} - A^p_{\alpha,\beta} + gc_u{}^p{}_v A^u_\alpha A^v_\beta\big] - \tfrac{1}{2}F_p^{\alpha\beta}F^p_{\alpha\beta}\big\} \tag{107}$$

and agree to construe A_p^μ and $F_p^{\mu\nu} = -F_p^{\nu\mu}$ to be *independent* fields. We then have a *pair* of Lagrange equations

$$\begin{aligned} \left\{\partial_\mu \frac{\partial}{\partial F^r_{\rho\sigma,\mu}} - \frac{\partial}{\partial F^r_{\rho\sigma}}\right\}\mathcal{L}_{\text{free gauge}} &= 0 \\ \left\{\partial_\mu \frac{\partial}{\partial A^r_{\nu,\mu}} - \frac{\partial}{\partial A^r_\nu}\right\}\mathcal{L}_{\text{free gauge}} &= 0 \end{aligned}$$

The former yields (105) *as a field equation*, while the later reproduces (106.1).

The free motion of the gauge field system is, according to (106), described by a coupled system of *non-linear* partial differential equations. The system is—owing to the presence of the current-like s-term on the right hand side—"self-excited." That the latter phenomenon is an artifact of non-commutivity (i.e., of the non-Abelian character of the gauge group) is most vividly evident in (106.3).

[45] See again (1–31).

To describe, finally, the dynamics of the full locally $SU(2)$-invariant Dirac theory we assemble

$$\begin{aligned}\mathcal{L}_2 = \hbar c\Big[\tfrac{1}{2}i\Big\{\tilde{\psi}\boldsymbol{\Gamma}^\alpha\big[\psi_{,\alpha} - igA^p_\alpha\boldsymbol{\Sigma}_p\psi\big] \\ -\big[\tilde{\psi}_{,\alpha} + ig\tilde{\psi}A^p_\alpha\boldsymbol{\Sigma}_p\big]\boldsymbol{\Gamma}^\alpha\psi\Big\} - \varkappa\tilde{\psi}\psi\Big] + \mathcal{L}_{\text{free gauge}}\end{aligned} \tag{108}$$

and from

$$\Big\{\partial_\mu\frac{\partial}{\partial\tilde{\psi}_{,\mu}} - \frac{\partial}{\partial\tilde{\psi}}\Big\}\mathcal{L}_2 = 0 \quad \text{and its adjoint}$$
$$\Big\{\partial_\mu\frac{\partial}{\partial F^r_{\rho\sigma,\mu}} - \frac{\partial}{\partial F^r_{\rho\sigma}}\Big\}\mathcal{L}_2 = 0$$
$$\Big\{\partial_\mu\frac{\partial}{\partial A^r_{\nu,\mu}} - \frac{\partial}{\partial A^r_\nu}\Big\}\mathcal{L}_2 = 0$$

obtain (compare (72))

$$\left.\begin{aligned}\big[\boldsymbol{\Gamma}^\mu(\boldsymbol{\partial}_\mu - ig\boldsymbol{A}_\mu) + i\varkappa\boldsymbol{I}\big]\psi = 0 \quad \text{and its adjoint} \\ F^p_{\mu\nu} = \partial_\mu A^p_\nu - \partial_\nu A^p_\mu + gc_u{}^p{}_v A^u_\mu A^v_\nu \\ \partial_\mu F^{\mu\nu}_p = \tfrac{1}{c}(J^\nu_p + s^\nu_p)\end{aligned}\right\} \tag{109}$$

where s^ν_p are the self-interaction currents first encountered at (106), and where the currents J^ν_p can in terms of the fluxes introduced at (91) be described

$$\begin{aligned}J^\nu_p &\equiv eQ^\nu_p \quad \text{with} \quad e \equiv g\hbar c \\ &= ec\tfrac{1}{2}\big\{\tilde{\psi}\boldsymbol{\Gamma}^\nu\boldsymbol{\Sigma}_p\psi + \tilde{\psi}\boldsymbol{\Sigma}_p\boldsymbol{\Gamma}^\nu\psi\big\}\end{aligned} \tag{110}$$

The *global* $SU(2)$-invariance of the Lagrangian $\mathcal{L}_0$ from which we started led to $\partial_\nu J^\nu_p = 0$, but with the adjustment $\mathcal{L}_0 \to \mathcal{L}_2$ those conservation laws have been lost; in their place one has

$$\partial_\nu\mathcal{J}^\nu_p = 0 \quad \text{with} \quad \mathcal{J}^\nu_p \equiv J^\nu_p + s^\nu_p \tag{111.1}$$

which can be read as an immediate consequence of the antisymmetry of $F^{\mu\nu}_p$, and speak to the conservation of

$$\mathcal{J}_p = \int \mathcal{J}^0_p\, dx^1dx^2dx^3 \tag{111.2}$$

At (110) we set $g = \frac{e}{\hbar c}$ to maximize the "electromagnetic appearance" of our results, but abandon any notion that e may have something to do with *electric* charge: e is a new kind of coupling constant—"$SU(2)$-charge." Notice that e serves to describe also the strength of the *self*-interaction, which (as was previously remarked) is a symptom of the non-Abelian character of the gauge group. Gauge theory—conceived by Shaw[8] to be a theory of field interactions—has become now a theory also of intricately structured *self*-interactions.

Dirac theory with local SU(N) gauge invariance. The hard work lies now behind us; we have now only to retrace the argument of the preceding section and to make adjustments at those few points where we drew on properties *specific* to $SU(2)$. We begin (compare (86)) by preparing the canvas; i.e., by assembling the $4N$-component complex field

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \vdots \\ \psi^N \end{pmatrix} \quad \text{with} \quad \psi^a = \begin{pmatrix} \psi^a_1 \\ \psi^a_2 \\ \psi^a_3 \\ \psi^a_4 \end{pmatrix} \quad : \quad a = 1, 2, \ldots, N \tag{112.1}$$

and by writing

$$\mathcal{L}_0(\psi, \tilde{\psi}, \partial\psi, \partial\tilde{\psi}) = \hbar c\Big[\tfrac{1}{2}i\big\{\tilde{\psi}\boldsymbol{\Gamma}^\alpha\psi_{,\alpha} - \tilde{\psi}_{,\alpha}\boldsymbol{\Gamma}^\alpha\psi\big\} - \tilde{\psi}\boldsymbol{\varkappa}\psi\Big] \tag{112.2}$$

with

$$\boldsymbol{\Gamma}^\mu \equiv \boldsymbol{I}_N \otimes \boldsymbol{\gamma}^\mu \quad \text{and} \quad \boldsymbol{\varkappa} \equiv \begin{pmatrix} \varkappa_1 & 0 & \ldots & 0 \\ 0 & \varkappa_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \varkappa_N \end{pmatrix} \otimes \boldsymbol{I}_4 \tag{112.3}$$

From the block design of those matrices it follows that the component fields ψ^a are uncoupled in (112.2), which could be written

$$\mathcal{L}_0(\psi, \tilde{\psi}, \partial\psi, \partial\tilde{\psi}; \boldsymbol{\varkappa}) = \sum_{a=1}^{N} \mathcal{L}_0(\psi^a, \tilde{\psi}^a, \partial\psi^a, \partial\tilde{\psi}^a; \varkappa_a)$$

Let $\boldsymbol{s}$ be $N \times N$, unimodular and unitary; the matrices $\boldsymbol{s}$ comprise the "natural elements" of $SU(N)$, while the matrices $\boldsymbol{S} \equiv \boldsymbol{s} \otimes \boldsymbol{I}_4$ give rise to a $4N$-dimensional unimodular unitary representation of $SU(N)$. One can always write

$$\boldsymbol{s} = e^{i\boldsymbol{h}} \quad \text{where } \boldsymbol{h} \text{ is } N \times N \text{ traceless hermitian} \tag{112.1}$$

Write

$$\boldsymbol{h} = h^1\boldsymbol{\sigma}_1 + h^2\boldsymbol{\sigma}_2 + \cdots h^{\mathcal{N}}\boldsymbol{\sigma}_{\mathcal{N}} \quad : \quad \mathcal{N} \equiv N^2 - 1 \tag{112.2}$$

where $\big\{\boldsymbol{\sigma}_p : p = 1, 2, \ldots, \mathcal{N}\big\}$ refer now to an *arbitrary* basis in the vector space $\mathcal{V}_{\mathcal{N}}$ of such (N-dimensional traceless hermitian) matrices. Necessarily there exist real structure constants $c_p{}^r{}_q$ such that

$$[\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_q] = ic_p{}^r{}_q\boldsymbol{\sigma}_r \tag{112.3}$$

and from which we construct the $\mathcal{N} \times \mathcal{N}$ matrices $\mathbb{C}_r \equiv \|ic_r{}^m{}_n\|$ which were seen at (81) to provide the "adjoint representation" of (112.3):

$$[\mathbb{C}_p, \mathbb{C}_q] = ic_p{}^r{}_q\mathbb{C}_r \tag{113}$$

We agree to use

$$k_{pq} = \mathrm{tr}\,\mathbb{C}_p\mathbb{C}_q \quad : \quad \text{elements of } \Bbbk \equiv \|k_{pq}\| \tag{114}$$

and the elements k^{pq} of $\Bbbk^{-1}$ to lower/raise "gauge indices."

Ascend now from N to $4N$ dimensions, it follows straightforwardly from $\boldsymbol{S} \equiv \boldsymbol{s}\otimes\boldsymbol{I}_4$ and properties of the Kronecker product that

$$\boldsymbol{S} = e^{i\boldsymbol{H}} \quad \text{with} \quad \boldsymbol{H} = h^1\boldsymbol{\Sigma}_1 + h^2\boldsymbol{\Sigma}_2 + \cdots h^{\mathcal{N}}\boldsymbol{\Sigma}_{\mathcal{N}} \tag{115.1}$$

where

$$\boldsymbol{\Sigma}_p \equiv \boldsymbol{\sigma}_p\otimes\boldsymbol{I}_4 \quad \text{is } 4N\times 4N \text{ traceless hermitian} \tag{115.2}$$

The structure constants still serve

$$[\boldsymbol{\Sigma}_p, \boldsymbol{\Sigma}_q] = ic_p{}^r{}_q\boldsymbol{\Sigma}_r \tag{115.3}$$

Look now to the response of $\mathcal{L}_0$ to $\psi \longrightarrow \boldsymbol{S}\psi$: we have

$$\tilde{\boldsymbol{S}}\boldsymbol{\Gamma}^\mu\boldsymbol{S} = (\tilde{\boldsymbol{s}}\otimes\boldsymbol{I}_4)(\boldsymbol{I}_N\otimes\boldsymbol{\gamma}^\mu)(\boldsymbol{s}\otimes\boldsymbol{I}_4) = \tilde{\boldsymbol{s}}\boldsymbol{s}\otimes\boldsymbol{\gamma}^\mu = (\boldsymbol{I}_N\otimes\boldsymbol{\gamma}^\mu) = \boldsymbol{\Gamma}^\mu$$

but (by the same line of reasoning)

$$\tilde{\boldsymbol{S}}\boldsymbol{\varkappa}\boldsymbol{S} = \boldsymbol{\varkappa} \quad \text{if and only if all the } \varkappa_a \text{ are set equal}$$

This we do, rendering $\mathcal{L}_0$ "globally $SU(N)$-invariant by design," and Noether hands us a collection of $\mathcal{N} \equiv N^2 - 1$ conservation laws

$$\partial_\mu Q_r^\mu = 0 \text{ with } Q_r^\mu \equiv \tfrac{1}{2}c\tilde{\psi}(\boldsymbol{\Gamma}^\mu\boldsymbol{\Sigma}_r + \boldsymbol{\Sigma}_r\boldsymbol{\Gamma}^\mu)\psi \;:\; r = 1, 2, \ldots, \mathcal{N} \tag{116.1}$$

which are of a design encountered most recently at (110). We note in passing that the manifest global $U(1)$-symmetry of the theory leads a conservation law of similar design:

$$\partial_\mu Q_0^\mu = 0 \text{ with } Q_0^\mu \equiv \tfrac{1}{2}c\tilde{\psi}(\boldsymbol{\Gamma}^\mu\boldsymbol{\Sigma}_0 + \boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^\mu)\psi \tag{116.0}$$

where $\boldsymbol{\Sigma}_0$ is but a fancy way of saying $\boldsymbol{I}_N$.[46]

To achieve invariance under *local* $SU(N)$—of which

$$\begin{aligned} \psi \longrightarrow \psi' = &\boldsymbol{S}(x)\psi \\ &\boldsymbol{S}(x) = e^{ig\boldsymbol{\Omega}(x)} \quad : \quad \boldsymbol{\Omega}(x) = \Omega^r(x)\boldsymbol{\Sigma}_r \text{ traceless hermitian} \end{aligned} \tag{117.1}$$

[46] Note that we are now *not* in position to write equations so explicit as (91), since those reflect special properties of a specific basis (the Pauli basis).

is the initial/defining symptom— we make the "minimal coupling" adjustment

$$\begin{aligned}
&\mathcal{L}_0(\psi,\tilde\psi,\partial\psi,\partial\tilde\psi)\\
&\downarrow\\
\mathcal{L}_1 =&\mathcal{L}_0(\psi,\tilde\psi,\mathcal{D}\psi,\mathcal{D}\tilde\psi)\\
=&\,\hbar c\Big[\tfrac{1}{2}i\big\{\tilde\psi\boldsymbol{\Gamma}^\alpha(\psi_{,\alpha}-ig\boldsymbol{A}_\alpha\psi)-(\tilde\psi_{,\alpha}+ig\tilde\psi\boldsymbol{A}_\alpha)\boldsymbol{\Gamma}^\alpha\psi\big\}-\tilde\psi\boldsymbol{\varkappa}\psi\Big]
\end{aligned}$$

and expand the meaning of (117.1) to include (compare (8) and (93))

$$\boldsymbol{A}_\mu\longrightarrow\boldsymbol{A}'_\mu=\boldsymbol{S}\boldsymbol{A}_\mu\boldsymbol{S}^{-1}+i\tfrac{1}{g}\,\boldsymbol{S}_{,\mu}\boldsymbol{S}^{-1}\tag{117.2}$$

so as to achieve

$$\mathcal{D}_\mu\psi\longrightarrow(\mathcal{D}_\mu\psi)'=\boldsymbol{S}(\mathcal{D}_\mu\psi)\tag{117.3}$$

"Gauge fields"—the $\mathcal{N}$-tuple of vector fields $A^r_\mu(x)$—enter the discussion when we allow ourselves to write

$$\boldsymbol{A}_\mu(x)=A^r_\mu(x)\boldsymbol{\Sigma}_r\tag{118}$$

and are evidently basis-contingent constructs.

It is as a step preparatory to launching the gauge fields into dynamical motion that—appropriating the intricate argument which led us to (101), an argument which is seen to be not at all specific to $SU(2)$, or to the selection of any specific basis (though with the loss of (98.2) we lose the possibility of using the cross product to express our results, as we did at (103) and (106.2))—we construct

$$\boldsymbol{F}_{\mu\nu}=F^r_{\mu\nu}\boldsymbol{\Sigma}_r\equiv(\partial_\mu\boldsymbol{A}_\nu-\partial_\nu\boldsymbol{A}_\mu)-ig\,(\boldsymbol{A}_\mu\boldsymbol{A}_\nu-\boldsymbol{A}_\nu\boldsymbol{A}_\mu)\tag{119.1}$$

$$F^r_{\mu\nu}=(\partial_\mu A^r_\nu-\partial_\nu A^r_\mu)+gc_p{}^r{}_qA^p_\mu A^q_\nu\tag{119.2}$$

—the point being that $\boldsymbol{F}_{\mu\nu}$ responds to gauge transformation (117.2) by the very simple rule

$$\boldsymbol{F}_{\mu\nu}\longrightarrow\boldsymbol{F}'_{\mu\nu}=\boldsymbol{S}\boldsymbol{F}_{\mu\nu}\boldsymbol{S}^{-1}\tag{120.1}$$

We are placed thus in position to assemble an $\mathcal{L}_2$ formally identical[47] to (108), and obtain coupled field equations formally identical to (109).

In a more complete account of this subject it would become natural at this point to inquire into

- the construction of the *stress-energy tensor* of the gauged theory, and details of energy-momentum trade-off between the ψ-field and the gauge fields;
- Belinfante *symmetrization* of the stress-energy tensor (if required);
- *spin* of the gauge field system;
- motion of the center of mass of the gauge field system.

And, of course, it would be instructive to collect *illustrative solutions* of the field equations. I prefer, however, to look to other matters:

[47] I say "formally identical" because the implicit $\sum_p$ in (108) ran on $\{1,2,3\}$, but runs in the general case on $\{1,2,\ldots,\mathcal{N}\}$. Moreover, the $\boldsymbol{\Sigma}_p$ in (108) are taken to refer specifically to the Pauli basis in $\mathcal{V}_3$, but refer now to an arbitrary basis in $\mathcal{V}_\mathcal{N}$.

The argument which gave (102.2) leads in the more general case to the conclusion that (120.1) can be expressed

$$\vec{F}_{\mu\nu} \longrightarrow \vec{F}_{\mu\nu}^{\,\prime} = \mathbb{R}\vec{F}_{\mu\nu} \tag{120.2}$$

where

$$\vec{F}_{\mu\nu} \equiv \begin{pmatrix} F^1_{\mu\nu} \\ F^2_{\mu\nu} \\ \vdots \\ F^{\mathcal{N}}_{\mu\nu} \end{pmatrix} \quad \text{and} \quad \mathbb{R} \equiv e^{ig\mathbb{Z}(x)} \quad \text{with} \quad \mathbb{Z} \equiv \Omega^r \mathbb{C}_r$$

We know from (83.2) that

$$\text{the real matrix } i\mathbb{Z} \text{ is } \Bbbk\text{-antisymmetric: } (i\mathbb{Z})^{\mathsf{T}} = -(i\mathbb{Z})$$

and from this it follows that

$$\text{the real matrix } \mathbb{R} \equiv e^{ig\mathbb{Z}} \text{ is } \Bbbk\text{-orthogonal: } \mathbb{R}^{\mathsf{T}}\Bbbk\,\mathbb{R} = \Bbbk$$

If, in particular, the basis matrices $\{\mathbb{C}_r\}$ are, by contrived pre-arrangement, *tracewise orthonormal* then $\Bbbk = \mathbb{I}_{\mathcal{N}}$, and $\mathbb{R}$ becomes an element of the rotation group $O(\mathcal{N})$. But

$SU(N)$ is an $\mathcal{N} \equiv (N^2-1)$-parameter group, while
$O(\mathcal{N})$ is an $\frac{1}{2}\mathcal{N}(\mathcal{N}-1) = \frac{1}{2}(N^2-1)(N^2-2)$-parameter group

and from data tabulated below

N	$\mathcal{N} \equiv (N^2-1)$	$\frac{1}{2}\mathcal{N}(\mathcal{N}-1)$
2	3	3
3	8	28
4	15	105
$\vdots$	$\vdots$	$\vdots$

we infer on numerological grounds that the correspondence

$$\boldsymbol{S} = \exp\big\{ig\Omega^p\boldsymbol{\Sigma}_p\big\} \in SU(N) \quad \longleftrightarrow \quad \mathbb{R} = \exp\big\{ig\Omega^p\mathbb{C}_p\big\} \in O(\mathcal{N})$$

which (familiarly) serves to associate elements of $SU(2)$ with elements of $O(3)$, can in more general cases $N > 2$ serve only to associate elements of $SU(N)$ with elements of a *subgroup* of $O(\mathcal{N})$. We note in passing that there do, however, exist other cases in which $\mathcal{N}(N)$ is at least *triangular*:

$$\begin{aligned}
\mathbf{2}^2-1 &= 3=\Delta(2) \quad \text{where } \Delta(n)\equiv\sum_{k=1}^n k \text{ is the } n^{\text{th}} \text{ triangular number}\\
\mathbf{4}^2-1 &= 15=\Delta(5)\\
11^2-1 &= 120=\Delta(15)\\
23^2-1 &= 528=\Delta(32)\\
\mathbf{64}^2-1 &= 4095=\Delta(90)\\
134^2-1 &= 17955=\Delta(189)\\
373^2-1 &= 139128=\Delta(527)\\
&\vdots
\end{aligned}$$

and since $O(n)$ is a $\Delta(n-1)$-parameter group it is at least conceivable that an association of (say) the form $SU(4)\longleftrightarrow O(6)$ is possible. Relatedly, the theory of Clifford algebras inspires interest in numbers of the form 2^p-1, and Ramanujan has observed that in three and only three cases is *such* a number triangular. Each of those cases appears (boldface) in the preceding list; the $p=2$ and $p=4$ are of well-established physical importance (Pauli algebra, Dirac algebra) and it seems to me plausible that the final case $p=12$ may also possess latent physical significance. But it is difficult to manage an algebra with 4095 $\mathbf{\Sigma}$-matrices, and my occasional efforts to develop that hunch have thus far been fruitless. Returning now to less speculative matters...

Bringing (120.1) to the field equation $\partial_\mu F_p^{\mu\nu}=\frac{1}{c}\mathcal{J}_p^\nu$ we infer that the conserved net current $\boldsymbol{\mathcal{J}}^\nu\equiv\mathcal{J}^{p\nu}\mathbf{\Sigma}_p$ responds to local gauge transformation by the complicated rule

$$\begin{aligned}
\boldsymbol{\mathcal{J}}^\nu\longrightarrow\boldsymbol{\mathcal{J}}'^\nu &= \boldsymbol{S}_{,\mu}\boldsymbol{S}^{-1}\cdot\boldsymbol{F}'^{\mu\nu}+\boldsymbol{S}\boldsymbol{\mathcal{J}}^\nu\boldsymbol{S}^{-1}+\boldsymbol{F}'^{\mu\nu}\cdot\boldsymbol{S}(\boldsymbol{S}^{-1})_{,\mu}\\
&= \boldsymbol{S}\boldsymbol{\mathcal{J}}^\nu\boldsymbol{S}^{-1}+[\boldsymbol{S}_{,\mu}\boldsymbol{S}^{-1},\boldsymbol{F}'^{\mu\nu}]
\end{aligned}\tag{121.1}$$

Its response to *global* gauge transformation is, however, simple: the commutator drops away (because $\boldsymbol{S}_{,\mu}=\mathbf{0}$), leaving

$$\boldsymbol{\mathcal{J}}^\nu\longrightarrow\boldsymbol{\mathcal{J}}'^\nu=\boldsymbol{S}\boldsymbol{\mathcal{J}}^\nu\boldsymbol{S}^{-1}\tag{121.21}$$

which can be written

$$\vec{\mathcal{J}}^{\,\nu}\longrightarrow\vec{\mathcal{J}}'^{\nu}=\mathbb{R}\vec{\mathcal{J}}^{\,\nu}\tag{121.22}$$

It was this circumstance (together with the circumstance that in $SU(2)$ theory the 3×3 matrix $\mathbb{R}\in O(3)$) which led Yang & Mills to the satisfying conclusion that "total isotopic spin" $\vec{\mathcal{J}}\equiv\int\vec{\mathcal{J}}^{\,0}dx^1dx^2dx^3$ is a *vector*, which responds to (global) gauge transformation by "rotation in isotopic spin space."

It is (recall (110)) through

$$\boldsymbol{J}=ec\tfrac{1}{2}\{\tilde{\psi}\boldsymbol{\Gamma}^\nu\mathbf{\Sigma}^p\psi+\tilde{\psi}\mathbf{\Sigma}^p\boldsymbol{\Gamma}^\nu\psi\}\mathbf{\Sigma}_p\tag{122}$$

that the gauge fields sense the presence of the Dirac field ψ. The right side of the preceding equation is easily seen to be basis-independent, but I have been unable to discover any natural way to formulate (122) which does not make incidental *reference* to a basis... which strikes me as curious.

General observations, and some topics which might be included in a more comprehensive account of gauge field theory. In the preceding discussion we took "stacked copies of the Dirac equation" as our point of departure. We could instead have taken "stacked copies of the Klein-Gordon equation" or "stacked copies of the Procca equation"... and—particularly if we worked in canonical formalism—would be led to results formally identical (or nearly so) to the results now in hand (though the specific meaning of the $\boldsymbol{\Gamma}$-matrices would vary from case to case).[48]

Elementary calculus supplies the statement

$$(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)(\text{any nice function}) = 0$$

which we abbreviate $(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu) = 0$. But from the definition (10)

$$\mathcal{D}_\mu \equiv \partial_\mu - igA_\mu(x)$$

it follows on the other hand that

$$\mathcal{D}_\mu\mathcal{D}_\nu - \mathcal{D}_\nu\mathcal{D}_\mu = -ig(A_{\nu,\mu} - A_{\mu,\nu})$$

while in the non-Abelian case

$$\boldsymbol{\mathcal{D}}_\mu \equiv \boldsymbol{\partial}_\mu - ig\boldsymbol{A}_\mu(x)$$

we obtain

$$\begin{aligned} \boldsymbol{\mathcal{D}}_\mu\boldsymbol{\mathcal{D}}_\nu - \boldsymbol{\mathcal{D}}_\nu\boldsymbol{\mathcal{D}}_\mu &= -ig\big\{(\boldsymbol{A}_{\nu,\mu} - \boldsymbol{A}_{\mu,\nu}) - ig[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]\big\} \\ &= -ig\boldsymbol{F}_{\mu\nu} \quad \text{by (119.1)} \end{aligned} \tag{123}$$

We are not yet in position to comment on the deeper significance of this striking result, except to remark that it makes transparently clear how $\boldsymbol{F}_{\mu\nu}$ acquired its especially simple gauge transformation properties: it inherited them from $\boldsymbol{\mathcal{D}}_\mu$.

In Maxwellian electrodynamics we learn that it is from the sourceless equations

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \quad \text{and} \quad \boldsymbol{\nabla}\times\boldsymbol{E} + \partial_0\boldsymbol{B} = \boldsymbol{0} \tag{124.1}$$

—which is to say: from

$$\partial_\mu G^{\mu\nu} = 0 \quad \text{with} \quad G_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{124.2}$$

which can be expressed alternatively as a quartet of "windmill sum" relations

$$\epsilon^{\mu\nu\alpha\beta}\partial_\nu F_{\alpha\beta} = 0 \tag{124.3}$$

—that we acquire license to write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{125}$$

[48] Could we proceed similarly from "stacked copies of the Hamilton-Jacobi equation" or "stacked copies of the relativistic free particle equation," and thus produce non-Abelian generalizations of the theories developed on pp. 8–20 and pp. 20–26? Multi-component field systems are commonplace in field theory, but what might be the physical interpretation of "stacked copies of the classical mechanics of a particle"? Could such formalism be associated with the classical physics of particles with internal degrees of freedom ("spin")? These are questions to which I hope to return on another occasion.

And that when one *does* write (125) then equations (124) become *automatic*. In non-Abelian gauge field theory ("generalized electrodynamics") one has only to introduce (123) into the following instance

$$[\boldsymbol{\mathfrak{D}}_\nu,[\boldsymbol{\mathfrak{D}}_\alpha,\boldsymbol{\mathfrak{D}}_\beta]]+[\boldsymbol{\mathfrak{D}}_\alpha,[\boldsymbol{\mathfrak{D}}_\beta,\boldsymbol{\mathfrak{D}}_\nu]]+[\boldsymbol{\mathfrak{D}}_\beta,[\boldsymbol{\mathfrak{D}}_\nu,\boldsymbol{\mathfrak{D}}_\alpha]]=\mathbf{0}$$

of Jacobi's identity to obtain

$$[\boldsymbol{\mathfrak{D}}_\nu,\boldsymbol{F}_{\alpha\beta}]+[\boldsymbol{\mathfrak{D}}_\alpha,\boldsymbol{F}_{\beta\nu}]+[\boldsymbol{\mathfrak{D}}_\beta,\boldsymbol{F}_{\nu\alpha}]=\mathbf{0} \tag{126}$$

or (more compactly)

$$\epsilon^{\mu\nu\alpha\beta}[\boldsymbol{\mathfrak{D}}_\nu,\boldsymbol{F}_{\alpha\beta}]=\epsilon^{\mu\nu\alpha\beta}[\boldsymbol{\partial}_\nu,\boldsymbol{F}_{\alpha\beta}]-ig\epsilon^{\mu\nu\alpha\beta}[\boldsymbol{A}_\nu,\boldsymbol{F}_{\alpha\beta}]=\mathbf{0}$$

In the Abelian (i.e., in the Maxwellian) case the second term on the right drops away, and the surviving first term, when allowed to operate on the function 1, gives back (124.3). It will appreciated that equations (126) are not field equations but *identities*, satisfied in every instance and necessarily by $\boldsymbol{F}_{\mu\nu}$ (which is to say: by $\boldsymbol{A}_\mu$) in consequence of the manner in which those objects were defined. Equations (126) play within gauge field theory a role analogous (and abstractly identical) to the roll played by the so-called "Bianchi identities" in general relativity.[49]

It can be argued that the exterior calculus provides the language of choice for developing formal (and some computational) properties of the theory of Abelian gauge fields (Maxwellian electrodynamics).[50] The question arises: can a "generalized exterior calculus" be devised which serves equally well to illuminate the essentials of non-Abelian gauge field theory?

Maxwellian electrodynamics is well-known to be invariant with respect to "duality rotation"—an internal symmetry of which

$$\begin{aligned}\boldsymbol{E}\longrightarrow\boldsymbol{E}'&=\boldsymbol{E}\cos\theta+\boldsymbol{B}\sin\theta\\ \boldsymbol{B}\longrightarrow\boldsymbol{B}'&=\boldsymbol{B}\cos\theta-\boldsymbol{E}\sin\theta\end{aligned}$$

captures the simplified essence.[51] Can a similar symmetry be identified in the non-Abelian case?

It was a $U(1)\otimes SU(2)$ gauge theory which Glashow-Weinberg and Salam used in the late 1960's to achieve unification of the electromagnetic and weak interactions (see Quigg's Chapter 6), a $SU(3)$ gauge theory which (at about the

---

[49] See Michio Kaku, *Quantum Field Theory: A Modern Introduction* (1993), p. 297; M. E. Peskin & D. V. Schroeder, *An Introduction to Quantum Field Theory* (1995), p. 500 and/or the index of any good general relativity text.

[50] See my "Electrodynamical applications of the exterior calculus," (1996).

[51] See §7 in the material just cited; also pp. 327–331 in ELECTRODYNAMICS (1972) and p. 51 in Chris Quigg's *Gauge Theories of the Strong, Weak, and Electromagnetic Interactions* (1983).

same time) called colored quarks into being and resulted in the creation of QCD (quantum chromodynamics; see Quigg's Chapter 8), and the so-called Standard Model—which unites the whole shebang and accounts satisfactorally for most of the observational evidence—is a $U(1) \otimes SU(2) \otimes SU(3)$ gauge theory. It would be interesting—but a major undertaking—to trace the classical outlines[52] of that work. One would come away from such an exercise with a sense of how one finds "wiggle-room" within the fairly rigid framework provided by the gauge field idea. . . and of what contortions Nature herself appears to require; more particularly, one would acquire a sense of how difficult it is to endow gauge fields with mass, and of how wonderfully ingenious (if in some respects still unsatisfactory) has been the effort to do so (Quigg's Chapter 5). But for discussion of those topics I must—for now—be content to refer my readers to the abundant literature.[53]

[52] By "classical outlines" I mean "up to the point of quantization." It is, of course, quantization which lends physical significance the theory. But it opens a can of mathematical worms which have no place in an account of the elements of *classical* field theory.

[53] I have made references to Quigg,[51] who is often especially clear, and supplies good bibliographic information, but one should also consult Chapter 11 in Griffiths[4] and relevant paragraphs in (say) Kaku and Peskin & Schroeder[49]. The literature is, as I say, vast; for a random taste of its riches see the essay "Secret symmetry: an introduction to spontaneous symmetry breakdown and gauge fields" in S. Coleman, *Aspects of Symmetry* (1985).