# Fast evaluation of the Witten zeta function

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## 1. Fast expansion for certain arguments.

Define

$$\mathcal{W}(r,s,t) := \sum_{m,n\ge 1} \frac{1}{m^r}\frac{1}{n^s}\frac{1}{(m+n)^t}.$$

This explicit summation is characteristically slow to converge. A fast evaluation may be effected via a free parameter $X \in (0,1)$, and the following formula:

*When neither $r$ nor $s$ is a positive integer,*

$$\Gamma(t)\mathcal{W}(r,s,t) = \sum_{m,n\ge 1} \frac{\Gamma(t,(m+n)X)}{m^r n^s (m+n)^t}$$

$$+\sum_{u,v\ge 0} (-1)^{u+v} \frac{\zeta(r-u)\zeta(s-v)X^{u+v+t}}{u!v!(u+v+t)}$$

$$+\Gamma(1-r)\sum_{q\ge 0}(-1)^q \frac{\zeta(s-q)X^{r+q+t-1}}{q!(r+q+t-1)}$$

$$+\Gamma(1-s)\sum_{q\ge 0}(-1)^q \frac{\zeta(r-q)X^{s+q+t-1}}{q!(s+q+t-1)}$$

$$+\Gamma(1-r)\Gamma(1-s)\frac{X^{r+s+t-2}}{r+s+t-2}.$$

(When one or both of $r, s$ is an integer, a different formula with a few more terms applies.)

One observes the pole in $\mathcal{W}$ at $r+s+t=2$, with residue $\Gamma(1-r)\Gamma(1-s)/\Gamma(t)$. Also, in the limit $t \to 0$ we see that the residual term is just the first ($u = v = 0$) term of the $u, v$ summation, and so $\mathcal{W}(r,s,0) = \zeta(r)\zeta(s)$ is verified. Moreover, one may use the $X$-formula in various sanity-checking modes, as follows.

1) Varying $X$ within the interval $(0,1)$ should yield an invariant $\mathcal{W}$, as is so for any valid free-parameter expansion.

2) One may verify numerically the Zagier triangle identity

$$\mathcal{W}(r,s,t) = \mathcal{W}(r-1,s,t+1) + \mathcal{W}(r,s-1,t+1).$$

3) A typical numerical value from the $X$ formula is, for $X = 4/5$ (an efficient choice)

$$\mathcal{W}(\pi,\pi,\pi) \approx 0.121784932649073172392415831466446\ldots$$

4) A typical evaluation near the pole is, for $d := 200001/300000$,

$$\mathcal{W}(d,d,d) = 529982.9016524962105\ldots$$

## 2. General analytic expansion.

The above expansion for $\mathcal{W}$ is illegal for either $r, s$ a positive integer, because 1) The $\zeta(1)$ evaluation is illegal, and 2) the $\Gamma(1-r)$ or $\Gamma(1-s)$ is also illegal. However, the singularities do cancel, and we can write a general formula. For real number $p$, define a coefficient $A_p$ according to whether $p$ be a positive integer:

$$A_p := \Gamma(1-p); \quad p \notin Z^+,$$

$$:= \frac{(-1)^{p-1}}{\Gamma(p)} H_{p-1}; \quad p \in Z^+,$$

where $H_k = \sum_{j=1}^{k} 1/j$ is the $k$-th harmonic number, with $H_0 := 0$. Similarly, define

$$B_p := 0; \quad p \notin Z^+,$$

$$:= \frac{(-1)^p}{\Gamma(p)}; \quad p \in Z^+.$$

Then a general formula is obtained as

*For general $r, s$, whether integer or not,*

$$\Gamma(t)\mathcal{W}(r,s,t) = \sum_{m,n\ge 1} \frac{\Gamma(t,(m+n)X)}{m^r n^s (m+n)^t}$$

$$+ \sum_{u,v\ge 0}{}^{\prime} (-1)^{u+v} \frac{\zeta(r-u)\zeta(s-v)X^{u+v+t}}{u!v!(u+v+t)}$$

$$+ \sum_{q\ge 0}{}^{\prime} (-1)^q \frac{\zeta(r-q)X^{s+q+t-1}}{q!}\left(\frac{A_s + B_s \log X}{(s+q+t-1)} - \frac{B_s}{(s+q+t-1)^2}\right)$$

$$+ \sum_{q\ge 0}{}^{\prime} (-1)^q \frac{\zeta(s-q)X^{r+q+t-1}}{q!}\left(\frac{A_r + B_r \log X}{(r+q+t-1)} - \frac{B_r}{(r+q+t-1)^2}\right)$$

$$+X^{r+s+t-2}\left(\frac{(A_r + B_r\log X)(A_s + B_s \log X)}{r+s+t-2} - \frac{A_r B_s + A_s B_r + 2B_r B_s \log X}{(r+s+t-2)^2} + \frac{2B_r B_s}{(r+s+t-2)^3}\right).$$

The idea here is that the notation

$$\sum{}^{\prime}$$

means that we *avoid* any $\zeta(1)$ evaluations entirely. The extra complexity involving the $A, B$ coefficients and $\log X$ arises from said singularity avoidance.

It is not hard to see that the above formula for general $r, s$ reduces to our first formula when neither $r$ nor $s$ is a positive integer (being that all $B$ coefficients vanish).

It is believed that this general formula provides also an analytic continuation of $\mathcal{W}$, as it can converge even for $r, s, t$ triples for which the defining $\mathcal{W}$ sum does not.

A verification of the general formula obtains with $X := 4/5$, and a summation limit of 100 on every summation index, with the numerical result

$$\mathcal{W}(2,2,1) \approx 0.843825435164482457400074423599148639993\ldots,$$

which agrees with J. Borwein's formula

$$\mathcal{W}(2,2,1) = 2\zeta(2)\zeta(3) - 3\zeta(5)$$

to 40 places.

A suggestion that the general formula provides an analytic continuation is embodied in the numerical evaluation

$$\mathcal{W}(-3,-3,1/2) \approx 0.0051112406\ldots$$

**References.**

Crandall, R. E. and Buhler, J. P. 1995, "On the evaluation of Euler sums," Experimental Mathematics, 3, 4, 275-285