Mansuripur's paradox

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A recent article by Mansuripur claims that the Lorentz force law is incompatible with special relativity. We discuss the "paradox" on which this claim is based. The resolution depends on whether one assumes a "Gilbert" model for the magnetic dipole (separated monopoles) or the standard "Ampère" model (a current loop). The former case was treated in these pages many years ago; the latter, as several authors have noted, constitutes an interesting manifestation of "hidden momentum."
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I. INTRODUCTION

On May 7, 2012, a remarkable article appeared in *Physical Review Letters*.[1] The author, Masud Mansuripur, claimed to offer "incontrovertible theoretical evidence of the incompatibility of the Lorentz [force] law with the fundamental tenets of special relativity," and concluded that "the Lorentz law must be abandoned." The Lorentz law,

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})], \tag{1}$$

tells us the force $\mathbf{F}$ on a charge q moving with velocity $\mathbf{v}$ through electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. Together with Maxwell's equations, it is the foundation on which all of classical electrodynamics rests. If it is incorrect, 150 years of theoretical physics is in serious jeopardy.

Such a provocative proposal was bound to attract attention. *Science*[2] published a full-page commentary, and within days several rebuttals were posted.[3] Critics pointed out that since the Lorentz force law can be embedded in a manifestly covariant formulation of electrodynamics, it is guaranteed to be consistent with special relativity,[4] and some of them identified the specific source of Mansuripur's error: neglect of "hidden momentum." Nearly a year later *Physical Review Letters* published four rebuttals,[5] and *Science* printed a follow-up article declaring the "purported relativity paradox resolved."[6]

Mansuripur's argument is based on a "paradox" that was explored in this journal by Victor Namias and others[7] many years ago: a magnetic dipole moving through an electric field can experience a torque, with no accompanying rotation. In Sec. II, we introduce Mansuripur's version of the paradox, in simplified form, and explain Namias's resolution. The latter is based on a "Gilbert" model of the dipole (separated magnetic monopoles); it does not work for the (realistic) "Ampère" model (a current loop). For Amperian dipoles, the resolution involves "hidden" momentum, so in Sec. III, we discuss the physical nature of this often-misunderstood phenomenon. Mansuripur himself treated the dipole as the point limit of a magnetized object, so in Sec. IV, we repeat the calculations in that context (for both models), and confirm our earlier results. In Sec. V, we discuss the Einstein–Laub force law, which Mansuripur proposed as a replacement for the Lorentz law and in Sec. VI, we offer some comments and conclusions.

II. GILBERT DIPOLES: NAMIAS'S RESOLUTION

First the paradox: In $\mathcal{S}'$ (the "proper" frame), there is an ideal magnetic dipole $\mathbf{m} = m_0\,\hat{\mathbf{x}}$ at $(0, 0, d)$, and a point charge q at the origin, both at rest. The torque on $\mathbf{m}$ is (obviously) zero. Now examine the same configuration in $\mathcal{S}$ (the "lab" frame), with respect to which $\mathcal{S}'$ moves at constant speed v in the z-direction (Fig. 1). In $\mathcal{S}$ the (moving) point charge generates electric and magnetic fields,

$$\mathbf{E}(x, y, z, t) = \frac{q}{4\pi\epsilon_0}\frac{\gamma}{R^3}[x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + (z - vt)\,\hat{\mathbf{z}}], \tag{2}$$

$$\mathbf{B}(x, y, z, t) = \frac{q}{4\pi\epsilon_0}\frac{v\gamma}{c^2R^3}(-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}), \tag{3}$$

($\gamma \equiv 1/\sqrt{1-(v/c)^2}$, $R \equiv \sqrt{x^2 + y^2 + \gamma^2(z - vt)^2}$), and the (moving) magnetic dipole acquires an electric dipole moment[8]

$$\mathbf{p} = \frac{1}{c^2}(\mathbf{v} \times \mathbf{m}) = \frac{1}{c^2}vm_0\,\hat{\mathbf{y}}. \tag{4}$$

The torque on the dipole is

$$\mathbf{N} = (\mathbf{m} \times \mathbf{B}) + (\mathbf{p} \times \mathbf{E}) = \frac{qm_0}{4\pi\epsilon_0}\frac{v}{c^2d^2}\,\hat{\mathbf{x}} \tag{5}$$

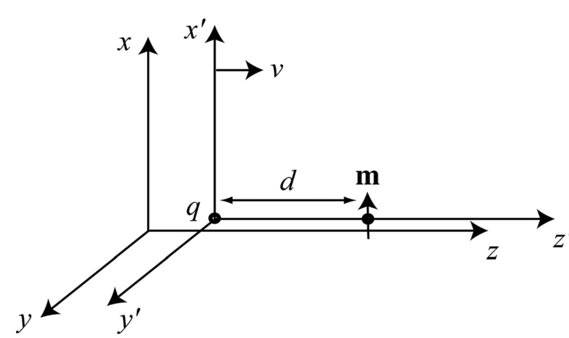

Fig. 1. Electric charge q and magnetic dipole m in proper (primed) and lab (unprimed) frames.

(by Lorentz transformation, $d = \gamma(z - vt)$; the magnetic contribution is zero because $\mathbf{B}$ vanishes on the z-axis). The torque is zero in one inertial frame, but *non* zero in the other! Mansuripur concludes that the Lorentz force law (on which Eq. (5) is predicated) is inconsistent with special relativity.

This "paradox" was resolved years ago by Victor Namias.[7] The standard torque formulas ($\mathbf{p} \times \mathbf{E}$ and $\mathbf{m} \times \mathbf{B}$) apply to dipoles *at rest*, but they do not hold, in general, for dipoles in motion. Suppose we model the magnetic dipole as separated monopoles (Fig. 2). The "Lorentz force law" for a magnetic monopole q^* reads[9]

$$\mathbf{F} = q^*[\mathbf{B} - (1/c^2)\mathbf{v} \times \mathbf{E}], \tag{6}$$

so the torque[10] on a moving dipole $\mathbf{m} = q^*(\mathbf{r}_+ - \mathbf{r}_-)$ is

$$\begin{aligned}\mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-)\\ &= (\mathbf{m} \times \mathbf{B}) - \frac{1}{c^2}\mathbf{m} \times (\mathbf{v} \times \mathbf{E}).\end{aligned} \tag{7}$$

But $\mathbf{m} \times (\mathbf{v} \times \mathbf{E}) = \mathbf{v} \times (\mathbf{m} \times \mathbf{E}) + (\mathbf{m} \times \mathbf{v}) \times \mathbf{E}$, so

$$\begin{aligned}\mathbf{N} &= (\mathbf{m} \times \mathbf{B}) - \frac{1}{c^2}(\mathbf{m} \times \mathbf{v}) \times \mathbf{E} - \frac{1}{c^2}\mathbf{v} \times (\mathbf{m} \times \mathbf{E})\\ &= (\mathbf{m} \times \mathbf{B}) + (\mathbf{p} \times \mathbf{E}) - \frac{1}{c^2}\mathbf{v} \times (\mathbf{m} \times \mathbf{E}).\end{aligned} \tag{8}$$

There is a third term, missing in Eq. (5), which (it is easy to check) exactly cancels the offending torque; the net torque is zero in both frames.

III. AMPÈRE DIPOLES: HIDDEN MOMENTUM

Namias believed that his formula [Eq. (8)] applies just as well to an Ampère dipole as it does to a Gilbert dipole. He was mistaken. An Ampère dipole in an electric field carries "hidden" momentum,[11]

$$\mathbf{p}_h = \frac{1}{c^2}(\mathbf{m} \times \mathbf{E}). \tag{9}$$

Because it is crucial in understanding the resolution to Mansuripur's paradox, we pause to review the derivation of this formula using a simple model.

Imagine a rectangular loop of wire carrying a steady current. Picture the current as a stream of noninteracting positive charges that move freely within the wire.[12] When a uniform electric field $\mathbf{E}$ is applied (Fig. 3), the charges accelerate up the left segment and decelerate down the right one. *Question:* What is the total momentum of all the charges in the loop? The left and right segments cancel, so we need

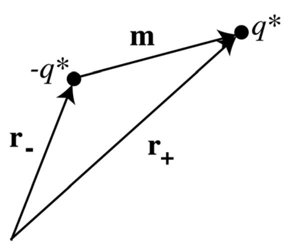

Fig. 2. A "Gilbert" magnetic dipole.

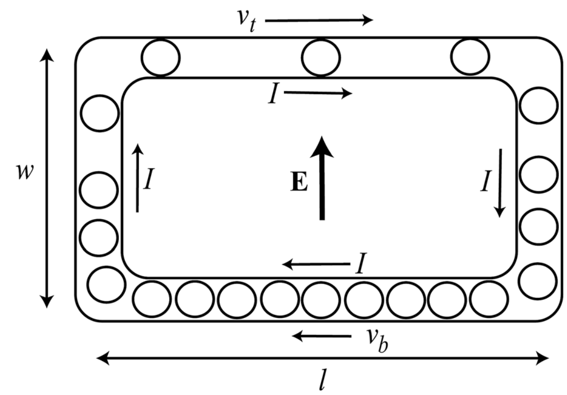

Fig. 3. Rectangular current loop in an external electric field.

only consider the top and bottom. Say there are N_t charges in the top segment, going to the right at speed v_t, and N_b charges in the lower segment, going to the left at (slower) speed v_b. The *current* ($I = \lambda v$) is the same in all four segments (otherwise charge would be piling up somewhere). Thus

$$I = \frac{qN_t}{l}v_t = \frac{qN_b}{l}v_b, \quad \text{so} \quad N_t v_t = N_b v_b = \frac{Il}{q}, \tag{10}$$

where q is the charge of each particle and l is the length of the rectangle. *Classically,* the momentum of a single particle is $p = mv$, where m is its mass, so the total momentum (to the right) is

$$p_{\text{classical}} = mN_t v_t - mN_b v_b = m\frac{Il}{q} - m\frac{Il}{q} = 0, \tag{11}$$

as one would certainly expect (after all, the loop as a whole is not moving). But *relativistically* the momentum of a particle is $p = \gamma mv$ and we get

$$p_{\text{relativistic}} = \gamma_t m N_t v_t - \gamma_b m N_b v_b = \frac{mIl}{q}(\gamma_t - \gamma_b), \tag{12}$$

which is *not* zero, because the particles in the upper segment are moving faster. In fact, the gain in energy (γmc^2), as a particle goes up the left side, is equal to the work done by the electric force qEw, where w is the height of the rectangle, so

$$\gamma_t - \gamma_b = \frac{qEw}{mc^2}, \quad \text{and hence} \quad p_{\text{rel}} = \frac{IlEw}{c^2}. \tag{13}$$

Now Ilw is the magnetic dipole moment of the loop; as vectors, $\mathbf{m}$ points into the page, and $\mathbf{p}$ is to the right, so

$$\mathbf{p}_{\text{rel}} = \frac{1}{c^2}(\mathbf{m} \times \mathbf{E}). \tag{14}$$

This is the "hidden" momentum of Eq. (9).

The term "hidden momentum" was coined by Shockley;[11] it was an unfortunate choice. The phenomenon itself was first studied in the context of static electromagnetic systems with nonzero field momentum ($\mathbf{p}_{\text{field}} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d^3\mathbf{r}$). In such configurations, the hidden momentum exactly cancels the field momentum ($\mathbf{p}_h = -\mathbf{p}_{\rm field}$), leaving a total of zero, as required by the "center of energy theorem."[13] This has created the impression that hidden momentum is something artificial and ad hoc—invented simply to rescue an abstract theorem.[14] Nothing could be further from the truth. Hidden momentum is perfectly ordinary relativistic mechanical momentum, as the example above indicates; it occurs in systems with internally moving parts, such as current-carrying loops, and it is "hidden" only in the sense that it not associated with motion of the object as a whole. A *Gilbert* dipole in an electric field, having no moving parts, harbors *no* hidden momentum (and the fields—with the crucial delta-function term in **B** included—carry no compensating momentum).[15]

Returning to the configuration in Fig. 1, the hidden momentum in $\mathcal{S}'$ is

$$\mathbf{p}_h = \frac{1}{c^2}\left[(m_0\,\hat{\mathbf{x}}) \times \left(\frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\,\hat{\mathbf{z}}\right)\right] = -\frac{qm_0}{4\pi\epsilon_0 c^2 d^2}\,\hat{\mathbf{y}}. \tag{15}$$

Because $\mathbf{p}_h$ is perpendicular to $\mathbf{v}$ and transverse components are unaffected by Lorentz transformations, this is also the hidden momentum in $\mathcal{S}$. It is constant (in time), so there is no associated force. But the hidden *angular* momentum,

$$\mathbf{L}_h = \mathbf{r} \times \mathbf{p}_h, \tag{16}$$

is *not* constant (in the lab frame) because $\mathbf{r}$ is changing. In fact,

$$\frac{d\mathbf{L}_h}{dt} = \mathbf{v} \times \mathbf{p}_h = \frac{qm_0}{4\pi\epsilon_0}\frac{v}{c^2 d^2}\,\hat{\mathbf{x}}. \tag{17}$$

This increase in angular momentum requires a torque

$$\mathbf{N} = \frac{qm_0}{4\pi\epsilon_0}\frac{v}{c^2 d^2}\,\hat{\mathbf{x}}, \tag{18}$$

and this is precisely what we found in Eq. (5).

Recapitulating: In the Gilbert model, there is an extra term in the torque formula [Eq. (8)]; the total torque is zero, there is no hidden angular momentum, and nothing rotates. In the Ampère model, there is no third term in the torque formula [Eq. (5)];[16] the torque is *not* zero and drives the increasing hidden angular momentum—but still nothing rotates.[17] It helps to separate the angular momentum into two types: "overt" (associated with actual rotation) and "hidden" (so called because it is *not* associated with any overt rotation of the object). Torque is the rate of change of the *total* angular momentum,

$$\mathbf{N} = \frac{d\mathbf{L}_o}{dt} + \frac{d\mathbf{L}_h}{dt}. \tag{19}$$

In both models $d\mathbf{L}_o/dt = 0$. In the Gilbert model $\mathbf{N}$ and $d\mathbf{L}_h/dt$ are also zero; in the Ampère model they are equal but nonzero.

IV. MAGNETIZED MATERIALS

It is of interest to see how this resolution plays out in Mansuripur's formulation of the problem. He treats the dipole as a magnetized medium and calculates the torque directly from the Lorentz force law, without invoking $\mathbf{p} \times \mathbf{E}$ or $\mathbf{m} \times \mathbf{B}$. In the proper frame, he writes the magnetization as

$$\mathbf{M}'(x', y', z', t') = m_0\,\delta(x')\delta(y')\delta(z' - d)\,\hat{\mathbf{x}}. \tag{20}$$

Now, $\mathbf{M}$ and the polarization $\mathbf{P}$ constitute an antisymmetric second-rank tensor

$$P^{\mu\nu} = \begin{pmatrix} 0 & cP_x & cP_y & cP_z \\ -cP_x & 0 & -M_z & M_y \\ -cP_y & M_z & 0 & -M_x \\ -cP_z & -M_y & M_x & 0 \end{pmatrix}, \tag{21}$$

whose transformation rule is[18]

$$P_z = P'_z, \quad P_x = \gamma\left(P'_x + \frac{v}{c^2}M'_y\right), \qquad P_y = \gamma\left(P'_y - \frac{v}{c^2}M'_x\right), \tag{22}$$

$$M_z = M'_z, \quad M_x = \gamma(M'_x - vP'_y), \quad M_y = \gamma(M'_y + vP'_x) \tag{23}$$

(for motion in the z-direction). In the present case, then, the magnetization and polarization in the "lab" frame are

$$\mathbf{M}(x, y, z, t) = m_0\,\delta(x)\delta(y)\delta(z - vt - d/\gamma)\hat{\mathbf{x}}, \tag{24}$$

$$\mathbf{P}(x, y, z, t) = \frac{m_0 v}{c^2}\delta(x)\delta(y)\delta(z - vt - d/\gamma)\hat{\mathbf{y}}. \tag{25}$$

According to the Lorentz law, the force density is

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}, \tag{26}$$

where $\rho = -\nabla\cdot\mathbf{P}$ is the bound charge density and $\mathbf{J} = \partial\mathbf{P}/\partial t + \nabla\times\mathbf{M}$ is the sum of the polarization current and the bound current density. Using Eqs. (2), (3), (24) and (25), we obtain

$$\mathbf{f} = -(\nabla\cdot\mathbf{P})\mathbf{E} + (\nabla\times\mathbf{M})\times\mathbf{B} + \frac{\partial\mathbf{P}}{\partial t}\times\mathbf{B} = -\frac{qm_0 v}{4\pi\epsilon_0 c^2}\frac{d}{R^3}\delta(x)\delta'(y)\delta(z - vt - d/\gamma)\,\hat{\mathbf{z}} \tag{27}$$

(where a prime denotes the derivative). The net force on the dipole is

$$\mathbf{F} = \int \mathbf{f}\,dx\,dy\,dz = \frac{qm_0 vd}{4\pi\epsilon_0 c^2}\frac{d}{dy}\left[\frac{1}{(y^2 + d^2)^{3/2}}\right]\Bigg|_{y=0}\hat{\mathbf{z}} = 0. \tag{28}$$

Meanwhile, the torque density is

$$\mathbf{n} = \mathbf{r}\times\mathbf{f} = -\frac{qm_0 vd}{4\pi\epsilon_0 c^2}\frac{y}{R^2}\delta(x)\delta'(y)\delta(z - vt - d/\gamma)\,\hat{\mathbf{x}}, \tag{29}$$

so the net torque on the dipole is

$$\mathbf{N} = \int \mathbf{n}\,dx\,dy\,dz = -\frac{qm_0 vd}{4\pi\epsilon_0 c^2}\left\{-\frac{d}{dy}\left[\frac{y}{(y^2 + d^2)^{3/2}}\right]\right\}\Bigg|_{y=0}\hat{\mathbf{x}} = \frac{qm_0 v}{4\pi\epsilon_0 c^2 d^2}\,\hat{\mathbf{x}}, \tag{30}$$

confirming Eq. (5). This is the torque required to account for the increase in hidden angular momentum.

What if we run Mansuripur's calculation for a dipole made out of magnetic monopoles? The bound charge, bound current, and magnetization current are[19]

$$\rho_b^* = -\nabla\cdot\mathbf{M}, \quad \mathbf{J}_b^* = -c^2\nabla\times\mathbf{P}, \quad \mathbf{J}_p^* = \frac{\partial\mathbf{M}}{\partial t}, \tag{31}$$

so the force density on the magnetic dipole [again invoking Eqs. (2), (3), (24) and (25)] is[20]

$$\begin{aligned}\mathbf{f} &= \rho^*\mathbf{B} - \frac{1}{c^2}\mathbf{J}^*\times\mathbf{E}\\ &= -(\nabla\cdot\mathbf{M})\mathbf{B} + (\nabla\times\mathbf{P})\times\mathbf{E} - \frac{1}{c^2}\left(\frac{\partial\mathbf{M}}{\partial t}\right)\times\mathbf{E}\\ &= 0.\end{aligned} \tag{32}$$

The total force is again zero, but this time so too is the torque density ($\mathbf{n} = \mathbf{r}\times\mathbf{f}$) and hence the total torque. As before, the torque is zero in the Gilbert model—and there is no hidden angular momentum.

V. THE EINSTEIN-LAUB FORCE LAW

Having concluded that the Lorentz force law is unacceptable, Mansuripur proposes to replace Eq. (27) with an expression based on the Einstein–Laub law,[21]

$$\begin{aligned}\mathbf{f}_{\rm EL} &= (\mathbf{P}\cdot\nabla)\mathbf{E} + \frac{\partial\mathbf{P}}{\partial t}\times(\mu_0\mathbf{H}) + (\mathbf{M}\cdot\nabla)\mu_0\mathbf{H}\\ &\quad - \frac{1}{c^2}\frac{\partial\mathbf{M}}{\partial t}\times\mathbf{E}\\ &= \frac{m_0 q v\gamma}{4\pi\epsilon_0 c^2}\frac{1}{R^3}\delta(x)\delta(y)[2\delta(z-vt-d/\gamma)\\ &\quad -(z-vt)\delta'(z-vt-d/\gamma)]\,\hat{\mathbf{y}}.\end{aligned} \tag{33}$$

The total force on the dipole still vanishes,

$$\mathbf{F}_{\rm EL} = \frac{m_0 q v\gamma}{4\pi\epsilon_0 c^2}\,\hat{\mathbf{y}}\left\{\frac{2}{d^3} + \frac{1}{\gamma^3}\frac{d}{dz}\left[\frac{1}{(z-vt)^2}\right]\Bigg|_{z-vt=d/\gamma}\right\} = 0. \tag{34}$$

The torque density should be $\mathbf{r}\times\mathbf{f}_{\rm EL}$,

$$\begin{aligned}\mathbf{n}_{\rm EL} &= -\frac{m_0 q v\gamma}{4\pi\epsilon_0 c^2}\frac{z}{R^3}\delta(x)\delta(y)[2\delta(z-vt-d/\gamma)\\ &\quad -(z-vt)\delta'(z-vt-d/\gamma)]\,\hat{\mathbf{x}},\end{aligned} \tag{35}$$

giving a total torque

$$\begin{aligned}\mathbf{N}_{\rm EL} &= -\frac{m_0 q v\gamma}{4\pi\epsilon_0 c^2}\,\hat{\mathbf{x}}\left\{\frac{2(vt+d/\gamma)}{d^3} + \frac{1}{\gamma^3}\frac{d}{dz}\left[\frac{z}{(z-vt)^2}\right]\right\}\\ &= -\frac{m_0 q v}{4\pi\epsilon_0 c^2 d^2}\,\hat{\mathbf{x}}\end{aligned} \tag{36}$$

(the derivative is again evaluated at $z - vt = d/\gamma$). It's not zero! In fact, it's *minus* the "Lorentz" torque given in Eq. (30). But Mansuripur argues that, "To guarantee the conservation of angular momentum, [Eq. (35)] must be supplemented…" with extra terms,

$$\mathbf{n}'_{\rm EL} = \mathbf{n}_{\rm EL} + (\mathbf{P}\times\mathbf{E}) + (\mathbf{M}\times\mathbf{B}). \tag{37}$$

In our case, the extra terms are

$$(\mathbf{P}\times\mathbf{E}) + (\mathbf{M}\times\mathbf{B}) = \frac{m_0 q v}{4\pi\epsilon_0 c^2 d^2}\times\delta(x)\delta(y)\delta(z-vt-d/\gamma)\,\hat{\mathbf{x}}, \tag{38}$$

and their contribution to the total torque is

$$\int[(\mathbf{P}\times\mathbf{E}) + (\mathbf{M}\times\mathbf{B})]\,dx\,dy\,dz = \frac{m_0 q v}{4\pi\epsilon_0 c^2 d^2}\,\hat{\mathbf{x}}, \tag{39}$$

which is just right to cancel Eq. (36), yielding a net torque of zero (which Mansuripur takes to be the correct answer).

What are we to make of this argument? In the first place, the Einstein–Laub force density was derived assuming that the medium is at rest,[21] which in this case it is not. More important, the magnetization terms implicitly assume a Gilbert model for the magnetic dipole,

$$(\mathbf{M}\cdot\nabla)\mathbf{B} = -\nabla\times(\mathbf{M}\times\mathbf{B}) + (\mathbf{B}\cdot\nabla)\mathbf{M} - (\nabla\cdot\mathbf{M})\mathbf{B}; \tag{40}$$

as long as the magnetization is localized, the first two terms yield vanishing surface integrals,[22] leaving $-(\nabla\cdot\mathbf{M})\mathbf{B} - (1/c^2)[(\partial\mathbf{M}/\partial t)\times\mathbf{E}]$ for the net force density on the object, the same as in the Gilbert model [Eq. (32)].[23] There may be some contexts in which the Einstein–Laub force law is valid and useful, but this is not one of them. Mansuripur is quite explicit in writing that the magnetic dipole he has in mind is "a small, charge neutral loop of current," which is to say, an Ampère dipole.

VI. CONCLUSION

The resolution of Mansuripur's "paradox" depends on the model for the magnetic dipole:

- If it is a Gilbert dipole (made from magnetic monopoles), the third term in Namias's formula [Eq. (8)] supplies the missing torque. In Mansuripur's formulation (using a polarizable medium), it comes from a correct accounting of the bound charge/current [Eq. (31)]. The net torque is zero in the lab frame, just as it is in the proper frame.
- If it is an Ampère dipole (an electric current loop), the third term in Namias's equation is absent, and the torque on the dipole is *not* zero. It is, however, just right to account for the increasing hidden angular momentum in the dipole.

In either model, the Lorentz force law is entirely consistent with special relativity.

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[1]M. Mansuripur, "Trouble with the Lorentz Law of Force: Incompatibility with special relativity and momentum conservation," Phys. Rev. Lett. **108**, 193901-1–4 (2012).

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[4]The details are worked out in Cross and Vanzella (Ref. 3).

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[6]"Paradox Lost," Science **339**, 496 (2013); A. Cho, "Purported relativity paradox resolved," ⟨http://scim.ag/Lorpara⟩ (2013). Mansuripur himself does not accept this verdict, though he does appear to have softened his assertions somewhat: M. Mansuripur, "Mansuripur Replies," Phys. Rev. Lett. **110**, 089405-1 (2013).

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[8]See, for instance, V. Hnizdo, "Magnetic dipole moment of a moving electric dipole," Am. J. Phys. **80**, 645–647 (2012).

[9]See, for example, D. J. Griffiths, *Introduction to Electrodynamics*, 4th ed. (Pearson, Boston, 2013), Eq. (7.69).

[10]We calculate all torques (in the lab frame) with respect to the origin. But because the net force on the dipole is zero in all cases, it does not matter—we could as well use *any* fixed point, including the (instantaneous) position of the dipole.

[11]W. Shockley and R. P. James, "'Try simplest cases' discovery of 'hidden momentum' forces on 'magnetic currents,'" Phys. Rev. Lett. **18**, 876–879 (1967); W. H. Furry, "Examples of momentum distributions in the electromagnetic field and in matter," Am. J. Phys. **37**, 621–636 (1969); L. Vaidman, "Torque and force on a magnetic dipole," Am. J. Phys. **58**, 978–983 (1990); V. Hnizdo, "Conservation of linear and angular momentum and the interaction of a moving charge with a magnetic dipole," Am. J. Phys. **60**, 242–246 (1992); V. Hnizdo, "Hidden mechanical momentum and the field momentum in stationary electromagnetic and gravitational systems," Am. J. Phys. **65**, 515–518 (1997).

[12] This is of course an unrealistic model for an actual current-carrying wire. Vaidman (Ref. 11) explores more plausible models, but the result is unchanged.

[13]If the center of energy of a closed system is at rest, the total momentum of the system must be zero. See, for example, S. Coleman and J. H. Van Vleck, "Origin of 'Hidden Momentum Forces' on Magnets," Phys. Rev. **171**, 1370–1375 (1968); M. G. Calkin, "Linear momentum of the source of a static electromagnetic field," Am. J. Phys. **39**, 513–516 (1971).

[14]Mansuripur variously calls hidden momentum an "absurdity" (M. Mansuripur, "Resolution of the Abraham-Minkowski controversy," Opt. Commun. **283**, 1997–2005 (2010), p. 1999), a "problem" to be "solved" (Ref. 1), and "as applied to magnetic materials… an unnecessary burden" (Ref. 6).

[15]D. J. Griffiths, "Dipoles at rest," Am. J. Phys. **60**, 979–987 (1992). The magnetic field of a point dipole is $\mathbf{B} = (\mu_0/4\pi)(1/r^3)[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] + \alpha\mu_0\mathbf{m}\delta^3(\mathbf{r})$, where $\alpha = 2/3$ for Ampère dipoles and $-1/3$ for Gilbert dipoles. The delta function term leads, for example, to hyperfine splitting in the ground state of hydrogen, and provides experimental confirmation that the proton is an Ampére dipole.

[16]We do not know a simple way to prove this directly, but we will confirm it implicitly in Sec. IV.

[17]This is certainly not the first time such issues have arisen. How can there be a torque in the lab frame, when there is none in the proper frame? See J. D. Jackson, "Torque or no torque? Simple charged particle motion observed in different inertial frames," Am. J. Phys. **72**, 1484–1487 (2004). How can there be a torque, with no accompanying rotation? See D. G. Jensen, "The paradox of the L-shaped object," Am. J. Phys. 57, 553–555 (1989).

[18]See, for instance, Ref. 9, Eq. (12.118).

[19]The minus sign in $\mathbf{J}_b^*$ is due to the switched sign in "Ampère's law" for magnetic monopoles [see Ref. 9, Eq. (7.44)]. Note that in Eq. (31) [and hence also Eq. (32)], **M** and **P** are the densities of the dipole moments of magnetic monopoles and magnetic-monopole currents, respectively.

[20]There is some dispute as to the correct form of the Lorentz force law for magnetic monopoles in the presence of polarizable and magnetizable materials, but not when (as here) the polarization/magnetization is itself due to monopoles. See K. T. McDonald, "Poynting's Theorem with Magnetic Monopoles," (11 pp), ⟨http://puhep1.princeton.edu/ mcdonald/examples/poynting.pdf⟩ (2013).

[21]A. Einstein and J. Laub, "Über die im elektromagnetischen Felde auf ruhende Körper ausgeübten ponderomotorischen Kräfte," Ann. Phys. (Leipzig) **26**, 541–550 (1908); English translation in The Collected Papers of Albert Einstein, Vol. 2 (Princeton U.P., Princeton, NJ, 1989). In evaluating the force, Mansuripur uses the field H due to q (see Eq. (12b) in Ref. 1); his $\mu_0\mathbf{H}$ is our **B** [Eq. (3)].

[22]The ith component of the second term is $(\mathbf{B}\cdot\nabla)M_i = \nabla\cdot(M_i\mathbf{B}) - M_i\nabla\cdot\mathbf{B} = \nabla\cdot(M_i\mathbf{B})$.

[23]B. D. H. Tellegen, "Magnetic-dipole models," Am. J. Phys. **30**, 650–652 (1962). Tellegen's force (6), which assumes a Gilbert magnetic dipole, can be obtained by integrating the magnetization-dependent terms in the Einstein–Laub force density in which it is assumed that $\mathbf{M} = \mathbf{m}\,\delta^3(\mathbf{r}-\mathbf{r}_0)$. Another derivation of the force on a Gilbert magnetic dipole is by A. D. Yaghjian, "Electromagnetic forces on point dipoles," IEEE Antenna. Prop. Soc. Symp. 4, 2868–2871 (1999).