# A Catalog of Hidden Momenta

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**Abstract**

Electromagnetic fields carry momentum: $\mathbf{P}_{\text{em}} = \epsilon_0 \int (\mathbf{E}\times\mathbf{B})\, d\tau$. But if the center-of-energy of a (localized) system is at rest, its *total* momentum must be zero. The compensating term has come to be called "hidden" momentum: $\mathbf{P}_{\text{h}} = -\mathbf{P}_{\text{em}}$. It is (typically) ordinary *mechanical* momentum, relativistic in nature, and is "hidden" only in the sense that it is not associated with motion of the system as a whole—only with that of its constituent parts. This article develops a catalog of field momenta and hidden momenta for ideal electric and magnetic dipoles—both the "standard" variety made from electric charges and currents, and the "anomalous" variety made from hypothetical magnetic monopoles and their currents—in the presence of electric and magnetic fields (which themselves may be produced by "standard" or "anomalous" sources).

## 1 Electric and Magnetic Dipoles

In the static case, Maxwell's equations read:

$$\begin{array}{llll} (a) & \boldsymbol{\nabla}\cdot\mathbf{E} = (1/\epsilon_0)\rho, & (c) & \boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}, \\ (b) & \boldsymbol{\nabla}\cdot\mathbf{B} = 0, & (d) & \boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}, \end{array} \tag{1}$$

where $\rho$ is the electric charge density and $\mathbf{J}$ is the electric current density. In a world with magnetic monopoles there would also exist electromagnetic fields sourced by magnetic charges ($\tilde{\rho}$) and their currents ($\tilde{\mathbf{J}}$):

$$\begin{array}{llll} (a) & \boldsymbol{\nabla}\cdot\tilde{\mathbf{E}} = 0, & (c) & \boldsymbol{\nabla}\times\tilde{\mathbf{E}} = -\mu_0\tilde{\mathbf{J}}, \\ (b) & \boldsymbol{\nabla}\cdot\tilde{\mathbf{B}} = \mu_0\tilde{\rho}, & (d) & \boldsymbol{\nabla}\times\tilde{\mathbf{B}} = \mathbf{0}. \end{array} \tag{2}$$

(I'll use a tilde to designate these "anomalous" sources and fields, to distinguish them from the "standard" variety associated with ordinary electric charge.) It follows (by applying the divergence to Equations 1($d$) and 2($c$)) that the electric and magnetic currents are divergenceless:

$$\boldsymbol{\nabla}\cdot\mathbf{J}=0,\qquad \boldsymbol{\nabla}\cdot\tilde{\mathbf{J}}=0 \tag{3}$$

(the associated charges are locally conserved).

The force on an electric charge ($q$) is given by the Lorentz force law:

$$\mathbf{F}=q\left[\mathbf{E}'+(\mathbf{v}\times\mathbf{B}')\right], \tag{4}$$

where $\mathbf{E}'=\mathbf{E}+\tilde{\mathbf{E}}$ is the *total* electric field (standard plus anomalous), and $\mathbf{B}'=\mathbf{B}+\tilde{\mathbf{B}}$. Likewise the force on a magnetic monopole ($\tilde{q}$) is

$$\mathbf{F}=\tilde{q}\left[\mathbf{B}'-\epsilon_0\mu_0(\mathbf{v}\times\mathbf{E}')\right]. \tag{5}$$

(The slight asymmetry in all these formulas is an unfortunate artifact of the SI system, and would not appear in Gaussian units.)

The fields can be expressed in terms of scalar and vector potentials:[1]

$$\begin{aligned}\mathbf{E}&=-\boldsymbol{\nabla}V, &\qquad \mathbf{B}&=\boldsymbol{\nabla}\times\mathbf{A},\\ \tilde{\mathbf{E}}&=-\boldsymbol{\nabla}\times\tilde{\mathbf{A}}, &\qquad \tilde{\mathbf{B}}&=-\boldsymbol{\nabla}\tilde{V}.\end{aligned} \tag{6}$$

Adopting the gauge condition

$$\boldsymbol{\nabla}\cdot\mathbf{A}=0,\qquad \boldsymbol{\nabla}\cdot\tilde{\mathbf{A}}=0, \tag{7}$$

Maxwell's equations become

$$\begin{aligned}\nabla^2 V&=-\frac{1}{\epsilon_0}\rho, &\qquad \nabla^2\mathbf{A}&=-\mu_0\mathbf{J},\\ \nabla^2\tilde{V}&=-\mu_0\tilde{\rho}, &\qquad \nabla^2\tilde{\mathbf{A}}&=-\mu_0\tilde{\mathbf{J}}.\end{aligned} \tag{8}$$

For localized charge and current configurations it follows that

$$V(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau',\quad \mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau', \tag{9}$$

$$\tilde{V}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\tilde{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau',\quad \tilde{\mathbf{A}}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\tilde{\mathbf{J}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau'. \tag{10}$$

The electric dipole moment of an electric charge distribution ($\rho$) is defined by

$$\mathbf{p}\equiv\int\mathbf{r}\rho\,d\tau, \tag{11}$$

[1]Many of the equations in Section 1 are taken from D. J. Griffiths, *Am. J. Phys.* **60**, 979 (1992), and are recapitulated here in order to make the paper self-contained. However, I have reversed an unfortunate sign convention in the definition of $\tilde{\mathbf{A}}$.

where $\mathbf{r}$ is the vector from the origin to the volume element $d\tau$. The magnetic dipole moment of an electric current configuration ($\mathbf{J}$) is

$$\mathbf{m} \equiv \frac{1}{2}\int (\mathbf{r} \times \mathbf{J})\, d\tau. \tag{12}$$

I shall call these "standard" dipoles, to distinguish them from the hypothetical "anomalous" variety associated with monopole charges and currents:

$$\tilde{\mathbf{m}} \equiv \int \mathbf{r}\tilde{\rho}\, d\tau \tag{13}$$

and

$$\tilde{\mathbf{p}} \equiv -\frac{\epsilon_0\mu_0}{2}\int (\mathbf{r} \times \tilde{\mathbf{J}})\, d\tau. \tag{14}$$

If (as we shall always assume) the dipoles are *neutral*,

$$\int \rho\, d\tau = 0, \quad \int \tilde{\rho}\, d\tau = 0, \tag{15}$$

then $\mathbf{p}$ and $\tilde{\mathbf{m}}$ are independent of the choice of origin. It follows from Equation 3 that

$$\begin{aligned} 0 = \int r_i(\boldsymbol{\nabla} \cdot \mathbf{J})\, d\tau = \int r_i(\nabla_j J_j)\, d\tau = \int \nabla_j(r_i J_j)\, d\tau - \int (\nabla_j r_i) J_j\, d\tau \\ = -\int \delta_{ji} J_j\, d\tau = -\int J_j\, d\tau \end{aligned} \tag{16}$$

(repeated indices are to be summed from 1 to 3; we assume that charge and current distributions are *localized*, so all boundary terms coming from integration by parts vanish). Thus

$$\int \mathbf{J}\, d\tau = \mathbf{0}, \tag{17}$$

and in this case the standard magnetic dipole moment ($\mathbf{m}$) is independent of origin. By the same reasoning,

$$\int \tilde{\mathbf{J}}\, d\tau = \mathbf{0}, \tag{18}$$

and hence $\tilde{\mathbf{p}}$ is independent of origin.

Similarly,

$$\begin{aligned} 0 = \int r_i r_j(\nabla_k J_k)\, d\tau = -\int [\nabla_k(r_i r_j)] J_k\, d\tau = -\int (r_i\delta_{jk} + r_j\delta_{ik}) J_k\, d\tau \\ = -\int (r_i J_j + r_j J_i)\, d\tau. \end{aligned} \tag{19}$$

On the other hand, from Equation 12,

$$\begin{aligned}\epsilon_{ijk} m_k &= \frac{1}{2}\left(\epsilon_{ijk}\epsilon_{klm}\right)\int r_l J_m \, d\tau = \frac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\int r_l J_m \, d\tau \\ &= \frac{1}{2}\int \left(r_i J_j - r_j J_i\right) \, d\tau = \int r_i J_j \, d\tau. \end{aligned} \tag{20}$$

By the same token,

$$\epsilon_{ijk}\tilde{p}_k = -\mu_0\epsilon_0 \int r_i \tilde{J}_j \, d\tau. \tag{21}$$

In general, the charge and current configurations constituting a physical dipole will be distributed over some finite region of space. However, we shall from now on confine our attention to "ideal" dipoles, localized at a single point. More precisely, we are interested in the *limiting case*, in which the size of the dipole shrinks to zero.

To compute the potential of such a dipole, we note that

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}'}. \tag{22}$$

For an *ideal* dipole at the origin, the charge (or current) distribution at $\mathbf{r}'$ vanishes except at $\mathbf{r}' \to 0$, so (in Equations 9 and 10) we may safely confine our attention to the region $r' \ll r$, for which the binomial expansion gives

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{1}{r}\left(1 + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right). \tag{23}$$

In the case of a standard electric dipole,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{1}{r}\int \rho(\mathbf{r}')\, d\tau' + \frac{\mathbf{r}}{r^3}\cdot\int \mathbf{r}'\rho(\mathbf{r}')\, d\tau'\right\}, \tag{24}$$

or (using Equations 11 and 15),

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}. \tag{25}$$

Similarly, the potential of a nonstandard magnetic dipole is

$$\tilde{V}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\tilde{\mathbf{m}}\cdot\mathbf{r}}{r^3}. \tag{26}$$

For a standard magnetic dipole,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\frac{1}{r}\int \mathbf{J}(\mathbf{r}')\, d\tau' + \frac{1}{r^3}\int (\mathbf{r}\cdot\mathbf{r}')\mathbf{J}(\mathbf{r}')\, d\tau'\right\}, \tag{27}$$

or (using Equations 17 and 20):

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}. \tag{28}$$

Likewise, for an anomalous electric dipole:

$$\tilde{\mathbf{A}}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\frac{\tilde{\mathbf{p}}\times\mathbf{r}}{r^3}. \tag{29}$$

To determine the field of a dipole, we take the gradient or curl of the pertinent potential (Equation 6). This requires some care, however, because the dipole potentials are very singular at the origin. In general,[2]

$$\nabla_i\left(\frac{r_j}{r^3}\right) = \frac{1}{r^3}\left(\delta_{ij} - 3\frac{r_i r_j}{r^2}\right) + \frac{4\pi}{3}\delta_{ij}\,\delta^3(\mathbf{r}), \tag{30}$$

so for any constant vector $\mathbf{a}$,

$$\nabla\left(\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right) = \frac{1}{r^3}\left(\mathbf{a} - 3\frac{\mathbf{r}(\mathbf{a}\cdot\mathbf{r})}{r^2}\right) + \frac{4\pi}{3}\mathbf{a}\,\delta^3(\mathbf{r}), \tag{31}$$

and

$$\nabla\times\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = -\frac{1}{r^3}\left(\mathbf{a} - 3\frac{\mathbf{r}(\mathbf{a}\cdot\mathbf{r})}{r^2}\right) + \frac{8\pi}{3}\mathbf{a}\,\delta^3(\mathbf{r}). \tag{32}$$

Using these two identities we find

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(3\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{p})}{r^2} - \mathbf{p}\right) - \frac{\mathbf{p}}{3\epsilon_0}\,\delta^3(\mathbf{r}), \tag{33}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left(3\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{m})}{r^2} - \mathbf{m}\right) + \frac{2\mu_0\mathbf{m}}{3}\,\delta^3(\mathbf{r}), \tag{34}$$

$$\tilde{\mathbf{E}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(3\frac{\mathbf{r}(\mathbf{r}\cdot\tilde{\mathbf{p}})}{r^2} - \tilde{\mathbf{p}}\right) + \frac{2\tilde{\mathbf{p}}}{3\epsilon_0}\,\delta^3(\mathbf{r}), \tag{35}$$

$$\tilde{\mathbf{B}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left(3\frac{\mathbf{r}(\mathbf{r}\cdot\tilde{\mathbf{m}})}{r^2} - \tilde{\mathbf{m}}\right) - \frac{\mu_0\tilde{\mathbf{m}}}{3}\,\delta^3(\mathbf{r}). \tag{36}$$

The delta-function terms are often left out, because one is usually interested in the field at some remove from the dipole; what remains has the same form in all four cases. This "universal" part holds outside a sphere of vanishingly small radius; the delta-function describes the field inside this sphere. Although the latter contributes only at one point, it is essential for the internal consistency of the theory.

It is useful to note that this entire theory is invariant under the following duality transformation:

[2]C. P. Frahm, *Am. J. Phys.* **51**, 826 (1983), Equation 6.

| | | | |
|---|---|---|---|
| $\rho \to \frac{1}{c}\tilde{\rho}$ | $\mathbf{J} \to \frac{1}{c}\tilde{\mathbf{J}}$ | $\tilde{\rho} \to -c\rho$ | $\tilde{\mathbf{J}} \to -c\mathbf{J}$ |
| $\mathbf{E} \to c\tilde{\mathbf{B}}$ | $\mathbf{B} \to -\frac{1}{c}\tilde{\mathbf{E}}$ | $\tilde{\mathbf{E}} \to c\mathbf{B}$ | $\tilde{\mathbf{B}} \to -\frac{1}{c}\mathbf{E}$ |
| $V \to c\tilde{V}$ | $\mathbf{A} \to \frac{1}{c}\tilde{\mathbf{A}}$ | $\tilde{V} \to -\frac{1}{c}V$ | $\tilde{\mathbf{A}} \to -c\mathbf{A}$ |
| $\mathbf{p} \to \frac{1}{c}\tilde{\mathbf{m}}$ | $\mathbf{m} \to -c\tilde{\mathbf{p}}$ | $\tilde{\mathbf{p}} \to \frac{1}{c}\mathbf{m}$ | $\tilde{\mathbf{m}} \to -c\mathbf{p}$ |

## 2 Field Momentum

The (linear) momentum in electromagnetic fields is

$$\mathbf{P}_{\text{em}} = \epsilon_0 \int (\mathbf{E}' \times \mathbf{B}')\, d\tau \tag{37}$$

(the fields could be standard or anomalous, or—in principle—some of each). It is often more convenient to express this equation in terms of potentials; the resulting formula depends on the nature of the sources:

1. **Standard electric and magnetic fields:** $\mathbf{E} = -\boldsymbol{\nabla} V$, so

$$\begin{aligned}\mathbf{P}_{\text{em}} &= -\epsilon_0 \int [(\boldsymbol{\nabla} V) \times \mathbf{B}]\, d\tau = -\epsilon_0 \left[\int \boldsymbol{\nabla} \times (V\mathbf{B})\, d\tau - \int V(\boldsymbol{\nabla} \times \mathbf{B})\, d\tau\right] \\ &= \mu_0\epsilon_0 \int V(\mathbf{r})\mathbf{J}(\mathbf{r})\, d\tau. \end{aligned} \tag{38}$$

On the other hand, since[3] (Equation 9)

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d\tau' \quad \text{and} \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d\tau',$$

$$\begin{aligned}\mathbf{P}_{\text{em}} &= \mu_0\epsilon_0 \frac{1}{4\pi\epsilon_0} \int\int \frac{\rho(\mathbf{r}')\mathbf{J}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|}\, d\tau'\, d\tau = \int \left\{\frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r})}{|\mathbf{r}' - \mathbf{r}|}\, d\tau\right\} \rho(\mathbf{r}')\, d\tau' \\ &= \int \rho(\mathbf{r})\mathbf{A}(\mathbf{r})\, d\tau. \end{aligned} \tag{39}$$

[3]Of course, you could go back to Equation 37 and insert $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, but it's a little easier to work from Equation 38.

2. **Standard electric field and anomalous magnetic field:** The first line of Equation 38 still holds, but since $\boldsymbol{\nabla} \times \tilde{\mathbf{B}} = \mathbf{0}$ the field momentum is zero:
$$\mathbf{P}_{\text{em}} = \mathbf{0}. \tag{40}$$

3. **Anomalous electric and magnetic fields:**
$$\begin{aligned}\mathbf{P}_{\text{em}} &= \epsilon_0 \int (\tilde{\mathbf{E}} \times \tilde{\mathbf{B}})\, d\tau = -\epsilon_0 \int (\tilde{\mathbf{E}} \times \boldsymbol{\nabla}\tilde{V})\, d\tau = -\epsilon_0 \int \tilde{V}(\boldsymbol{\nabla} \times \tilde{\mathbf{E}})\, d\tau \\ &= \mu_0\epsilon_0 \int \tilde{V}(\mathbf{r})\, \tilde{\mathbf{J}}(\mathbf{r})\, d\tau, \end{aligned} \tag{41}$$
and since (Equation 10)
$$\tilde{V}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\tilde{\rho}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d\tau' \quad \text{and} \quad \tilde{\mathbf{A}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\tilde{\mathbf{J}}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d\tau',$$
$$\mathbf{P}_{\text{em}} = \mu_0\epsilon_0 \int \tilde{\rho}(\mathbf{r})\tilde{\mathbf{A}}(\mathbf{r})\, d\tau. \tag{42}$$

4. **Anomalous electric field and standard magnetic field:**
$$\begin{aligned}\mathbf{P}_{\text{em}} &= \epsilon_0 \int \tilde{\mathbf{E}} \times (\boldsymbol{\nabla} \times \mathbf{A})\, d\tau \\ &= -\epsilon_0 \int \left[\mathbf{A} \times (\boldsymbol{\nabla} \times \tilde{\mathbf{E}}) + (\mathbf{A} \cdot \boldsymbol{\nabla})\tilde{\mathbf{E}} + (\tilde{\mathbf{E}} \cdot \boldsymbol{\nabla})\mathbf{A}\right] d\tau \\ &= \mu_0\epsilon_0 \int (\mathbf{A} \times \tilde{\mathbf{J}})\, d\tau. \end{aligned} \tag{43}$$
I used the fact that the $i$th component of $\int (\mathbf{A} \cdot \boldsymbol{\nabla})\tilde{\mathbf{E}}\, d\tau$ is
$$\int \mathbf{A} \cdot \boldsymbol{\nabla}(\tilde{E}_i)\, d\tau = -\int (\boldsymbol{\nabla} \cdot \mathbf{A})\tilde{E}_i\, d\tau = 0, \tag{44}$$
and the same goes for $\int (\tilde{\mathbf{E}} \cdot \boldsymbol{\nabla})\mathbf{A}\, d\tau$. Finally, using the now-familiar trick,
$$\begin{aligned}\mathbf{P}_{\text{em}} &= \mu_0\epsilon_0 \int (\mathbf{A} \times \tilde{\mathbf{J}})\, d\tau = \mu_0\epsilon_0 \int\int \frac{\mu_0}{4\pi} \frac{\mathbf{J}(\mathbf{r}') \times \tilde{\mathbf{J}}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|}\, d\tau'\, d\tau \\ &= \mu_0\epsilon_0 \int (\mathbf{J} \times \tilde{\mathbf{A}})\, d\tau. \end{aligned} \tag{45}$$

I summarize these results in the following table:

| | |
|---|---|
| $\mathbf{E}$ and $\mathbf{B}$ | $\mathbf{P}_{\rm em} = \mu_0\epsilon_0 \int V\,\mathbf{J}\,d\tau = \int \rho\,\mathbf{A}\,d\tau$ |
| $\mathbf{E}$ and $\tilde{\mathbf{B}}$ | $\mathbf{P}_{\rm em} = \mathbf{0}$ |
| $\tilde{\mathbf{E}}$ and $\mathbf{B}$ | $\mathbf{P}_{\rm em} = \mu_0\epsilon_0 \int (\mathbf{A}\times\tilde{\mathbf{J}})\,d\tau = -\mu_0\epsilon_0 \int (\tilde{\mathbf{A}}\times\mathbf{J})\,d\tau$ |
| $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ | $\mathbf{P}_{\rm em} = \mu_0\epsilon_0 \int \tilde{V}\,\tilde{\mathbf{J}}\,d\tau = \mu_0\epsilon_0 \int \tilde{\rho}\,\tilde{\mathbf{A}}\,d\tau$ |

Now we will use these formulas to determine the field momenta of electric and magnetic dipoles (both standard and anomalous) in external electric and magnetic fields (both standard and anomalous). The dipoles are at rest (we might as well put them at the origin), and since they occupy an infinitesimal volume we can expand the external potentials:

$$V(\mathbf{r}) = V(\mathbf{0}) + \mathbf{r}\cdot(\boldsymbol{\nabla}_0 V) = V(\mathbf{0}) - \mathbf{r}\cdot\mathbf{E}(\mathbf{0});\quad \mathbf{A}(\mathbf{r}) = \mathbf{A}(\mathbf{0}) + (\mathbf{r}\cdot\boldsymbol{\nabla}_0)\mathbf{A};\tag{46}$$
$$\tilde{V}(\mathbf{r}) = \tilde{V}(\mathbf{0}) + \mathbf{r}\cdot(\boldsymbol{\nabla}_0 \tilde{V}) = \tilde{V}(\mathbf{0}) - \mathbf{r}\cdot\tilde{\mathbf{B}}(\mathbf{0});\quad \tilde{\mathbf{A}}(\mathbf{r}) = \tilde{\mathbf{A}}(\mathbf{0}) + (\mathbf{r}\cdot\boldsymbol{\nabla}_0)\tilde{\mathbf{A}}.\tag{47}$$

(Here $\boldsymbol{\nabla}_0$ means "evaluate the derivatives at $\mathbf{r} = \mathbf{0}$," and we don't need any higher-order terms.)

1. **Standard electric dipole in standard magnetic field.** Use Equation 39:

$$\begin{aligned}\mathbf{P}_{\rm em} &= \int \rho(\mathbf{r})[\mathbf{A}(\mathbf{0}) + (\mathbf{r}\cdot\boldsymbol{\nabla}_0)\mathbf{A}]\,d\tau\\ &= \left\{\int \rho(\mathbf{r})\,d\tau\right\}\mathbf{A}(\mathbf{0}) + \left(\left\{\int \mathbf{r}\rho(\mathbf{r})\,d\tau\right\}\cdot\boldsymbol{\nabla}_0\right)\mathbf{A}\\ &= (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{A}.\end{aligned}\tag{48}$$

   (I dropped the subscript on $\boldsymbol{\nabla}$; it is to be evaluated at the location of the dipole.)

2. **Standard magnetic dipole in standard electric field.** Use Equation 38:

$$\mathbf{P}_{\rm em} = \mu_0\epsilon_0\left\{V(0)\int \mathbf{J}(\mathbf{r})\,d\tau - E_j(\mathbf{0})\int \mathbf{r}_j\mathbf{J}\,d\tau\right\}.\tag{49}$$

But $\int \mathbf{J}\, d\tau = \mathbf{0}$ and $\int r_i\, J_j\, d\tau = \epsilon_{ijk} m_k$, so the $j$th component is

$$P_{\text{em}j} = -\mu_0\epsilon_0 E_i(\mathbf{0}) \int \mathbf{r}_i J_j\, d\tau = -\mu_0\epsilon_0 E_i(\mathbf{0})\epsilon_{ijk} m_k. \tag{50}$$

(I used Equation 20) and therefore

$$\mathbf{P}_{\text{em}} = \mu_0\epsilon_0(\mathbf{E}\times\mathbf{m}). \tag{51}$$

3. **Standard electric dipole in anomalous magnetic field, or anomalous magnetic dipole in standard electric field.** Equation 40 says

$$\mathbf{P}_{\text{em}} = \mathbf{0}. \tag{52}$$

4. **Anomalous electric dipole in anomalous magnetic field.** Use Equation 41:

$$\begin{aligned}
\mathbf{P}_{\text{em}} &= \mu_0\epsilon_0 \int [\tilde{V}(\mathbf{0}) - \mathbf{r}\cdot\tilde{\mathbf{B}}(\mathbf{0})]\tilde{\mathbf{J}}\, d\tau. \\
P_{\text{em}i} &= -\mu_0\epsilon_0 \tilde{B}_j(\mathbf{0}) \int r_j \tilde{J}_i\, d\tau = -\mu_0\epsilon_0 \tilde{B}_j(\mathbf{0}) \left(-\frac{1}{\mu_0\epsilon_0}\right) \epsilon_{jik}\tilde{p}_k \\
&= -\epsilon_{ijk}\tilde{B}_j(\mathbf{0})\tilde{p}_k, \\
\mathbf{P}_{\text{em}} &= -\tilde{\mathbf{B}}\times\tilde{\mathbf{p}}.
\end{aligned} \tag{53}$$

5. **Anomalous magnetic dipole in anomalous electric field.** Use Equation 42:

$$\begin{aligned}
\mathbf{P}_{\text{em}} &= \mu_0\epsilon_0 \int [\tilde{\mathbf{A}}(\mathbf{0}) + (\mathbf{r}\cdot\boldsymbol{\nabla}_0)\tilde{\mathbf{A}}]\,\tilde{\rho}(\mathbf{r})\, d\tau. \\
P_{\text{em}i} &= \mu_0\epsilon_0(\nabla_{0j}\tilde{A}_i) \int r_j \tilde{\rho}(\mathbf{r})\, d\tau = \mu_0\epsilon_0(\nabla_{0j}\tilde{A}_i)\tilde{m}_j \\
\mathbf{P}_{\text{em}} &= \mu_0\epsilon_0(\tilde{\mathbf{m}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{A}}.
\end{aligned} \tag{54}$$

6. **Anomalous electric dipole in standard magnetic field.** Use Equation 43:

$$\begin{aligned}
\mathbf{P}_{\text{em}} &= \mu_0\epsilon_0 \int [\mathbf{A}(\mathbf{0}) + (\mathbf{r}\cdot\boldsymbol{\nabla}_0)\mathbf{A}] \times \tilde{\mathbf{J}}(\mathbf{r})\, d\tau. \\
P_{\text{em}i} &= \mu_0\epsilon_0\epsilon_{ijk}(\nabla_{0l}A_j) \int r_l \tilde{J}_k\, d\tau = \mu_0\epsilon_0\epsilon_{ijk}(\nabla_{0l}A_j) \left(-\frac{1}{\mu_0\epsilon_0}\right) \epsilon_{lkm}\tilde{p}_m \\
&= -\left(\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm}\right)(\nabla_{0l}A_j)\tilde{p}_m = -(\nabla_0\cdot\mathbf{A})\tilde{p}_i + (\nabla_{0i}A_j)\tilde{p}_j, \\
\mathbf{P}_{\text{em}} &= \tilde{p}_j\boldsymbol{\nabla}A_j = \boldsymbol{\nabla}(\tilde{\mathbf{p}}\cdot\mathbf{A}) = \tilde{\mathbf{p}}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\tilde{\mathbf{p}}\cdot\boldsymbol{\nabla})\mathbf{A} \\
&= (\tilde{\mathbf{p}}\times\mathbf{B}) + (\tilde{\mathbf{p}}\cdot\boldsymbol{\nabla})\mathbf{A}.
\end{aligned} \tag{55}$$

(The form $\boldsymbol{\nabla}(\tilde{\mathbf{p}}\cdot\mathbf{A})$ is tidy but dangerous: the derivative does *not* act on $\tilde{p}$, only on $\mathbf{A}$.)

7. **Standard magnetic dipole in anomalous electric field.** Use Equation 45:

$$\mathbf{P}_{\rm em} = -\mu_0\epsilon_0 \int [\tilde{\mathbf{A}}(\mathbf{0}) + (\mathbf{r}\cdot\boldsymbol{\nabla}_0)\tilde{\mathbf{A}}] \times \mathbf{J}(\mathbf{r})\, d\tau.$$

$$P_{{\rm em}\,i} = -\mu_0\epsilon_0\epsilon_{ijk}(\nabla_{0l}\tilde{A}_j)\int r_l J_k\, d\tau = -\mu_0\epsilon_0\epsilon_{ijk}(\nabla_{0l}\tilde{A}_j)\epsilon_{lkm}m_m$$
$$= -\mu_0\epsilon_0\,(\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm})\,(\nabla_{0l}\tilde{A}_j)m_m = \mu_0\epsilon_0(\nabla_{0i}\tilde{A}_j)\tilde{m}_j,$$
$$\mathbf{P}_{\rm em} = \mu_0\epsilon_0 m_j\boldsymbol{\nabla}\tilde{A}_j = \mu_0\epsilon_0\boldsymbol{\nabla}(\mathbf{m}\cdot\tilde{\mathbf{A}}) = \mu_0\epsilon_0[\mathbf{m}\times(\boldsymbol{\nabla}\times\tilde{\mathbf{A}}) + (\mathbf{m}\cdot\boldsymbol{\nabla})\tilde{\mathbf{A}}]$$
$$= \mu_0\epsilon_0[-\mathbf{m}\times\tilde{\mathbf{E}} + (\mathbf{m}\cdot\boldsymbol{\nabla})\tilde{\mathbf{A}}]. \quad (56)$$

Again, I summarize the results in a table:

| | | | |
|---|---|---|---|
| $\mathbf{p}$ in $\mathbf{B}$ | $\mathbf{P}_{\rm em} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{A}$ | $\tilde{\mathbf{p}}$ in $\mathbf{B}$ | $\mathbf{P}_{\rm em} = (\tilde{\mathbf{p}}\times\mathbf{B}) + (\tilde{\mathbf{p}}\cdot\boldsymbol{\nabla})\mathbf{A}$ |
| $\mathbf{m}$ in $\mathbf{E}$ | $\mathbf{P}_{\rm em} = -\mu_0\epsilon_0(\mathbf{m}\times\mathbf{E})$ | $\tilde{\mathbf{m}}$ in $\mathbf{E}$ | $\mathbf{P}_{\rm em} = \mathbf{0}$ |
| $\mathbf{p}$ in $\tilde{\mathbf{B}}$ | $\mathbf{P}_{\rm em} = \mathbf{0}$ | $\tilde{\mathbf{p}}$ in $\tilde{\mathbf{B}}$ | $\mathbf{P}_{\rm em} = \tilde{\mathbf{p}}\times\tilde{\mathbf{B}}$ |
| $\mathbf{m}$ in $\tilde{\mathbf{E}}$ | $\mathbf{P}_{\rm em} = -\frac{1}{c^2}[\mathbf{m}\times\tilde{\mathbf{E}} - (\mathbf{m}\cdot\boldsymbol{\nabla})\tilde{\mathbf{A}}]$ | $\tilde{\mathbf{m}}$ in $\tilde{\mathbf{E}}$ | $\mathbf{P}_{\rm em} = \mu_0\epsilon_0(\tilde{\mathbf{m}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{A}}$ |

# 3 Hidden Momentum

Now, there is a general theorem[4] in special relativity that says "if the center of energy of a localized system is at rest, then the total momentum is zero." In the cases we are considering (stationary dipoles in static fields) the center of energy is certainly not moving, and yet the *field* momentum is *not* zero, as we have seen. Evidently there must be some *other* momentum, equal and opposite to $\mathbf{P}_{\rm em}$. This other momentum has come to be called "hidden" momentum,[5] though there is nothing secret about it—in the present context it is perfectly ordinary mechanical momentum, relativistic in nature, and "hidden" only in the sense that it is not associated with motion of the object (here, the dipole) as a whole, but rather with its internally moving parts.

[4] S. Coleman and J. H. Van Vleck, *Phys. Rev.* **171**, 1370 (1968); M. G. Calkin, *Am. J. Phys.* **39**, 513 (1971).

[5] W. Shockley and R. P. James, *Phys. Rev. Lett.* **18**, 876 (1967); O. Costa de Beauregard, *Phys. Lett.* **A 24**, 177 (1967). For a history, and comprehensive references, see K. T. McDonald: http://physics.princeton.edu/~mcdonald/examples/hiddendef.pdf (2018).

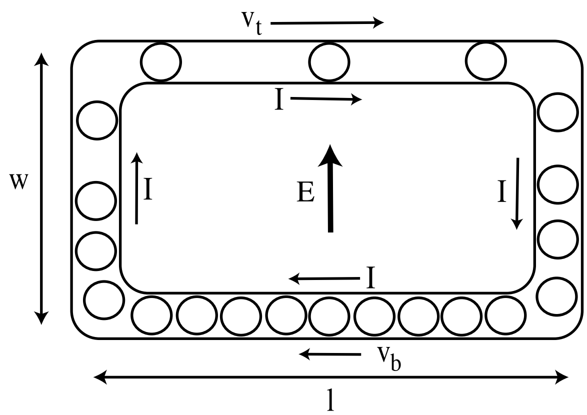


Figure 1: The Penfield-Haus model.

The most illuminating example of hidden momentum goes back to Penfield and Haus in the mid 1960's.[6] Imagine a rectangular loop of wire, carrying a steady current $I$ in the presence of a uniform electrostatic field $\mathbf{E}$ (Figure 1). Picture the current as a resistanceless flow of free positive charges,[7] each with charge $q$ and mass $m$. The electric field accelerates them as they ascend the left side, and slows them down as they descend the right side. Accordingly, their speed is greater along the top segment than at the bottom: $v_t > v_b$. On the other hand, they are farther apart in the top segment, so there are more of them at the bottom: $N_b > N_t$. The current (which, remember, is constant around the loop), is

$$I = \frac{N_t q}{l} v_t = \frac{N_b q}{l} v_b \quad \Rightarrow \quad N_t v_t = N_b v_b = \frac{Il}{q}. \tag{57}$$

The net (relativistic) momentum of the charges—to the right—is

$$P_{\rm h} = \gamma_t N_t m v_t - \gamma_b N_b m v_b = \frac{Ilm}{q}(\gamma_t - \gamma_b). \tag{58}$$

Now, the kinetic energy gained in ascending the left leg is equal to the work done by the electric force:

$$\gamma_t mc^2 - \gamma_b mc^2 = qEw \quad \Rightarrow \quad \gamma_t - \gamma_b = \frac{qEw}{mc^2}, \tag{59}$$

[6] P. Penfield, Jr. and H. A. Haus, *Electrodynamics of Moving Media* (M.I.T. Press, Cambridge, MA, 1967), page 214-216.

[7] This is not, of course, a realistic model of an actual current-carrying wire, but it makes the essential point with a minimum of extraneous baggage. For other models see L. Vaidman, *Am. J. Phys.* **58**, 978 (1990); V. Hnizdo, *Am. J. Phys.* **65**, 92 (1997).

so

$$P_{\rm h} = \left(\frac{Ilm}{q}\right)\left(\frac{qEw}{mc^2}\right) = \frac{1}{c^2}(Ilw)E. \tag{60}$$

But $Ilw$ is the magnetic dipole moment of the loop; it points into the page. So

$$\mathbf{P}_{\rm h} = \frac{1}{c^2}(\mathbf{m}\times\mathbf{E}) \qquad [\mathbf{m} \text{ in } \mathbf{E}]. \tag{61}$$

This is the hidden momentum of the configuration—it is nothing but the net mechanical momentum of the charges constituting the current. It is independent of the *size* of the dipole (as long as $\mathbf{E}$ is constant over its area), so it applies in particular to an ideal (point) dipole. And it is just right to cancel the field momentum (Equation 51)

$$\mathbf{P}_{\rm em} = -\mu_0\epsilon_0(\mathbf{m}\times\mathbf{E}),$$

as required by the center-of-energy theorem.

Notice that the Penfield-Haus mechanism applies to particles *in motion*. If, for example, this were an *anomalous* magnetic diplole, made from monopoles at rest, there would be *no* hidden momentum (in a standard electric field):

$$\mathbf{P}_{\rm h} = \mathbf{0} \qquad [\tilde{\mathbf{m}} \text{ in } \mathbf{E}]. \tag{62}$$

But in that case there's no *field* momentum either (Equation 52), and the total is again zero.

What if $\mathbf{E}$ is *not* uniform over the current loop? Consider a segment $d\mathbf{l}$; its momentum is

$$d\mathbf{P} = \gamma(\lambda_m\,dl)\mathbf{v} = \gamma\frac{\lambda_m}{\lambda_e}\lambda_e v\,d\mathbf{l} = \gamma(\alpha I)\,d\mathbf{l}, \tag{63}$$

where $\lambda_m$ is the mass (of the moving charges) per unit length, $\lambda_e$ is their charge per unit length, and $\alpha$ is the mass-to-charge ratio). For the whole loop, then,

$$\mathbf{P}_{\rm h} = \alpha I\oint\gamma\,d\mathbf{l}. \tag{64}$$

Picking as the reference point for potential some convenient spot $\mathcal{O}$ on the loop,

$$\gamma(\mathbf{r}) = \gamma_0 + \int_{\mathcal{O}}^{\mathbf{r}}\frac{dW}{mc^2} = \gamma_0 + \frac{q}{mc^2}\int_{\mathcal{O}}^{\mathbf{r}}\mathbf{E}\cdot d\mathbf{l} = \gamma_0 - \frac{1}{\alpha c^2}V(\mathbf{r}), \tag{65}$$

where $\gamma_0$ is the value at $\mathcal{O}$, and $dW$ is the work done on a charge as it advances by $d\mathbf{l}$ along the loop. But

$$\oint\gamma_0\,d\mathbf{l} = \gamma_0\oint d\mathbf{l} = \mathbf{0}, \tag{66}$$

and hence[8]

$$\mathbf{P}_{\rm h} = -\frac{I}{c^2}\oint V(\mathbf{r})\,d\mathbf{l}. \tag{67}$$

[8]For volume currents Equation 67 becomes $\mathbf{P}_{\rm h} = -\frac{I}{c^2}\int V\mathbf{J}\,d\tau$, and we see immediately that it cancels the field momentum (Equation 38). If $\mathbf{E}$ is *uniform* over the current region, then $V(\mathbf{r}) = V(\mathbf{0}) - \mathbf{E}\cdot\mathbf{r}$, and (using Equation 20) $\mathbf{P}_{{\rm h}_j} = \mu_0\epsilon_0 E_i\int r_i J_j\,d\tau = \mu_0\epsilon_0 E_i\epsilon_{ijk}m_k = \mu_0\epsilon_0(\mathbf{m}\times\mathbf{E})_j$, so we recover Equation 61.

For example, if the source of the electric field is an ordinary electric dipole, **p**, (I'm changing the reference point, but the closed line integral is independent of any added constant)

$$\mathbf{P}_\mathrm{h} = -\frac{I}{c^2}\frac{1}{4\pi\epsilon_0}\oint \frac{[\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|^3}\,d\mathbf{l} = -\frac{\mu_0}{4\pi}\oint \mathbf{I}\,\frac{[\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|^3}\,dl, \tag{68}$$

where $(\mathbf{r}-\mathbf{r}')$ is the vector from the dipole (at $\mathbf{r}'$) to the point $\mathbf{r}$. (It doesn't matter whether you associate the vector with $I$ or with $dl$, since they are in the same direction: $I\,d\mathbf{l} = \mathbf{I}\,dl$.) We can express this result in terms of the vector potential (due to the current loop) at the location of the electric dipole:

$$\mathbf{A}(\mathbf{r}') = \frac{\mu_0}{4\pi}\oint \frac{\mathbf{I}(\mathbf{r})}{|\mathbf{r}'-\mathbf{r}|}\,dl. \tag{69}$$

Thus

$$\big[(\mathbf{p}\cdot\boldsymbol{\nabla}')\mathbf{A}(\mathbf{r}')\big]_i = \frac{\mu_0}{4\pi}\oint I_i p_j \nabla'_j\left(\frac{1}{|\mathbf{r}'-\mathbf{r}|}\right)dl = -\frac{\mu_0}{4\pi}\oint I_i p_j \frac{(\mathbf{r}'-\mathbf{r})_j}{|\mathbf{r}'-\mathbf{r}|^3}\,dl, \tag{70}$$

or

$$[(\mathbf{p}\cdot\boldsymbol{\nabla}')\mathbf{A}(\mathbf{r}')] = -\frac{\mu_0}{4\pi}\oint \mathbf{I}\,\frac{[\mathbf{p}\cdot(\mathbf{r}'-\mathbf{r})]}{|\mathbf{r}'-\mathbf{r}|^3}\,dl. \tag{71}$$

So the hidden momentum is

$$\mathbf{P}_\mathrm{h} = -(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{A}, \qquad [\mathbf{p} \text{ in } \mathbf{B}], \tag{72}$$

which once again is just right to cancel the field momentum (Equation 48).

The same argument applies to an anomalous *magnetic* dipole in an anomalous *electric* field, except that what does the work is now the magnetic force ($\mathbf{F} = \tilde{q}\,\tilde{\mathbf{B}}$) acting on the particles in the monopole current loop:

$$\mathbf{P}_\mathrm{h} = -\frac{1}{c^2}(\tilde{\mathbf{m}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{A}} \qquad [\tilde{\mathbf{m}} \text{ in } \tilde{\mathbf{E}}]. \tag{73}$$

Alternatively, you can get Equation 73 by applying the duality transformation to Equation 72. Likewise, from Equations 61 and 62,

$$\mathbf{P}_\mathrm{h} = -(\tilde{\mathbf{p}}\times\tilde{\mathbf{B}}) \qquad [\tilde{\mathbf{p}} \text{ in } \tilde{\mathbf{B}}], \tag{74}$$

$$\mathbf{P}_\mathrm{h} = \mathbf{0} \qquad [\mathbf{p} \text{ in } \tilde{\mathbf{B}}]. \tag{75}$$

The original Penfield-Haus model made no reference to the *source* of the electrostatic field—they presumably took it to be some collection of stationary electric charges (perhaps in the form of a surrounding parallel-plate capacitor). The hidden momentum in the magnetic dipole itself would be the same (Equation 61) if the electric field were due to a current of magnetic monopoles $\left(\frac{1}{c^2}(\mathbf{m}\times\tilde{\mathbf{E}})\right)$. However, in that case there would *also* be hidden momentum residing in the monopole current (the monopoles accelerating and decelerating in response to the magnetic field of the electric current loop). The latter is given by Equation 73 (for the momentum *in the monopole current loop* it doesn't matter whether the source of the magnetic field is standard or anomalous). Combining the two we get

$$\mathbf{P}_{\rm h} = \frac{1}{c^2}[(\mathbf{m} \times \tilde{\mathbf{E}}) - (\mathbf{m} \cdot \boldsymbol{\nabla})\tilde{\mathbf{A}}] \qquad (\mathbf{m} \text{ in } \tilde{\mathbf{E}}). \tag{76}$$

By the same argument (or by invoking the duality transformation)

$$\mathbf{P}_{\rm h} = -[(\tilde{\mathbf{p}} \times \mathbf{B}) + (\tilde{\mathbf{p}} \cdot \boldsymbol{\nabla})\mathbf{A}] \qquad (\tilde{\mathbf{p}} \text{ in } \mathbf{B}). \tag{77}$$

The following catalog summarizes these results:

| | | |
|---|---|---|
| $\mathbf{p}$ in $\mathbf{B}$ | $\mathbf{P}_{\rm h} = -(\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{A}$ | resides in source of $\mathbf{B}$ |
| $\mathbf{m}$ in $\mathbf{E}$ | $\mathbf{P}_{\rm h} = \frac{1}{c^2}(\mathbf{m} \times \mathbf{E})$ | resides in $\mathbf{m}$ |
| $\mathbf{p}$ in $\tilde{\mathbf{B}}$ | $\mathbf{P}_{\rm h} = \mathbf{0}$ | (nothing moving) |
| $\mathbf{m}$ in $\tilde{\mathbf{E}}$ | $\mathbf{P}_{\rm h} = \frac{1}{c^2}[\mathbf{m} \times \tilde{\mathbf{E}} - (\mathbf{m} \cdot \boldsymbol{\nabla})\tilde{\mathbf{A}}]$ | resides in $\mathbf{m}$ and source of $\tilde{\mathbf{E}}$ |
| $\tilde{\mathbf{p}}$ in $\mathbf{B}$ | $\mathbf{P}_{\rm h} = -[(\tilde{\mathbf{p}} \times \mathbf{B}) + (\tilde{\mathbf{p}} \cdot \boldsymbol{\nabla})\mathbf{A}]$ | resides in $\tilde{\mathbf{p}}$ and source of $\mathbf{B}$ |
| $\tilde{\mathbf{m}}$ in $\mathbf{E}$ | $\mathbf{P}_{\rm h} = \mathbf{0}$ | (nothing moving) |
| $\tilde{\mathbf{p}}$ in $\tilde{\mathbf{B}}$ | $\mathbf{P}_{\rm h} = -(\tilde{\mathbf{p}} \times \tilde{\mathbf{B}})$ | resides in $\tilde{\mathbf{p}}$ |
| $\tilde{\mathbf{m}}$ in $\tilde{\mathbf{E}}$ | $\mathbf{P}_{\rm h} = -\frac{1}{c^2}(\tilde{\mathbf{m}} \cdot \boldsymbol{\nabla})\tilde{\mathbf{A}}$ | resides in source of $\mathbf{E}$ |

In each case the hidden momentum is just right to cancel the field momentum.

## 4 Interacting Dipoles

As an application, suppose that the field is itself due to another dipole. There are four possibilities: (1) $\mathbf{p}$ and $\mathbf{m}$, (2) $\mathbf{p}$ and $\tilde{\mathbf{m}}$, (3) $\tilde{\mathbf{p}}$ and $\mathbf{m}$, and (4) $\tilde{\mathbf{p}}$ and

$\tilde{\mathbf{m}}$. We could regard the first as a standard magnetic dipole in the electric field of a standard electric dipole:

$$\begin{aligned}\mathbf{P}_{\rm h} &= \frac{1}{c^2}(\mathbf{m}\times\mathbf{E}) = \mu_0\epsilon_0\mathbf{m}\times\left\{\frac{1}{4\pi\epsilon_0}\frac{[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}]}{r^3} - \frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r})\right\}\\ &= \frac{\mu_0}{4\pi r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})(\mathbf{m}\times\hat{\mathbf{r}}) - (\mathbf{m}\times\mathbf{p})\right] - \frac{\mu_0}{3}(\mathbf{m}\times\mathbf{p})\,\delta^3(\mathbf{r}), \qquad (78)\end{aligned}$$

(where $\mathbf{r}$ is the vector from one dipole to the other), or as a standard electric dipole in the magnetic field of a standard magnetic dipole:

$$\begin{aligned}\mathbf{P}_{\rm h} &= -(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{A} = -(\mathbf{p}\cdot\boldsymbol{\nabla})\frac{\mu_0}{4\pi}\frac{(\mathbf{m}\times\mathbf{r})}{r^3},\\ P_{{\rm h}i} &= -\frac{\mu_0}{4\pi}\epsilon_{ijk}p_l m_j\nabla_l\left(\frac{r_k}{r^3}\right) = -\frac{\mu_0}{4\pi}\epsilon_{ijk}p_l m_j\left[\frac{1}{r^3}\left(\delta_{lk} - 3\frac{r_l r_k}{r^2}\right) + \frac{4\pi}{3}\delta_{lk}\,\delta^3(\mathbf{r})\right]\\ &= -\frac{\mu_0}{4\pi}\left\{\frac{[-3(\mathbf{p}\cdot\hat{\mathbf{r}})(\mathbf{m}\times\hat{\mathbf{r}})_i + (\mathbf{m}\times\mathbf{p})_i]}{r^3} + \frac{4\pi}{3}(\mathbf{m}\times\mathbf{p})_i\,\delta^3(\mathbf{r})\right\}, \qquad (79)\end{aligned}$$

and we recover Equation 78.

In the same way, we obtain the hidden momentum in the other three cases. These results are summarized in the following table:

| | | |
|---|---|---|
| $\mathbf{p},\mathbf{m}$ | $\mathbf{P}_{\rm h} = \frac{\mu_0}{4\pi r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})(\mathbf{m}\times\hat{\mathbf{r}}) - (\mathbf{m}\times\mathbf{p})\right] + \frac{\mu_0}{3}(\mathbf{p}\times\mathbf{m})\,\delta^3(\mathbf{r})$ | resides in $\mathbf{m}$ |
| $\mathbf{p},\tilde{\mathbf{m}}$ | $\mathbf{P}_{\rm h} = \mathbf{0}$ | nothing moving |
| $\tilde{\mathbf{p}},\mathbf{m}$ | $\mathbf{P}_{\rm h} = \frac{\mu_0}{4\pi r^3}\left\{3[(\tilde{\mathbf{p}}\times\mathbf{m})\cdot\hat{\mathbf{r}}]\hat{\mathbf{r}} - (\tilde{\mathbf{p}}\times\mathbf{m})\right\} - \frac{\mu_0}{3}(\tilde{\mathbf{p}}\times\mathbf{m})\,\delta^3(\mathbf{r})$ | resides in $\tilde{\mathbf{p}}$ and $\mathbf{m}$ |
| $\tilde{\mathbf{p}},\tilde{\mathbf{m}}$ | $\mathbf{P}_{\rm h} = -\frac{\mu_0}{4\pi r^3}\left[3(\tilde{\mathbf{m}}\cdot\hat{\mathbf{r}})(\tilde{\mathbf{p}}\times\hat{\mathbf{r}}) - (\tilde{\mathbf{p}}\times\tilde{\mathbf{m}})\right] + \frac{\mu_0}{3}(\tilde{\mathbf{p}}\times\tilde{\mathbf{m}})\,\delta^3(\mathbf{r})$ | resides in $\tilde{\mathbf{p}}$ |

## 5 Spherical Shell Models

In this paper I have treated ideal (point) dipoles, whose fields include the subtle delta function terms. It is embarrassingly easy to get these "contact" contributions wrong, and wise to check one's results using a finite model. What if we picture the dipoles as spherical shells, of radius $R$, carrying appropriate surface charges ($\sigma$) or currents ($\mathbf{K}$)?[9] Letting $v \equiv \frac{4}{3}\pi R^3$ be the volume of the sphere:

[9] If you prefer, think of them as uniformly polarized or uniformly magnetized solid spheres, but this raises diverting questions about the correct formula for the field momentum inside a material medium (Abraham versus Minkowski), which I would like to avoid.

1. **Standard electric dipole, p:** $\sigma = \dfrac{(\mathbf{p}\cdot\hat{\mathbf{r}})}{v}$,

$$V(\mathbf{r}) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{(\mathbf{p}\cdot\hat{\mathbf{r}})}{r^2}, & (r > R), \\ \\ \dfrac{(\mathbf{p}\cdot\mathbf{r})}{3\epsilon_0 v}, & (r < R). \end{cases} \tag{80}$$

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]}{r^3}, & (r > R), \\ \\ -\dfrac{\mathbf{p}}{3\epsilon_0 v}, & (r < R). \end{cases} \tag{81}$$

2. **Standard magnetic dipole, m:** $\mathbf{K} = \dfrac{(\mathbf{m}\times\hat{\mathbf{r}})}{v}$,

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{(\mathbf{m}\times\hat{\mathbf{r}})}{r^2}, & (r > R), \\ \\ \dfrac{\mu_0(\mathbf{m}\times\mathbf{r})}{3v}, & (r < R). \end{cases} \tag{82}$$

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]}{r^3}, & (r > R), \\ \\ \dfrac{2\mu_0\mathbf{m}}{3v}, & (r < R). \end{cases} \tag{83}$$

3. **Anomalous magnetic dipole, $\tilde{\mathbf{m}}$:** $\tilde{\sigma} = \dfrac{(\tilde{\mathbf{m}}\cdot\hat{\mathbf{r}})}{v}$,

$$\tilde{V}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{(\tilde{\mathbf{m}}\cdot\hat{\mathbf{r}})}{r^2}, & (r > R), \\ \\ \dfrac{\mu_0(\tilde{\mathbf{m}}\cdot\mathbf{r})}{3v}, & (r < R). \end{cases} \tag{84}$$

$$\tilde{\mathbf{B}}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{[3(\tilde{\mathbf{m}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \tilde{\mathbf{m}}]}{r^3}, & (r > R), \\ \\ -\dfrac{\mu_0\tilde{\mathbf{m}}}{3v}, & (r < R). \end{cases} \tag{85}$$

4. **Anomalous electric dipole, $\tilde{\mathbf{p}}$:** $\tilde{\mathbf{K}} = -c^2 \dfrac{(\tilde{\mathbf{p}} \times \hat{\mathbf{r}})}{v}$,

$$\tilde{\mathbf{A}}(\mathbf{r}) = \begin{cases} -\dfrac{1}{4\pi\epsilon_0}\dfrac{(\tilde{\mathbf{p}} \times \hat{\mathbf{r}})}{r^2}, & (r > R), \\ \\ -\dfrac{(\tilde{\mathbf{p}} \times \mathbf{r})}{3\epsilon_0 v}, & (r < R). \end{cases} \tag{86}$$

$$\tilde{\mathbf{E}}(\mathbf{r}) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{[3(\tilde{\mathbf{p}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \tilde{\mathbf{p}}]}{r^3}, & (r > R), \\ \\ \dfrac{2\tilde{\mathbf{p}}}{3\epsilon_0 v}, & (r < R). \end{cases} \tag{87}$$

As $R \to 0$, $1/v \to \delta^3(\mathbf{r})$, and we recover the ideal dipole fields (Equations 33-36).

Now suppose the sphere is *both* an electric dipole (either kind) *and* a magnetic dipole (either type).[10] Let's first calculate the field momentum for each combination. The external contribution ($r > R$) is the same for all of them; letting $\mathbf{a} \equiv (\mathbf{p} \times \mathbf{m})$:

$$\begin{aligned} \mathbf{P}_{\rm em}^{\rm out} &= \epsilon_0 \frac{1}{4\pi\epsilon_0}\frac{\mu_0}{4\pi}\int \frac{[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] \times [3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]}{r^6}\,d\tau \\ &= \frac{\mu_0}{(4\pi)^2}\int \frac{[3(\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - 2\mathbf{a}]}{r^6}\,d\tau. \end{aligned} \tag{88}$$

Setting the $z$ axis along $\mathbf{a}$, so that $\mathbf{a} = a\,\hat{\mathbf{z}}$, $\mathbf{a}\cdot\hat{\mathbf{r}} = a\cos\theta$, and $\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$, the $\phi$ integral kills the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components, leaving

$$\mathbf{P}_{\rm em}^{\rm out} = \frac{\mu_0}{(4\pi)^2}(a\,\hat{\mathbf{z}})(2\pi)\int_R^\infty \frac{1}{r^4}\,dr \int_0^\pi (3\cos^2\theta - 2)\sin\theta\,d\theta \tag{89}$$

$$= -\frac{\mu_0}{4\pi}\frac{(\mathbf{p}\times\mathbf{m})}{3R^3}. \tag{90}$$

The internal contribution ($r < R$) is trivial (since the fields are uniform), but different for the (four) different configurations:

1. <u>$\mathbf{p}$ and $\mathbf{m}$</u>: $\mathbf{P}_{\rm em}^{\rm in} = \epsilon_0\left(-\dfrac{\mathbf{p}}{3\epsilon_0 v}\right)\times\left(\dfrac{2\mu_0\mathbf{m}}{3v}\right)v = -\dfrac{2\mu_0(\mathbf{p}\times\mathbf{m})}{9v}$.

$$\mathbf{P}_{\rm em} = -\frac{\mu_0}{4\pi}\frac{(\mathbf{p}\times\mathbf{m})}{R^3}. \tag{91}$$

2. <u>$\mathbf{p}$ and $\tilde{\mathbf{m}}$</u>: $\mathbf{P}_{\rm em}^{\rm in} = \epsilon_0\left(-\dfrac{\mathbf{p}}{3\epsilon_0 v}\right)\times\left(-\dfrac{\mu_0\tilde{\mathbf{m}}}{3v}\right)v = \dfrac{\mu_0}{9v}(\mathbf{p}\times\tilde{\mathbf{m}})$.

$$\mathbf{P}_{\rm em} = \mathbf{0}. \tag{92}$$

[10]The "standard" case ($\mathbf{p}$ and $\mathbf{m}$) was introduced by R. H. Romer, *Am. J. Phys.*, **63**, 777 (1995).

3. <u>$\tilde{\mathbf{p}}$ and $\mathbf{m}$:</u> $\mathbf{P}_{\rm em}^{\rm in} = \epsilon_0 \left(\frac{2\tilde{\mathbf{p}}}{3\epsilon_0 v}\right) \times \left(\frac{2\mu_0 \mathbf{m}}{3v}\right) v = \frac{4\mu_0(\tilde{\mathbf{p}} \times \mathbf{m})}{9v}$.

$$\mathbf{P}_{\rm em} = \frac{\mu_0}{4\pi} \frac{(\tilde{\mathbf{p}} \times \mathbf{m})}{R^3}. \tag{93}$$

4. <u>$\tilde{\mathbf{p}}$ and $\tilde{\mathbf{m}}$:</u> $\mathbf{P}_{\rm em}^{\rm in} = \epsilon_0 \left(\frac{2\tilde{\mathbf{p}}}{3\epsilon_0 v}\right) \times \left(-\frac{\mu_0 \tilde{\mathbf{m}}}{3v}\right) v = -\frac{2\mu_0}{9v}(\tilde{\mathbf{p}} \times \tilde{\mathbf{m}})$.

$$\mathbf{P}_{\rm em} = -\frac{\mu_0}{4\pi} \frac{(\tilde{\mathbf{p}} \times \tilde{\mathbf{m}})}{R^3}. \tag{94}$$

Now let's calculate the hidden momentum in each configuration:

1. <u>$\mathbf{p}$ and $\mathbf{m}$:</u> I would *like* to use the Penfield-Haus formula (Equation 61), but that assumes the electric field is uniform over the current region. In this case the field *is* uniform *inside* the sphere, but right at the surface (where the current is located) $\mathbf{E}$ is discontinuous, and the external field is *not* uniform. You can finesse this problem by a trick: make the magnetic sphere ever-so-slightly *smaller* than the electric sphere; then the electric field really *is* uniform over the current, and we get[11]

$$\mathbf{P}_{\rm h} = \frac{1}{c^2}(\mathbf{m} \times \mathbf{E}) = \mu_0\epsilon_0 \left(\mathbf{m} \times \left(\frac{-\mathbf{p}}{3\epsilon_0 v}\right)\right) = \frac{\mu_0}{4\pi} \frac{(\mathbf{p} \times \mathbf{m})}{R^3}. \tag{95}$$

2. <u>$\tilde{\mathbf{p}}$ and $\tilde{\mathbf{m}}$:</u> using the duality transformation and Equation 98

$$\mathbf{P}_{\rm h} = \frac{\mu_0}{4\pi} \frac{(\tilde{\mathbf{p}} \times \tilde{\mathbf{m}})}{R^3}. \tag{96}$$

3. <u>$\mathbf{p}$ and $\tilde{\mathbf{m}}$:</u> nothing is moving, so

$$\mathbf{P}_{\rm h} = \mathbf{0}. \tag{97}$$

4. <u>$\tilde{\mathbf{p}}$ and $\mathbf{m}$:</u> In this case there is hidden momentum in *both* spheres. Using the trick (making the magnetic sphere slightly smaller than the electric sphere—you can do it the other way, of course, but you get the same answer), the hidden momentum in the electric current is

$$\mathbf{P}_{\rm h}^{e} = \frac{1}{c^2}(\mathbf{m} \times \tilde{\mathbf{E}}) = \mu_0\epsilon_0 \left(\mathbf{m} \times \left(\frac{2\tilde{\mathbf{p}}}{3\epsilon_0 v}\right)\right) = -2\,\frac{\mu_0}{4\pi} \frac{(\tilde{\mathbf{p}} \times \mathbf{m})}{R^3}. \tag{98}$$

But the magnetic field in the vicinity of the monopole current is *not* uniform, and we must use Equation 64 (or rather, its analog for a monopole current ($\tilde{I}$) in a standard magnetic field $\mathbf{B}$):

$$\mathbf{P} = \tilde{\alpha}\tilde{I} \oint \gamma\, d\mathbf{l}. \tag{99}$$

[11] If this bothers you, go back to Equation 67 (or rather, its analog for surface currents), $\mathbf{P}_{\rm h} = -\frac{1}{c^2}\int V\mathbf{K}\,da$; you get the same answer either way.

In this case Equation 65 becomes

$$\gamma(\mathbf{r}) = \gamma_0 + \frac{1}{\tilde{\alpha} c^2} \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{B} \cdot d\mathbf{l}. \tag{100}$$

We might as well choose our axes so that $\tilde{\mathbf{p}}$ points in the $z$ direction. I'll first calculate the hidden momentum in a single ring of monopole current, $\tilde{I}$, at $z = R\cos\theta$, with radius $R\sin\theta$.

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]}{R^3}, \qquad d\mathbf{l} = R\sin\theta\, d\phi\, \hat{\boldsymbol{\phi}}, \tag{101}$$

and (setting the reference point directly above the $x$ axis),

$$\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{B}\cdot d\mathbf{l} = \frac{\mu_0}{4\pi} \frac{R\sin\theta}{R^3} \int_0^{\phi} [3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]\cdot\hat{\boldsymbol{\phi}}\, d\phi = -\frac{\mu_0 \sin\theta}{4\pi R^2} \int_0^{\phi} (\mathbf{m}\cdot\hat{\boldsymbol{\phi}})\, d\phi.$$

Now

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \quad \text{so} \quad \mathbf{m}\cdot\hat{\boldsymbol{\phi}} = -m_x\sin\phi + m_y\cos\phi,$$

and therefore

$$\begin{aligned}\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{B}\cdot d\mathbf{l} &= -\frac{\mu_0\sin\theta}{4\pi R^2}\int_0^{\phi} (-m_x\sin\phi + m_y\cos\phi)\, d\phi \\ &= -\frac{\mu_0\sin\theta}{4\pi R^2}\left[m_x(\cos\phi - 1) + m_y\sin\phi\right]. \end{aligned} \tag{102}$$

The hidden momentum in this ring is

$$\begin{aligned}\mathbf{P}_{\rm h}^{\rm ring} &= \frac{\tilde{I}}{c^2}\left(-\frac{\mu_0\sin\theta}{4\pi R^2}\right)\oint \left[m_x(\cos\phi - 1) + m_y\sin\phi\right] R\sin\theta\, d\phi\,\hat{\boldsymbol{\phi}} \\ &= -\frac{\tilde{I}\mu_0\sin^2\theta}{4\pi c^2 R}\int_0^{2\pi} \left[m_x(\cos\phi - 1) + m_y\sin\phi\right](-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}})\, d\phi \\ &= -\frac{\tilde{I}\mu_0\sin^2\theta}{4\pi c^2 R}\left(-\pi m_y\,\hat{\mathbf{x}} + \pi m_x\,\hat{\mathbf{y}}\right). \end{aligned} \tag{103}$$

Now we integrate over all the rings that cover the monopole current sphere, using $\tilde{I} \to -|\tilde{\mathbf{K}}| R\, d\theta$ and $\tilde{\mathbf{K}} = -c^2(\tilde{\mathbf{p}}\times\hat{\mathbf{r}})/v$

$$\begin{aligned}\mathbf{P}_{\rm h}^{\rm m} &= \frac{\mu_0}{4c^2}\left(-m_y\,\hat{\mathbf{x}} + m_x\,\hat{\mathbf{y}}\right)\int_0^{\pi}\sin^2\theta \frac{c^2|\tilde{\mathbf{p}}\times\hat{\mathbf{r}}|}{v}\, d\theta \\ &= \frac{\mu_0}{4v}\left(-m_y\,\hat{\mathbf{x}} + m_x\,\hat{\mathbf{y}}\right)\tilde{p}\int_0^{\pi}\sin^3\theta\, d\theta = \frac{\mu_0}{4v}\left(-m_y\,\hat{\mathbf{x}} + m_x\,\hat{\mathbf{y}}\right)\tilde{p}\left(\frac{4}{3}\right) \\ &= \frac{\mu_0}{3v}(\tilde{\mathbf{p}}\times\mathbf{m}) = \frac{\mu_0}{4\pi}\frac{(\tilde{\mathbf{p}}\times\mathbf{m})}{R^3}. \end{aligned} \tag{104}$$

Finally, combining Equations 98 and 104,

$$\mathbf{P}_{\rm h} = \mathbf{P}_{\rm h}^{e} + \mathbf{P}_{\rm h}^{m} = -\frac{\mu_0}{4\pi}\frac{(\tilde{\mathbf{p}}\times\mathbf{m})}{R^3}. \tag{105}$$

These results confirm the contact terms in the table on page 15.[12] As always, the hidden momentum is equal and opposite to the field momentum. There is nothing *surprising* in any of this, but it is gratifying to see it work out in explicit detail.

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[12]Since the two spheres coincide we are *only* checking the contact term. On the other hand, if we *separate* the spheres (by a distance greater than the sum of their radii) there is really nothing to check, since the fields are precisely those of an ideal dipole.