Chaotic Waterwheel

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A.1 The Lorenz Attractor plotted in $x - y - z$ space.
Abstract

The study of chaotic systems is a relatively new science, one that is considered to be a great revolution in twentieth century physics along with Quantum Mechanics and Relativity. Since interest in Chaos Theory exploded in the 1960s, researchers have discovered that chaos describes many physical phenomena that we experience in our day to day lives, from dripping faucets to circadian rhythms, electrical circuits, chemical reactions, population growths, satellites in our solar system, and the weather. In this thesis, I discuss methods used to examine chaotic systems in phase space and point out the features that these phase portraits have in common in an attempt to educate the reader in a basic, qualitative manner. I then use these methods to carry out an analysis of a chaotic waterwheel that I built which emulates the behavior of the famous Lorenz system, widely considered to be the seminal chaotic system.
Introduction

Most people who have never studied chaos theory suffer from a gross ambiguation of the word “chaos”. In our common usage, the term refers to something that completely lacks any sort of order. We hear always of a chaotic desk being littered with disorganized papers, or we read of a riotous group of people breaking out in chaos. In fact, the word originates from the Greek notion of the disorderly mess of the elements which composed the universe before order was imposed, before the Earth came into existence and spawned life. Another misconception is that chaos refers to random behavior. However, in the physical sciences “chaos” refers not to a system in which disorder reigns superior, but to one in which the order is extremely intricate. Anyone who has gazed upon a fractal image has seen the extent to which the order in a chaotic system is present. So before we begin, let us not be confused by our previous understanding of the word chaos. We now mean it to imply the supremely intricate workings of a system which abides fully by the basic laws of physics.

Chaos theory is a relatively new science that has come to be considered one of the three great revolutions in twentieth century physics along with quantum mechanics and relativity. Before chaos theory, people thought that every problem of classical mechanics was predictable, provided you had the tools to take the necessary measurements or the mathematical skills to solve the system. The belief was that Newton's laws governed these systems in a deterministic way and we simply had to pay close attention to all the minute details in order to be able to predict the behavior of a system. Unpredictable systems were those that, while governed by deterministic laws, had too many variables to measure and keep track of, such as turbulent motion in fluids and the weather. We now know that despite the deterministic laws which govern certain phenomena, their behavior can be utterly unpredictable. No matter how many measurements we make or how much we reduce the noise in certain systems, knowledge of the parameters and information concerning a system may never allow us to fully predict the outcomes of what have come to be known as chaotic systems.

The first inklings of chaos theory were encountered by Poincaré at the turn of the
twentieth century in his attempts to address the famous three-body problem. This problem consists of a set of differential equations concerning the motion and interaction of three massive bodies in space under the influence of Newtonian mechanics. In 1887 King Oscar II of Sweden threw a contest in which anyone who could solve the problem, or come closest to solving it and in the meantime make a significant contribution to classical mechanics, would win a monetary prize. Poincaré tried his hand at the three-body problem and encountered solutions that no one had previously dealt with or discussed: he found orbits that were nonperiodic, yet neither soared off to infinity, nor settled into a fixed point.[1] He came across a formation known as Poincaré’s nightmare: the first glimpses of a chaotic solution. Unfortunately, he had only pen and paper, and approximating solution techniques such as perturbation theory at his disposal. Although Poincaré didn’t actually solve the problem, he did better than anyone else in trying and won King Oscar’s prize. After the competition, however, Poincaré did not know how to completely solve the three-body monster lacking the aid of a computational godsend which is the modern computer. Chaos theory lay dormant for decades. It is most commonly thought that Edward Lorenz was the man who reintroduced Chaos Theory to the scientific community in the middle of the twentieth century, after a revelation of his own.

In the early 1960s, Edward Lorenz was studying the peculiarities of the weather at MIT.[2] It was the dawn of the age of computers and this new technology was about to open up a field of study virtually off limits to those who came before it. Lorenz designed a computer program to simulate a very simple model of the weather which was essentially composed of 12 variables, a set of rules to abide by, and a “go” button. The program would spit out reels and reels of numbers that corresponded to various changing weather variables such as atmospheric pressure, wind speeds, temperatures, etc. Lorenz would run these simulations without being able to predict the outcome, despite the deterministic laws he had imposed on the system.[3] The system was deterministic, yet unpredictable.

One day, while running his simulations, Lorenz decided to recreate one of the runs. He looked back at a printout of some previous data, recorded the numbers for each parameter somewhere in the middle of the run, typed them into his computer and let his simulation go. As the story goes, he walked down to the end of the hallway to fill his coffee cup, and when he returned was surprised to see that the simulation had not reenacted the old simulation, but had set his fanciful weather on a vastly different course. What he had expected to observe was the same progression of events that he had witnessed earlier with the same set of states for each parameter.
Confounded, he sifted through the details and working of his program, looking for
the bugs that must have caused this ambiguation. After finding none, he finally
realized that while the program printed out data to only three decimal places, the
internal machinery of his computer was working with six decimal places, a very small
difference. When he thought he had typed in the exact same parameters as his
previous run, he had actually dropped a few mere decimal places and was off by just
a few hundred thousandths. Looking back at his newly acquired data, he saw that
the simulation repeated the old course of events for a short while, but then wildly
diverged.[3] The system seemed to be incredibly sensitive to small changes in the
initial conditions.

After some years of work, Lorenz simplified his system down to a set of three
nonlinear equations that described the currents induced in a convective cell of liquid
when heated from the bottom. Although these equations had only three variables and
were governed by deterministic laws, they behaved in an unpredictable, aperiodic way
and retained an extreme sensitivity to initial conditions. The unpredictability was
not due to an inability to measure all of the associated variables and parameters, but
rather to some inherent quality of the system itself.

This came as an exciting new revelation. Up until this point in history it had
been widely agreed by the scientific community that the outcomes of systems under
the influence of deterministic laws could be predicted with great accuracy over long
periods of time. In Newtonian dynamics, a small range of inputs was supposed to
yield a small range of outputs. For example, if one tossed a softball across a field
toward third base from the pitchers mound, one knew that it would land near third
base. Throwing it with an ever so slightly greater velocity would yield a similar
outcome. One never had to fear that this small change in initial conditions could end
with the ball smacking the firstbaseman in the head. But what Lorenz’s revelation
indicated was that certain systems which obey deterministic rules just as the softball
does could work in quite a dissimilar fashion. Changing the initial conditions just a
tad really could result in a completely different and unexpected outcome.

In fact, in 1986 Sir James Lighthill, then the Lucasian Chair of Mathematics at
Trinity College, Cambridge, wrote an article titled, “The recently recognized failure
of predictability in Newtonian dynamics”[4] which addressed this newfound revelation
and discovery of chaos theory. He wrote,

We are all deeply conscious today that the enthusiasm of our forebears
for the marvellous achievements of Newtonian mechanics led them to make
generalizations in this area of predictability which indeed, we may have
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generally tended to believe before 1960, but which we now recognize were false. We collectively wish to apologize for having misled the general educated public by spreading ideas about the determinism of systems satisfying Newton’s laws of motion that, after 1960, were to be proved incorrect.

Lorenz’s discovery and development of chaos theory were quite monumental, and led to an entirely new understanding of the dynamics of our world.

Chaotic systems abide by the same laws as predictable systems. In fact, chaotic systems are predictable in the short term, but for larger timeframes it becomes impossible to know which state the system will occupy. With the weather, for example, one can say with a good amount of certainty whether it will be cloudy in a couple of days, but to predict the weather a few weeks from now cannot be done with much accuracy.

The weather, however, is a pretty complicated system. There are many parameters to take note of as well as many influences on these parameters. One somewhat expects such a multi-faceted, complex system to be difficult to predict, although it is certainly deterministic. What is even more curious are deterministic systems that seem fairly simple, with only a few parameters to keep track of, or only involving a few equations, but exhibit chaotic behavior. What makes these deterministic systems unpredictable is their extreme sensitivity to initial conditions.

All of this is further complicated by the fact that one can never make a truly accurate measurement of anything. If one was to measure the length of a table, the degree of precision allowed would be limited by their measuring tool. Using a meter stick, for example, can only approximate to the closest millimeter, although the table’s length is certainly not an exact denomination of millimeters. Even if you had the very best measuring tool, eventually Heisenberg’s Uncertainty Principle[5] would come into play and make it impossible to make a perfectly accurate measurement.

In other words, some precision in a measurement is always lost. Usually this doesn’t matter, and in linear and periodic systems, a good approximation of the initial conditions will suffice for a prediction of how the system will unfold. If one was trying to predict where ones softball will land in a field, a reasonable approximation of initial position and velocity will do the trick. If the measurement is off a bit, the predicted landing spot will be more or less localized to one specific area of the field.

Chaotic systems, however, are not linear, nor periodic. If the initial measurements are off by just a tad, which they inherently will be, the outcome at a later specified time will not be localized around a specific point. This will be illustrated more clearly
later on, after we’ve described more details of chaos theory. For now, I will give an illuminating definition of chaos from Steven Strogatz’s wonderful book, *Nonlinear Dynamics and Chaos*, “Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions”.[2]

What I will examine in this thesis are the workings of a seemingly simple system that abides by only a few deterministic laws: a chaotic waterwheel that serves as a mechanical analogue to the convection currents studied by Lorenz. It is governed by a set of three differential equations, but the behavior that unfolds according to these equations is quite complex, unpredictable, and chaotic. The waterwheel which I have built is governed by the Lorenz equations themselves, the seminal equations of chaos theory. The chaotic waterwheel was developed by Lorenz and improved upon by Malkus.[2] Chapter 1 discusses methods used to analyze dynamic systems in phase space and explores the characteristics of nonlinear systems. Chapter 2 outlines the construction of a chaotic waterwheel. Chapter 3 then derives the equations that govern our system and maps them onto Lorenz’s famous equations. And finally, Chapter 4, uses nonlinear time series analysis in order to reconstruct an attractor and qualitatively determine whether the waterwheel’s behavior is chaotic.
Chapter 1

A Qualitative Approach to Examining Chaotic Systems

To fully understand the dynamics of the water wheel, we must first discuss the tools and methods used in the analysis of dynamical systems, such as phase space and attractors. With all dynamical systems, we consider two things: the state of the system and the dynamic that rules it. We can then use phase space to examine the evolution of a system. We will discuss some common features of phase space plots of chaotic systems and explore how these features account for behavior we typically characterize as chaotic, namely aperiodicity and sensitive dependence on initial conditions. We begin with phase space.

1.1 Phase Space

1.1.1 The geometry of differential equations

If we were to consider the nonlinear differential equation

\[ \dot{x} = \sin x \]

we could plot it as illustrated in Fig. 1.1. Wherever the curve crosses the x-axis there is a fixed point because \( \dot{x} = 0 \) at those locations on the x-axis. We imagine a phase fluid that flows along the x-axis to the right when \( \dot{x} \) is greater than zero and flows to the left when \( \dot{x} \) is less than zero. When flows on either side of a fixed point direct the phase fluid towards the point, it is called a stable fixed point; these are marked with solid black dots. When the flows are away from the point, the point is called unstable,
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Figure 1.1: Examining the nonlinear differential equation $\dot{x} = \sin x$. Here, the phase space is the one dimensional line $x$. Points where the curve crosses the $x$-axis are called fixed points. The solid fixed points are stable and the hollow fixed points are non stable.

marked with empty dots. By denoting the direction of the flow, we have represented our differential equation as a vector field on the $x$-axis. The one dimensional $x$-axis is the phase space of this system.

Now that we have examined our differential equation in phase space, we can make some qualitative statements about its solution, $x(t)$. For example, we know that if $x(0) = \pi/4$, $x$ will get larger faster and faster until it reaches $\pi/2$ and the curve of $x(t)$ in a plot of $x$ vs. $t$ will be concave up. Once $x$ passes $\pi/2$, $x(t)$ will continue to grow, but now the rate at which it grows will slow down and its curve will be concave down because $\dot{x}$ has reached its maximum and is now decreasing. $x$ will approach $\pi$ asymptotically from here. It will not stray away from $\pi$ because there is a stable fixed point here. For $x > \pi$ the flow is to the left, tending to decrease $x(t)$. Flows on either side of the fixed point $x^* = \pi$ are toward $\pi$. Through our qualitative assessment, we have arrived at an understanding of the shape of the solution $x(t)$ to our differential equation, as seen in Fig. 1.2.

1.1.2 The geometry of dynamical systems

We can also use phase space to examine the behavior of dynamical systems, which are governed by differential equations with more than one variable. In phase space, we represent all the possible states of a system geometrically as individual points in the
1.1. Phase Space

space, and demonstrate how the states progress in time according to the dynamics, or laws of physics, that govern the system. In phase space, the finish line waves backward but approaches cheerfully! The dimensions in phase space must be equal to the number of initial conditions we need to specify for a unique solution. Every point in phase space represents a single state of the system.

Now we will bestow some physical significance on phase space plots by considering an example with whose behavior we are familiar: a pendulum. The differential equation governing the undamped pendulum is $\ddot{x} + \frac{g}{L} \sin x = 0$, where $g$ is the acceleration due to gravity and $L$ is the length of the pendulum. We need two pieces of information to describe the state of the pendulum: initial position and initial velocity. If we knew only where (the position, $x(t)$, at which) the pendulum bob was located at a given time, we would not know which direction it was swinging without knowing the velocity, $\dot{x}(t)$. The dimensions of phase space are often equal to twice the number of degrees of freedom of a mechanical system, one for its position in that dimension and another for its velocity component in that dimension. Since the pendulum has only one degree of freedom, its phase space plot has two dimensions. Therefore, we will plot horizontal displacement, $x$, against velocity, $\dot{x}$, of the pendulum bob. In Fig. 1.3, we see on the left the phase space plot of a simple, dissipative pendulum. Imagine a pendulum that starts its swing after being drawn over to the left. Once you let go, the pendulum bob gains speed in the positive direction and reaches its peak velocity at the bottom of its swing. Once it passes this midpoint, its velocity decreases until it stands still at the top of its swing on the right side. It then swings back to the left, gaining velocity again, this time negative because it’s swinging to the left, and so on. We say that when the velocity is positive (the point is above the $x$-axis), the flow is
Figure 1.3: Phase space plots of two pendulums. We plot horizontal position of the pendulum bob versus velocity. On the left is the phase space plot of a dissipative pendulum (one subject to friction). The green point at the origin is called a point attractor. On the right is the plot of a driven pendulum. In the case shown, its motion tends toward a limit cycle attractor (pictured in green).

to the right, and when $\dot{x}$ lies below the $x$-axis, the flow is to the left. Because the pendulum will eventually stop swinging due to the dissipative effects of friction, it spirals in toward the origin. This spiral is called the pendulum’s trajectory in phase space. A trajectory represents the behavior or evolution of a system over time; by following the trajectory, we follow the states in which a system exists in consecutive points in time and can see in which state or states the system ultimately ends up. Trajectories in dissipative systems usually asymptotically approach some structure which is the system’s attractor. After a sufficient amount of time the system’s trajectory will essentially be on the attractor. We then say that the point that exists at the origin of the plot on the left in Fig. 1.3 (pictured in green) is a stable fixed point, or a point attractor because the trajectory is drawn toward it from every initial state on the phase space plot. Furthermore, the trajectory of the dissipative pendulum will asymptotically reach this point and remain there unless it comes under the influence of another outside force, such as a hand reaching in to give it another swing.

On the right of Fig. 1.3, we see the phase space plot of an ideal, frictionless pendulum. It swings freely back and forth without any dissipative effects, and therefore follows a limit cycle (pictured in green) because it will keep swinging indefinitely, repeating the exact same oscillation during every period. A limit cycle attractor is
one step up in complexity from a point attractor, and applies to any system that falls into regular periodic motion, whereas point attractors describe systems that eventually come to a standstill. A driven pendulum will follow a limit cycle: its initial conditions may place it apart from the limit cycle at first, if it starts off with too great or small an angle or velocity, but dissipation will quickly reduce the influence of the initial conditions until only the drive determines the system dynamics. For a periodic drive mechanism, the pendulum will eventually experience regular periodic motion and be on a limit cycle attractor.

The phase space plots of certain dynamical systems include not only attractors, but also repellors, under the influence of which a trajectory is directed away from these points. The trajectories can then soar out to infinity or back towards other attractors.

In three or more dimensions of phase space, we sometimes encounter three dimensional torus (donut-shaped) attractors, which describe systems under the influence of two separate oscillations, kind of like a limit cycle in three dimensions of phase space rather than two. Recall that the more degrees of freedom or variables a dynamical system has, the more pieces of information we need to solve the system, and the more dimensions it has in phase space. For this reason attractors can take on the form of many-dimensional tori, the dimensions, of course, equal to the number of variables involved with the system, and needed to specify a unique solution.

When we consider more complicated systems with more than one dimension in phase space, we sometimes see not only attractors and repellors, but also saddle points as shown in Fig. 1.4, where trajectories are not strictly attracted to or repelled from a point, but both, depending on which direction the trajectory is coming from. If we had a pendulum hanging directly between two magnets of different strengths anchored on the floor beneath, there would be some point in the pendulum’s swing where it could be perfectly balanced between the two magnets, offset somewhat from the center. A saddle point would exist here, because if the pendulum was knocked ever so slightly it would become off balance and go toward one magnet or the other, depending in which direction it was knocked. A trajectory leading in to a point is called an *inset*, and, vice versa, a trajectory leading outward is an *outset*. When we have more than one saddle point, we may begin to observe heteroclinic structure, in which the outset of one saddle point leads directly to the inset of another saddle point.

Strange attractors have an even more complicated structure. Strange attractors are characteristic of dissipative chaotic systems. In these systems, trajectories are not lead onto limit cycles where they will experience periodic motion, nor to fixed points
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Figure 1.4: Saddle point. Saddle points are half-stable fixed points that both attract and repel trajectories. Depending on where they begin, some trajectories lead in, and others lead out.

where they come to a standstill. They begin at some point in phase space determined by their initial conditions and make their way onto an attractor, just like any other system. The portion of the trajectory that leads a chaotic system onto its attractor is called a *transient*. The transient can be somewhat periodic, but once the trajectory reaches its chaotic attractor, the system experiences aperiodic motion.

1.2 Chaotic Attractors

Before chaos theory exploded in the 1960s, many mathematicians and physicists thought that any sort of chaotic or apparently random system, such as turbulence in fluids or the behavior of the weather, must be described by an almost-infinite dimensional phase space. These systems were too complicated to completely portray and analyze in phase space, but in theory, it was believed that if we could gather all of that information we would be able to unfold and predict the behavior of these systems (as discussed in the introduction). This is why it came as a surprise in the 1960s that a system with only a few degrees of freedom, which existed in a low-dimensional phase space, could exhibit chaotic behavior. Lorenz’s equations comprised such a system, and was one of the first of this class to be widely examined by mathematicians.\[6\]
1.2.1 Saddle points and aperiodicity

When we encounter saddle points or saddle limit cycles in phase space, our dynamic system will neither settle into periodicity, nor become motionless. A trajectory might be drawn inwards, looking somewhat periodic for a while as it approaches the saddle, and then get shot out to a different area of phase space, away from the attractive grasp of the saddle. When more than one such saddles exist in a phase portrait of a system, they may become tangled, resulting in a complicated aperiodic motion. Trajectories can be slingshotted between saddles indefinitely. This is what happens in the Lorenz attractor, which doesn’t have the shape of a point, a cycle, or a torus, but rather some convoluted object in three dimensions with two holes embedded inside each wing. It can be illuminating to imagine the actual shape of the Lorenz attractor. This is shown in Abraham & Shaw’s attempt to draw the Lorenz attractor, in Fig. 1.5. Systems such as these exhibit aperiodic motion. The trajectory will circle

![Lorenz Attractor Diagram]

Figure 1.5: An attempt to draw the Lorenz attractor. Here we see the shape of the attractor with rigidly defined boundaries. (Abraham & Shaw Dynamics–The Geometry of Behavior, Part Two: Chaotic Behavior[7] p. 88, reprinted with permission from Ralph Abraham).

... one hole a few (and unpredictable number of) times, then the other hole, and back again indefinitely. We encounter a very intricate structure underlying the motion of the system, where trajectories never cross and therefore never repeat a cycle. Such attractors have a fractal dimension, squeezing in an infinite number of uncrossing paths into a finite region of phase space.\(^1\) This is accomplished through a technique that topologists call *stretching and folding.*

\(^1\)For more on fractal dimensions, see Mandelbrot’s *The Fractal Geometry of Nature.*[8]
1.2.2 Stretching and folding

A common analogy for the stretching and folding that occurs in chaotic attractors is the kneading of bread.[9] If we were to place a dot of food coloring on a ball of dough and then proceed to knead it, it would get stretched and folded with each knead. The area covered by the dye would eventually spread out until it covered a large area of dough, although it started in only a very small area. The blue dye here is analogous to a set of initial conditions, which spread out according to the trajectories of each initial state. In chaotic systems, these initial conditions spread out to cover a wide range of states at some later time, just as the food coloring will be spread out over a large volume of the dough, so much so that the entire dough will, after some time, have a lightly colored tint to it. This stretching and folding process is illustrated in Fig. 1.6 below, from Crutchfield, Farmer, Packard, and Shaw’s article, *Chaos.*[9] In the construction of the Roessler attractor, it’s as if the trajectories have first been stretched apart (seen in the top left of Fig. 1.6), then folded together like a bedsheets (bottom left) and then wrapped around to insert the trajectories back near their initial paths.

![Figure 1.6: A demonstration of the stretching and folding mechanism in the Roessler attractor, from Crutchfield, Farmer, Packard, & Shaw’s article, Chaos,[9] p. 51, reprinted with permission from Norman Packard. The Roessler attractor is considered one of the simplest chaotic attractors, somewhat resembling a Moebius strip.](image-url)
1.3 Sensitive dependence on initial conditions

The concept of sensitive dependence on initial conditions is sometimes thought of as synonymous with chaotic systems. Chaotic systems are unpredictable precisely because of this attribute. A small set of initial conditions maps to a quite large set of later conditions. This is illustrated well in Fig. 1.7, from Crutchfield, Farmer, Packard, & Shaw’s article.[9] Because our measurement tools can never allow us to make a perfectly accurate measurement with an infinite degree of precision, we will always start with not just one initial condition, but some small set of initial conditions, or cloud of uncertainty, which we incorrectly localize at one point when we record it. In other words, we can only really say that the initial condition lies somewhere inside some sphere of uncertainty (for example, the red dot in the top left panel of Fig. 1.7), but we cannot say exactly where inside the sphere we began. This localized set first stretches into wispy red lines, and after a sufficient amount of time, has spread over the entire attractor. Now we are not sure which of these red points represents the final state of our system because we cannot say precisely where our original initial condition \((x, y, z)\) was located, only that it was somewhere inside the sphere of uncertainty. So as far as our measurement tools and the Heisenberg Uncertainty Principle[5] will allow, any one of the 10,000 red points in the bottom right panel of Fig. 1.7 could describe our system at this point in time, and we cannot know which.

Fig. 1.8 gives another illustration of this sensitive dependence on initial conditions. Here we plot only one variable of the Lorenz system against time, rather than all three, and see how a just a few minute degrees of precision can make all the difference in determining the state occupied by a chaotic system at a later time. With the green line we plot the evolution of Lorenz’s \(x\) starting from an initial condition with eight decimal places. With the blue line, we plot the evolution of \(x\) from that same initial condition with the last five degrees of precision cut off. This number is only ever so slightly offset from the green line’s initial condition by a few mere hundred thousandths, but the path it follows diverges exponentially from the green path after only ten seconds.

If our measurement leans just slightly in one direction, which it inevitably will, due not only to the impossibility of infinitely precise measuring tools but also to Heisenberg’s Uncertainty Principle, then we will be unable to say which trajectory it has actually followed, and if we try, will end up with a wildly different outcome than that which was truly pursued by the system. In this sense, chaos is deterministic
unpredictability. Once a chaotic system starts running, the outcome is predetermined by the initial conditions, but unpredictability lies in our inability to exactly locate this initial state. This inability, mind you, is owed not only to our crude human
1.4 The Lorenz Attractor

The Lorenz equations are:

\[ \dot{x} = \sigma(y - x) \]  \hspace{1cm} (1.1)  

\[ \dot{y} = \rho x - y - x z \]  \hspace{1cm} (1.2)  

\[ \dot{z} = xy - \beta z. \]  \hspace{1cm} (1.3)  

where \( x, y, \) and \( z \) are variables and \( \sigma, \rho, \) and \( \beta \) are parameters greater than zero. \( \sigma \) is called the Prandtl number, \( \rho \) is the Rayleigh number, and \( \beta \) is some quantity without a name. The Rayleigh number is important because it represents the ratio of forcing to damping in the Lorenz system. Specifically, it refers to the ratio of the temperature at the bottom to the temperature at the top of a convective cell. When the bottom methods of measurement, but more importantly to the inherent impossibility of ever making an infinitely precise statement of some object’s position and momentum.

Figure 1.8: A plot of \( x \) vs. \( t \) of Lorenz’s \( x \)-variable, demonstrating exponential sensitivity to initial conditions. The green line shows the evolution of \( x \) when started at an initial condition with eight decimal places, whereas the blue line shows the evolution from the same number with the last five digits cut off, a mere difference of a few hundred thousandths, which diverges wildly from the green trajectory after only ten seconds.
temperature increases, the system is driven harder and the Rayleigh number increases. For very small values of $\rho$, the Lorenz system remains motionless. Convection currents will not occur if there is no temperature gradient from the bottom to the top of the convective cell: it is now just a box of still water. The attractor for this regime is a stable fixed point. If $\rho$ is increased a bit, convection currents begin occurring and the Lorenz system enters a regime of transient chaos; the currents would take an exponentially long time to settle down onto a fixed point. As $\rho$ is further increased, the convection rolls speed up and, at a certain critical value around $\rho = 24$, become chaotic. For certain regimes with a large $\rho$, the Lorenz system is periodic and follows a limit cycle. The attractor in this regime has the topological equivalence of a circle. Any loop that closes back on itself has the topological equivalence of a circle. The trajectory will go around this loop indefinitely. This is illustrated in Fig. 1.9.

By varying the parameters of the Lorenz system, we have altered its topology in phase space. As we increase the parameter $\rho$, we enter into several different regimes of behavior: from stationary to chaotic to periodic and back and forth and in between these regimes as we continue to increase $\rho$. We say that a bifurcation occurs when a parameter is varied such that the topology of the phase space plot changes. The
Lorenz system has several bifurcations, however, we will not discuss them in much detail because the quantitative theory is irrelevant to our present analysis.\footnote{For a more detailed discussion of bifurcations in the Lorenz equations, see Colin Sparrow’s \textit{The Lorenz Equations: Bifurcations, Chaos, and Strange Attractor}, in which he meticulously discusses parameter variation and bifurcation plots.}

Now that we have a basic understanding of the topology and behavior of chaotic systems, we march onward toward the experiment itself. A chaotic waterwheel is not only an achievable construction due to its purely mechanical nature, but also a near perfect analogue the the Lorenz system.
Chapter 2

Experimental Setup

Figure 2.1: My chaotic waterwheel. Sitting on top of the axle of the wheel is the encoder which I used to collect data.

In this chapter, we will lay out in detail, the steps taken to build the chaotic waterwheel and discuss the electronic equipment and computer programs used to collect data. To build the wheel, I used my own mechanical innovation under the guidance of Greg Eibel, the physics machinist. Nearly every piece was made from scratch or foraged from the bowels of the science buildings. First, I needed to find
something that spun nicely, with minimal rotational friction. I considered using a record player, an old hard drive, and a bicycle wheel. The problem with record players was that they don’t quite spin uniformly when tilted at an angle. Hard drives were ruled out because their large mass contributes to a large and unwanted moment of inertia, as I wanted the wheel to be able to switch spin direction with ease. I decided a bicycle wheel was the best option, so I went to a bike shop and chose a smallish 16 1/2" BMX wheel with incredibly smooth bearings. My next step was to figure out what kind of cups to use around the perimeter. I searched the biology storeroom and some plastics stores for containers, but wasn’t sure which size to pick. I ended up choosing 50 cc syringes from the bio storeroom for several reasons. First of all, they not only had uniform holes in the bottoms, but also short pipes connected to each hole to simulate Poiseuille flow which would save me some work and possible inconsistency in drilling the holes and attaching the pipes myself. Secondly, they were free, as they had been donated to the biology department, who had virtually no use for them. I took 56 of these syringes and attached them around the perimeter of the wheel using a metal synching band, arranging them as close together as possible to minimize spilling. I shaved off either side of the top rim of each syringe to make them fit more snugly.

![Figure 2.2: Top view of the syringes. I shaved off either side of the finger pulls at the tops of each syringe so that they would fit more snugly together.](image)

I then headed down to the physics machine shop to build a stand that could hold on the wheel at a variable angle. I machined an aluminum cylinder to fit around the bolt going through the axle of the wheel, and tightened it onto the bolt using set screws. I then drilled and threaded a hole in the other side of the cylinder so that it could be bolted down to a pair of hinged boards that would act as my variable angle stand.

---

1Poiseuille flow describes the flow of a liquid through a tube. For more a more detailed description, see Eckert’s *The dawn of fluid dynamics.*[10]
Next I needed to build a manifold to dispense water evenly into the top fourth of the cups (14 of the 56) symmetrically from the highest point on the wheel. I took a piece of \( \frac{1}{2} \)" copper tubing, bent it to have the same curvature as the perimeter of the wheel, drilled 14 uniform, evenly spaced holes in the tube, clamped the ends, and soldered a separate piece of \( \frac{1}{4} \)" tubing in a T in the top of the manifold for water inflow. I joined this T to a plastic tube that I connected to the faucet in my thesis office. I then mounted the manifold onto the wheel stand.

I then needed to make a velocity dependent brake. I first thought that a viscous drag brake was my best bet, but these are hard to construct from scratch. My thoughts turned to a magnetic eddy current brake.\(^2\) I ended up using a non-ferrous disk which was fixed directly to the wheel so that it would spin with it. I glued sixteen quite powerful, large, and twelve smaller rare earth magnets\(^3\) to another non-ferrous disk, and fixed this disk to the non-spinning aluminum post that housed the wheel’s axle such that it could slide up and down the post so that its proximity to the spinning disk was variable. The stationary magnets induce eddy currents in the spinning disk which in turn produce magnetic fields that oppose the changing magnetic flux. This provides a braking force on the spinning disk. The braking force can be altered by changing the proximity of the stationary magnets to the spinning disk, imposing on it either a stronger or weaker magnetic field. This brake worked well, and I was finally able to achieve non-periodic motion from my wheel. The time had come to collect data.

I needed some sort of device to detect the angular velocity of the wheel. I found an encoder with a rotating hub that could detect the speed and direction of the wheel. Because the axle of a bicycle wheel does not spin, I machined another piece to fit over the top of the axle which I glued down to the hub so that it would spin with the wheel. The encoder then fit easily over this piece of aluminum to take data on the direction and magnitude of the wheel’s angular velocity.

The encoder consists of two perforated disks, one slightly offset from the other. By determining which disk is lagging behind the other, we can tell which direction the encoder hub is rotating. The signal from the encoder enters the “encoder box” (built by Bob Ormund) which offers a convenient way to interface the encoder and computer. The box contains a HCL-2016 Quadrature Decoder/Counter chip that continuously counts pulses from the encoder in real time and stores this data in a

\(^2\)For a detailed explanation of magnetic braking, see Wiederick et al.’s paper, Magnetic braking: Simple theory and experiment.\(^1\)

\(^3\)The magnets were manufactured by Magcraft, product ID: NSN0604 Rare Earth Disc Magnets. The dimensions of the large magnets are 1" by 1/8", and the small magnet are 1/2" by 1/16"
2^{16} bit memory chip. Additional benefits of the encoder box include digital noise filters, optical isolation techniques to protect against power surges or the misplugging of wires, and transmission line techniques to keep the signal clean. (See Appendix B and the supplemental materials CD for a detailed description and circuitry of the encoder box.)

The encoder box is hooked up to a National Instruments USB-6009 8 input, 14bits, Multifunction I/O card which in turn connects to a PC. On the PC, we run a LabView program which collects data from the 16-bit chip one byte at a time, reading the high byte, then the low byte, etc. It then gives us two pieces of data for every count: the time at which the count was collected and the angle in degrees at which the wheel was positioned when the count was measured.

With all of this equipment set up, I was ready to begin playing around with different behaviors of the wheel by altering the angle of inclination and proximity of the magnets to the nonferrous disk. By experimenting with various forcing to damping ratios, I located regimes for periodic and nonperiodic spinning. Once I found good parameters for both periodic and chaotic behavior, I began taking data in each regime.
Chapter 3

Equations Specific to the Waterwheel

3.1 Equations for our system

Now we must construct a set of equations that govern the motion of the waterwheel. Here we will follow the approach of Strogatz in his wonderful book *Nonlinear Dynamics and Chaos*. He models the water flow and cups as a continuous system rather than discrete streams into discrete cups, although you can obtain the same set of equations using alternative methods. First, we identify the components contributing to the motion of our system as water flow, gravity, and damping forces such as the magnetic brake. Then we can determine how exactly each one contributes.

3.1.1 Conservation of Mass and Water flow: Mass change of water in the wheel due to rotation of the wheel, water inflow, and outflow.

Here we recognize that mass must be conserved while water is going into the wheel and coming out of it. If we examine a small portion of space through which the wheel rotates into and out of, say a slice from an angle $\theta_1$ to a second angle $\theta_2$ ($\theta = 0$ at the top of the wheel), we can see that there are three contributors to the change in water of the portion of the wheel we are examining: the wheel is rotating, water is being poured into the cups, and water is leaking out of the holes in the bottoms of
Chapter 3. Equations Specific to the Waterwheel

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{\text{total}}(t)$</td>
<td>total mass of water in the wheel</td>
<td>kg</td>
</tr>
<tr>
<td>$m(\theta, t)$</td>
<td>mass distribution of water around the wheel’s perimeter</td>
<td>kg/rad</td>
</tr>
<tr>
<td>$\theta$</td>
<td>angle in our lab frame</td>
<td>rad</td>
</tr>
<tr>
<td>$\omega(t)$</td>
<td>angular velocity of the wheel</td>
<td>rad/sec</td>
</tr>
<tr>
<td>$Q$</td>
<td>water inflow rate</td>
<td>kg/sec</td>
</tr>
<tr>
<td>$g_o$</td>
<td>acceleration due to gravity</td>
<td>m/s²</td>
</tr>
<tr>
<td>$R_p$</td>
<td>radius of the pipes at the bottom of the syringes</td>
<td>m</td>
</tr>
<tr>
<td>$l_p$</td>
<td>length of the pipes at the bottom of the syringes</td>
<td>m</td>
</tr>
<tr>
<td>$v_w$</td>
<td>viscosity of water</td>
<td>cm²/s²</td>
</tr>
<tr>
<td>$\tau$</td>
<td>torque on the wheel</td>
<td>N · m</td>
</tr>
<tr>
<td>$I$</td>
<td>moment of inertia of the wheel</td>
<td>kg · m²/s</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>angle of inclination of the wheel</td>
<td>deg</td>
</tr>
<tr>
<td>$r$</td>
<td>radius of the wheel</td>
<td>m</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>strength of the magnetic brake</td>
<td>kg · m²/s</td>
</tr>
<tr>
<td>$v$</td>
<td>translational velocity of the wheel</td>
<td>m/s</td>
</tr>
<tr>
<td>$a$</td>
<td>translational acceleration of the wheel</td>
<td>m/s²</td>
</tr>
<tr>
<td>$a_n, b_n$</td>
<td>amplitude coefficients in Fourier series of mass</td>
<td>m</td>
</tr>
<tr>
<td>$q_n$</td>
<td>amplitude coefficients in Fourier series of inflow</td>
<td>kg/s</td>
</tr>
</tbody>
</table>

Table 3.1: Symbols used in our model of the wheel.

the cups. Then we have

$$\frac{dM_{\text{total}}}{dt} = \text{Rotation} + \text{Inflow} + \text{Outflow}$$

**Rotation.** We first consider the motion of the water in the wheel into and out of our slice. Let $dM_{R_1} = m(\theta_1, t) \omega dt$ be the water rotating into our slice of space and let $dM_{R_2} = -m(\theta_2, t) \omega dt$ be the water rotating out of our slice, where $M$ is the total mass of water in all of the cups, $m(\theta, t)$ is the mass distribution of the water around the perimeter of the wheel and $\omega$ is the angular velocity of the wheel. Then Eq. 3.1 describes the change of mass in a slice of the wheel due to rotation:

$$\frac{dM_{\text{rotation}}}{dt} = \omega(t)[m(\theta_1, t) - m(\theta_2, t)] = -\omega \int_{\theta_1}^{\theta_2} \frac{\partial m}{\partial \theta} d\theta. \quad (3.1)$$
3.1. Equations for our system

Figure 3.1: A slice of the wheel. This schematic illustrates a portion of the wheel we are examining in order to construct equations concerning the rotation of the wheel and the amount of water in our frame of reference.

Outflow. Now we can think about the water leaking out of each of the cups around the wheel. Each of the cups has a pipe at the bottom, making Poiseuille flow possible. This means that the rate at which water leaks out of the cup depends on the mass of the water in the cup.

\[
\frac{dM_{\text{outflow}}}{dt} = -k \int_{\theta_1}^{\theta_2} m \, d\theta, \quad k = \frac{g_c R_p^2}{8v_w l}
\]  

(3.2)

where \( m \) is the mass of the water in the cup, \( k \) is the standard fluid dynamics equation for the rate at which water will flow out of the cups,\(^1\) and \( g_c = 9.81 \text{ m/s}^2 \), \( R_p \) is the radius of the pipe, \( v_w \) is the viscosity of water, \( 0.010 \text{ cm}^2 \text{s}^{-1} \), and \( l \) is the length of the pipe.

Inflow. The inflow of water, \( Q \), is easy because we can directly control it. It’s dependent on nothing other than the flow rate we impose on the manifold. So we have

\[
\frac{dM_{\text{inflow}}}{dt} = Q.
\]  

(3.3)

Total mass change. Putting all of the mass change equations together, we get

\[
\frac{dM_{\text{total}}}{dt} = \int_0^{2\pi} \frac{\partial m}{\partial t} \, d\theta = \int_0^{2\pi} \left( -\omega \frac{\partial m}{\partial \theta} + Q - km \right) \, d\theta
\]

\(^1\)This can be derived from Eckert,[10] p.108.
or, equivalently,
\[
\frac{\partial m}{\partial t} = -\omega \frac{\partial m}{\partial \theta} - km + Q. 
\] (3.4)

And, total mass is conserved: \( M_{total} = \frac{Q_{total}}{k} \).

### 3.1.2 Torque on the wheel.

We will now discuss the driving and damping forces on the wheel’s motion. The driving force is gravity acting on the water in the cups. The damping forces are due to the magnetic brake and the spinning up of water that is falling straight downward without any initial angular momentum. Let \( I \) be the moment of inertia of the wheel. Then

\[
\tau = I \dot{\omega} = \text{gravitational torque} + \text{damping torque}. 
\]

![Figure 3.2: View of the waterwheel from the side. The waterwheel is inclined at an angle \( \alpha \).](image)

**Gravity.** Because the wheel is inclined at some angle, \( \alpha \), gravity pulls down on the cups with a force

\[
F_{\text{gravity}} = mg_0 \sin \alpha 
\]
3.1. Equations for our system

and exerts a torque with magnitude

\[ \tau_{\text{gravity}} = \vec{r} \times \vec{F}_{\text{gravity}} \]

\[ \tau_{\text{gravity}} = r \, m \, g_0 \, \sin \alpha \, \sin \theta \, d\theta \]  

(3.5)

where \( r \) is the radius from the wheel’s axle to the center of the cups.

_Damping due to the brake._ The magnetic brake opposes the motion of the wheel, and the strength with which it opposes the motion is dependent on the angular velocity of the wheel (i.e. the faster the wheel spins, the more braking force the magnets exert on the wheel to slow it down).\(^2\) The damping torque is given by

\[ \tau_{\text{brake}} = -\gamma \omega \]  

(3.6)

where \( \gamma \) is some parameter associated with the strength of the brake.

_Damping due to the speeding up of water with no angular velocity._ Because we are taking water that is presumably motionless in the horizontal direction (has no angular momentum) and squirting it into our moving cups, we are giving up some of the wheel’s angular velocity and bestowing it upon the water in order to conserve the angular momentum of our system. Hence, in doing so, we exert a small damping force on the wheel. Then the damping force exerted by the water on the entire wheel is

\[ F_{\text{water}} = ma = Qv = Q\omega r \]

where \( r \), again, is the radius of the wheel, and the torque that this still water exerts on the wheel is

\[ \tau_{\text{water}} = \vec{r} \times \vec{F} = -Q\omega r^2. \]  

(3.7)

_Total torque._ Now we can add up the torque due to both damping forces and gravity to obtain an equation of total torque:

\[ \tau_{\text{total}} = -\gamma \omega - Q\omega r^2 + g_0 \sin \alpha r \int_{\theta_1}^{\theta_2} m(\theta, t) \sin \theta \, d\theta \]

or

\[ \tau_{\text{total}} = -\nu \omega + gr \int_{\theta_1}^{\theta_2} m(\theta, t) \sin \theta \, d\theta \]  

(3.8)

where \( \nu = \gamma + Qr^2 \) and \( g = g_0 \sin \alpha \).

\(^2\)For a detailed discussion of the magnetic break, see Wiederick et al.’s article, *Magnetic braking: Simple theory and experiment.*[11]
So to summarize, the two equations governing the wheel are Eqs. 3.4 and 3.8:

\[ I_{\text{total}}\dot{\omega} = \nu \omega + gr \int m \sin \theta d\theta \]
\[ \dot{m} = Q - km - \omega \frac{\partial m}{\partial \theta} \]

3.1.3 Amplitude Equations

Because the mass of water in the wheel is periodic in \( \theta \) (i.e. the wheel is round), we can write the equations that describe our system (mass change and torque) as a Fourier series as follows. First we’ll expand our mass term \( m(\theta, t) \) as a sum of various harmonics where \( a_n \) and \( b_n \) are amplitude coefficients for each sine and cosine term, respectively:

\[ m(\theta, t) = \sum_{n=0}^{\infty} [a_n(t) \sin n\theta + b_n(t) \cos n\theta]. \] \hspace{1cm} (3.9)

We can also write the inflow as a Fourier series:

\[ Q(\theta) = \sum_{n=0}^{\infty} q_n \cos n\theta \] \hspace{1cm} (3.10)

with no sine terms because water is added symmetrically to the top of the wheel by the manifold (recall that we count our \( \theta \) from the highest point on the perimeter). Now we can substitute Eqs. 3.9 and 3.10 into Eq. 3.4 for total mass change and Eq. 3.8 for total torque to get a set of ODEs for the amplitudes of the different harmonics of our system. After expanding the mass change equation with these Fourier series, we get

\[ \frac{\partial}{\partial t} \left[ \sum_{n=0}^{\infty} a_n(t) \sin n\theta + b_n(t) \cos n\theta \right] = -\omega \frac{\partial}{\partial \theta} \left[ \sum_{n=0}^{\infty} a_n(t) \sin n\theta + b_n(t) \cos n\theta \right] - k \left[ \sum_{n=0}^{\infty} a_n(t) \sin n\theta + b_n(t) \cos n\theta \right] + \sum_{n=0}^{\infty} q_n \cos n\theta \]

When we take the derivative with respect to theta, the first term on the right hand side becomes

\[ -\omega \left[ \sum_{n=0}^{\infty} a_n(t)n \cos n\theta - b_n(t)n \sin n\theta \right]. \]
Then we can collect the sine and cosine terms separately by orthogonality of the functions. For the sine coefficients we have

\[ \frac{\partial}{\partial t} a_n = \omega b_n n - ka_n \]  

(3.11)

and for cosine we have

\[ \frac{\partial}{\partial t} b_n = -\omega a_n n - kb_n + q_n. \]  

(3.12)

Inserting the same Fourier series into Eq. 3.8 for the torque, we get

\[
\tau_{\text{total}} = I \dot{\omega} = -\nu \omega + gr \int_0^\pi \Sigma_{n=0}^\infty [a_n(t) \sin n\theta + b_n(t) \cos n\theta] \sin \theta \, d\theta \\
= -\nu \omega + gr \int_0^\pi a_1 \sin^2 \theta \, d\theta \\
= \nu \omega + gra_1 \pi 
\]

when we integrate from 0 to \( \pi \). Because the sine and cosine functions inside the integral are orthogonal, only one term makes it past the integration. We end up getting only \( a_1 \) in the resulting torque equation!

Then by looking at Eqs. 3.11 and 3.12, we can see that \( a_1, b_1, \) and \( \omega \) form a closed system and are decoupled from all \( a_n \) and \( b_n \) where \( n \neq 1 \); we can solve the system without them. Then we can throw out all Eqs. 3.11 and 3.12 where \( n \neq 1 \), and we end up with only three equations

\[ \dot{a}_1 = \omega b_1 - ka_1 \]  

(3.13)

\[ \dot{b}_1 = -\omega a_1 - kb_1 + q_1 \]  

(3.14)

\[ \dot{\omega} = -\nu \omega + gra_1 \pi. \]  

(3.15)

Although it is not yet obvious, these equations amazingly have the same exact structure of the Lorenz equations! Later on we will map our three equations onto the Lorenz Equations in order to see how both sets of parameters line up.

### 3.1.4 Finding fixed points in our system

Finding the fixed points is a simple way to understand a little bit more about the solution to our system, composed of Eqs. 3.13, 3.14, and 3.15 (as we did with our
differential equation $\dot{x} = \sin x$ in the beginning of Chapter 1). To find the fixed points in our system, we begin by setting the derivatives in these equations to zero, yielding:

$$a_1 = \frac{\omega b_1}{k}$$  \hspace{1cm} (3.16)

$$a_1 = \frac{q_1 - kb_1}{\omega}$$  \hspace{1cm} (3.17)

$$a_1 = \frac{\nu \omega}{gr\pi}.$$  \hspace{1cm} (3.18)

Equating Eqs. 3.16 and 3.17 allows us to solve for $b_1$:

$$b_1 = \frac{q_1 k}{\omega^2 + k^2}.$$  \hspace{1cm} (3.19)

Then equating Eqs. 3.16 and 3.18 gives us

$$\frac{\omega b_1}{k} = \frac{\nu \omega}{gr\pi}$$

which implies either $\omega = 0$ or $b_1 = k\nu/gr\pi$. We’ll now examine these two regimes, one in which $\omega = 0$ and the other in which $\omega \neq 0$.

1. $\omega = 0$ In this regime, the wheel has no angular velocity and is at rest. If $\omega = 0$, then this implies that $a_1 = 0$ and $b_1 = q_1/k$, so the wheel is balanced symmetrically by the water in each cup. Water is squirting into the cups and leaking out of them symmetrically.

2. $\omega \neq 0$ This regime corresponds to a spinning wheel. If $\omega \neq 0$, then

$$b_1 = \frac{q_1 k}{\omega^2 + k^2} = k\nu/gr\pi$$

and

$$\omega^2 = \frac{\pi grq_1}{\nu} - k^2.$$  \hspace{1cm} (3.20)

Then there are two possibilities within this regime: $+\omega$ and $-\omega$, in which the wheel is spinning at a constant angular velocity. Because angular velocity cannot be imaginary, $\frac{\pi grq_1}{\nu} - k^2$ must be positive, thus it must be true that

$$\frac{\pi grq_1}{\nu k^2} > 1.$$  \hspace{1cm} (3.20)

The group of parameters on the left hand side of Eq. 3.20 corresponds to the Rayleigh number. It is a ratio of forcing vs. damping in our system. The parameters in
the numerator, \( g \) and \( q_1 \) (gravity and inflow), represent the driving of the wheel, whereas the parameters in the denominator, \( \nu \) and \( k \) (damping forces and outflow), represent the dissipation of the wheel’s motion. In relation to the convection currents studied by Lorenz, the Rayleigh number represents the ratio of temperature at the top and bottom of a convective cell.[2] When the Rayleigh number is small, the system which it is describing stands still; the driving contributors are not great enough to overcome the damping contributors. As it increases, the regarded system begins its motion, and transient chaos ensues, corresponding to the slow roll of a convection current heated with a low temperature. This motion is not quite periodic, but not yet chaotic. One would have to wait an exponentially long time before the system would settle onto a fixed point. As the Rayleigh number increases past a critical value, chaotic motion begins in both the waterwheel and convection currents. This is easily observed in a pot of water boiling on high heat on the stove. At an even higher Rayleigh number, convection currents remain chaotic while the waterwheel is forced into periodic motion. As we will see later, at a very high angle of inclination with a low breaking force (high effective \( g \) with a low \( \nu \)), the wheel spins once to the left, once to the right, and so on. At this point, the waterwheel is no longer a perfect mechanical analogue to convection currents. Our task in the experiment will be to find the right range for our forcing to damping ratio so that the wheel spins chaotically.

### 3.2 Mapping the waterwheel equations onto the Lorenz equations

Recall Lorenz’s equations[6]:

\[
\dot{x} = \sigma(y - x) \tag{3.21}
\]

\[
\dot{y} = \rho x - y - xz \tag{3.22}
\]

\[
\dot{z} = xy - \beta z \tag{3.23}
\]

We can see that the waterwheel equations (3.13, 3.14, and 3.15) are somewhat similar to the Lorenz equations, and will soon be able to see that the waterwheel is a mechanical analogue to Lorenz’s system of convection currents. The Lorenz equations have variables \( x, y, \) and \( z, \) and parameters \( \sigma, \rho, \) and \( \beta, \) whereas our equations have
variables $\omega$, $a_1$, and $b_1$, and parameters $\nu$, $g$, $r$, $\pi$, $I$, $k$, and $q_1$. However, $r$, $\pi$, and $k$ are fixed or already chosen, and we only have control over $\nu$ by altering the strength of the brake, $g$ by altering the effective torque that gravity imposes on the wheel by altering the angle $\alpha$ at which the wheel is inclined, $I$, which depends on $\nu$ and $g$, and $q_1$ by altering the rate of inflow. Our task is to match each of our equations to one of the Lorenz equations in order to figure out how the parameters of each system line up. We can pretty much see right off which of our equations is analogous to each Lorenz equation by examining the position of each variable. Our $\omega$ should be Lorenz’s $x$, our $a_1$ his $y$, and our $b_1$ his $z$.

\[
\dot{\omega} = -\nu \omega + \frac{gra_1 \pi}{I} \quad \rightarrow \quad \dot{x} = \sigma(y - x) \\
\dot{a}_1 = \omega b_1 - ka_1 \quad \rightarrow \quad \dot{y} = \rho x - y - xz \\
\dot{b}_1 = -\omega a_1 - kb_1 + q_1 \quad \rightarrow \quad \dot{z} = xy - \beta z
\]

Because Lorenz’s equation are dimensionless, we must also make ours dimensionless. Let’s begin from the top. First, we’ll make a relation between $x$ and $\omega$ that will help us map one to the other:

\[
x = x_0 + \omega/\omega_0.
\]

We divide $\omega$ by some constant with units $\omega$, which we’ll call $\omega_0$, in order to render this term dimensionless. We add $x_0$ as an offset. Then

\[
\omega = \omega_0(x - x_0).
\]

We do the same for the remaining variables:

\[
y = y_0 + a_1/A \quad \rightarrow \quad a_1 = A(y - y_0) \\
z = z_0 + b_1/B \quad \rightarrow \quad b_1 = B(z - z_0)
\]

We must also make a form of dimensionless time

\[
t^* = t/\tau \quad \rightarrow \quad t = t^* \tau
\]

and

\[
\frac{d}{dt} = \frac{1}{\tau} \frac{d}{dt^*}.
\]
3.2. Mapping the waterwheel equations onto the Lorenz equations

We now have an army of free parameters \( x_0, \omega_0, y_0, A, z_0, B \) and \( \tau \).

For our first equation, we start with our dimensionless \( x \), and take the derivative with respect to dimensionless time \( t^* \), and start plugging in:

\[
\frac{dx}{dt^*} = \tau \frac{d}{dt}(x_0 + \omega/\omega_0)
\]

\[
= \tau \frac{d\omega}{\omega_0}
\]

\[
= \frac{t}{\omega_0} \left(-\nu \omega + \frac{gra_1 \pi}{I}\right)
\]

\[
= \frac{t}{\omega_0} \left(-\nu \omega_0 (x - x_0) + \frac{gr \pi A(y - y_0)}{I}\right)
\]

\[
= \frac{-\nu \omega x}{I} + \frac{\nu \omega x_0}{I} + \frac{gr \pi A y}{I \omega_0} - \frac{gr \pi A y_0}{I \omega_0}.
\]

We are only interested in the first and third terms, the \( x \) and \( y \) terms, respectively, as dictated by Lorenz’s first equation, Eq. 3.21. We set the other terms to zero, and comparing this with Eq. 3.21, we get the conditions:

1. \( \frac{\nu \omega_0}{I} \frac{gr \pi A y_0}{I \omega_0} = 0 \)
2. \( \frac{\nu \omega_0}{I} \frac{gr \pi A y}{I \omega_0} = \sigma \to A = \frac{\nu \omega_0}{gr \pi}, \sigma = \frac{\nu \omega_0}{I} \).

Now for the mapping of \( a_1 \) onto \( y \), we proceed similarly:

\[
\frac{dy}{dt^*} = \tau \frac{d}{dt}(y_0 + a_1/A)
\]

\[
= \frac{\tau da_1}{A}
\]

\[
= \frac{\tau}{A} \left(\omega b_1 - k a_1\right)
\]

\[
= \frac{\tau}{A} \left(\omega_0 (x - x_0) B(z - z_0) - kA(y - y_0)\right)
\]

\[
= \frac{\tau}{A} \left(\omega_0 Bxz - \omega_0 Bx_0 z - \omega_0 Bx_0 z_0 + \omega_0 Bx_0 z_0 - kAy + kAy_0\right)
\]

\[
= \frac{\tau \omega_0 Bxz}{A} - \frac{\tau \omega_0 Bx_0 z}{A} - \frac{\tau \omega_0 Bx_0 z_0}{A} + \frac{\tau \omega_0 Bx_0 z_0}{A} - \left(\tau ky + \tau ky_0\right).
\]

Here we are interested in the first, second, and fifth terms, the \( xz, x, \) and \( y \) terms, respectively, as they appear in Eq. 3.22. We get more conditions:

3. \( \frac{\tau \omega_0 Bx_0 z}{A} + \frac{\tau \omega_0 Bx_0 z_0}{A} + \tau ky_0 = 0 \to x_0 = y_0 = 0 \)
4. \( \tau = \frac{1}{k} \to \tau k = \beta = 1 \)
5. \[ \frac{\tau}{A} \omega_0 B = -1 \]

6. \[ -\frac{\tau}{A} \omega_0 B z_0 = \rho \quad \rightarrow \quad z_0 = \rho \]

And finally, we map \( b_1 \) to \( z \):

\[
\frac{dz}{dt} = \tau \frac{d}{dt} \left( z_0 + \frac{b_1}{B} \right) = \frac{\tau}{B} db_1 = \frac{\tau}{B} \left( -\omega a_1 - kb_1 + q_1 \right)
\]

\[
= \frac{\tau}{B} \left[ (-\omega_0 (x - x_0))(A(y - y_0)) - kB(z - z_0) + q_1) \right]
\]

\[
= \frac{\tau}{B} \left[ -\omega_0 x A y + \omega_0 x A y_0 + \omega_0 x_0 A y - \omega_0 x_0 A y_0 - kBz + kBz_0 + q_1 \right]
\]

\[
= \frac{-\tau \omega_0 A x y}{B} + \frac{\tau \omega_0 x A y_0}{B} \quad \frac{\tau \omega_0 x_0 A y}{B} \quad \frac{-\tau \omega_0 x_0 A y_0}{B} - \frac{kz}{B} + \tau k z_0 + \frac{\tau q_1}{B}.
\]

We are interested in keeping the first and fifth terms, \( xy \) and \( z \), respectively, as per Eq. 3.23, and our last conditions are:

7. \[ \frac{\tau \omega_0 x A y_0}{B} \quad - \frac{\omega_0 x A y}{B} \quad - \frac{\tau \omega_0 x_0 A y_0}{B} + \tau k z_0 + \frac{\tau q_1}{B} = 0 \]

(We know that the first three terms are zero because they each contain either \( x_0 \) or \( y_0 \), which we’ve already established are equal to zero in condition 3.)

Therefore, condition 7 \[ \rightarrow \quad kz_0 + \frac{q_1}{B} = 0 \]

8. \[ -\frac{\tau \omega_0 A}{B} = 1 \quad \text{(with condition 5)} \quad \rightarrow \quad A^2 = B^2, \quad A = -B \]

9. (Conditions 2, 7, and 8) \[ \rightarrow \quad z_0 = \frac{q_1}{kB} = \frac{q_1}{kA} = \frac{q_1 g \tau^2 \pi}{kB \omega_0} \]

10. \[ \tau k = \beta = 1 \quad \text{(with condition 8)} \quad \rightarrow \quad \tau = \frac{1}{k} = \frac{1}{\omega_0}, \quad \omega_0 = k \]

To summarize, our mapping has given us information about what our free parameters \( x_0, \omega_0, y_0, A, z_0, B \) and \( \tau \) must be, and even better, the relation between Lorenz’s parameters and ours. These relations can be seen in Table 3.2. We can see now that our equations are equivalent to Lorenz’s equations. We must keep in mind which Lorenz parameters we can adjust by changing corresponding parameters of our system and which Lorenz parameters are fixed. \( \beta \) is always one and we cannot adjust it for the waterwheel. Recall that \( \rho \) represents the forcing to damping ratio of our system. We suspected that it could be adjusted by changing the angle of inclination of the waterwheel and the strength of the magnetic brake. We can now see that \( \rho \)
3.2. Mapping the waterwheel equations onto the Lorenz equations

<table>
<thead>
<tr>
<th>Lorenz and Scaling parameters</th>
<th>Waterwheel parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0 = y_0 = 0$</td>
<td>$z_0 = \rho = \frac{q_0 g_0 \sin \alpha r \pi}{k^2 \nu}$</td>
</tr>
<tr>
<td>$\omega_0 = k$</td>
<td></td>
</tr>
<tr>
<td>$\tau = 1/k$</td>
<td></td>
</tr>
<tr>
<td>$A = -B = \nu k / g r \pi$</td>
<td>$\sigma = \nu / k I$</td>
</tr>
<tr>
<td>$\beta = \tau k = 1$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.2: How the waterwheel parameters map to the Lorenz and scaling parameters we used.

really is dependent on $\alpha$ and $\nu$. Recall that $\nu$ depends on the magnetic brake strength $\gamma$ and is given by $\nu = \gamma + Q r^2$. We can also alter the Lorenz parameter $\sigma$ by changing the damping strength $\nu$. 
Chapter 4

Data Analysis

In this chapter, we analyze data taken in two parameter regimes of the chaotic water-wheel: one in which the motion of the wheel seemed to be periodic and one in which it seemed to spin chaotically, changing directions in an unpredictable way. To determine whether the wheel’s behavior was chaotic, we will reconstruct the attractors that our system followed in both regimes and discuss them qualitatively. Then we will investigate the similarities between our system and Lorenz’s by visually comparing the attractors specific to each system, as well as observing the behavior of each system in similar regimes. As discussed earlier, we know that the Lorenz system goes between periodic and chaotic behavior for different parameter values of $\rho$, and we will see whether our system does so in a similar way. Finally, we will construct a power spectrum of the rotational frequencies of the wheel in order to more closely examine the periodicity of each regime.

4.1 Attractor Reconstruction

Recall that in order to specify a point in phase space, a phase vector is typically used, specifying the value of each coordinate involved. This means that for an $n$-dimensional system, we must measure each of these $n$ coordinates to construct a system’s trajectory in phase space, if we are to go about it the puritanical way. However, it can be difficult in experiments to measure all of the concerned coordinates in phase space. The experimentalist generally ends up with a time series consisting of a sequence of scalar measurements of just one coordinate. Just so in our case, we collect data on $\theta$, the angular position of the wheel at evenly spaced increments of time. We can then take the derivative of the curve fit described by experimental data, as discussed in more detail further on in this chapter, to get a measurement of
angular velocity, $\omega$. Hence, we too end up with data of just one coordinate in our three coordinate system. A wonderful trick for using this data to get a picture of the trajectory in phase space is time series reconstruction. The idea is that knowledge of what one variable is doing should be sufficient information to get an idea of what the system is doing as a whole. In fact, there are several ways of reconstructing an attractor in phase space. Norman Packard wrote in his 1980 paper, Geometry from a Time Series,[13]

We have found that beginning with a time series obtained by sampling a single coordinate of \{the Roessler equations\}, one can obtain a variety of three independent quantities which appear to yield a faithful phase-space representation of the dynamics of the original $x$, $y$, $z$ space. One possible set of three such quantities is the value of the coordinate with its values at two previous times, e.g., $x(t)$, $x(t - \tau)$, and $x(t - 2\tau)$. Another set obtained by making the time delays small, and taking the differences is $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$.

This is good news for us, as it provides us with a clever and relatively simple way of plotting the trajectory of our system in phase space. We will choose the time delay reconstruction because it most closely and accurately lines up with our data. If we were to choose the reconstruction method that utilizes a coordinate along with its first and second derivatives, we would be introducing a larger error than necessary into our calculations, since taking derivatives amplifies noise.

Our time series consists of a sequence of scalar measurements of one coordinate $\omega(t)$ which depends on the state of our system, taken at regular time intervals $\Delta t$. By choosing an appropriate lag time $\tau$, we will construct a space with the coordinates, $(\omega(t), \omega(t - \tau))$. These coordinates make up our delay vector in the two dimensional space in which we will reconstruct an attractor. The phase space object we construct will generally be equivalent to the actual, traditionally obtained trajectory as long as we use a large enough dimension $m$ (in our case, we have chosen $m = 2$) of delay coordinate space to plot our reconstruction. Interestingly, the embedding theorems (theorems which address the question of topological equivalence between the object we reconstruct using time delay and the original phase space trajectory) guarantee that in a time series with ideal, noise-free measurements, the constructed delay vectors are equivalent to the original phase space vectors for some appropriate dimension $m$ of delay coordinate space and lag time, $\tau$.[14] It has been shown that a possibly fractal attractor of box-counting dimension $d$ can always be reconstructed with $m$
4.2 MatLab Program

time delayed versions of one coordinate, where \( m \) is an integer greater than \( 2d \).[15] For \( m \) less than \( 2d \), we obtain a projection of the attractor.

We have decided on reconstructing a two dimensional space, but the question remains as to what value we ought to choose for \( \tau \). In our analysis, we started with a lag time of 1 ms and increased this number until we achieved a good looking attractor. Remember that we are constructing a projection of the attractor in two dimensional space, whereas a traditional attractor space would call for at least three dimensions because our system has three variables. With too small of a lag time, the projected attractor looks squished, and by increasing the lag time we are merely spreading out the projection. This process is somewhat analogous to changing the angle at which we are examining a three dimensional object in two dimensions. If we make our eyes level with a tabletop on which a dinner plate sits and try to guess its shape, it appears flattened and we get a poor idea of what the actual three dimensional shape of the plate is. It could be square or round or pentagram shaped and we wouldn’t be able to tell. If we lift our eye level up to an appropriate height, we can get a better look at the plate’s shape. Similarly, by slowly increasing our lag time \( \tau \), we achieve a better projection of the three dimensional attractor in two dimensional space. We ultimately used 20 time-steps of length 0.1 seconds, a 2 second delay, in reconstructing the attractors for our data.

4.2 MatLab Program

To analyze the data files I gathered in LabView, I used MatLab. My raw data files consisted of two columns: time at which each data pont was taken, which I fixed at 100 ms intervals, and the angle at which the wheel was located when the measurement was taken. Because the encoder box has a 16-bit memory chip, once the \( 2^{16} \) data slots have been filled a large jump occurs in the angular position data column. The MatLab program first removes these jumps by locating these large differences and adding or subtracting them, depending on whether it was an upward or downward jump, corresponding to which direction the wheel was spinning when the jump occurred. At this point, we could take the derivative with respect to time, to get angular velocity, but that would amplify the noise in the data. Instead we instruct MatLab to take a few intermediary steps.

Another obstacle in our data acquisition method was that LabView had a hard time sampling at exactly every 100 ms, so some data points were closer together or further apart in time than others. Data points occurred not every 100 ms, but
rather had time gaps that were sometimes around 85 ms or 120 ms. The MatLab program was designed to resample these data points to space them evenly at 100 ms intervals. It does this constructing a spline (a fit where the first and second derivatives are continuous) for the entire data set and using this spline to make the best approximation of the angle at which the encoder was sitting at evenly spaced time intervals of 100 ms.

The MatLab program takes the derivative of this new resampled data curve with respect to time, to get the angular velocity in degrees per second, \( \omega = \frac{\text{angle}}{dt} \). However, this derivative tremendously amplifies noise. We utilize the fact that our data is oversampled to remove the noise to a large extent. We do this by taking a moving average of the \( \omega \) data set with an averaging window that is 15 data points wide. Then we plot this smoothed data with respect to time.

MatLab then plots angular velocity vs. angular velocity delayed, and in this way we reconstruct an attractor. We can choose different delay times in order to achieve the best projection of our attractor. We can also plot, alongside the reconstructions using our variable \( \omega(t) \), a reconstruction of the Lorenz attractor from numerical simulations using the same method. Recall from chapter 3 that our \( \omega \) is like Lorenz’s \( x(t) \). We can use MatLab to numerically solve Lorenz’s equations for \( x(t) \) and plot a similar time delay reconstruction of his three dimensional attractor in two dimensional space.

I played with the forcing to damping ratio (\( \rho \)) of the wheel by altering its angle of inclination (higher inclination corresponds to a stronger effective driving force of gravity) as well as the proximity of the stationary magnets to the nonferrous disk (changing the strength of the eddy current brake). The wheel didn’t appear to be affected much by slight alterations of the eddy current brake. It tended to either exert a damping force on the wheel or not be affecting it much at all. I began treating the brake as more of an “on/off” system rather than a variable one. For the data below, I left the brake at a constant position, and let the parameter \( \rho \) be altered solely by the angle of inclination. Through altering the angle of inclination systematically, I found regimes where the wheel appeared to be periodic, as well as others where the wheel exhibited much aperiodicity. I took data in both regimes for several hours. Data from both regimes can be seen in the following sections.

### 4.3 Periodic Data

For this data, the wheel was inclined at an angle of 21° (high \( \rho \)). Fig. 4.1 shows a zoomed-in view of a plot of angular velocity vs. time. This figure shows only a section
of the data because the data collection ran for several hours and the plot extends very far horizontally, in time. An upward spike indicates a spin in the direction we will call “forward”, and a downward spike corresponds to a spin in the “backward” direction. We can see that the motion of the wheel appears to be roughly periodic, spinning forward once, then backward once, and so on. However, the data in Fig. 4.1 is not perfectly periodic; the amplitude of each spike changes very slightly with every spin. This deviation from exact periodicity can be attributed to noise in the system as well as the fact that the wheel was built by human hands (mine).

Figure 4.1: Periodic Data. A plot of angular velocity vs. time, taken with the wheel inclined at an angle of $21^\circ$.

In the time delay reconstruction in Fig. 4.2, we see that the trajectory made by the wheel in this regime roughly followed a limit cycle. The limit cycle looks more like a thread that has been unravelled a bit than a perfect, topologically one-dimensional loop, which is in line with the changing amplitude of the spikes in Fig. 4.1. This can be attributed, again, to noise in the system and a tendency of the wheel to drift slightly in one direction rather than repeating the exact same cycle. This drifting tendency is perhaps due to an imbalance in the wheel. Nevertheless, this attractor displays an important feature of this regime. The trajectory travels once around the top half of the structure in Fig. 4.2, then once around the bottom half, and so on without crossing through the center. This motion corresponds to exactly one spin in each direction consecutively for the duration of the data run.
4.4 Chaotic Data

For the data below, the wheel was inclined at an angle of $12^\circ$ (a lower $\rho$). In Fig. 4.3, we see another zoomed-in section of a plot of angular velocity vs. time. Again, each upward spike corresponds to the wheel spinning forward, and each downward spike corresponds to the wheel spinning backwards. It can easily be seen that the number of consecutive spins in each direction keeps changing. The wheel appears to be spinning aperiodically.

Figure 4.3: Chaotic Data. A plot of angular velocity vs. time, taken with the wheel inclined at an angle of $12^\circ$. 
4.4. Chaotic Data

Let’s examine the time delay reconstruction of the attractor for this regime, shown in Fig. 4.4. This attractor looks much less like a limit cycle. There appears to be a more intricate web of paths crossing through the center. This crossing in the center corresponds to a changing period. Recall that in the attractor pictured in Fig. 4.2, for our periodic data, the loop corresponded to the wheel spinning forward once, then backward once. The crossing in the center of the attractor in Fig. 4.4 corresponds to changes in the number of spins in each direction.

Figure 4.4: Our Chaotic Attractor. A trajectory that roughly follows a chaotic attractor similar in topology to the Lorenz Attractor in a plot (using time delay reconstruction) of angular velocity vs. angular velocity delayed 2 seconds.

But what’s even more illuminating is to compare it to a similarly constructed projection of the Lorenz attractor, found in Fig. 4.5. This attractor was made with the time delay reconstruction method of Lorenz’s system using the $x$ variable, which, as we showed in Chapter 3, is analogous to our variable $\omega$. For this projection of the Lorenz attractor in two dimensions, we chose a $\rho$ of 26.5, which we’ll recall from Fig. 1.9 in Chapter 1, corresponds to a chaotic regime of the Lorenz system. We chose a delay time of 3 seconds. The scaling differs between Figs. 4.5 and 4.4 because while our equations are similar to Lorenz’s equations, they were scaled, using our free parameters, in Chapter 3.

Let’s further examine the chaotic and periodic regimes by plotting a power spectrum of their frequencies of rotation, shown in Fig. 4.6. Along the bottom, the horizontal axis is frequency of rotation and the vertical axis represents how often the wheel was rotating at that frequency. The green line was constructed using the data
which looked periodic. The wheel’s power spectrum is dominated by one frequency, characteristic of periodic motion, indicating that the data taken at the wheel’s inclination of 12° really was periodic. The blue line represents the chaotic data. Its power spectrum is spread over a large distribution of frequencies, characteristic of chaotic rotation.

We have successfully found both a chaotic and periodic regime for our system. Additionally, our chaotic regime corresponded to a small \( \rho \) and our periodic regime corresponded to a larger \( \rho \), which is analogous to the Lorenz system. However, I did not obtain a numerical measurement for our \( \rho \), which I could have done with more time and patience using Table 3.2, shown at the end of Chapter 3. Had I done this, I could have more carefully and quantitatively compared the regime changes and bifurcation locations between our system and the Lorenz system. However, I am fairly certain that I would have obtained uninteresting results lacking a perfect coordination of the two systems for two reasons. First, our wheel has irreconcilable flaws in its construction, as evident in the stray from perfect periodicity evident in Fig. 4.1 of our periodic data. And second, the analogy of the wheel as a mechanical replication of the Lorenz system breaks down at high values of \( \rho \).[2] Nevertheless, the data presented here shows that through changing parameters associated with the wheel, specifically \( \rho \), we can change the topology of the attractor and behavioral regimes specific to our system, but also that a simple, self-constructed piece of machinery can exhibit chaotic
Figure 4.6: A plot of the frequencies exhibited by both regimes, chaotic and periodic. The horizontal axis show the frequencies at which the wheel rotates, and the vertical axis is the power spectral density, i.e. how often the wheel spun at each of the frequencies on the horizontal axis. The green line represents the periodic regime, where the wheel spent most time spinning at a frequency of about .057 Hz. The blue line represents the chaotic regime, during which the wheel spun at a variety of frequencies centering around about .037 Hz.

motion with a great likeness to that of the Lorenz attractor, which is what we were striving for.
Conclusion

We have successfully built and analyzed a chaotic machine. We have seen that by varying certain parameters, we can bring the wheel into periodic and chaotic regimes, changing the topology of its attractor. We have reconstructed a projection of the attractor characteristic of our system. However, further investigation of the behavior of the chaotic waterwheel could be made. A future researcher could carry out a more quantitative analysis. For example, a bifurcation plot could be made as either the angle of the wheel or the braking force was slowly and meticulously altered. The magnetic break could also be improved to achieve a wider range of braking forces. Additionally, perfect periodicity could perhaps be achieved through a fine tuning of the wheel.

Hopefully, this thesis is approachable and understandable to a reasonably educated and interested person, and can clear up some common misconceptions about the great science of chaos theory. Chaos theory is relevant to nearly every field studied by modern scientists. While the other great physics revolutions of the twentieth century deal with incredibly fast or incredibly small (and often both) regimes, chaos theory treats phenomena from everyday life with which most people are familiar. It contributes to our understanding of phenomena such as population growth, arrhythmic hearts, circadian rhythms, dripping faucets, weather, circuitry; the list goes on and on. Chaos theorists are constantly finding new ways to apply their science. They even ponder large, abstract, philosophical quandries. In his book, *Chaos: Making a New Science,*[3] Gleick writes

The first chaos theorists, the scientists who set the discipline in motion, shared certain sensibilities. They had an eye for pattern, especially pattern that appeared on different scales at the same time. They had a taste for randomness and complexity, for jagged edges and sudden leaps. Believers in chaos—and they sometimes call themselves believers, or converts, or evangelists—speculate about determinism and free will, about evolution, about the nature of conscious intelligence. They feel that they are turning
back a trend in science toward reductionism, the analysis of systems in
terms of their constituent parts: quarks, chromosomes, or neurons. They
believe that they are looking for the whole.

Chaos theorists bring their theory into nearly every aspect of life. In reading books
and papers on chaos, one inevitably finds interjections such as this one, quoted below.
Crutchfield et al. wrote in their article *Chaos* in the Scientific American.[9]

> Even the process of intellectual progress relies on the injection of new
ideas and on new ways of connecting old ideas. Innate creativity may
have an underlying chaotic process that selectively amplifies small fluctua-
tions and molds them into macroscopic coherent mental states that are
experienced as thoughts. In some cases the thoughts may be decisions,
or what are perceived to be the exercise of will. In this light, chaos pro-
vides a mechanism that allows for free will within a world governed by
deterministic laws.

In order to understand the ways in which chaos is present in these more com-
plicated areas, it is important to first understand the basics. The goal of my thesis
was to familiarize myself and the reader with chaos theory through the workings of a
relatively simple system. Now that we have acquired the tools necessary to examine
chaotic systems, we can move onward to further educate ourselves in more advanced
chaos theory.
Appendix A

Plotting a solution of the Lorenz equations in Mathematica

We use Mathematica’s NDSolve to find a numerical solution to the Lorenz equations in Mathematica. Recall that the Lorenz equations are

\[\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xz - bz
\end{align*}\]

We can insert whatever parameters we want for the values of \(\sigma\), \(r\), and \(b\). Here, we’ve chosen \(\sigma = 3\), \(r = 26.5\), and \(b = 1\). Then we can plot the numerical solution in rainbow color!

```mathematica
NDSolve[\{x'[t] == -3 (x[t] - y[t]), y'[t] == -x[t] z[t] + 26.5 x[t] - y[t], z'[t] == x[t] y[t] - z[t], x[0] == z[0] == 0, y[0] == 1\}, \{x, y, z\}, \{t, 0, 200\}, MaxSteps -> Infinity];
ParametricPlot3D[Evaluate[\{x[t], y[t], z[t]\} /. %], \{t, 0, 200\}, PlotPoints -> 10000, ColorFunction -> (ColorData["Rainbow"][#4] &)]
```
Figure A.1: The Lorenz Attractor plotted in $x - y - z$ space.
Appendix B

Supplemental Materials

On the attached CD, I have included three folders. One contains all of the MatLab programs I used to analyze my data, generate plots and solve the Lorenz equations. Each program contains a description of what it does at the top of the file. Another folder contains a copy of the LabView program I used to collect data, as well as image files of the block diagram, front panel, and description of the program elements. A third folder contains diagrams related to the circuitry of the encoder box, built by Bob Ormund, which acted as an interface between the encoder and the computer running the LabView program. If one wished to take data using the chaotic waterwheel, one would find everything they needed to do so on the supplemental CD.
References


