Fast evaluation of the Witten zeta function

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1. Fast expansion for certain arguments.

Define

\[ W(r, s, t) := \sum_{m,n \geq 1} \frac{1}{m^r} \frac{1}{n^s} \frac{1}{(m+n)^t}. \]

This explicit summation is characteristically slow to converge. A fast evaluation may be effected via a free parameter \( X \in (0,1) \), and the following formula:

When neither \( r \) nor \( s \) is a positive integer,

\[
\begin{align*}
\Gamma(t)W(r, s, t) &= \sum_{m,n \geq 1} \frac{\Gamma(t, (m+n)X)}{m^rn^s(m+n)^t} \\
&+ \sum_{u,v \geq 0} (-1)^{u+v} \frac{\zeta(r-u)\zeta(s-v)X^{u+v+t}}{u!v!(u+v+t)} \\
&+ \Gamma(1-r) \sum_{q \geq 0} (-1)^q \frac{\zeta(s-q)X^{r+q+t-1}}{q!(r+q+t-1)} \\
&+ \Gamma(1-s) \sum_{q \geq 0} (-1)^q \frac{\zeta(r-q)X^{s+q+t-1}}{q!(s+q+t-1)} \\
&+ \Gamma(1-r)\Gamma(1-s) \frac{X^{r+s+t-2}}{r+s+t-2},
\end{align*}
\]

(When one or both of \( r, s \) is an integer, a different formula with a few more terms applies.)

One observes the pole in \( W \) at \( r+s+t = 2 \), with residue \( \Gamma(1-r)\Gamma(1-s)/\Gamma(t) \). Also, in the limit \( t \to 0 \) we see that the residual term is just the first \((u = v = 0)\) term of the \( u, v \) summation, and so \( W(r, s, 0) = \zeta(r)\zeta(s) \) is verified. Moreover, one may use the \( X \)-formula in various sanity-checking modes, as follows.
1) Varying $X$ within the interval $(0, 1)$ should yield an invariant $W$, as is so for any valid free-parameter expansion.

2) One may verify numerically the Zagier triangle identity

$$W(r, s, t) = W(r - 1, s, t + 1) + W(r, s - 1, t + 1).$$

3) A typical numerical value from the $X$ formula is, for $X = 4/5$ (an efficient choice)

$$W(\pi, \pi, \pi) \approx 0.121784932649073172392415831466446\ldots$$

4) A typical evaluation near the pole is, for $d := 200001/300000$,

$$W(d, d, d) = 529982.9016524962105\ldots$$

2. General analytic expansion.

The above expansion for $W$ is illegal for either $r, s$ a positive integer, because 1) The $\zeta(1)$ evaluation is illegal, and 2) the $\Gamma(1 - r)$ or $\Gamma(1 - s)$ is also illegal. However, the singularities do cancel, and we can write a general formula. For real number $p$, define a coefficient $A_p$ according to whether $p$ be a positive integer:

$$A_p := \Gamma(1 - p); \quad p \notin \mathbb{Z}^+,$$

$$:= \frac{(-1)^{p-1}}{\Gamma(p)}H_{p-1}; \quad p \in \mathbb{Z}^+,$$

where $H_k = \sum_{j=1}^{k} 1/j$ is the $k$-th harmonic number, with $H_0 := 0$. Similarly, define

$$B_p := 0; \quad p \notin \mathbb{Z}^+,$$

$$:= \frac{(-1)^p}{\Gamma(p)}; \quad p \in \mathbb{Z}^+.$$

Then a general formula is obtained as
For general $r, s$, whether integer or not,

$$
\Gamma(t) \mathcal{W}(r, s, t) = \sum_{m,n \geq 1} \frac{\Gamma(t, (m+n)X)}{m^rn^s(m+n)^t} \\
+ \sum_{u,v \geq 0} (-1)^u \zeta(r-u) \zeta(s-v) X^{u+v+t} \frac{\zeta(r-q) X^{s+q+t-1}}{q!} \left( \frac{A_s + B_s \log X}{(s+q+t-1)} - \frac{B_s}{(s+q+t-1)^2} \right) \\
+ \sum_{q \geq 0} (-1)^q \zeta(s-q) X^{r+q+t-1} \frac{\zeta(r-q) X^{r+q+t-1}}{q!} \left( \frac{A_r + B_r \log X}{(r+q+t-1)} - \frac{B_r}{(r+q+t-1)^2} \right) \\
+ X^{r+s+t-2} \frac{(A_r + B_r \log X)(A_s + B_s \log X)}{r+s+t-2} - \frac{A_rB_s + A_sB_r + 2B_rB_s \log X}{(r+s+t-2)^2} + \frac{2B_rB_s}{(r+s+t-2)^3}.
$$

The idea here is that the notation $\sum'$ means that we avoid any $\zeta(1)$ evaluations entirely. The extra complexity involving the $A, B$ coefficients and $\log X$ arises from said singularity avoidance.

It is not hard to see that the above formula for general $r, s$ reduces to our first formula when neither $r$ nor $s$ is a positive integer (being that all $B$ coefficients vanish).

It is believed that this general formula provides also an analytic continuation of $\mathcal{W}$, as it can converge even for $r, s, t$ triples for which the defining $\mathcal{W}$ sum does not.

A verification of the general formula obtains with $X := 4/5$, and a summation limit of 100 on every summation index, with the numerical result

$$
\mathcal{W}(2, 2, 1) \approx 0.8438254351644824574000744235991486399930 \ldots,
$$

which agrees with J. Borwein’s formula

$$
\mathcal{W}(2, 2, 1) = 2\zeta(2)\zeta(3) - 3\zeta(5)
$$

to 40 places.

A suggestion that the general formula provides an analytic continuation is embodied in the numerical evaluation

$$
\mathcal{W}(-3, -3, 1/2) \approx 0.0051112406 \ldots
$$

References.